Universidade Estadual de Londrina

WESLEI BERNARDINO FONTANA

EFFECTIVE THEORIES FOR FRACTON PHASES

Londrina
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Tese apresentada ao Departamento de Física da Universidade Estadual de Londrina, como requisito parcial a obtenção do título de Doutor em física.

Orientador: Prof. Dr. Pedro Rogerio Sergi Gomes

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To my mom, Derli Bernardino

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"Bear in mind that the wonderful things that you learn in your schools are the work of many generations. All this is put into your hands as your inheritance in order that you may receive it, honor it, add to it, and one day faithfully hand it on to your children."

Albert Einstein.

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Resumo

O objetivo desta tese encontrar descrições efetivas de fases fractonicas. Fractons representam um novo tipo extico de matéria quântica na qual as excitações fundamentais carecem da habilidade de se mover livremente, uma característica não compartilhada por nenhuma quasi partícula previamente conhecida. Em tais sistemas, os estados ligados das excitações fundamentais podem adquirir mobilidade restrita, caracterizando o que agora é conhecido como fratons do tipo I, mas em casos particulares mesmo os estados ligados permanecem imóveis, estes são chamados de fratons do tipo II. Além disso, esses sistemas so quase topológicos, no sentido que algumas de suas propriedades possuem uma dependência topológica, mas não se "desvinculam" inteiramente da geometria. Buscamos entender esses sistemas em termos de uma teoria quântica de campos. Em princípio, isso parece uma tarefa bastante difícil, uma vez que parece impossível acomodar todas as características dos sistemas fractonicos na linguagem de uma teoria contínua; no entanto, fomos capazes de encontrar descrições efetivas que capturam as propriedades fundamentais de tais sistemas. Usamos representações das matrizes de Dirac da álgebra de Clifford para construir modelos de fractons na rede e sua teoria do tipo de Chern-Simons correspondente. Como exemplo, construímos uma generalização do modelo de Chamon em D dimensẽspaciais, (com D mpar) e sua teoria efetiva com dimensão do espao-tempo igual a (D + 1). A descrição do contínuo possui uma matriz K anti-simétrica semelhante à da construção hierarquica do efeito Hall quântico. As cargas do grupo de calibre são conservadas em sub-variedades que garantem o comportamento fractonico. A construção se estende a qualquer modelo de fractons na rede construído a partir de operadores comutantes e com produtos tensoriais de spin-1/2 como graus de liberdade nos sítios. Continuando nossos esforos, tambm construímos também uma teoria de campo efetiva para um modelo de fracton do tipo II a partir do código Haah na rede. A teoria topológica efetiva no é dada exclusivamente em termos de uma ação; deve ser complementado com uma condição que seleciona estados físicos. Sem a restrição, a ação descreve apenas um fracton do tipo I. A restrição surge de uma condição de que os operadores de cubo na rede se multiplicam a identidade e , esta, no pode ser implementada de forma consistente na teoria do contínuo a nível operatorial, mas apenas em uma forma mais fraca, em termos de elementos de matriz de estados físicos.

Palavras-chave: Fractons. Fases topológicas. Chern-Simons. Teorias Efetivas.

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Abstract

The purpose of this thesis is to find effective descriptions of fracton phases. Frac- tons represent an exotic new type of quantum matter in which its single fundamen- tal excitations lack the ability move, a feature not shared by any quasiparticle pre- viously known. In such systems, bound states of the fundamental excitations can acquire restricted mobility, characterizing what is now called fractons of type-I, but in particular cases even the bound states remain immobile, these are fractons of type- II. In addition to that, these systems are quasi-topological, meaning that some of its properties have a topological dependence but they not "decouple" entirely from the geometry. We want to understand these systems in the framework of a quantum field the- ory. In principle, this seems like a rather hard task since it seems impossible to accomodate all the features of fractonic systems in the language of a continuum the- ory, nonetheless, we are able to find effective descriptions that captures the essential physics of such systems. We use Dirac matrix representations of the Clifford algebra to build fracton mod- els on the lattice and their effective Chern-Simons-like theory. As an example, we build a generalization of the fracton Chamon model in odd D spatial dimensions and their (D + 1) spacetime dimensional effective theory. The continuum description possesses an anti-symmetric K matrix resembling that of hierarchical quantum Hall states. The gauge charges are conserved in sub-dimensional manifolds which ensures the fractonic behavior. The construction extends to any lattice fracton model built from commuting projectors and with tensor products of spin-1/2 degrees of freedom at the sites. Continuing our efforts, we also construct an effective field theory for a type-II fracton starting from the Haah code on the lattice. The effective topological theory is not given exclusively in terms of an action; it must be supplemented with a con- dition that selects physical states. Without the constraint, the action only describes a type-I fracton. The constraint emerges from a condition that cube operators multiply to the identity, and it cannot be consistently implemented in the continuum theory at the operator level, but only in a weaker form, in terms of matrix elements of physical states. Informed by these studies and starting from the opposite end, i.e., the con- tinuum, we discuss a Chern-Simons-like theory that does not need a constraint or projector, and yet has no mobile excitations. Whether this continuum theory admits a lattice counterpart remains unanswered.

Keywords: Fracton. Topological Phases. Chern-Simons. Effective Theories.

Lista de ilustrações

Sumário

APÊNDICE E – PURE TYPE-II CONTINUM FRACTON MODEL 105

1 Introduction

The field of condensed matter physics attempts to understand and classify all possible phases of matter, but at the same time we search for new phases that contradicts the contemporary paradigms. These goals are evidently contradictory, but it is in these contradictions that we can push the field forward. The most recent example of a new class of systems that displays a phase that still lacks comprehension, is what is now named by fractons (For a review, see (1)). These system possess very unusual properties, that we are unable to capture with the usual approaches used to categorize previous observed phases, such as Landau's symmetry breaking theory (2) and the classification via topological orders (3).

Fracton systems possess quasiparticles that are unable to move in response to an applied force(4, 5, 6, 7, 8). However, due to some particular details of the model in question, fractons can sometimes form a bound state which supports mobility in some direction. Another way for a individual fracton to move is at the cost of generating new fractons from the vacuum on each step of its motion. Of course, without an external source of energy to compensate for the additional excitations being generated, the isolated fracton will tend to remain immobile. Fracton phases also possess a robust topological ground state degeneracy (GSD), although, unlike the usual models with topological order, the ground state degeneracy for fractonic models depends not only on the topology of the underlying manifold but also on some geometrical aspects of the lattice (in general, the GSD scales with the linear size of the system).

These properties, suggest to us that we can split the fracton phases into two categories, defined as follows:

- *Fractons of type-I*: Fracton excitations appears on the corners of membrane operators; multiple fractons can bind together and form a composite particle which presents mobility within sub-dimensional manifolds. Examples of such are (4, 6, 9).

- *Fractons of type-II*: Fracton excitations appears on the corners of fractal objects. No mobile bound states are allowed and thus all excitations are strictly localized. Examples are (5, 10).

Furthermore, it is this inherent immobility of the isolated fractons that leads to slow dynamical behavior, which endows fractons with glassy dynamics (4, 11, 12). Moreover, the restricted mobility and slow dynamical relaxation of fractonic excitations opens a window, at the practical level, for building quantum memories (13, 14, 15).

In addition, fractonic phases have connections with elasticity theory (16, 17) and gravitational physics (18).

It is agreed nowadays that the first model presenting fractonic behavior was due to the studies in spin models with glassy dynamics due to Chamon (4). In the modern language, the Chamon model is an example of a fracton of type-I, where two fractons can bind together to form dipoles, and these dipoles can move along certain directions. The first example of a fracton of type-II was built with the goal of creating a self correcting quantum memory. This model is currently known as the Haah code (5). The Haah code possess a fractal structure and the excitations are located in the corners of these fractal objects.

The importance of such models was not immediately clear, and it was not until the work of Vijay, Haah and Fu (6, 7) that was noted that these models belonged to a much larger class of fractonic systems and, therefore, fundamentally representing a new phase of matter. Much of the progress in fractonic systems was made by studying exactly solvable spin lattice models, as the ones already mentioned, since their properties appears unnatural for continuums descriptions, for example:

- Gapped fracton phases possess a ground state degeneracy that depends on the linear size of the system[1]. In terms of the entropy, this essentially means that $S = \log(GSD) \sim L$.

The fact that the entropy depends on the perimeter instead of the volume is already surprising enough, suggesting that fractons have some sort of unconventional holography description (19, 20).[2] But also, any continuum description of a fracton phase must have an infinite number of ground states, since $L \to \infty$ in the continuum. This dependency on lattice parameters also signals an infrared/ultra-violet (UV/IR) mixing, i.e., fracton phases do not exhibit the usual decoupling between high-energy and low-energy physics.

- Fracton phases exhibit exotic global symmetries.

Some of these exotic symmetries are known to be subsystem symmetries (21). A subsystem symmetry consists of a set of independent symmetry operations over a subspace of dimension $d < D$ of the total space, with D being the dimension of the total space. For example, a three dimensional system $D = 3$ may support symmetries acting along planes $d = 2$ or lines $d = 1$ in addition to a local symmetry $d = 0$ (gauge symmetry). Such symmetries are sometimes referred to as *gauge-like symmetries* and can lead to additional charge conservations over the set of subdimensional spaces.

[1] In some cases, like in the Haah code, the degeneracy might have a more complicated form.

[2] By unconventional holography we simply mean that it differs from the known idea of holographic principle in which the entropy scales with the area of the system instead of the volume.

All these characteristics make continuum descriptions of fracton systems very unlikely, but despite that, we will try to find a framework in which is possible to find such continuum descriptions. Some progress have already been made in this direction. Slagle and Kim (22) proposed a theory that captures the low-energy properties of the X-cube model (7, 23), where they explicitly start from the microscopic description of the X-cube model and obtain a gauge invariant theory in the continuum. Much of the work in this thesis draws inspiration from (22).

The motivation for finding these effetive descriptions for fracton models (despite the fact that they represent a new challenge) comes from a simplicity argument. The effective theories enable much further progress (22, 24, 25, 26, 27, 28, 29), since many aspects of fracton phases are captured in a somewhat simpler way. The restricted mobility, for example, is encoded as conservation laws along sub-dimensional manifolds of the total space, in addition to the global conserved charge along the entire volume of the system. The conservation of charge in sub-manifolds can be traced to conservation of higher moment multipoles (dipoles, quadrupoles and so on), which is equivalent to the conservation of vector charges. This is a feature of the higher rank gauge theories approches (8, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42), which in general, are gapless models. Gapped fracton models can also be obtained from these higher rank gauge theories via the Higgs mechanism (43, 44). In fact, we are not restricted only to the Higgs mechanism. Gapped 3D fractons can also be obtained by stacking layers (37, 45, 46, 47, 48, 49, 50) or glueing (51, 52, 53) known (2+1)-dimensional topological models.

In a different route, You *et al.* (25) presented a Chern-Simons-like theory with vector gauge fields that possess sub-dimensional conservation laws, hence conserving some higher order multipole (in the particular case of (25), conservation of dipoles). This Chern-Simons-like theory obtained in (25) is gapped and, when discretized to a lattice, it coincides with the Chamon fracton model introduced in (4). The connection of fracton models and Chern-Simons theories is appealing, since it suggests that one can generalize these descriptions in order to describe classes of gapped fractons, in a similar fashion as classes of quantum Hall states can be described by Chern-Simons theories (54).

The approach of Chern-Simons-like theories motivated us to build families of such theories in order to describe a higher dimensional version of the Chamon model (55). These families of Chern-Simons-like theories are described by an anti-symmetric K-matrix and associated charge vectors. The effective theories are obtained from a microscopic model, where n spin $1/2$ degrees of freedom (qubits) are placed at the sites of the lattice. The qubits are represented in the Dirac basis of the Clifford algebra. The Dirac representation with 2^n - dimensional matrices is a natural choice for the lattice models. The fracton nature of these models can be seen as a simple

consequences of the lattice connectivity and the Clifford algebra. Also, the effective theory can be obtained in a simple way by means of a bosonization-like procedure. This procedure is discussed extensively along this thesis, in special in Chapter. 3.

Although the procedure we introduced in (55) aimed at describing the type-I fracton phase given by the Chamon model, our construction is generic enough and it can be used to obtain the effective theories associated with other fracton models. The only requirement for the validity of our procedure is that the microscopic Hamiltonian is a sum of commuting projectors[3] built from tensor products of spin 1/2 operators. With such a generic start, this bosonization-like procedure can be used to analyze many others lattice models, such as those with sub-system protected topological (SSPT) phases (21, 48) and type-II fracton phases (5), with the latter being the more natural route for our efforts.

For the case of type-II fracton phases some extra care is needed in order to obtain a consistent effective theory. One of the main features of such phases, as already mentioned, is that all excitations are strictly localized and are completely immobile, few attempts have been made in order to write down an effective theory for the Haah code (24, 29), but they seem unable to capture the immobility of the excitations. Both of these models support the existence of a gauge invariant Wilson line that can be interpreted as the trajectory of an excitation along the direction where the line extends. In fact, with our approach (56), the immobility of the excitations is not immediate, an extra ingredient is needed. This extra information comes as a selection criteria that distinguishes between mobile and immobile quasiparticles. This allow us to write down a consistent continuum model for the type-II phase and also gives us a hint that (from the continuum theory perspective) type-II fractons are embedded into type-I fracton phases. The detailed discussion of this construction is postponed until Chapter. 4. In Chapter 2, we give a brief introduction to what is topological order and discuss the example of Kitaev toric code. In Chapters 3 and 4 is where lies the novelty of this thesis. The discussions in there are mainly adapted from (55, 56). In chapter 3, we discuss the lattice description of the Chamon model and propose an effective theory in terms of a Chern-Simons-like action. In Chapter 4, we analyze the type-II fracton model, Haah code, and show that it can also be described by a Chern-Simons-like theory. We close the thesis at Chapter 5 with some final remarks and open questions for future investigation.

[3] Saying that the Hamiltonian is built from commuting projectors is equivalent to say that all terms in the Hamiltonian commute among themselves.

2 Topological systems

In this chapter we briefly discuss the notion of topological order and focus on a pratical example: Kitaev's toric code. One of the simplest models that enlightens the properties of topological ordering. We also use the prescription proposed in (55) to find the effective theory of the toric code. This will serve as a warm up for the more complicated cases in the following chapters.

2.1 Topological order

Topological order emerged in trying to understand the fractional quantum Hall systems (FQH) (57, 58). The necessity of a new framework to distinguish Hall phases was because the traditional many body theory could not be applied really well. Due to the strong interactions and correlations that are present in quantum Hall phases, Landau's fermi liquid was not enough, and most strikingly, FQH systems contains several different phases at zero temperature with the same symmetries, which makes impossible to use symmetry arguments to distinguish among these phases, therefore, Landau's symmetry breaking theory was not a good fit.

A new framework was needed to distinguish between all this new phenomena that was emerging, and for that, many new concepts were introduced, quantum/topological order was one of them. The idea is: since FQH states can not be described using symmetry arguments, some new kind of order must be playing a important role, the notion of topological order was then introduced (54, 59). None of the usual concepts as symmetry breaking, long range order or local order parameters applies to topological order, instead, they are characterized by quantities such as; the number of degenerate ground states (60), the statistical properties of quasi particles (61) and the properties of the edge states(62, 63).

Another important aspect of topological order is the notion of entanglement entropy (64, 65) the topological entanglement entropy is a negative constant correction to the area law (the Von Neumann entropy is bounded from above to be proportional to the area of the boundary in which the system exists). For example, in two dimensional systems the entropy $S(\rho) = aL - \gamma$, with a a constant and L the perimeter of the boundary, γ is the entanglement entropy, which is insensitive to microscopic details and remains constant under deformations of the Hamiltonian, as long as the deformations does not close the energy gap.

The topological ground state degeneracy is also robust against any perturbations (66). Therefore, the GSD of a topologically ordered system is a universal property that one can use to distinguish between different phases. Topological orders are better

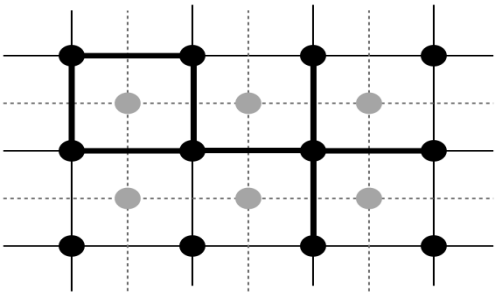


Figura 1 – The star and plaquette operators represented on a square lattice by the highlighted links.

understood with some explicitly examples, in the following we discuss the toric code that can be characterized using the tools we just discussed.

2.2 Toric code: Lattice description

In this section we introduce the toric code model, first proposed by Kitaev in his seminal paper (67). The toric code was first proposed as a stabilizer quantum code for quantum computation purposes (67, 68). The model is built on a lattice with periodic boundary conditions, forming a 2-Torus, on each link of the lattice we place a qubit, the stabilizer operators of the model are related to plaquettes and stars on the lattice. These stabilizer operators can be put together in such a way to form a Hamiltonian with local interactions. In the following we shall make a brief presentation of the model and later present a connection with a continuum field theory, more precisely a BF theory.
To start, we introduce the Hamiltonian

$$H_{TC} = -J_e \sum_s A_s - J_m \sum_p B_p \,, \tag{2.1}$$

where the operators A_s and B_p are related to vertices and plaquettes on the lattice. These operators are defined as

$$A_s = \prod_j \sigma_j^x, \quad B_p = \prod_j \sigma_j^z, \tag{2.2}$$

as one can see, as well, in Fig.1. The $\sigma_j^{x,y,z}$ operators live on the links of the lattice and acts non trivially only in the Hilbert space defined at the position j, i.e.,

$$\sigma_j^{x,y,z} = \left(\prod_{i \neq j} \otimes_i \mathbb{1} \right) \otimes \sigma_{i=j}^{x,y,z}. \tag{2.3}$$

Note that the stars and plaquettes commute, i.e., $[A_s, B_p] = 0$, which implies that they can be diagonalized simultaneously, and the Hamiltonian can be solved exactly. The

commutation follows from Fig.1, by noticing that each star and plaquette will always share an even number of links (either 0 or 2). Also, from the Pauli matrices structure, it is trivial to note that A_s and B_p in (2.2) have the nice property

$$A_s^2 = B_p^2 = \prod_{i \in S} \otimes_i \mathbb{1}_2. \tag{2.4}$$

Where S represents the set that contains all the points in the lattice. This tell us that each star and plaquette can be labeled by two eigenvalues, ± 1. Then, the Hamiltonian (2.1) is minimized by the condition

$$A_s \left|\xi_0\right\rangle = \left|\xi_0\right\rangle, \quad B_p \left|\xi_0\right\rangle = \left|\xi_0\right\rangle, \quad \forall v, p. \tag{2.5}$$

The conditions in (2.5) defines a subspace $\mathcal{G}$ of ground states. The energy of the ground states can be found by acting with the Hamiltonian,

$$H \left|\xi_0\right\rangle = -(n_v + n_p) \left|\xi_0\right\rangle, \tag{2.6}$$

where n_v and n_p accounts for the number of stars and plaquettes that picks the $+1$ eigenvalue. To make further progress we choose to work on a spin basis, which is to say, define spin variables $s_i = \pm 1$ that labels each configuration of the system. For each configuration $\{s\}$ assign a flux, given by

$$\omega_p(s) = \prod_{j \in p} s_j, \tag{2.7}$$

each configuration with $\omega_p = -1$ at some plaquete p will be called a *vortex* at p. The fluxes (2.7) will be used to label the degenerate ground states of the Hamiltonian (2.1). The explicit ground states can be found as follows. Since we are working on a spin basis, the ground state wavefunction can be decomposed into a linear combination on this basis, i.e.,

$$\left|\psi_0\right\rangle = \sum_s c_s \left|s\right\rangle. \tag{2.8}$$

The plaquette condition $B_p \left|\psi_0\right\rangle = \left|\psi_0\right\rangle$ give us

$$\sum_s \omega_p(s) c_s \left|s\right\rangle = \sum_s c_s \left|s\right\rangle, \tag{2.9}$$

which forces us to restrict the sum to configurations with $\omega_p(s) = +1$, which amounts to say that the ground state contains no vortices. The star operators acts on this basis by flipping the spins, then the star condition $A \left|\psi_0\right\rangle = \left|\psi_0\right\rangle$ holds only if the constants c_s of the linear combination (2.8) are equal on each of the orbits[1] of the star operators. More important to us now, is that the star operators preserves the Wilson loops on the torus. Define a loop as

$$\omega_l(s) = \prod_{j \in L} s_j, \quad L = l_1, l_2. \tag{2.10}$$

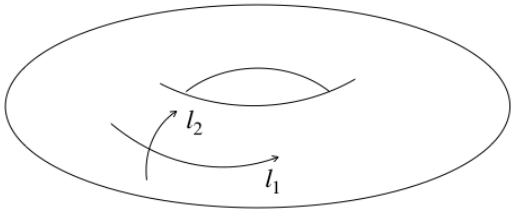


Figura 2 – The two non trivial cycles on the 2-Torus.

Where $L = l_1, l_2$ represents the two cycles around the 2-Torus, as shown in Fig.2. Each star will share either 0 or 2 links with the cycles l_1, l_2, therefore, the Wilson loops defined in (2.10) are preserved under the action of the star operators. The lesson from all of this is that the Wilson loops (2.10) give us good labels to characterize the ground state for the toric code model. Then, we can consider the following state

$$|\Psi_0\rangle = \sum_{s:\omega_p(s)=+1} c_{\omega_{l_1}\omega_{l_2}} |s\rangle . \tag{2.11}$$

The state $|\Psi\rangle$ is the ground state of the toric code model. From (2.10) we see that $\omega_l = \pm 1$, and thus the ground state $|\Psi\rangle$ have a four fold degeneracy. As we mentioned in the appendix (A.1.1), this is a topological degeneracy.
Another way of obtaining that the ground state of the theory is 4-fold degenerate is to notice that the stars and plaquette operators obeys the relations

$$\prod_s A_s = 1, \quad \prod_p B_p = 1. \tag{2.12}$$

This imposes two constraints on the model. Thus, on a lattice with n qubits, there are only $n-2$ independent stabilizers A_s and B_p, therefore, the dimensionality of the ground state Hilbert space defined by the conditions (2.5) is $\dim(\mathcal{H}_{gs}) = 2^{n-(n-2)} = 4$. Once again, we see that the ground state is 4-fold degenerate. This degeneracy is also robust, as discussed in the appendix (A.1.1), and therefore, can not be lifted by any local perturbations.

The excitations of the model come in two varieties, eletric and magnetic excitations. To find the electric charge, we introduce the eletric path operator as follows

$$W_l^{(e)} = \prod_{j \in l} \sigma_j^z, \tag{2.13}$$

where l is a path on the lattice connecting two sites s_1 and s_2. This operator clearly commutes with every plaquette B_p on the lattice and also with every star A_s except those stars which have a link connected to one of the end points s_1, s_2. In this scenario,

[1] The orbit $orb(x)$ of an element $x \in X$ is defined as $orb(x) = G * x$, with G being a group that acts on the set X. More precisely, $G(x) = \{gx \in X : g \in G\}$.

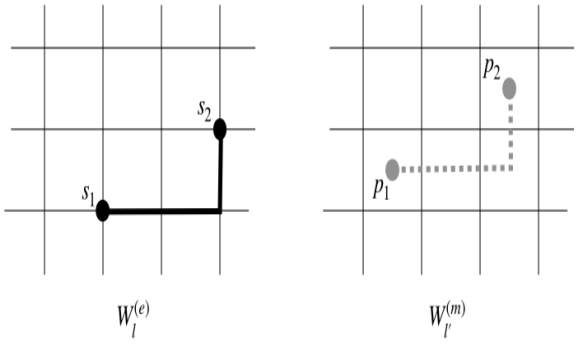


Figura 3 – Electric and magnetic excitations of the toric code.

the operator $W_l^{(e)}$ will anti commute with that specific star, i.e.,

$$W_l^{(e)} A_{s_i} = -A_{s_i} W_l^{(e)}, \quad s_i = \{s_1, s_2\} \tag{2.14}$$

Thus, if we consider the state

$$|\Psi_{s_1,s_2}\rangle = W_l^{(e)} |\Psi_0\rangle . \tag{2.15}$$

Then, the state $|\Psi_{s_1,s_2}\rangle$ will be an eigenstate of the Hamiltonian (2.1), with electric charges located at s_1 and s_2 each of which costs an amount of energy equals to $2J_e$ relatively to the ground state. In a similar reasoning, we can build an magnetic path operator as

$$W_l^{(m)} = \prod_{j \in l'} \sigma_j^x, \tag{2.16}$$

where l' is a path on the dual lattice connecting dual sites p_1 and p_2. In a similar fashion, the operators $W_{l'}^{(m)}$ will commute with every star A_s on the lattice and with every plaquette B_p except with the plaquettes which are centered at the positions p_1 and p_2, thus

$$W_{l'}^{(m)} B_{p_i} = -B_{p_i} W_{l'}^{(m)}, \quad p_i = \{p_1, p_2\} . \tag{2.17}$$

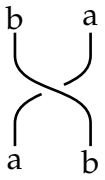
Again, we can build a state

$$|\Psi_{p_1,p_2}\rangle = W_{l'}^{(m)} |\Psi_0\rangle , \tag{2.18}$$

which is also an eigenstate of the Hamiltonian (2.1) with magnetic charges located at p_1 and p_2 each of which costs an amount of energy equals to $2J_m$ relatively to the ground state. These excitations are depicted in Fig.3.

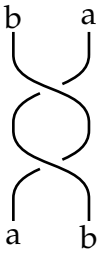
To close this section, we discuss about the braiding and fusion properties of the toric code. In order to discuss the braiding of excitations in the toric code, we introduce the braiding operator, R_{ab}, this operator acts on a pair of excitations by

exchanging their positions, pictorially, we can represent it as

$R_{ab} =$ 

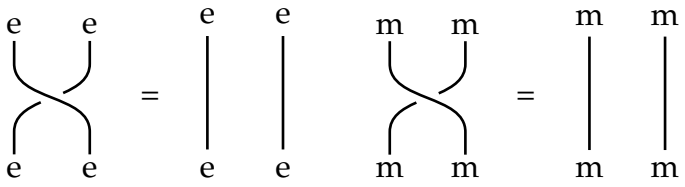
, (2.19)

if the two excitations can be distinguished, R_{ab} does not have a invariant meaning. In the case of distinguishable particles, we need to use the *mutual statistics*

$R_{ba} \cdot R_{ab} =$ 

, (2.20)

with these notions at hand we can examine the braiding properties of the toric code. Since the eletric and magnetic path operators commute among themselves, we can immediatelly read the bosonic self statistics equations.



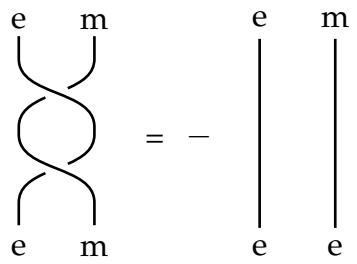
(2.21)

To compute the mutual statistics of the different species of particles, we need to analyze the following: take a state $|\xi\rangle$ that contains a magnetic vortice at a position p_1, now using the electric path operator take a electric charge e and move it around the position p_1 where the vortice is defined, the resulting state will be

$$|\xi\rangle = \left(\prod_{j\in l} \sigma_j^z\right) |\xi\rangle = \left(\prod_{p \text{ inside } l} B_p\right) |\xi\rangle \ , \tag{2.22}$$

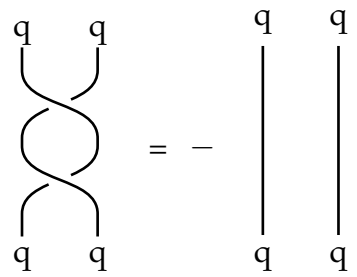
where we have used an analog of the stokes theorem to write the second equality. This allow us to, instead of computing the product of operators around the loop we can compute the product of the internal plaquettes. Since we assume that there is a single magnetic vortice inside the loop l and we know that $B_{p_1} |\xi\rangle = -|\xi\rangle$, the net result of taking the electric excitation around the vortice is that $|\xi\rangle \to -|\xi\rangle$, which we

can illustrate as



(2.23)

Using these braiding properties, we can show that the toric code also exhibit another kind of excitation, a boud state $q = e \times m$, which behaves as a fermion. This can be represented using our braiding diagrams as



(2.24)

To end this section we discuss the fusion properties of the toric code. To do that we first have to be more precise with what mean by a "particle". We take the definition given in (68), each type of particle corresponds to a super selection sector, which is a representation of the local operator algebra. Two particles are of the same type, $a \sim b$, if there is some local operator that acts on a finite region and can trasform a into b.

As an example, consider the case of two particles of the type e (electric charges). We can always find a electric path operator, $W_l^{(e)}$ that acts on this configuration and leave us with no charges, i.e., two particles of the type e are equivalent to the vacuum. Thus, we say that two particles of type e can fuse together in order to leave us with the vacuum, this is represented by

$$e \times e = 1\,, \tag{2.25}$$

the 1 notation represents the vacuum sector. In a similar fashion, we can write down all the fusion rules for the toric code. We are left with

$$\begin{aligned} e \times e = 1\,, \quad e \times m = q\,, \\ m \times m = 1\,, \quad e \times q = m\,, \\ q \times q = 1\,, \quad m \times q = e\,. \end{aligned}$$

2.3 Continuum theory for the Toric code

We can sistematically obtain the effective theory for the Toric code using the map introduced in (55). The map consists of assigning to each lattice operator $\Gamma^{(I,\alpha)}$ a combination of vector gauge fields A as

$$\Gamma^{(I,\alpha)}_{\mathbf{x}} = \exp\left(i\, T^{(I,\alpha)}_a\, K_{ab}\, A_b(\mathbf{x})\right) , \tag{2.26}$$

where the matrix indices $a, b = 1, 2$, the label $\alpha = 1, 2$ indicates the different local operators of the model (for the toric code, the star and plaquette operators) and the I indice is related to the Pauli operators in the lattice, this connection will be clarified soon.

The star and plaquette operators of the toric code are defined in terms of σ^x and σ^z. To make the connection with the Dirac operators $\Gamma^{(I,\alpha)}$ we consider the matrices $\gamma^I \in S^{(1)} = \{\sigma^x, \sigma^z, \sigma^y\}$. The matrices $\gamma^I \in S^{(1)}$ can be represented in an exponential form as in (2.26) but with a different set of T-vectors. We associate each γ^I with a vector $t^{(I)}_a$ belonging to *the principal basis*

$$\gamma^I_{\mathbf{x}} = \exp\left(i\, t^{(I)}_a\, K_{ab}\, A_b(\mathbf{x})\right) . \tag{2.27}$$

The *principal basis* is defined as $t^{(I)}_a = \delta^I_a$ for $I = 1, 2$ and obeys $\sum_I t^{(I)}_a = 0$. This last condition follows from the fact that the elements in $S^{(1)}$ multiply to the identity ($\gamma^1\gamma^2\gamma^3 = 1$) and it determines the last vector $t^{(3)}_a$ in terms of the others.

We now wish to proceed and construct the effective theory for the toric code. In order to do that we first note that the map (2.26) furnishes a field representation of a lattice variable at a specific position $\mathbf{x}$, which is not the case for the operators in the toric code. The $T^{(I,\alpha)}$ introduced in (2.26) are linear combinations of the *principal basis*, in the particular case of the toric code, the two sets of $T^{(I,\alpha)}$ with $\alpha = 1, 2$ will be two copies of the principal basis. According to (2.2), the star and plaquette operator are defined on the links of the lattice, which essentially means that each spin operator of the model depends on two points (the end points of each link), in order to capture this feature we redefine the map (2.26) slightly as

$$\Gamma^{(I,\alpha)}_{\mathbf{x}_i\mathbf{x}_j} = \exp\left(i\, T^{(I,\alpha)}_a\, K_{ab} \int_{\mathcal{C}} dx^{(i)}\, A^{(i)}_a(\mathbf{x})\right) , \tag{2.28}$$

where $\mathcal{C}$ represents the link with endpoints x_i and x_j. In (2.28) we promoted the local field $A(\mathbf{x})$ to an extended object $\int_{\mathcal{C}} dx^{(i)} A^{(i)}_a$ such as the ones we have in the lattice theory. From (2.28) we can immediately read the operators

$$\Gamma^{(1,1)}_{x_i\, x_j} = \exp\left(i \int_{\mathcal{C}} dx^{(i)} A^{(i)}_2\right) , \quad \Gamma^{(2,2)}_{x_i\, x_j} = \exp\left(-i \int_{\mathcal{C}} dx^{(i)} A^{(i)}_1\right) , \tag{2.29}$$

where $\Gamma^{(1,1)}$ is the representation of the σ^x Pauli matrix in the star operator and $\Gamma^{(2,2)}$ is the representation for the σ^z Pauli matrix in the plaquette operator.

Explicitly, we can consider the star A_s and plaquette B_p operators defined as

$$\begin{aligned} A_s &= \Gamma^{(1,1)}_{\mathbf{x},\mathbf{x}+\hat{x}}\,\Gamma^{(1,1)}_{\mathbf{x}+\hat{x},\mathbf{x}+\hat{x}+\hat{y}}\Gamma^{(1,1)}_{\mathbf{x}+\hat{x}+\hat{y},\mathbf{x}+\hat{y}}\Gamma^{(1,1)}_{\mathbf{x}+\hat{y},\mathbf{x}}\,, \\ B_p &= \Gamma^{(2,2)}_{\mathbf{x},\mathbf{x}+\hat{x}}\,\Gamma^{(2,2)}_{\mathbf{x}+\hat{x},\mathbf{x}+\hat{x}+\hat{y}}\Gamma^{(2,2)}_{\mathbf{x}+\hat{x}+\hat{y},\mathbf{x}+\hat{y}}\Gamma^{(2,2)}_{\mathbf{x}+\hat{y},\mathbf{x}}\,. \end{aligned} \tag{2.30}$$

We start by writing down the continuum representation of the star operator, the same line of reasoning applies for the plaquette one. With (2.29) we can write A_s as

$$A_s = \exp\left[i\left(\int_{\mathbf{x}}^{\mathbf{x}+\hat{x}} dx A_2^{(1)} + \int_{\mathbf{x}+\hat{x}}^{\mathbf{x}+\hat{x}+\hat{y}} dy A_2^{(2)} - \int_{\mathbf{x}+\hat{x}+\hat{y}}^{\mathbf{x}+\hat{y}} dx A_2^{(1)} - \int_{\mathbf{x}+\hat{y}}^{\mathbf{x}} dy A_2^{(2)}\right)\right], \tag{2.31}$$

we stress that the integration over the star operator has to be thought as a loop in the dual lattice. The minus signs appearing in the expression above comes from the change in the orientation as we run along the loop. The lenght of each link is exactly the lattice spacing ℓ. Using a mid-point prescription to approximate the integrals as

$$\int_{x_i}^{x_j} dx\, A(x) = A(x^*)\ell\,, \tag{2.32}$$

where x_i and x_j are the endpoints of the link and x^* is the midpoint between x_i and x_j. In what follows we will set the lattice spacing to one, i.e., $\ell = 1$. Under this approximation, (2.31) can be written as

$$\begin{aligned} A_s &= \exp\left[i\left(A_2^{(1)}\left(\mathbf{x}+\frac{\hat{x}}{2}\right) + A_2^{(2)}\left(\mathbf{x}+\hat{x}+\frac{\hat{y}}{2}\right)\right) - A_2^{(1)}\left(\mathbf{x}+\hat{y}+\frac{\hat{x}}{2}\right) - A_2^{(2)}\left(\mathbf{x}+\frac{\hat{y}}{2}\right)\right], \\ &= \exp\left[i\left(\partial_x A_2^{(2)} - \partial_y A_2^{(1)} + \dots\right)\right]. \end{aligned} \tag{2.33}$$

With a similar reasoning we can obtain the plaquette operator as

$$B_p = \exp\left[-i\left(\partial_x A_1^{(2)} - \partial_y A_1^{(1)} + \dots\right)\right]. \tag{2.34}$$

These can be written in a more compact form as

$$\begin{aligned} A_s &= \exp\left(i\,\epsilon^{ij}\,D_a^{(i,1)}\,K_{ab}\,A_b^{(j)}\right), \\ B_p &= \exp\left(i\,\epsilon^{ij}D_a^{(i,2)}\,K_{ab}\,A_b^{(j)}\right), \end{aligned} \tag{2.35}$$

where $D_a^{i,1} = T_a^{(1,1)}\partial_i$ and $D_a^{i,2} = T_a^{(2,2)}\partial_i$ and ϵ^{ij} is the two-dimensional Levi-Civita symbol. The commutation relations of the lattice operators (2.28) imposes the following commutation relation among the A-fields

$$\left[A_a^{(i)}(\mathbf{x}),\, A_b^{(j)}(\mathbf{x}')\right] = i\pi\left(K^{-1}\right)_{ab}\epsilon^{ij}\delta\left(\mathbf{x}-\mathbf{x}'\right). \tag{2.36}$$

Explicitly, the non-trivial commutations are

$$\left[A_2^{(2)}(\mathbf{x}),\, A_1^{(1)}(\mathbf{x}')\right] = -i\pi\delta\left(\mathbf{x}-\mathbf{x}'\right), \quad \left[A_1^{(2)}(\mathbf{x}),\, A_2^{(1)}(\mathbf{x}')\right] = i\pi\delta\left(\mathbf{x}-\mathbf{x}'\right). \tag{2.37}$$

The Hamiltonian (2.1) can thus be written as

$$H_{TC} = -J_e \int d^2x \, \cos\left(\epsilon^{ij} D_a^{(i,1)} K_{ab} A_b^{(j)}\right) - J_m \int d^2x \, \cos\left(\epsilon^{ij} D_a^{(i,2)} K_{ab} A_b^{(j)}\right) . \quad (2.38)$$

The ground state of the theory corresponds to the case in which all the cosines are maximized, this is captured as constraints in the following Lagrangian

$$L = \frac{1}{\pi}\left[-A_1^{(1)}\partial_0 A_2^{(2)} + A_1^{(2)}\partial_0 A_2^{(1)} + A_0^{(1)}\left(\partial_x A_2^{(2)} - \partial_y A_2^{(1)}\right) + A_0^{(2)}\left(\partial_x A_1^{(2)} - \partial_y A_1^{(1)}\right)\right] , \quad (2.39)$$

this Lagrangian corresponds to a BF theory. We can put this into a more familiar form by naming the field components as

$$A_a^{(i)} = \begin{pmatrix} A_1^{(1)} & A_1^{(2)} \\ A_2^{(1)} & A_2^{(2)} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} , \quad A_0^{(\alpha)} = \begin{pmatrix} A_0^{(1)} \\ A_0^{(2)} \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} . \quad (2.40)$$

In terms of these variables, the Lagrangian assumes the familiar form

$$L = \frac{1}{\pi}\left[-a_1\partial_0 b_2 + a_2\partial_0 b_1 + a_0\left(\partial_x b_2 - \partial_y b_1\right) + b_0\left(\partial_x a_2 - \partial_y a_1\right)\right] ,$$

$$L = \frac{1}{\pi}\epsilon^{\mu\nu\rho} a_\mu \partial_\nu b_\rho , \quad \mu, \nu, \rho = 0, 1, 2 . \quad (2.41)$$

As usual, the theory is invariant under the set of $U(1)$ gauge transformations

$$\begin{pmatrix} \delta a_1 \\ \delta a_2 \end{pmatrix} = \begin{pmatrix} \partial_x \zeta^{(1)} \\ \partial_y \zeta^{(2)} \end{pmatrix} , \quad \begin{pmatrix} \delta b_1 \\ \delta b_2 \end{pmatrix} = \begin{pmatrix} \partial_x \zeta^{(1)} \\ \partial_y \zeta^{(2)} \end{pmatrix} , \quad (2.42)$$

$$\delta a_0 = \partial_0 \zeta^{(1)} , \quad \delta b_0 = \partial_0 \zeta^{(2)} . \quad (2.43)$$

In order to examine the ground state degeneracy of the theory (2.41) we consider the solutions for (a_i, b_i) of the constraints that describes the ground state

$$a_i(\mathbf{x}, t) = \Phi_i^{(1)}(t) + \partial_i \phi^{(1)}(\mathbf{x}) , \quad (2.44)$$

$$b_i(\mathbf{x}, t) = \Phi_i^{(2)}(t) + \partial_i \phi^{(2)}(\mathbf{x}) . \quad (2.45)$$

Plugging these solutions into the Lagrangian (2.41) allow us to write down the following

$$L = -\frac{1}{\pi}\epsilon^{ij}\Phi_i^{(1)}(t)\partial_0 \Phi_j^{(2)}(t) , \quad (2.46)$$

The fields $\Phi^{(1)}$ and $\Phi^{(2)}$ obeys the commutation rules

$$\left[\Phi_2^{(2)}, \Phi_1^{(1)}\right] = -i\pi , \quad \left[\Phi_1^{(2)}, \Phi_2^{(1)}\right] = i\pi . \quad (2.47)$$

We can define the line operators

$$W_1 = e^{i\Phi_1^{(1)}} , \quad W_2 = e^{i\Phi_2^{(2)}} , \qquad \tilde{W}_1 = e^{i\Phi_1^{(2)}} , \quad \tilde{W}_2 = e^{i\Phi_2^{(1)}} , \quad (2.48)$$

which anti commutes due to the commutation rules in (2.47), i.e.,

$$W_1 W_2 = -W_2 W_1 , \qquad \tilde{W}_1 \tilde{W}_2 = -\tilde{W}_2 \tilde{W}_1 . \tag{2.49}$$

The ground state degeneracy corresponds to the size of the representation given by $(W, \tilde{W})$. Clearly, since they anti commute, each pair corresponds to a representation that is 2-fold degenerate, therefore, the GSD associated with the theory (2.46) is $GSD = 4$, matching the result obtained for the microscopic model.

3 Chamon Model

The Chamon model (4, 9) is considered to be the first fracton model introduced in the literature. It was build as a generalization of the Wen's model (69) in an attempt to study quantum glassiness in strongly correlated systems. In this chapter we discuss its lattice version and later we find an effective description for it.

3.1 Lattice description

The spin model can be built as follows, consider a lattice Λ with periodic conditions and with linear sizes L_x, L_y, L_z. Sites on this lattice are labeled as $u = (ijk) \in \Lambda$, a site is said even (odd) if $i + j + k =$ even (odd). It is important to note that the parity of a given site is only defined for periodic boundary conditions if all the lattice sizes are even, otherwise one could have a transformation $i \to i + L_x$ that changes the parity of the site $u = (ijk)$. With this parity notion, we can break our original lattice into two sublattices $\Lambda = \Lambda_{even} \oplus \Lambda_{odd}$, each of the sublattices can be identified with a FCC lattice. Now, we place a qubit at every even site of the lattice, thus having $n = \frac{1}{2} L_x L_y L_z$ total qubits. The qubits can be represented using Pauli operators, $X = \sigma^x$, $Y = \sigma^y$, $Z = \sigma^z$, and they act on sites $u \in \Lambda_{even}$.

Let $\hat{x}_i$, $i = 1, 2, 3$ be the canonical basis vectors for $\mathbb{R}^3$. Then, introduce a stabilizer operator for every odd site of the lattice

$$\mathcal{O}_{\mathbf{x}} = X_{\mathbf{x}+\hat{x}} X_{\mathbf{x}-\hat{x}} Y_{\mathbf{x}+\hat{y}} Y_{\mathbf{x}-\hat{y}} Z_{\mathbf{x}+\hat{z}} Z_{\mathbf{x}-\hat{z}} \,. \tag{3.1}$$

It is easy to check that these operators commute among themselves,

$$\mathcal{O}_{\mathbf{x}} \mathcal{O}_{\mathbf{x}'} = \mathcal{O}_{\mathbf{x}'} \mathcal{O}_{\mathbf{x}} \,, \quad \forall\, \mathbf{x}, \mathbf{x}' \in \Lambda_{odd} \,. \tag{3.2}$$

This happens because each of these operators can only overlap along a line that shares the same qubit, or along two lines on which two minus signs are acquired and the net result is that they also commute.

The Hamiltonian of the model is written in terms of the commuting stabilizers operators

$$H = - \sum_{\mathbf{x} \in \Lambda_{odd}} \mathcal{O}_{\mathbf{x}} \,, \tag{3.3}$$

The ground state subspace, $\mathcal{G}$ is defined as follows,

$$\mathcal{G} = \{ |\psi\rangle : \mathcal{O}_{\mathbf{x}} |\psi\rangle = |\psi\rangle \} \,, \quad \forall\, \mathbf{x} \in \Lambda_{odd} \,. \tag{3.4}$$

Since each state in $|\psi\rangle \in \mathcal{G}$ minimizes the Hamiltonian (3.3), we can conclude that it is indeed the ground state subspace of H. The dimension $dim(\mathcal{G})$ give us the

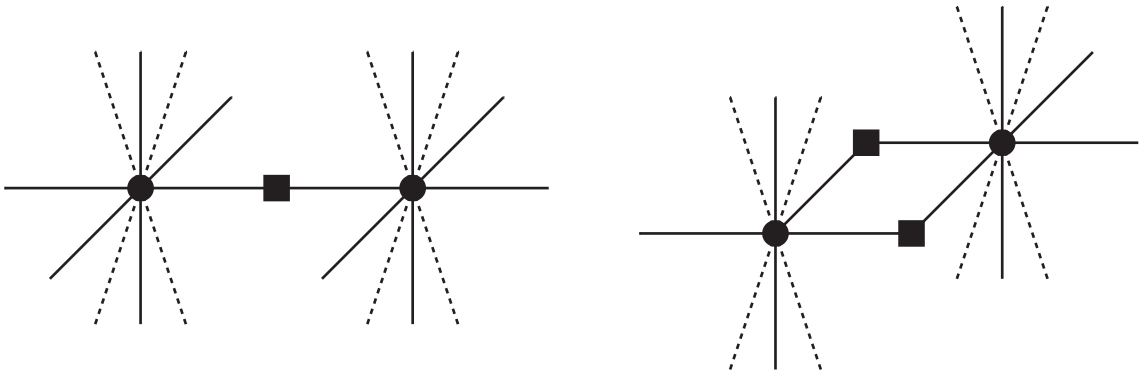


Figura 4 – The stabilizers operators can share only one or two legs, in either case, the net result is that they commute. This is valid in any number of odd spatial dimensions.

ground state degeneracy (GSD) of the model, and one can show that the GSD for the Chamon model depends not only on the topology of the underlying manifold (in the present case a 3-torus), but also depends on geometrical aspects (lattice discretization), in general, the GSD can be written as

$$dim(\mathcal{G}) = GSD = 2^{4g}\,, \quad g = gcd(p_x,\, p_y,\, p_z)\,, \tag{3.5}$$

where the numbers p_x, p_y, p_z are related to the way we parametrize our lattice, in the present case they relate with the spatial lenghts L_x, L_y, L_z as

$$L_x = 2p_x\,, \quad L_y = 2p_y\,, \quad L_z = 2p_z\,. \tag{3.6}$$

In this sense, the Chamon model is not a topological theory in the standard meaning of the terminology, since its properties does not depend entirely in the topological aspects of the underlying manifold but also on some geometrical aspects of the lattice. In the modern days it is understood that the Chamon model is an example of a Fracton model (55, 25), it has some properties that are often seen in topological systems, such as having a GSD that is exponentially insensitive against local perturbations while the spectral gap above the ground state is robust against these perturbations, but the geometrical dependence on the GSD and the restricted mobility of some excitations are a new feature that it is not common in standard topological phases.

The present model can be generalized to an odd number of spatial dimensions by using the Dirac representation of the Clifford algebra. Consider a spacetime with dimensions $d = D + 1$, for $D = 2n + 1$ we consider a representation of the Clifford algebra of dimension $2^n \times 2^n$, i.e., the Dirac matrices γ^I, $I = 1, \ldots, D$. They satisfy the constraint $\prod_I^D \gamma^I = i^n$.

To generalize the Chamon model to D dimensions, we proceed as follows: Start with an hypercubic lattice Λ that can be written as $\Lambda = \Lambda_{even} \oplus \Lambda_{odd}$, i.e., as

before, the lattice admits two sub-lattices with even sites and odd sites. The degrees of freedom of the model are placed on the even sites of the lattice, and we represent the degrees of freedom by $\Gamma^{(I,\alpha)}_{\mathbf{x}}$ and they are products of the Dirac matrices. The $\alpha = 1, \ldots, n$ indice indicates several species of operators, since the Hilbert space is 2^n dimensional, we need n different operators to gap the theory. The case $\alpha = 1$ is a special one, and we call it *principal configuration*. In the principal configuration we have $\Gamma^{(I,1)}_{\mathbf{x}} = \gamma^I_{\mathbf{x}}$.

A general Γ operator can be parametrized in terms of integer valued vectors $T^{(I,\alpha)}_a$, $a = 1, \ldots, 2n$

$$\Gamma^{(I,\alpha)} = \left(\gamma^1\right)^{T^{(I,\alpha)}_1} \left(\gamma^2\right)^{T^{(I,\alpha)}_2} \ldots \left(\gamma^{2n}\right)^{T^{(I,\alpha)}_{2n}} . \tag{3.7}$$

The $T^{(I,\alpha)}_a$ are only defined mod(2) since the γ's obeys $\left(\gamma^I\right)^2 = 1$. The principal configuration has to be chosen such that the lattice operators obeys the proper commutation relations. In the present case this can be achivied with the following choice

$$T^{(I,\alpha)}_a = t^{(I)}_a = \delta^I_a , \quad I = 1, \ldots, 2n , \quad \text{and} \quad T^{(2n+1,1)}_a = t^{(2n+1)}_a = -\sum_I^{2n} t^{(I)}_a . \tag{3.8}$$

The fact that the product of all γ's in the lattice is proportional to the identity tie these principal configuration vectors to obey $\sum_{I=1}^{D} t^(I)_a = 0$. The hyper-octahedra operators live on the odd sub-lattice and are now defined as

$$\mathcal{O}^{(\alpha)}_{\mathbf{x}} = \prod_{I=1}^{D} \Gamma^{(I,\alpha)}_{\mathbf{x}+\hat{a}_I} \Gamma^{(I,\alpha)}_{\mathbf{x}-\hat{a}_I} , \quad \alpha = 1, \ldots, \frac{D-1}{2} . \tag{3.9}$$

Due to the properties of the $\Gamma^{(I,\alpha)}$, the $\mathcal{O}^{(\alpha)}$ also obeys $\left(\mathcal{O}^{(\alpha)}_{\mathbf{x}}\right)^2 = 1$. Using these operators we can build the following Hamiltonian

$$H = -\sum_{\alpha=1}^{(D-1)/2} \left(g_\alpha \sum_{\mathbf{x}} \mathcal{O}^{\alpha}_{\mathbf{x}} \right) , \tag{3.10}$$

where we consider all couplings to be positive, i.e., $g_\alpha > 0$. In a similar fashion such as in the 3 dimensional case, we choose these operators to be such that they commute

$$\left[\mathcal{O}^{(\alpha)}_{\mathbf{x}}, \mathcal{O}^{(\beta)}_{\mathbf{x}'}\right] = 0 , \quad \forall\, \alpha, \beta \quad \text{and} \quad \mathbf{x}, \mathbf{x}' . \tag{3.11}$$

For such operators we say that the Hamiltonian is a sum of *commuting projectors*, and there as many commuting projectors as there are degrees of freedom, up to constraints that will be responsible for the ground state degeneracy.

The Γ's trivially commute when they are defined at the same site $\mathbf{x}$ or when they do not share any site whatsoever. When they share only one site they also

commute, since they will only share sites with the same kind of Γ's, the only non-trivial commutation arises from the case where they share two sites, as one can see in Fig.(10). For this case, the operators are allowed to commute or anti-commute, which can be represented as

$$\Gamma^{(I,\alpha)}\,\Gamma^{(J,\beta)} = (-1)^{\eta^{IJ}_{\alpha\beta}}\,\Gamma^{(J,\beta)}\,\Gamma^{(I,\alpha)}\,. \tag{3.12}$$

The $\eta^{IJ}_{\alpha\beta}$ can be either 0 or 1. The desired commutations can be achivied if $\eta^{IJ}_{\alpha\beta} = \eta^{JI}_{\alpha\beta}$. The three dimensional case we saw earlier can be described with this formulation with the simple choice

$\Gamma^{(I,\alpha)}$	$I =$ 1	2	3
$\alpha = 1$	$\gamma^1 = \sigma^1$	$\gamma^2 = \sigma^2$	$\gamma^3 = \sigma^3$

. (3.13)

The next non trivial case we can write down is for $D = 5$, where we use the 4-dimensional representation for the Dirac matrices

$\Gamma^{(I,\alpha)}$	$I =$ 1	2	3	4	5
$\alpha = 1$	γ^1	γ^2	γ^3	γ^4	γ^5
$\alpha = 2$	$\gamma^3\gamma^5$	$\gamma^4\gamma^5$	$\gamma^1\gamma^5$	$\gamma^2\gamma^5$	γ^5

. (3.14)

These Γ's obeys the condition (3.12) and hence they yield hyper-octahedra commuting operators $\mathcal{O}^{(1)}_{\mathbf{x}}$ and $\mathcal{O}^{(2)}_{\mathbf{x}}$ with $\mathbf{x} \in \Lambda_{odd}$.

The ground state of the model consists of all states where $\mathcal{O}^{(\alpha)}_{\mathbf{x}} = +1$, excitations (or defects) are the states on which these operators assume the -1 value. In the following, we argue that the topological degeneracy of the higher dimensional Chamon model is at least $2^{(D-1)2^{(D-3)L}}$ with L the linear size of the system, and that the model is indeed fractonic, i.e., that there is no local operator that is capable of moving a single defect.

The sets of γ operators can be built as follows. Start with matrices that are products of Pauli operators

$$\gamma^{(n)}_{\mu_1\mu_2\ldots\mu_n} = \sigma_{\mu_1}\sigma_{\mu_2}\ldots\sigma_{\mu_n}\,, \quad \mu_i = 0, \ldots, 3\,. \tag{3.15}$$

Where we are using that $\sigma_0 = 1$. For $n = 1$ we can obtain a set of anti commuting matrices that are simply the Pauli operators, i.e., γ^1_I, $I \in S^1 = \{1, 2, 3\}$.[1]

For $n = 2$ we can build the matrices using the previous set. Take $\gamma^{(2)}_{I3} = \gamma^{(1)}_I \otimes \sigma_3$, this will give us three matrices, then we add to this sub set the following two matrices: γ_{01} and γ_{02}, such that we can define $\gamma^{(2)}_I$, $I \in S^{(2)}$ $\{13, 23, 33, 01, 02\}$. These are all anti commuting matrices.

[1] The notation $\{\ldots j \ldots\}$ simply means that we are considering the given σ_j pauli operator.

The set of general anti commuting matrices can be constructed by induction, consider that we have $(2n-1)$ anti commuting matrices $\gamma_I^{(n-1)}$, $I \in S^{(n-1)}$, we build $2n-1$ matrices as

$$\gamma_{I3}^{(n)} = \gamma_I^{(n-1)} \otimes \sigma_3 \,, \tag{3.16}$$

then, add to this set of matrices the following two

$$\gamma_{00\ldots01}^{n} \,, \quad \gamma_{00\ldots02}^{(n)} \,. \tag{3.17}$$

This procedure give us $2n-1+2 = 2n+1$ matrices $\gamma_I^{(n)}$, $I \in S^{(n)} = \left\{ J3 | J \in S^{(n-1)} \right\} \bigcup \{00\ldots01, 00\ldots02\}$. The whole set of matrices obeys the following property

$$\prod_{I \in S^{(n)}} \gamma_I^{(n)} = \mp i \, \gamma_{0\ldots0}^{(n)} \,, \tag{3.18}$$

this simply means that any matrix in the set can be written as the product of all the others $2n$ matrices. It is also good to stress that this is not the only set we can build, we could have taken σ_2 instead of σ_3 in (3.16), then instead of adding to the set the matrices $\gamma_{0\ldots01}$ and $\gamma_{0\ldots02}$ we should have added the matrices $\gamma_{0\ldots01}$ and $\gamma_{0\ldots03}$, this would again yield another equivalent set of γ's with the same properties.

Also, they have additional properties:

1. **The identity is the only tensor product of Pauli matrices that commutes with all the Dirac matrices**. In other words, only the matrix $\gamma_J^{(n)}$, $J = 00\ldots0$, can commute the $2n+1$ matrices $\gamma_I^{(n)}$ with $I \in S^{(n)}$.

   To show this property, suppose that there is a matrix $\gamma_J^{(n)}$ that commutes with all the $2n+1$ matrices. This J must be of the form $J = j0$ for $\gamma_J^{(n)}$ to commute with both $\gamma_{0\ldots0\,1}^{(n)}$ and $\gamma_{0\ldots0\,2}^{(n)}$. Therefore,

   $$[\gamma_J^{(n)}, \gamma_I^{(n)}] = 0, \quad \forall I \in S^{(n)} \quad \Leftrightarrow \quad [\gamma_j^{(n-1)}, \gamma_i^{(n-1)}] = 0, \quad \forall i \in S^{(n-1)} \,.$$

   We can use this recursion all the way to $n = 1$, where only $\gamma_0^{(1)}$ commutes with the $\gamma_i^{(1)}, i \in S^{(1)}$, and conclude that J must be $J = 00\ldots0$, i.e., all the entries must be 0.

2. **The set of matrices $\gamma_I^{(n)}$ with $I \in S^{(n)}$ is maximal, i.e., no other matrix can be added to the set that anticommutes with those already in**. The statement is true for $n = 1$: the matrices $\gamma_I^{(1)}$ with $I \in S^{(1)}$ are the three Pauli matrices, leaving no other option to include that would anticommute with these three.

   Now suppose that the statement is true up to $n-1$; let us analyze the consequences for when we consider n.

   Suppose by contradiction that there exists a $J \notin S^{(n)}$ such that $\gamma_J^{(n)}$ anticommutes with all the $\gamma_I^{(n)}$ with $I \in S^{(n)}$. Let us break the problem in four cases, and show impossibility in all cases.

- $J = j0$
  This is the simplest case: $\gamma^{(1)}_{j0}$ commutes with both $\gamma^{(n)}_{0\ldots0\,1}$ and $\gamma^{(n)}_{0\ldots0\,2}$, so $J = j0$ cannot be added to the set.
- $J = j1$ (the case $J = j2$ is analogous)
  This case is also simple: $\gamma^{(1)}_{j1}$ commutes with $\gamma^{(n)}_{0\ldots0\,1}$, so $J = j1$ cannot be added to the set.
- $J = j3$
  $\gamma^{(1)}_{j3}$ anticommutes with $\gamma^{(n)}_{0\ldots0\,1}$ and $\gamma^{(n)}_{0\ldots0\,2}$, so we should only consider the anticommutation with the other $2n-1$ matrices $\gamma^{(n)}_{i3}$, for $i \in S^{(n-1)}$. But because $S^{(n-1)}$ is maximal, there is no new $j \notin S^{(n-1)}$ to add.

We thus conclude that the set of $2n+1$ matrices $\gamma^{(n)}_I$ with $I \in S^{(n)}$ is maximal.

3. **There is no matrix $\gamma^{(n)}_J$ that commutes with $2n$ of the matrices $\gamma^{(n)}_I$ with $I \in S^{(n)}$. Therefore, defects cannot be created in only one direction**. Phrasing it differently, this states that there is no line defects on the model. This result will allow us to argue that we can construct a fracton model.

   The statement is true for $n = 1$: there is no matrix $\gamma^{(1)}_J$ that commutes with two of the matrices $\gamma^{(1)}_I$ with $I \in S^{(1)}$, because no one Pauli matrix commutes with two Pauli matrices.

   Now suppose that the statement is true up to $n-1$; let us analyze the consequences for when we consider n.

   Let us break the problem in four cases:

   - $J = j0$
     In this case, the commutation with $\gamma^{(n)}_{0\ldots0\,1}$ and $\gamma^{(n)}_{0\ldots0\,2}$ comes for free. Therefore we reduce the problem to finding $\gamma^{(n-1)}_j$ that commutes with $2(n-1)$ matrices $\gamma^{(n-1)}_i$ with $i \in S^{(n-1)}$. Since there is no solution for this problem (the statement is true for the case with $n-1$), then there is no solution for the case with n either.
   - $J = j3$
     This is the simplest case; $\gamma^{(1)}_{j3}$ anticommutes with $\gamma^{(n)}_{0\ldots0\,1}$ and $\gamma^{(n)}_{0\ldots0\,2}$, so it is impossible that there are $2n$ other matrices that commute with $\gamma^{(n)}_J$ among the $\gamma^{(n)}_I$ with $I \in S^{(n)}$, since there are at most $2n+1-2 = 2n-1 < 2n$.
   - $J = j1$ (the case $J = j2$ is analogous)
     $\gamma^{(1)}_{j1}$ commutes with $\gamma^{(n)}_{0\ldots0\,1}$ and anticommutes with $\gamma^{(n)}_{0\ldots0\,2}$. So we need to find $2n-1$ additional matrices that commute with $\gamma^{(n)}_{j1}$ among the $\gamma^{(n)}_{i3}$ with

$i \in S^{(n-1)}$. This is equivalent to finding $2n-1$ matrices that anticommute with $\gamma_j^{(n-1)}$ among the $\gamma_i^{(n-1)}$ with $i \in S^{(n-1)}$. This is impossible since the set $S^{(n-1)}$ is maximal (see above).

The third property listed above is the one responsible for the restricted mobility that we have mentioned before.

At last, we show how to obtain the ground state degeneracy of the model. The ground state follows from the constraints

$$\prod_{\mathbf{x}\in\Lambda_{odd,\,k}} \mathcal{O}_{\mathbf{x}}^{(\alpha)} = 1\,, \quad \alpha = 1,\,\ldots,\,\frac{D-1}{2}, \quad k = 1,\,\ldots,\,2^{D-1}\,. \tag{3.19}$$

The k indice labels all the 2^{D-1} odd/even sub-lattices in our hypercubic lattice. From the fact that the $\Gamma^{(I,\alpha)}$ operators multiply to the identity we obtain the condition $\sum_{I=1}^{D} T_a^{(I,\alpha)} = 0$, this property will allow us to obtain an equivalent of the parity checks that were obtained in [terhal2011]. For clarity, we start with the three dimensional analisys of the ground state degeneracy and then we proceed to the higher dimensional case.

The cubic operators $\mathcal{O}^{(1)}$ are given in (3.1). Take the ground state sub space that can be defined as $\mathcal{G} = \left\{r \in \mathcal{B}\left(\Lambda_{odd}\right) : \prod_{\mathbf{x}\in\Lambda_{odd}} \mathcal{O}_{\mathbf{x}}^{r_{\mathbf{x}}} = 1\right\}$ with $\mathcal{B} = \{0,\,1\}$. We can plug the explicit form of the $\mathcal{O}_{\mathbf{x}}^{(1)}$ and we see that for $r \in \mathcal{G}$ it has to obey the following conditions

$$\begin{aligned} r_{\mathbf{x}+\hat{x}} \oplus r_{\mathbf{x}-\hat{x}} \oplus r_{\mathbf{x}+\hat{z}} \oplus r_{\mathbf{x}-\hat{z}} &= 0\,, \\ r_{\mathbf{x}+\hat{x}} \oplus r_{\mathbf{x}-\hat{x}} \oplus r_{\mathbf{x}+\hat{y}} \oplus r_{\mathbf{x}-\hat{y}} &= 0\,, \\ r_{\mathbf{x}+\hat{y}} \oplus r_{\mathbf{x}-\hat{y}} \oplus r_{\mathbf{x}+\hat{z}} \oplus r_{\mathbf{x}-\hat{z}} &= 0\,. \end{aligned} \tag{3.20}$$

For every $\mathbf{x} \in \Lambda_{even}$. Also, note that

$$r_{\mathbf{x}} = r_{\mathbf{x}+2\hat{x}} = r_{\mathbf{x}+2\hat{y}} = r_{\mathbf{x}+2\hat{z}}\,, \quad \forall\ \mathbf{x} \in \Lambda_{odd}\,. \tag{3.21}$$

Then, for any $r \in \mathcal{B}(\Lambda_{odd})$ that obeys the above, belongs to the sub space $\mathcal{G}$. More importantly, (3.21) allow us to think of the ground state sub space $\mathcal{G}$ as a repetition of four codes defined over the non overlapping sub lattices $\Lambda_{ijk} \in \Lambda_{odd} = \{\Lambda_{100},\,\Lambda_{010},\,\Lambda_{001},\,\Lambda_{111}\}$, where (ijk) are defined only mod(2). From this we obtain that $dim(\mathcal{G}) = 2^2 = 4$.

In higher dimensions we can proceed in a similar fashion. The parity checks will possess the general form

$$r_{\mathbf{x}+\hat{a}_I}^{(\alpha)} \oplus r_{\mathbf{x}-\hat{a}_I}^{(\alpha)} \oplus r_{\mathbf{x}+\hat{a}_J}^{(\alpha)} \oplus r_{\mathbf{x}-\hat{a}_J}^{(\alpha)} = 0\,, \quad \mathbf{x} \in \Lambda_{even}\,, \quad I,\,J = 1,\,2,\,\ldots,\,D\,. \tag{3.22}$$

Once again, the identities (3.21) can now be written as

$$r_{\mathbf{x}} = r_{\mathbf{x}+2\hat{a}_1} = \cdots = r_{\mathbf{x}+2\hat{a_D}}\,, \quad \mathbf{x} \in \Lambda_{odd}\,. \tag{3.23}$$

Once again, by taking linear combinations of the parity checks (3.22) we can show that the identities in (3.23) are satisfied. This allow us to regard the sub space $\mathcal{G}$ as 2^{D-1} repetition codes for each α configuration, from this we conclude that $dim(\mathcal{G}) = (D-1)2^{D-2}$, note that for $D = 3$ we obtain the previous result. A important point to note is that these results are valid only when the linear sizes of our lattice, $(p_1, p_2, \ldots, p_D)$, are such that $gcd(p_1, p_2, \ldots, p_D) = 1$, with $gcd(\ldots) = g$ standing for the *greatest common divisor*. For the cases where $g \neq 1$, the ground state degeneracy will depend on the linear size of the system, which is one important feature of fracton models.

For the more general case, on which $gcd(p_1, p_2, \ldots, p_D) = g$, the dimension of the sub space $\mathcal{G}$ can be written as $dim(\mathcal{G}) = (D-1)2^{(D-2)g}$. The key fact to understand this is to note that the parity checks in (3.22) can be used to find all $r_{\mathbf{x}}^{(\alpha)} \in \mathcal{G}(g, \ldots, g)$, where now the sub space $\mathcal{G}$ is defined as follows

$$\mathcal{G}(g, \ldots, g) = \left\{ r^{(\alpha)} \in \mathcal{B}(\Lambda_{odd}) : r^{(\alpha)}_{\mathbf{x}+\hat{a}_I} \oplus r^{(\alpha)}_{\mathbf{x}-\hat{a}_I} \oplus r^{(\alpha)}_{\mathbf{x}+\hat{a}_J} \oplus r^{(\alpha)}_{\mathbf{x}-\hat{a}_J} = 0, \forall\ (I, J) \in \Lambda_{even} \right\}. \tag{3.24}$$

Note that this is the generalization of the previous definition for the ground state sub space. Then, the idea is to take linear combinations of the parity checks in order to generate the solutions that belong to $\mathcal{G}$, we can start with two lines (e.g., x and y) and solving the parity checks will furnish a solution for the $(x-y)$-plane, this first case will generate $2 \times L/2$ logical qubits. We can proceed in a similar fashion, i.e., take the solutions for two planes and generate the solution in three dimensions, then take the solution for two three-dimensional surfaces and generate the solution on a four-dimensional manifold, and so on. Since the parity checks are all linear dependent, each new solution will generate one logical qubit per line and thus we end up with $2^{(D-1)}L/2$. The ground state sub space can, as before, be regarded as repetition of $2^{(D-1)}$ codes over the non overlapping sub lattices $\Lambda_{a_1 a_2 \ldots a_{2^{D-1}}}$, and with our previous line of thought, the higher dimensional ground state sub space can be written as products of the two dimensional ones,

$$\mathcal{G}(g, \ldots, g) = \mathcal{G}(g, g) \otimes \mathcal{G}(g, g) \otimes \cdots \otimes \mathcal{G}(g, g)\,. \tag{3.25}$$

To finally arrive at the answer, we again have to remember that there are $(D-1)/2$ configurations labeled by α, therefore the number of logical qubits generated are $(D-1)2^{(D-3)}L$, and thus the ground state degeneracy of the Chamon model in an odd dimensional space is given by

$$GSD = 2^{(D-1)2^{(D-3)}L}\,. \tag{3.26}$$

3.2 Effective theory

Now that we have discussed the properties of the lattice model we turn our attention on how to build a compatible effective theory. In order to do so we follow the procedure introduced in (55).

we begin by introducing a exponential map that connects the lattice operators $\gamma_{\mathbf{x}}^{(I,\alpha)}$ with some field $A(\mathbf{x})$,

$$\Gamma_{\mathbf{x}}^{(I,\alpha)} \equiv \exp\left(iT_a^{(I,\alpha)}\, K_{ab}\, A_b(\mathbf{x})\right)\,, \tag{3.27}$$

the matrix indices runs in the range $a, b = 1, \ldots, D-1$. Also, note that with this parametrization of the lattice operators, we introduce a symmetry

$$A_a \to Q_{ab}A_b\,, \quad K_{ab} \to (Q^\top)^{-1}_{am}K_{mn}Q^{-1}_{nb}\,, \quad T_a^{(I,\alpha)} \to T_b^{(I,\alpha)}Q_{ba}\,, \tag{3.28}$$

with Q being an arbitrary matrix. For the case of the principal configuration, in which $\Gamma^{(I,1)} = \gamma^I$ and also $T_a^{(I,1)} = t_a^{(I)}$, the map becomes

$$\gamma_{\mathbf{x}}^{(I)} = \exp\left(it_a^{(I)}\, K_{ab}\, A_b(\mathbf{x})\right)\,. \tag{3.29}$$

Since the γ's are the usual Dirac matrices, all the properties of the A fields and K matrices, can be drawn from the properties of the γ's. Using the BCH formula, we can express the product of γ's as

$$\gamma_{\mathbf{x}}^I\gamma_{\mathbf{x}'}^J = \exp\left(i\left[t_a^{(I)}K_{ab}A_b(\mathbf{x}),\, t_{a'}^{(J)}K_{a'b'}A_{b'}(\mathbf{x}')\right]\right)\gamma_{\mathbf{x}'}^J\gamma_{\mathbf{x}}^I\,, \tag{3.30}$$

To reproduce the correct algebra of the γ operators we impose the following equal time commutation rule for the A fields

$$\left[A_b(\mathbf{x}),\, A_{b'}(\mathbf{x}')\right] = i\pi(K^{-1})_{bb'}\delta_{\mathbf{x}\mathbf{x}'}\,. \tag{3.31}$$

Using the above commutation rule in (3.30) we see that the algebraic properties of the γ's translates to conditions among the t^I vectors and K matrices

$$t_a^{(I)}(K^\top)_{ab}t_b^{(J)} = \begin{cases} 1 & \mod(2)\,, \quad I \neq J \\ 0 & \mod(2)\,, \quad I = J \end{cases}\,. \tag{3.32}$$

This condition is particular to the principal basis $T_a^{(I,1)} = t_a^{(I)}$ and simply reflects the fact that the fundamental operators in the theory are anti commuting. The other objects for which $\alpha \geq 2$ does not need to share this property, indeed, the important feature for these objects is that they are such that the stabilizer operators $\mathcal{O}$ commute among themselves. Notice also that the condition (3.32) implies that the fields A_a are compact, since the shift

$$A_a \to A_a + 2\pi \sum_{j=1}^{D} t_b^{(J)}\, m_J\,, \quad m_J \in \mathbb{Z} \tag{3.33}$$

does not modify the $\gamma_{\mathbf{x}}^{(I)}$ due to the fact

$$\exp\left(2i\pi \sum_{J=1}^{D} t_a^{(I)} K_{ab} t_b^{(J)} m_J\right) = 1\,. \tag{3.34}$$

The conditions (3.32) imposes constraints on the form of the principal vectors and on the K matrix. Note that the condition on the second line of (3.32) forces the K matrix to be an anti-symmetric matrix, and since the inverse must exist (due to the commutation rule), we necessarily need $\det(K) \neq 0$. The determinant constraint forces the K matrix to be *even* $\times$ *even*, which tell us that this construction works only for odd spatial dimensions, since K is $D-1 \times D-1$ matrix.

Another useful relation emerges when we consideer the product of all the operators in our stabilizer $\mathcal{O}$, since the number of γ's will be even, the product of them all is proportional to the identity, with our parametrization this can be written as

$$1 = \gamma_1\gamma_2\ldots\gamma_D \sim \exp\left(i\sum_{I=1}^{D} t_a^{(I)} K_{ab} A_b\right) \exp\left(\frac{i\pi}{2}\frac{(D-1)D}{2}\right)\,, \tag{3.35}$$

this allow us to read a *neutrality condition*

$$\sum_{I=1}^{D} t_a^{(I)} = 0 \qquad \mod(2)\,. \tag{3.36}$$

Moreover, this property holds for any α configuration, since the stabilizer operators will keep multiplying to the identity (or at least proportional to the identity). Thus, we shall consideer the more general neutrality condition as

$$\sum_{I=1}^{D} T_a^{(I,\alpha)} = 0 \qquad \mod(2)\,, \quad \alpha = 1, \ldots, \frac{D-1}{2}\,. \tag{3.37}$$

The next step is to build the stabilizer operator in its continuum form. With the parametrization (3.27) we can write it down as

$$\mathcal{O}_{\mathbf{x}}^{(\alpha)} = \exp\left(i\sum_{I=1}^{D}\left(T_a^{(I,\alpha)} K_{ab} A_b(\mathbf{x}+\hat{a}_I) + T_a^{(I,\alpha)} K_{ab} A_b(\mathbf{x}-\hat{a}_I)\right)\right)\,. \tag{3.38}$$

We now analyze the conditions that should be imposed on the continuum objects such that the stabilizers $\mathcal{O}^{(\alpha)}$ have the correct properties. We know that two different stabilizer operators can share one or two links, in the case in which they share a single link the commutation is trivial and does not impose any constraint on the associated continuum version. When they share two links there is possibility for a non trivial structure.

Consider two stabilizer operators $\mathcal{O}_{\mathbf{x}}^{(\alpha)}$ and $\mathcal{O}_{\mathbf{x}+\hat{a}_I+\hat{a}_J}^{(\beta)}$, the requirement that they commute imposes the condition

$$C_{IJ}^{(\alpha\beta)} = 0\,, \tag{3.39}$$

where

$$C_{IJ}^{(\alpha\beta)} = T_a^{(I,\alpha)} K_{ab} T_b^{(J,\beta)} + T_a^{(J,\alpha)} K_{ab} T_b^{(I,\beta)} \,. \tag{3.40}$$

This object possess the symmetries $C_{IJ}^{(\alpha\beta)} = C_{JI}^{(\alpha\beta)}$ and $C_{IJ}^{(\alpha\beta)} = -C_{IJ}^{(\beta\alpha)}$. The symmetric property in IJ indices follows directly from its definition and the anti symmetry in $\alpha\beta$ it is due to the anti symmetry of the K matrix. In particular, from the anti symmetry in $\alpha\beta$ we automatically see that $C_{IJ}^{(\alpha\beta)}$ vanishes when $\alpha = \beta$, which is consistent with the known information that stabilizers of the same kind commute. We shall return to this point later.

By expanding the fields in (3.38) and taking the continuum limit of such expression, we can arrive at the following expression

$$\mathcal{O}_{\mathbf{x}}^{(\alpha)} = \exp\left(2i \sum_{I=1}^{D} T_a^{(I,\alpha)} K_{ab} A_b(\mathbf{x}) + i \sum_{I=1}^{D} T_a^{(I,\alpha)} K_{ab} \partial_I^2 A_b(\mathbf{x}) + \dots \right) \tag{3.41}$$

The physics of the Chamon model can be captured by the second order terms in the expansion, and therefore we stop at this order, although it is important to stress that this is a *post hoc* argument.

The neutrality condition ensures that the zeroth order term in (3.41) vanishes, which leave us with the following

$$\mathcal{O}_{\mathbf{x}}^{(\alpha)} = \exp\left(i \sum_{I=1}^{D} T_a^{(I,\alpha)} K_{ab} \partial_I^2 A_b(\mathbf{x}) + \dots \right) . \tag{3.42}$$

In this parametrization, the Hamiltonian (3.10) assumes the form

$$H \sim -2 \sum_{\alpha} g_\alpha \int d^D x \, \cos\left(M^{(\alpha)}(\mathbf{x}) \right) , \tag{3.43}$$

where

$$M^{(\alpha)}(\mathbf{x}) = \sum_{I=1}^{D} T_a^{(I,\alpha)} K_{ab} \partial_I^2 A_b(\mathbf{x}) \,. \tag{3.44}$$

Therefore, the ground state of the model corresponds to the case on which all the cosines are simultaneously pinned to 1, and thus $M^{(\alpha)}(\mathbf{x}) = 2\pi m^{(\alpha)}$ with $m^{(\alpha)} \in \mathbb{Z}$. Before we write down a Lagrangian let us put this flux $M^{(\alpha)}$ in a more convenient form. Using the neutrality condition (3.37) we can single out one direction and obtain a set of derivatives that is linearly independent, for example let us separate the last direction $I = D$, this amounts to use the neutrality condition to write down the $T_a^{(D,\alpha)}$ as a combination of all the other $D-1$ vectors, and thus $M^{(\alpha)}$ can be put in the form

$$M^{(\alpha)}(\mathbf{x}) = \sum_{I=1}^{D-1} T_a^{(I,\alpha)} K_{ab} D_I A_b(\mathbf{x}) \,, \quad D_I = \partial_I^2 - \partial_D^2 \,. \tag{3.45}$$

Now it is convenient to define a new derivative $\mathcal{D}_a^{(\alpha)}$, that we are going to name it as *fractonic derivative*. In terms of the fractonic derivatives, the flux can be written

as

$$M^{(\alpha)}(\mathbf{x}) = K_{ab}\mathcal{D}_a^{(\alpha)}A_b(\mathbf{x})\,, \quad \mathcal{D}_a^{(\alpha)} = \sum_{I=1}^{D-1} T_a^{(I,\alpha)}D_I\,. \tag{3.46}$$

The above form of the flux makes explicitly the invariance under a $U(1)$ local transformations

$$A_a(\mathbf{x}) \to A_a(\mathbf{x}) + \sum_{\alpha}\mathcal{D}_a^{(\alpha)}\zeta^{(\alpha)}(\mathbf{x})\,, \tag{3.47}$$

where $\zeta^{(\alpha)}(\mathbf{x}, t)$ is an arbitrary set of functions indexed by α. The invariance follows from the commutation properties of the stabilizer operators, as one can see from the following structure

$$\begin{aligned} K_{ab}\mathcal{D}_a^{(\alpha)}\mathcal{D}_b^{(\beta)} &= \sum_{I,J=1}^{D-1} T_a^{(I,\alpha)}K_{ab}T_b^{(J,\beta)}D_ID_J\,, \\ &= \sum_{I,J=1}^{D-1} C_{IJ}^{(\alpha\beta)}D_ID_J\,, \end{aligned} \tag{3.48}$$

which clearly vanishes, since the commutation property of the stabilizer operators requires that $C_{IJ}^{(\alpha\beta)} = 0$, therefore, the commutation of the lattice operators is closely related to the gauge invariance of the continuum theory. We shall make this point more explicitly in the next chapter, where we are going to discuss the Haah Model, in which the gauge structure and the commutations conditions are more complicated and a more in depht analysis is required.

With all these properties at hand, we are in shape to write down a effective theory for the ground state of the Chamon model. The effective action that captures all the essential physics is given by

$$\mathcal{A} = \int d^Dx\,dt\,\frac{1}{2\pi}\left(K_{ab}A_a\partial_0A_b + 2\sum_{\alpha}A_0^{(\alpha)}\,K_{ab}\mathcal{D}_a^{(\alpha)}A_b\right)\,. \tag{3.49}$$

The first term in the above action is necessary to ensure the proper commutation rules (3.31), the second term is the ground state constraints implemented by the set of Lagrange multipliers $A_0^{(\alpha)}$.

The requirement of full gauge invariance, fix the transformation of the $A_0^{(\alpha)}$ to be

$$A_0^{(\alpha)} \to A_0^{(\alpha)} + \partial_0\zeta^{(\alpha)}\,. \tag{3.50}$$

Therefore, we have obtained a *bona fide* gauge theory that resembles a Chern-Simons topological field theory, with the exception of the differential operators of our model possesses an intricate structure in comparison to usual Chern-Simons models, for that matter we shall refer to the action (3.49) as *Chern-Simons-like* action, in order to make a distinction with the usual CS theories.

We could as well have chosen to write down this theory in terms of the gauge invariant quantities, the *fractonic electric field* and a *fractonic magnetic field*,[2]

$$E_a = \partial_0 A_a - \sum_\alpha \mathcal{D}_a^{(\alpha)} A_0^{(\alpha)}\,, \quad B^{(\alpha)}_{a_1 a_2 \ldots a_{D-1}} = \epsilon_{a_1 a_2 \ldots a_{D-2} a_{D-1}} \mathcal{D}^{(\alpha)}_{D-2} A_{D-1}\,, \tag{3.51}$$

where $\epsilon_{a_1 a_2 \ldots a_{D-2} a_{D-1}}$ is the higher dimensional Levi-Civita symbol.

3.2.1 Beyond the principal configuration

In the previous section we introduced a promising candidate for the effective theory for the Chamon model, we have mentioned that the model consists of several commuting stabilizer operators that are labeled by an α index. This set of operators are mapped into α constraints which are distinguished by the $T^{(I,\alpha)}$ vectors. We have showed that for $\alpha = 1$ we can make a choice of basis, the principal basis, which is consistent with the algebra of the underlying lattice operators.

Now we want to construct a sistematic procedure to find all the $T^{(I,\alpha)}$ with $\alpha > 1$ which are consistent with the algebra of the lattice operators and thus form a good representation for the problem. In fact, the key ingredient is the quantity $C_{IJ}^{(\alpha\beta)}$ that encodes the commutations among the stabilizers.

We start with the following ansatz, consider that every representation with $\alpha > 1$ can be built from the principal configuration by means of a linear transformation $L^{(\alpha)}$, i.e.,

$$T_a^{(I,\alpha)} = \sum_M L^{(\alpha)}_{IM} t_a^{(M)}\,, \quad I, M = 1, \ldots, D-1\,, \quad T_a^{(D,\alpha)} = -\sum_{I=1}^{(D-1)} T_a^{(I,\alpha)}\,. \tag{3.52}$$

Note that since the principal configuration can be written as $t_a^{(I)} = \delta_a^I$ with δ_a^I being a Kronecker delta, this tell us that $T_a^{(I,\alpha)} = L_{Ia}^{(\alpha)}$. Plugging this information in the $C_{IJ}^{(\alpha\beta)}$ in (3.40) we obtain

$$\begin{aligned} C_{IJ}^{(\alpha\beta)} &= L_{Ia}^{(\alpha)} K_{ab} L_{Jb}^{(\beta)} + L_{Ja}^{(\beta)} K_{ab} L_{Ib}^{(\alpha)}\,, \\ C_{IJ}^{(\alpha\beta)} &= \left(L^{(\alpha)} K L^{(\beta)\top}\right)_{IJ} + \left(L^{(\beta)} K^\top L^{(\alpha)\top}\right)_{IJ}\,, \end{aligned} \tag{3.53}$$

to make it even more explicitly, we can write it as

$$C^{(\alpha\beta)} = L^{(\alpha)} K L^{(\beta)\top} + \left(L^{(\alpha)} K L^{(\beta)\top}\right)^\top\,. \tag{3.54}$$

Where we have ommited the IJ indices and make more explicitly the matrix nature of the objects. Therefore, if we can choose a matrix $S^{(\alpha\beta)}$ that is anti symmetric

[2] We are using explicitly the term *fractonic* to stress that these quantities are built using the fractonic derivative $\mathcal{D}_a$.

we can satisfy the vanishing condition of $C^{(\alpha\beta)} = 0$, and this will fix the allowed $L^{(\alpha)}$. Thus, we have

$$L^{(\alpha)} K L^{(\beta)\top} = S^{(\alpha\beta)} \,. \tag{3.55}$$

For a given choice of $S^{(\alpha\beta)}$ we can find a iterating solution for $L^{(\alpha)}$ as

$$L^{(\alpha)} = S^{(\alpha\beta)} \left(L^{(\beta)\top} \right)^{-1} K^{-1} \,. \tag{3.56}$$

Indeed, there are a myriad of choices for the $S^{(\alpha\beta)}$, for simplicity we choose to set $\beta = 1$ due to the fact that we know that $L^{(1)} = 1$, then for an arbitrary matrix $S^{(21)}$ we can find $L^{(2)}$, which then we plug in the above and find $L^{(3)}$ with some choice for $S^{(31)}$ and so on. In general, with $\beta = 1$, the set of solutions for $L^{(\alpha)}$ can be put in a simple form

$$L^{(\alpha)} = S^{(\alpha 1)} K^{-1} \,. \tag{3.57}$$

Also, there are compatibility conditions among these $S^{(\alpha\beta)}$ matrices, because one can reach $L^{(3)}$ from, for example, $L^{(1)}$ or $L^{(2)}$. By considering the general form $L^{(\alpha)} K L^{(\beta)\top}$ and plugging (3.57) in it, we find that the $S^{(\alpha\beta)}$ must be such that they obey

$$S^{(\alpha\beta)} = S^{(\alpha 1)} K^{-1} S^{(\beta 1)} \,. \tag{3.58}$$

Although the $S^{(\alpha\beta)}$ is an arbitrary matrix and, in fact, any anti symmetric choice would suffice, we already have a natural choice in our description, which is given by the matrix K[3]. Consider the $n \times n$

$$K_n = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ -1 & 0 & 1 & \dots & 1 \\ -1 & -1 & 0 & \dots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ -1 & -1 & \dots & 1 & 0 \end{pmatrix}_{n \times n} \,, \tag{3.59}$$

where n is related to the spatial dimension by the relation $D = 2n + 1$. Then, the $2n \times 2n$ matrix needed is simply K_{2n}. We can build the $S^{(\alpha 1)}$ matrices as follows, take the first matrix to be

$$S^{(11)} = K_{2n} \,, \tag{3.60}$$

and from this we can build the rest of them as

$$S^{(\alpha 1)} = (K_n)^{2\alpha - 3} \otimes 1_2 \,, \quad \alpha = 2, \dots, n \,. \tag{3.61}$$

It is easily seen that these matrices commute with K_{2n} and K_{2n}^{-1}, using this fact together with the anti symmetry of $A^{(\alpha 1)}$ and K_{2n} allow us to see that the $A^{(\alpha\beta)}$ is also anti symmetric, as is required by (3.58).

[3] For a more detailed discussion about determining the K-matrix elements see AppendixD

With these $A^{(\alpha 1)}$, we can proceed and obtain all the $L^{(\alpha)}$ matrices and built the corresponding $T^{(I,\alpha)}$ vectors, once we have the vectors it is straightforward to obtain the corresponding $\Gamma^{(I,\alpha)}$ operators in the lattice.

3.3 Properties and compatibility of the effective theory

In this section we shall analyze some features of the effective theory. First of all, let us note that there is some notion of level quantization in our model in a similar fashion with what happens in usual Chern-Simons theories.

Start from the condition (3.32) which ensures the proper algebra for the lattice operators. In the principal configuration, the $t_a^{(I)}$ vectors can be represented by a Kronecker delta δ_a^I, plugging this into the (3.32), we are able to read the following quantization for the elements of the K matrix

$$K_{IJ} = \text{odd}\,, \quad \text{if } I \neq J\,, \quad I,\, J = 1,\, \ldots,\, D-1\,. \tag{3.62}$$

Therefore, all elements of the K matrix must be an odd integer, consequently, non vanishing quantities. Although we have picked the principal configuration, in general, for any basis of the $t^{(I)}$ vectors we can obtain some notion of level quantization.

Another interesting fact of the effective theory emerges when one includes charged matter in the model. To do that we couple the A field with its own current J by means of $\sum_\alpha A_0^{(\alpha)} J_0^{(\alpha)} + A_a J_a$ into the action (3.49), we would be left with

$$\mathcal{A} = \int d^D x\, dt\, \frac{1}{2\pi} K_{ab} A_a \partial_0 A_b + \frac{1}{\pi} \sum_\alpha A_0^{(\alpha)}\, K_{ab} \mathcal{D}_a^{(\alpha)} A_b + \sum_\alpha A_0^{(\alpha)} J_0^{(\alpha)} + A_a J_a\,. \tag{3.63}$$

Requiring gauge invariance of this modified action lead us to a set of continuity equations

$$\partial_0 J_0^{(\alpha)} = \mathcal{D}_a^{(\alpha)} J_a\,, \tag{3.64}$$

from this continuity equation and due to the form of the fractonic derivatives, we can immediately read a conserved quantity by integrating over the entire space

$$\frac{d}{dt} \int d^D x\, J_0^{(\alpha)} = \int d^D\, \mathcal{D}_a^{(\alpha)} J_a = 0\,, \tag{3.65}$$

where we have assumed periodic boundary conditions over the spatial manifold.[4] This allow us to define a conserved quantity Q, defined as

$$Q^{(\alpha)} = \int d^D x\, J_0^{(\alpha)}\,. \tag{3.66}$$

But this is not the only conserved quantity in the model, in fact, due to the form of the fractonic derivative $\mathcal{D}_a^{(\alpha)}$ we have a few other possibilities. Indeed, charge

[4] One could also assume that the fields falls sufficiently fast at the boundaries. In any case, the right hand side vanishes with either one of these assumptions.

is also conserved along sub manifolds of the system. These extra conservation laws restricts the mobility of the excitations and also imposes constraints on the type of excitations that can be created.

Let us write the conservation laws in a more convenient form

$$\partial_0 J_0^{(\alpha)} = \sum_{I=1}^{D-1} T_a^{(I,\alpha)} D_I J_a \,, \tag{3.67}$$

$$= \sum_{I=1}^{D-1} D_I \mathcal{J}^{(\alpha)} \,, \quad \mathcal{J}^{(\alpha)} = T_a^{(I,\alpha)} J_a \,. \tag{3.68}$$

Introduce the coordinates $\hat{x}_{ID}^{\sigma_I} = \hat{a}_I + \sigma_I \hat{a}_D$, with $\sigma_I = \pm 1$. In these coordinates we can write the derivatives $D_I = \partial_I^2 - \partial_D^2$ as

$$D_I = (\partial_I + \partial_D)(\partial_I - \partial_D) = 4\partial_{ID}^{+}\partial_{ID}^{-} \,. \tag{3.69}$$

In these coordinates the continuity equation assume a convenient form

$$\partial_0 J_0^{(\alpha)} = 4 \sum_{I=1}^{D-1} \left(\partial_{ID}^{-}\partial_{ID}^{+}\right) \mathcal{J}^{(\alpha)} \,. \tag{3.70}$$

From the above we can immediately read 2^{D-1} extra conservation laws over $(D-1)$ sub manifolds labelled by $\left(x_{1D}^{\sigma_1}, x_{2D}^{\sigma_2}, \ldots, x_{(D-1)D}^{\sigma_{D-1}}\right)$. Indeed, integrating $J_0^{(\alpha)}$ over any of these $(D-1)$ sub manifolds allow us to obtain $D-1$ extra charges

$$Q^{(\alpha)}_{\sigma_1,\sigma_2,\ldots,\sigma_{D-1}} = \int dx_{1D}^{\sigma_1} dx_{2D}^{\sigma_2} \ldots dx_{(D-1)D}^{\sigma_{D-1}} J_0^{(\alpha)} \,, \tag{3.71}$$

these extra conservation laws are responsible for the mobility restrictions in the model. Any excitation that can be created in the Chamon model must obey the global conservation law (3.65) and simultaneously obey the extra conservations (3.70). Satisfying all these conditions restricts the allowed excitations that can be created, as well as how they can move, in fact the mobile bound states are one dimensional excitations (dipoles) and they can move in the direction perpendicular to the planes of conservation displayed in (3.71).

Naively, we could think that there are many other global conserved charges labeled by a function f, in fact, according to the continuity equation (3.70) any charge of the form

$$Q_f^{(\alpha)} = \int d^D x \, f(\mathbf{x}) \, J_0^{(\alpha)} \,, \tag{3.72}$$

with $f(\mathbf{x})$ obeying

$$\partial_{ID}^{+}\partial_{ID}^{-} f = 0 \quad \forall I \,, \tag{3.73}$$

is also globally conserved. But it turns out that the function f actually has to obey stronger conditions. The sub manifold conservations splits the constraint (3.73) into

$$\partial_{ID}^{+} f = 0 \,, \quad \partial_{ID}^{-} f = 0 \quad \forall I \,. \tag{3.74}$$

The above constraint is more restrictive and in fact the only solution is $f = constant$, therefore, the apparent extra global charge (3.72) is simply the charge (3.66) times a constant. Thus, we see that the conservation along sub manifolds forbids the existence of extra global charges. We are going to see a similar behavior when we discuss the type-II fracton phase described by the Haah model in the next chapter, in there, the lack of sub manifold conservations allows for f to have non trivial solutions which implies in the existence of infinitely many charges.

As a final property of the effetive theory, we discuss the ground state degeneracy associated with the action (3.49). From the lattice result (3.26) it is natural to expect that the ground state degeneracy of a continuum model would be infinite, since it has a geometric dependency on the linear system size. From the effective theory perspective we want to find a natural way in which we can regularize the theory such that we can keep track of this linear divergence. Luckily, the conservation laws over the $(D-1)$ sub manifolds provide us with a natural choice on how to regularize the theory. The basic idea is to consider a foliation of the manifold in layers that are stacked over the directions where movement is allowed (the directions perpendicular to the sub manifolds where charge is conserved).

To compute the degeneracy, it is sufficient to analyze only the kinectic part of the action

$$\mathcal{A} = \int dt\, d^{D-1}x\, \frac{1}{2\pi} K_{ab} A_a \partial_0 A_b + \dots\,, \tag{3.75}$$

Then, we make the following change of coordinates

$$\int dt\, d^{D-1}x \to \int dt \left(\prod_{I=1}^{D-1} dx_{ID}^{\sigma_I} \right) dx_\perp \mathcal{J}\,, \tag{3.76}$$

where $\mathcal{J}$ is the Jacobian of the transformation, which for this transformation is a mere constant and will not influence in any of our calculations, therefore, we will absorb it into the definition of $x_\perp$. We also consider that the system possess periodic boundary conditions along the new coordinates. Next we discretize the $x_\perp$ as

$$\int dx_\perp \ \to \sum_{i=1}^{N} 2p\,, \tag{3.77}$$

where $2p$ is the separation between the sub manifolds along the $x_\perp$ direction, and N is the number of such manifolds that belong to the stack. The choice of the planes being separated by a distance of $2p$, is to match with the microscopic description made in the beginning of the his chapter, see (3.6). The number of layers N and the linear size are tied by the relation $N = L/2p$.

Introducing this discretization, force us to re-scale the fields A_a in an appropriate way, i.e.,

$$A(x_{1D}^{\sigma_1}, \dots, x_{(D-1)D}^{\sigma_{D-1}}, x_\perp, t) \to \frac{1}{\sqrt{2p}} A^{(i)}(x_{1D}^{\sigma_1}, \dots, x_{(D-1)D}^{\sigma_{D-1}}, t)\,. \tag{3.78}$$

Then, the action (3.75) can be put into the following form

$$\mathcal{A} = \sum_{i=1}^{N} \int dt\, dx_{1D}^{\sigma_1} \dots dx_{(D-1)D}^{\sigma_{D-1}} \frac{1}{2\pi} K_{ab} A_a^{(i)} \partial_0 A_b^{(i)} + \dots . \tag{3.79}$$

In order to make further progress we make use of the transformations (3.28), these transformations allow us to put the K matrix into a block diagonal form

$$(Q^\top)^{-1} K Q^{-1} = diag \left\{ \begin{pmatrix} 0 & k_1 \\ -k_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & k_2 \\ -k_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & k_{\frac{D-1}{2}} \\ -k_{\frac{D-1}{2}} & 0 \end{pmatrix} \right\} . \tag{3.80}$$

Note that by doing so, the fields $A_a^{(i)}$ also have to change accordingly to (3.28), but this defines an equivalent theory and therefore should not affect the conclusion about the ground state degeneracy. In this new basis we can write the action as

$$\mathcal{A} = \sum_{i=1}^{N} \int dt\, dx_{1D}^{\sigma_1} \dots dx_{(D-1)D}^{\sigma_{D-1}} \sum_{j=1}^{(D-1)/2} \frac{k_j}{\pi} \tilde{A}_j^{(i)} \partial_0 \tilde{A}_{j+1}^{(i)} + \dots , \tag{3.81}$$

in this particular form we have the advantage that the action is simply a combination of $(D-1)/2$ decoupled pairs of fields. For the particular case of $D = 3$ we can explicitly write down the holonomies. For the general case of arbitrary D they are not straighforward due to the unusual form of the gauge transformations. Therefore, in $D = 3$ we have the gauge invariant lines

$$\exp\left(i \int_0^l dx_{13}^{\sigma} \tilde{A}_1^{(i)} \right) \quad \text{and} \quad \exp\left(i \int_0^l dx_{23}^{\sigma} \tilde{A}_2^{(i)} \right) , \tag{3.82}$$

Note that these holonomies are gauge invariant (up to boundary terms), since the gauge transformations are

$$\tilde{A}_1^{(i)} \to \tilde{A}_1^{(i)} + \partial_{13}^{+} \partial_{13}^{-} \zeta^{(i)} , \tag{3.83}$$

$$\tilde{A}_2^{(i)} \to \tilde{A}_2^{(i)} + \partial_{23}^{+} \partial_{23}^{-} \zeta^{(i)} . \tag{3.84}$$

We can also introduce a notion of large gauge transformation in a very straightforward way. Start by noting that under a gauge transformation, the holonomies (3.82) can change, at most, by a boundary term of the form

$$\exp\left(i\, \partial_{j3}^{-\sigma_j} \zeta^{(i)}|_{\partial R_j} \right) , \tag{3.85}$$

where ∂R_j represents the end points (boundary) of the line. The requirement that this term is proportional to the identity can be satisfied with the choice

$$\zeta^{(i)}|_{\partial R_j} = 2\pi n_j^{(i)}\, x_{jD}^{-\sigma_j} , \quad n_j^{(i)} \in \mathbb{Z} . \tag{3.86}$$

We see that for $n_j^{(i)} \neq 0$ the corresponding transformation is a *large gauge transformation*. It is straightforward to build the large gauge transformations that leave the holonomies invariant

$$\zeta^{(i)} = \sum_{j=1}^{(D-1)} 2\pi\, n_j^{(i)}\, x_{j3}^{+} x_{j3}^{-}\,, \tag{3.87}$$

Plugging (3.87) into the gauge transformation, we arrive at the desired large gauge transformations

$$\tilde{A}_1^{(i)} \to \tilde{A}_1^{(i)} + 2\pi\, n_1^{(i)}\,, \quad \text{and} \quad \tilde{A}_2^{(i)} \to \tilde{A}_2^{(i)} + 2\pi\, n_2^{(i)}\,. \tag{3.88}$$

Once again, we stress that these large gauge transformations are particular to the $D = 3$ case. For arbitrary D, the gauge transformations for the fields A becomes quite complicated and it is not obvious that these transformations can be found in a sistematic way.

Although, for the sole purpose of computing the ground state degeneracy of the D-dimensional model , it suffices to examine the field configurations satisfying the constraing $K_{ab}\, \mathcal{D}_a\, A_b = 0$. Thus, consider the solutions

$$\tilde{A}_j^{(i)}(\mathbf{x},\, t) = \frac{1}{l^{(D-1)/2}} a_j^{(i)}(t)\,, \tag{3.89}$$

with this, the action can be written as

$$\mathcal{A} = \sum_{i=1}^{(D-1)/2} \int dt\, \frac{k_j}{\pi} a_j^{(i)} \partial_0 a_{j+1}^{(i)}\,, \tag{3.90}$$

Clearly this quantum mechanical action is invariant under the shifts (up to boundary terms)

$$a_{j,j+1}^{(i)} \to a_{j,j+1}^{(i)} + \frac{\pi}{k_j}\,, \tag{3.91}$$

and the fields obeys the equal time commuation relation

$$\left[a_j^{(p)},\, a_{j+1}^{(p)}\right] = -\frac{i\pi}{k_j}\delta_{p\,p}\,. \tag{3.92}$$

The operator invariant under the above shift transformation is

$$\left(e^{i\, a_j^{(i)}}\right)^{2k_j}\,, \tag{3.93}$$

this suggest us to introduce the holonomies $e^{i\, a_{j,j+1}}$ which have following algebra

$$e^{ia_j^{(i)}}\, e^{ia_{j+1}^{(i)}} = e^{i\pi/k_j} e^{ia_{j+1}^{(i)}} e^{ia_j^{(i)}}\,, \tag{3.94}$$

from this we obtain that the degeneracy associated with each $(j, j+1)$ pair is $2k_j$, therefore the total degeneracy is given by

$$GSD(p) = \prod_{i=1}^{N} \left(\prod_{j=1}^{(D-1)/2} 2k_j \right) = \left(2^{\frac{D-1}{2}} Pf(K) \right)^N , \tag{3.95}$$

where $N = L/2$ is the number of layers on each of the 2^{D-1} stacks that we can build and $Pf(K)$ is the pfaffian of the matrix K. The degeneracy above is associated to a single stack of layers, which we label by p, the total degeneracy of the model can then be written as

$$GSD = \prod_{p=1}^{2^{D-1}} GSD(p) = \left(2^{\frac{D-1}{2}} Pf(K) \right)^{2^{D-1}N} . \tag{3.96}$$

In our construction $Pf(K) = 1$, and with this we finally arrive at the result that is compatible with (3.26)

$$GSD = 2^{(D-1)2^{D-3}L} . \tag{3.97}$$

3.4 Introducing Dynamics

In this section we consider the effect of dynamical terms in the the low-energy effective field theory (3.49) to study the spectrum of excitations. In particular, we focus on the three dimensional case of the Chamon model for simplicity. The analisys can be carried out straightforwardly to higher dimensions, although it will add an unecessary degree of complexity. We start by adding to the action an appropriate Maxwell-like term,

$$S = \int dt\, d^3x \left[\frac{1}{2g_E} F_{0a} F_{0a} + \frac{1}{4g_M} F_{ab} F_{ab} + \frac{1}{2\pi} K_{ab} A_a \partial_0 A_b + \frac{1}{\pi} A_0 K_{ab} \mathcal{D}_a A_b \right] , \tag{3.98}$$

where g_E and g_M are dimensionful couplings and

$$F_{0a} \equiv \partial_0 A_a - \mathcal{D}_a A_0 , \quad F_{ab} \equiv \mathcal{D}_a A_b - \mathcal{D}_b A_a , \tag{3.99}$$

is the fractonic field strenght, which is related to the fractonic electric and magnetic fields (3.51) in a similar fashion to the usual case in standard electrodynamics. A simple way to determine the spectrum of the excitations is to find the location of the poles of the propagator. To this, we first have to choose a convenient gauge fixing condition. Notice that we can fix a single gauge in three dimensions, since there is only a single gauge degree of freedom ζ. We will work with the choice $A_0 = 0$, for the sole purpose of computing the poles of the propagator this is the most convenient choice. Other choices of gauge fixing may make other physical aspects of the theory more transparent, but we do not investigate this in these notes.

With this gauge choice, the equations of motion for the A_a fields read

$$\left(-\frac{1}{g_E}\,\delta_{mb}\,\partial_0^2+\frac{1}{g_M}\,\left[\mathcal{D}_i\mathcal{D}_i\,\delta_{mb}-\mathcal{D}_m\,\mathcal{D}_b\right]+\frac{k}{\pi}\,\epsilon_{mb}\partial_0\right)A_b=0\,. \tag{3.100}$$

Note that we have assumed an arbitrary level k and re-wrote the K-matrix as $k\epsilon_{ab}$, where ϵ_{ab} is a two dimensional Levi-Civita symbol, with the normalization $\epsilon_{12}=1$. In momentum space, the equations of motion become

$$\left(\frac{1}{g_E}\,\delta_{mb}\,\omega^2-\frac{1}{g_M}\,\left[\mathcal{P}^2\,\delta_{mb}-\mathcal{P}_m\,\mathcal{P}_b\right]+\frac{ik\,\omega}{\pi}\,\epsilon_{mb}\right)A_b=0\,, \tag{3.101}$$

where $\mathcal{P}_m=t^I_m\,p^2_I$ and $\mathcal{P}^2=\mathcal{P}_m\mathcal{P}_m$. Explicitly, they can be written as

$$\mathcal{P}^2=p_1^2+p_2^2=\left(p_x^2-p_z^2\right)^2+\left(p_y^2-p_z^2\right)^2\,,\quad \mathcal{P}_m=\begin{pmatrix}p_1\\p_2\end{pmatrix}=\begin{pmatrix}p_x^2-p_z^2\\p_y^2-p_z^2\end{pmatrix}\,. \tag{3.102}$$

Note that (3.101) can be compactly written as $\Delta_{mb}\,A_b=0$. The propagator G_{bc} of the Δ-operator then obeys $\Delta_{mb}\,G_{bc}=\delta_{mc}$, therefore $G_{bc}=\left(\Delta^{-1}\right)_{bc}$. The poles of G_{bc} can then be read from the solutions of $\det(G)=0$. In the present case G_{bc} is simply a 2×2 matrix

$$G=\begin{pmatrix}\frac{\omega^2}{g_E}-\frac{p_2^2}{g_M} & \frac{p_1p_2}{g_M}+\frac{ik\omega}{\pi}\\ \frac{p_1p_2}{g_M}-\frac{ik\omega}{\pi} & \frac{\omega^2}{g_E}-\frac{p_1^2}{g_M}\end{pmatrix}\,, \tag{3.103}$$

and the determinant can be easily computed. From the condition $\det(G)=0$ we obtain the spectrum of the theory

$$\omega^2=g_E^2\frac{k^2}{\pi^2}+\frac{g_E}{g_M}\mathcal{P}^2\,. \tag{3.104}$$

Note is that in the limit $p_x,\,p_y,\,p_z\to 0$, the Chern-Simons-like term gives a contribution to the spectrum proportional to the level k, in a similar fashion to the topological mass contribution of the usual Maxwell Chern-Simons theory. Also, note in the limit of the pure Chern-Simons-like action, $g_E,\,g_M\to\infty$, we have that $\omega\to\infty$, and therefore the pure Chern-Simons-like action is infinitely gapped.

4 Haah model

The description presented in (55) can be extended to describe fractons of the type-II, although it is not so straightforward as we have shown in the last chapter. There are some subtleties that we have to take into account in order to obtain a compatible effective theory, but nonetheless, we are able to obtain a Chern-Simons-like description for the Haah code that captures the essential physics of the model. In this chapter we give a brief review of the lattice model (5) and then proceed to build the effective theory(56).

4.1 Lattice model

The Haah model is defined on the 3-dimensional cubic lattice Λ (5). Its Hamiltonian is written in terms of two types of cube operators, $C_{\mathbf{x}}^{(1)}$ and $C_{\mathbf{x}}^{(2)}$:

$$H_{\text{Haah}} = -\sum_{\mathbf{x}\in\,\Lambda^*} C_{\mathbf{x}}^{(1)} - \sum_{\mathbf{x}\in\,\Lambda^*} C_{\mathbf{x}}^{(2)}\,, \tag{4.1}$$

where $\mathbf{x}$ labels the cubes (or sites in the dual cubic lattice Λ^*). The cubes $C^{(1)}$ and $C^{(2)}$ are given by

$$\begin{aligned} C_{\mathbf{x}}^{(1)} &= XX_{\mathbf{x}-\hat{x}-\hat{y}-\hat{z}} IX_{\mathbf{x}-\hat{x}+\hat{y}-\hat{z}} IX_{\mathbf{x}+\hat{x}-\hat{y}-\hat{z}} IX_{\mathbf{x}-\hat{x}-\hat{y}+\hat{z}}\,, \\ &\times\ XI_{\mathbf{x}+\hat{x}+\hat{y}-\hat{z}} XI_{\mathbf{x}-\hat{x}+\hat{y}+\hat{z}} XI_{\mathbf{x}+\hat{x}-\hat{y}+\hat{z}} II_{\mathbf{x}+\hat{x}+\hat{y}+\hat{z}} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} C_{\mathbf{x}}^{(2)} &= II_{\mathbf{x}-\hat{x}-\hat{y}-\hat{z}} IZ_{\mathbf{x}-\hat{x}+\hat{y}-\hat{z}} IZ_{\mathbf{x}+\hat{x}-\hat{y}-\hat{z}} IZ_{\mathbf{x}-\hat{x}-\hat{y}+\hat{z}} \\ &\times\ ZI_{\mathbf{x}+\hat{x}+\hat{y}-\hat{z}} ZI_{\mathbf{x}-\hat{x}+\hat{y}+\hat{z}} ZI_{\mathbf{x}+\hat{x}-\hat{y}+\hat{z}} ZZ_{\mathbf{x}+\hat{x}+\hat{y}+\hat{z}}\,. \end{aligned} \tag{4.3}$$

They are also depicted in Fig.(5). The model possess a self duality in which is possible to obtain one cubic operator from another by means of a spatial inversion around the center of the cube, followed by a swap operation that exchanges the position in the tensor product of each qubit ($IX \to XI$) and finally a unitary rotation that realizes $X \to Z$ and $Z \to -X$. Therefore, the excitations of the model can be completely characterized by looking only at one sector of the model. It was shown (5) that this model does not support any string logical operator and thus all non trivial excitations are forbidden to move. This complete lack of mobility is one of the main features that characterizes the Haah code as a fracton type-II model.

The ground state subspace $\mathcal{G}$ of the model can be defined as

$$\mathcal{G} = \left\{ |\psi\rangle \,:\, C_{\mathbf{x}}^{(1)} |\psi\rangle = |\psi\rangle\,,\ C_{\mathbf{x}}^{(2)} |\psi\rangle = |\psi\rangle \right\}. \tag{4.4}$$

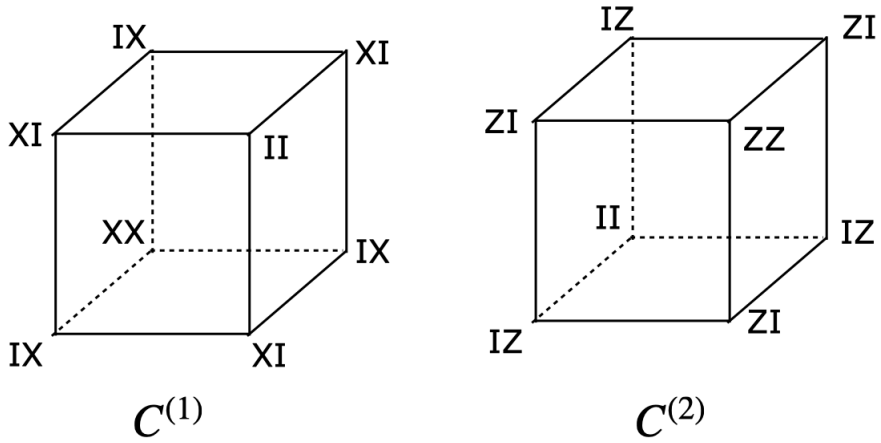


Figura 5 – Cubic operators for the Haah code. The symbols IX, IZ represents the tensor product between Pauli operators, i.e., $IX = I \otimes \sigma^x$, with I being the identity matrix.

Any excitation corresponds to a state in which $C_x^{(1)} |\psi'\rangle = -|\psi'\rangle$, such a state can be obtained by acting with a local operator in one site of the lattice, e.g., IX, this will flip the signs of four neighboring cubes such that a tetrahedral pattern is formed, as depicted in Fig.(7). These four excitations can then be separated by means of repeated application of local operators in a fractal pattern, Fig.(6) pictorially ilustrates the pattern in which the operators must act

Figura 6 – Flipping spins over a fractal shapped structure allow us to isolate single fracton excitations on its corners. (Figure from (1))

In the end, we can think of the excited states in the Haah model as being defects residing at the corners of a fractal geometrical structure, this excitations are completely immobile and therefore we denominate them as fractons.

The ground state degeneracy can be trick to find, Haah was able to write down some empirical formulas for a system with size $L \times L \times L$, although these formulas are somewhat complicated and not very enlightful. Nonetheless, it is possible to find

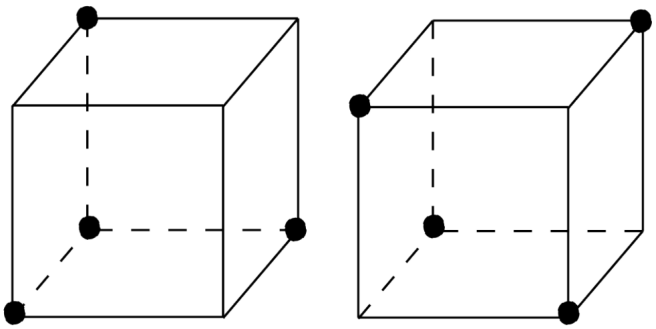


Figura 7 – Acting with a local operator creates defects in a tetrahedral pattern. The black dots indicate what are the spins being flipped.

an upper bound for the ground state degeneracy

$$\log_2(GSD) \leq 4L\,. \tag{4.5}$$

4.2 Effective description

One can construct the corresponding low-energy effective theory by following the procedure presented in (55). To this end, we write the cube operators in the Dirac basis. Consider the set of anti-commuting operators γ^I, $I \in S^{(2)}$, with $S^{(2)} = \{12, 22, 32, 01, 03\}$, where the notation ij represents the tensor product $\sigma_i \otimes \sigma_j$, with $\sigma_0 \equiv I$. In terms of this basis, the cubic operators (4.2) and (4.3) can be written as

$$\begin{aligned} C^{(1)}_{\mathbf{x}} &= \left(\gamma^1\gamma^5\right)^{\dagger}_{\mathbf{x}-\hat{x}-\hat{y}-\hat{z}} \left(\gamma^4\right)^{\dagger}_{\mathbf{x}-\hat{x}+\hat{y}-\hat{z}} \left(\gamma^4\right)^{\dagger}_{\mathbf{x}+\hat{x}-\hat{y}-\hat{z}} \left(\gamma^4\right)^{\dagger}_{\mathbf{x}-\hat{x}-\hat{y}+\hat{z}} \\ &\times \left(\gamma^2\gamma^3\right)_{\mathbf{x}+\hat{x}+\hat{y}-\hat{z}} \left(\gamma^2\gamma^3\right)_{\mathbf{x}-\hat{x}+\hat{y}+\hat{z}} \left(\gamma^2\gamma^3\right)_{\mathbf{x}+\hat{x}-\hat{y}+\hat{z}} II_{\mathbf{x}+\hat{x}+\hat{y}+\hat{z}} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} C^{(2)}_{\mathbf{x}} &= II_{\mathbf{x}-\hat{x}-\hat{y}-\hat{z}}\, \gamma^5_{\mathbf{x}-\hat{x}+\hat{y}-\hat{z}}\, \gamma^5_{\mathbf{x}+\hat{x}-\hat{y}-\hat{z}}\, \gamma^5_{\mathbf{x}-\hat{x}-\hat{y}+\hat{z}} \\ &\times \left(\gamma^1\gamma^2\right)^{\dagger}_{\mathbf{x}+\hat{x}+\hat{y}-\hat{z}} \left(\gamma^1\gamma^2\right)^{\dagger}_{\mathbf{x}-\hat{x}+\hat{y}+\hat{z}} \left(\gamma^1\gamma^2\right)^{\dagger}_{\mathbf{x}+\hat{x}-\hat{y}+\hat{z}} \left(\gamma^3\gamma^4\right)^{\dagger}_{\mathbf{x}+\hat{x}+\hat{y}+\hat{z}}\,. \end{aligned} \tag{4.7}$$

Our choice of conjugation of certain operators in both cubes is completely innocuous since all the lattice operators are Hermitian. This is simply a convenience for the description in terms of continuum fields, leading to more symmetrical forms for the T-vectors that will be constructed below. Which operators are conjugated or not reflects assignments of charges at the corners of the cube.

At this point we use the map introduced in the last chapter, which we re-write it here for convenience

$$\gamma^{(I,\alpha)}_{\mathbf{x}} \equiv \exp\left(i\, t^{(I,\alpha)}_a K_{ab} A_b(\mathbf{x})\right)\,. \tag{4.8}$$

where $a, b = 1, ..., 4$ (We need four independent fields A_a to realize a four dimensional representation of the Clifford algebra). Commutation relations between the fields $A_a(\mathbf{x})$ are set in order to reproduce the algebra of Dirac matrices,

$$[A_a(\mathbf{x}), A_b(\mathbf{x}')] \equiv i\pi (K^{-1})_{ab} \delta_{\mathbf{x},\mathbf{x}'}. \tag{4.9}$$

This, in turn, implies that the matrix K and the set of vectors $t_a^{(I)}$ must satisfy

$$t_a^{(I)} \, (K^\top)_{ab} \, t_b^{(J)} = \begin{cases} 1 \mod 2, & I \neq J \\ 0 \mod 2, & I = J, \end{cases} \tag{4.10}$$

to ensure that the representation in (4.8) reproduces properly the anticommutation relations of the $\gamma_{\mathbf{x}}^{(I)}$ matrices. For an antisymmetric matrix K, the condition in second line is satisfied exactly (not mod 2). A simple choice for K and $t_a^{(I)}$ is $K_{ab} \equiv +1$ if $a < b$ and the vectors $t_a^{(I)}$, $I = 1, \ldots, 5$ defined as $t_a^{(I)} \equiv \delta_a^I$ for $I = 1, \ldots, 4$ and $t_a^{(5)}$ defined so that the neutrality condition $\sum_{I=1}^5 t_a^{(I)} = 0$ is satisfied (this is equivalent to $\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5 = I$). These vectors identify the elements of $S^{(2)}$, and we refer to them as the *principal basis*.

The conditions (4.10) imply that the fields $A_a(\mathbf{x})$ in (4.8) are $U(1)$ compact, since the exponential is unchanged under the shifts

$$A_a \to A_a + 2\pi \sum_{J=1}^{3} t_a^{(J)} \, m_J \,, \quad m_J \in \mathbb{Z}. \tag{4.11}$$

Under these shifts the exponential changes by the factor

$$\exp\left(2\pi \, i \sum_{J=1}^{3} t_a^{(I)} \, K_{ab} \, t_b^{(J)} \, m_J \right) , \tag{4.12}$$

which is equal to one due to (4.10).

To each corner of the cubes we assign a $\Gamma^{(I,\alpha)}$ operator

$$\Gamma^{(I,\alpha)} \equiv \left(\gamma^1\right)^{T_1^{(I,\alpha)}} \left(\gamma^2\right)^{T_2^{(I,\alpha)}} \left(\gamma^3\right)^{T_3^{(I,\alpha)}} \left(\gamma^4\right)^{T_4^{(I,\alpha)}} , \tag{4.13}$$

where the index $\alpha = 1, 2$ indicates whether the operator belongs to either $C^{(1)}$ or $C^{(2)}$, and $T^{(I,\alpha)}$ can be written as a linear combination of the principal basis vectors. They have integer entries, which are defined only mod 2. (The freedom mod 2 arises from the fact that $(\gamma^I)^2 = I$.) We stress that the identification between $T^{(I,\alpha)}$ and $\Gamma^{(I,\alpha)}$ is unique on each cube.

The microscopic description has further properties that will be essential to arrive at a consistent low-energy effective field theory. These properties constrain the allowed T-vectors in the continuum theory [1]. They are summarized as follows:

[1] While the condition that the vectors $t^{(I)}$ are defined mod 2 are sufficient to ensure the proper commutation relations between Dirac matrices, the gauge invariance in the continuum depends that conditions like $\mathcal{T}_a^{(I,\alpha)} \, K_{ab} \, \mathcal{T}_b^{(J,\beta)} = 0$ are satisfied exactly, where $\mathcal{T}_a^{(I,\alpha)}$ are linear combinations of $T^{(I,\alpha)}$. These relations cannot be satisfied by the principal basis vectors $t_a^{(I)}$, but can be adjusted with $t^{(I)}$ mod 2. This is the reason we are forced to introduce the operators (4.13).

i **The $\Gamma^{(I,\alpha)}$ operators are equivalent to the γ^I, i.e.,**

$$T_a^{(I,\alpha)} = t_a^{(I)} \mod 2 \quad \text{and} \quad \sum_{I=1}^{5} T_a^{(I,\alpha)} = 0\,. \tag{4.14}$$

This condition allows for the entries of $T^{(I,\alpha)}$ to differ from the entries of $t^{(I)}$ up to the addition of an even integer, but the sum of them satisfies the neutrality condition exactly (not only mod 2), as the principal basis does.

ii **The cube operators commute, $[C_{\mathbf{x}}^{(\alpha)}, C_{\mathbf{x}'}^{(\beta)}] = 0$, for all $\alpha, \beta = 1, 2$, and $\mathbf{x}$, $\mathbf{x}'$.**

The commutation constrains the allowed T-vector to obey a set of relations. For example, the commutation relation $[C_{\mathbf{x}}^{(1)}, C_{\mathbf{x}-\hat{x}+\hat{y}+\hat{z}}^{(1)}] = 0$ implies that $\left(T^{(2,1)} + T^{(3,1)}\right) K\, T^{(4,1)} = 0 \mod 2$. There are in total 12 such relations that can be read systematically from the cubic operators. We list them explicitly in the Appendix

iii **The eight operators on the corners of each cube multiply to the identity**

This constraint is equivalent to requiring that the T-vectors corresponding to the operators at the corners of the cubes add to zero mod 2:

$$\begin{aligned} -T_a^{(1,1)} - T_a^{(5,1)} + 3\left(T_a^{(2,1)} + T_a^{(3,1)} - T_a^{(4,1)}\right) = 0 \qquad \mod 2\,, \\ -T_a^{(3,2)} - T_a^{(4,2)} + 3\left(T_a^{(5,2)} - T_a^{(1,2)} - T_a^{(2,2)}\right) = 0 \qquad \mod 2\,. \end{aligned} \tag{4.15}$$

Demanding that the conditions (ii) resulting from the commutation of the cubes vanish exactly (not only mod 2) ensures gauge invariance of the low-energy effective theory, similarly to the case of the type-I fractons studied in the last chapter. The following choices satisfies the conditions (i) and (ii):

$$T_a^{(1,1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad T_a^{(2,1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad T_a^{(3,1)} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad T_a^{(4,1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad T_a^{(5,1)} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \tag{4.16}$$

and

$$T_a^{(1,2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad T_a^{(2,2)} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad T_a^{(3,2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad T_a^{(4,2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad T_a^{(5,2)} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}. \tag{4.17}$$

However, these vectors do not satisfy condition (iii) exactly, but only mod 2. Therefore, we cannot implement (iii) at the operator level[2], instead we impose the conditions in

[2] In fact, conditions (i), (ii) and (iii) are mutually exclusive and cannot be simultaneously satisfied. Any set of T-vectors obeying two of the three conditions automatically breaks the third one.

(4.15) as constraints in the Hilbert space of the continuum theory. The specific form of the T-vectors is not unique. Other choices of these vectors are allowed as long they are compatible with conditions (i) and (ii). The constraint imposed by condition (iii) would also change accordingly, but there would not be any physical consequence in the continuum description. In this sense, there exists an equivalence class of T-vectors that defines a consistent effective theory.

The exponential map (4.8) allows us to write the cube operators in the continuum limit,

$$C_{\mathbf{x}}^{(\alpha)} \equiv \exp\left[i\,\mathcal{T}_a^{(1,\alpha)}\,K_{ab}\,A_b\right]\,\exp\left[i\,\left(\mathcal{T}_a^{(1,\alpha)}\,K_{ab}\,d_1\,A_b + \mathcal{T}_a^{(2,\alpha)}\,K_{ab}\,d_2\,A_b\right)\right] + h.c\,, \quad (4.18)$$

where the derivatives d_i are defined as

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{2}\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) \\ \partial_x\,\partial_y + \partial_y\,\partial_z + \partial_x\,\partial_z \end{pmatrix}. \quad (4.19)$$

The charge vectors $\mathcal{T}_a^{(i,\alpha)}$ are expressed as a combination of the $T_a^{(I,\alpha)}$. They are given explicitly by

$$\mathcal{T}^{(1,1)} = \begin{pmatrix} 0 \\ 4 \\ -4 \\ -2 \end{pmatrix}, \quad \mathcal{T}^{(2,1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathcal{T}^{(1,2)} = \begin{pmatrix} -6 \\ 6 \\ -4 \\ -4 \end{pmatrix}, \quad \mathcal{T}^{(2,2)} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}. \quad (4.20)$$

We remark that $\mathcal{T}^{(1,\alpha)}\,K\,\mathcal{T}^{(2,\alpha)} = 0$ and hence the terms in the exponentials in (4.18) commute.

The term in the first exponential in (4.18) originates from the product of the eight operators in the corners of the cubes in condition (iii) above. As the conditions in (4.15) cannot be enforced to vanish exactly in the continuum theory, we impose them in terms of matrix elements. In other words, we select the physical states via the conditions

$$\mathcal{T}_a^{(1,\alpha)}\,K_{ab}\,A_b\,|\text{phys}\rangle = 0, \quad \alpha = 1,2\,. \quad (4.21)$$

One might be concerned that using (4.21) to reduce the Hilbert space of the theory might lead to inconsistecies. We anticipate that this will not be the case in our description. The action we will derive in the following discussion describes only the ground state of the Haah code. Thus the physical subspace is trivially closed under time evolution, since any time dependent state $|\text{phys},\,t\rangle = e^{i\,E_0\,t}\,|\text{phys}\rangle$, with E_0 the energy of the ground state. This means that the constraint (4.21) is preserved under time evolution

$$\mathcal{T}_a^{(1,\alpha)}\,K_{ab}\,A_b\,|\text{phys},\,t\rangle = 0, \quad \alpha = 1,2\,. \quad (4.22)$$

In general, this procedure has the potential to lead to inconsistencies if one works with the full Hamiltonian (i.e., not only the ground state). In this case, the operator $\mathcal{A}^{(\alpha)} = \mathcal{T}_a^{(1,\alpha)} K_{ab} A_b$ in (4.21) no longer commutes with the Hamiltonian, and therefore (4.22) does not necessarily vanish. We emphasize that the need of the constraint (4.21) is a consequence of the fact that the conditions (i), (ii) and (iii) are mutually exclusive. In construction for the Chamon model in the last chapter (and in (55)), these conditions could be satisfied altogether and there was no need to impose any extra constraint.

Therefore, we shall consider the deformed (enlarged) theory obtained by omitting the first exponential in (4.18), and recover the physical subspace by means of (4.21). In general, the selection rule (4.21) does not define consistently a subspace which is closed under time evolution, however, we will use it only after the theory is properly projected onto the ground state (in the low-energy effective theory), where the closure under time evolution is trivial.

The redefined cube operators can be written as

$$C_{\mathbf{x}}^{(\alpha)} = \exp\left[iK_{ab}\,\mathcal{D}_a^{(\alpha)} A_b\right] + h.c\,, \tag{4.23}$$

where

$$\mathcal{D}_a^{(\alpha)} \equiv \sum_{i=1}^{2} \mathcal{T}_a^{(i,\alpha)} d_i\,. \tag{4.24}$$

In the continuum limit the Hamiltonian of the enlarged theory can be written as

$$H \sim -\sum_{\alpha} \int d^3x\, \cos\left(\mathcal{D}_a^{(\alpha)}\, K_{ab}\, A_b\right)\,. \tag{4.25}$$

The ground state corresponds to the situation where all the cosines are maximized. This can be enforced through a Lagrange multiplier $A_0^{(\alpha)}$ for each of the cubes. We thus arrive at the enlarged low-energy effective theory

$$S = \int d^3x\, dt \left[\frac{1}{2\pi} A_a\, K_{ab}\, \partial_0\, A_b + \frac{1}{\pi} A_0^{(\alpha)} K_{ab}\, \mathcal{D}_a^{(\alpha)}\, A_b\right]\,. \tag{4.26}$$

The first term of the action gives the equal-time commutation relation (4.9) while the second one corresponds to the ground state constraint. We recall that the low-energy physical subspace is obtained upon using (4.21). The action (4.26) is invariant under the gauge transformations

$$A_0^{(\alpha)} \to A_0^{(\alpha)} + \partial_0\, \zeta^{(\alpha)}\,, \tag{4.27}$$

$$A_a \to A_a + \mathcal{D}_a^{(\alpha)}\, \zeta^{(\alpha)}\,, \tag{4.28}$$

provided that $K_{ab}\, \mathcal{D}_a^{(\alpha)}\, \mathcal{D}_b^{(\beta)} = 0$, which follows directly from the requirements in (ii) above. In the following we shall discuss some properties of the effective field theory.

Explicitly, the gauge transformations can be written as

$$\begin{pmatrix} A_0^{(\alpha)} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \to \begin{pmatrix} A_0^{(\alpha)} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} + \begin{pmatrix} \partial_0 \zeta^{(\alpha)} \\ -6d_1\,\zeta^{(2)} + 2d_2\,\zeta^{(2)} \\ d_1\left(4\zeta^{(1)} + 6\zeta^{(2)}\right) - 2d_2\,\zeta^{(2)} \\ -4d_1\left(\zeta^{(1)} + \zeta^{(2)}\right) \\ -d_1\left(2\zeta^{(1)} + 4\zeta^{(2)}\right) + 2d_2\,\zeta^{(1)} \end{pmatrix}. \tag{4.29}$$

4.3 Aspects of the Effective Theory

4.3.1 Level Quantization

The matrix K entering the low-energy effective action (4.26) plays the role of a level, just as in the case of usual Chern-Simons theories. A natural question is whether it carries some notion of quantization. Coming from the lattice, the elements of the matrix K were chosen to be quantized. This was just a simple solution of the conditions in (4.10). Of course, the choice of $t_a^{(I)}$ and K is not unique. Therefore, we can think of the conditions in (4.10) as providing certain quantization conditions for the elements of the matrix K, given a set of vectors $t_a^{(I)}$. For example, with the vectors $t_a^{(I)}$ in the principal basis, the conditions (4.10) translate into the following quantization condition for the elements of the matrix K:

$$K_{IJ} = \text{odd}\,, \quad I \neq J \quad \text{and} \quad I, J = 1, \ldots, 4. \tag{4.30}$$

The next question is whether we can extract some notion of quantization exclusively from the effective theory (4.26) without making reference to the lattice, relying only on the possible existence of certain types of large gauge transformations. While this is expected in view of the fact that the level is quantized in our case, and also from the analogy with usual Chern-Simons theory the quantization of the level will unavoidably depends on some fundamental lenght scale (e.g., the lattice spacing), suggesting that the quantum theory need some fixed ultraviolet cut-off. The same problem appeared in (25). Understanding the role of this lenght dependence in the continuum theory is a very hard problem that remains unanswered at this point.

4.3.2 Excitations and Their Immobility in the Effective Field Theory

We can use the effective action (4.26) to study the low-energy properties of the Haah code after proper projection onto the physical subspace. We consider low-lying excitations parametrized by currents and couple them to the gauge fields according to $\int d^3x\,dt[J_0^{(\alpha)}\,A_0^{(\alpha)} + J_a\,A_a]$, which is gauge invariant provided that the current is conserved

$$\partial_0\,J_0^{(\alpha)} = \mathcal{D}_a^{(\alpha)}\,J_a\,. \tag{4.31}$$

This relation leads to infinitely many conserved charges (29)

$$Q_f^{(\alpha)} = \int d^3x\, f(\mathbf{x})\, J_0^{(\alpha)}\,, \tag{4.32}$$

provided that $\mathcal{D}_a^{(\alpha)} f(\mathbf{x}) = 0$. Note that the trivial case $f =$ constant gives the common notion of a global charge. Below we show that the projection onto the physical Hilbert space ensures the immobility of all excitations. To properly project onto the physical subspace consider the equation of motion of $A_0^{(\alpha)}$, which provides a "flux-attachment"relation:

$$J_0^{(\alpha)} = \frac{1}{\pi} K_{ab}\, \mathcal{D}_a^{(\alpha)}\, A_b. \tag{4.33}$$

We can use this relation to examine the corresponding conservation laws inside the physical subspace. Denoting the physical states generically as $|\text{phys}\rangle$, the relation (4.33) leads to

$$\begin{aligned} \frac{d}{dt} Q_f^{(\alpha)}\, |\text{phys}\rangle &= \int d^3x\, f(\mathbf{x})\, \frac{1}{\pi} K_{ab} \left[\mathcal{T}_a^{(1,\alpha)}\, d_1 + \mathcal{T}_a^{(2,\alpha)}\, d_2\right] \partial_0 A_b\ |\text{phys}\rangle \\ &= \int d^3x\, f(\mathbf{x})\, \frac{1}{\pi} K_{ab}\, \mathcal{T}_a^{(2,\alpha)}\, d_2\, \partial_0 A_b\ |\text{phys}\rangle\,, \end{aligned} \tag{4.34}$$

where we have used the constraint $\mathcal{T}_a^{(1,\,\alpha)}\, K_{ab}\, A_b\, |\text{phys}\rangle = 0$. Therefore, inside the physical subspace, the function $f(\mathbf{x})$ is less restricted than in the full Hilbert space, since it only needs to satisfy $d_2\, f = 0$, instead of $d_1\, f = d_2\, f = 0$. This, in turn, implies a more general set of conserved charges and consequently more restrictions on the mobility of the excitations, leading ultimately to the complete immobility of all quasiparticles.

To highlight the difference between the constrained and unconstrained cases, let us first consider solutions of $d_1\, f = d_2\, f = 0$, which can be generically writen as $f = (c_1\, l + c_0)\ h(u\,, v)$, where $h(u,\, v)$ is a harmonic function and the coordinates $(l,\, u,\, v)$ are explicitly given by

$$\hat{l} \equiv \frac{1}{\sqrt{3}} \left(\hat{x} + \hat{y} + \hat{z}\right)\,, \quad \hat{u} \equiv \frac{1}{\sqrt{2}} \left(\hat{y} - \hat{z}\right)\,, \quad \hat{v} \equiv \frac{1}{\sqrt{6}} \left(-2\hat{x} + \hat{y} + \hat{z}\right)\,. \tag{4.35}$$

For example, the density $J_0^{(\alpha)} = q^{(\alpha)} \delta(u) \delta(v)\, [\delta(l - l_0(t)) - \delta(l - l_1(t))]$ is such that for any function f of the form above, $Q_f^{(\alpha)} = c_1\, (l_0 - l_1)\, q^{(\alpha)}\, h(0,\, 0)$ is conserved provided that $l_0 - l_1$ is constant. The density $J_0^{(\alpha)}$ corresponds to a dipole moving along the (111) direction. Therefore, we find that the restrictions $d_1\, f = d_2\, f = 0$ allow for mobile excitations.

In contrast, when the condition $d_1\, f = 0$ is no longer required, the space of solutions for f is much less restricted (e.g., not forced to have at most linear dependence on l). For example, $f = c_1\, x$ implies dipole conservation along the (100) direction, and

similarly for dipole conservation along (010) and (001) directions with $f = c_2\, y$ and $f = c_3\, z$. The function $f = c_{11}\, x^2$ implies conservation of one of the components of the quadrupole tensor. In the appendix (B) we construct multinomial solutions of arbitrary order. These infinitely many conservation laws is what prevents mobility of excitations.

The mobility (or lack of thereof) can be understood in an equivalent way by means of gauge invariant line operators. In the full Hilbert space with both d_1 and d_2 operators, it is always possible to build a line operator $W = \exp\left(i \int_{\mathcal{C}} \tilde{A}_a\right)$, using a linear combination $\tilde{A}_a$ of the gauge fields, where $\mathcal{C}$ is a path along the (111) direction. For example, the linear combinations $\tilde{A}_1 = -\frac{1}{3}(A_1 + \frac{2}{3}A_2 + \frac{2}{3}A_3)$ and $\tilde{A}_2 = \frac{2}{3}(A_2 + \frac{1}{2}A_3 + \frac{1}{2}A_4)$ transform as

$$\delta\tilde{A}_1 = \partial_l^2 \zeta^{(2)} \quad \text{and} \quad \delta\tilde{A}_2 = \partial_l^2 \zeta^{(1)}, \tag{4.36}$$

so that the corresponding line operators are gauge invariant. They capture the motion of dipoles along the l-direction. Inside the physical subspace we no longer have the d_1 operator and, hence, is not possible to build these line operators. The only gauge invariant line operators are along the time direction ($\exp i \int_t A_0^{(\alpha)}$), describing immobile excitations.

Next, we study the local operators that create excitations inside the physical subspace. Consider a generic local operator $e^{i\mathsf{T}KA(\mathbf{x}')}$, where T corresponds to an integer-valued vector to be determined under the condition that the resulting state lies inside the physical subspace. This is met provided

$$\mathcal{T}^{(1,\alpha)}\, K\, \mathsf{T} = 0. \tag{4.37}$$

In this way, the state

$$|\mathsf{T}\rangle_{\text{phy}} \equiv e^{i\mathsf{T}\,K\,A(\mathbf{x}')}\ |0\rangle_{\text{phy}} \tag{4.38}$$

is physical. Consider a general vector T with $\mathsf{T} = (n_1, n_2, n_3, n_4)$, with $n_a \in \mathbb{Z}$. The conditions in (4.37) imply

$$2n_1 + 6n_2 + 6n_3 = 0 \quad \text{and} \quad 2n_1 + 2n_2 + 4n_3 - 4n_4 = 0. \tag{4.39}$$

Thus, any operator characterized by a nontrivial vector of the form

$$\mathsf{T} = (-3n + 3m,\ -n - m,\ 2n,\ m)\,, \quad n\,, m \in \mathbb{Z}\,. \tag{4.40}$$

is responsible for creating excitations in the physical subspace. Let us construct them explicitly. To this end, we compute the commutator

$$\begin{aligned}\left[J_0^{(\alpha)}(\mathbf{x})\,,\, e^{i\mathsf{T}\,K\,A(\mathbf{x}')}\right] &= \sum_{I=1}^{2} (\mathcal{T}^{(I,\alpha)}\, K\, \mathsf{T})\, e^{i\mathsf{T}\,K\,A(\mathbf{x}')} d_I\, \delta(\mathbf{x} - \mathbf{x}') \\ &= (\mathcal{T}^{(2,\alpha)}\, K\, \mathsf{T})\, e^{i\,\mathsf{T}\,K\,A(\mathbf{x}')} d_2\, \delta(\mathbf{x} - \mathbf{x}').\end{aligned} \tag{4.41}$$

Let us now determine the charge structure created by this local operator. By starting with a state $|0\rangle_{\rm phy}$ with no charge content, i.e., $J_0^{(\alpha)}(\mathbf{x})\,|0\rangle_{\rm phy}=0$, the charge content of the state $|\mathsf{T}\rangle_{\rm phy}$ is

$$J_0^{(\alpha)}(x)\;|\mathsf{T}\rangle_{\rm phy}=q^{(\alpha)}d_2\,\delta(\mathbf{x}-\mathbf{x}')\,|\mathsf{T}\rangle_{\rm phy}\,, \tag{4.42}$$

with the charges $q^{(\alpha)}$ defined as

$$q^{(\alpha)}\equiv\mathcal{T}^{(2,\alpha)}\,K\,\mathsf{T}. \tag{4.43}$$

For the two flavors of charges, one obtains

$$q^{(1)}=4(n-m)\quad\text{and}\quad q^{(2)}=4(m-2n). \tag{4.44}$$

We see that operators characterized by vectors T with $m-2n=0$ create physical excitations of flavor $\alpha=1$, while operators with $n-m=0$ produce excitations of flavor $\alpha=2$. Operators with both $n-m\neq 0$ and $m-2n\neq 0$ produce excitations of the two flavors simultaneously. With different discretization prescriptions for $d_2\,\delta(\mathbf{x}-\mathbf{x}')$ in (4.42), we can construct several charge arrangements back in the lattice, such as the tetrahedral configurations present in the original lattice model. For example, if we consider

$$\partial_i f(\mathbf{x})\to f(\mathbf{x}+\hat{x}_i)-f(\mathbf{x})\,, \tag{4.45}$$

in terms of the coordinates l, u, v, we obtain the configuration (a) in Fig. 8, since in these coordinates $d_2=\partial_l^2-(\partial_u^2+\partial_v^2)/2$. On the other hand, using the same discretization above, but in terms of coordinates x, y, z, we obtain the configuration depicted in (b) of Fig. 8, since $d_2=\partial_x\,\partial_y+\partial_y\,\partial_z+\partial_x\,\partial_z$. It is important to stress that since the charge $q^{(\alpha)}$ is only defined mod $2q^{(\alpha)}$, the black dots in Fig. 8 represents the positions of an odd number of charges, since $(2\mathbb{Z}+1)q^{(\alpha)}\sim q^{(\alpha)}$ and $2\mathbb{Z}q^{(\alpha)}\sim 0$. Naturally, the discretization procedure is not unique, since from the continuum perspective there is no *a priori* preferable way of discretizing derivatives. The above ones mimic the form of the excitations of the lattice model. Therefore, the effective field theory captures properly the spectrum of low-lying excitations of the lattice model.

4.3.3 Introducing dynamics

In this section we consider dynamical terms in the the low-energy effective field theory (4.26) to study the spectrum of excitations. We start by adding to the action an appropriate Maxwell-like term,

$$S=\int dt\,d^3x\left[\frac{1}{2g_E}\,F_{0a}\,F_{0a}+\frac{1}{4g_M}\,F_{ab}^{(\alpha)}\,F_{ab}^{(\alpha)}+\frac{1}{2\pi}\,K_{ab}\,A_a\,\partial_0\,A_b+\frac{1}{\pi}A_0^{(\alpha)}\,K_{ab}\,\mathcal{D}_a^{(\alpha)}\,A_b\right], \tag{4.46}$$

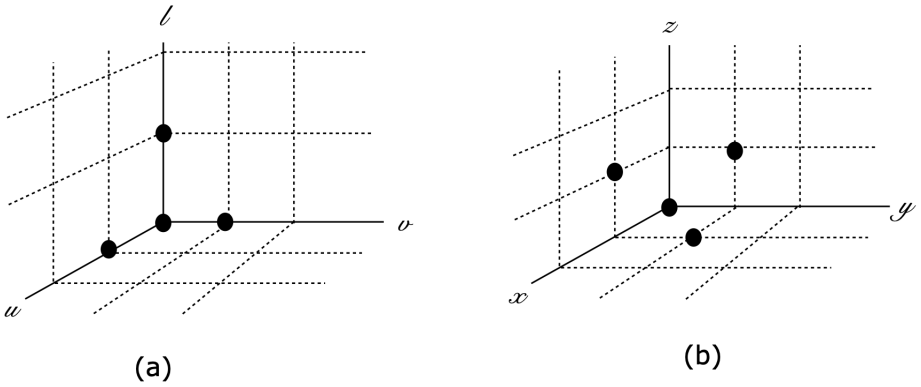


Figura 8 – Charge distributions created by the operator d_2. In configuration (a) we set the lattice spacing as $\epsilon_u = \epsilon_v = \epsilon_l/\sqrt{2}$, whereas in (b) we are considering equal lattice spacing.

where g_E and g_M are dimensionful couplings and

$$F_{0a} \equiv \partial_0 A_a - \mathcal{D}_a^{(\alpha)} A_0^{(\alpha)}\,, \quad F_{ab}^{(\alpha)} \equiv \mathcal{D}_a^{(\alpha)} A_b - \mathcal{D}_b^{(\alpha)} A_a\,. \tag{4.47}$$

are the field strenght components. A simple way to determine the spectrum of the excitations is to find the location of the poles of the propagator. To this, we first have to choose a convenient gauge fixing condition. Notice that we can fix two conditions, since there are two gauge degrees of freedom $\zeta^{(\alpha)}$. We will work with the choice $A_0^{(\alpha)} = 0$, for the sole purpose of computing the poles of the propagator this is the most convenient choice. Other choices of gauge fixing may make other physical aspects of the theory more transparent, but we do not investigate this in these notes. With this gauge choice, the equations of motion for the A_a fields read

$$\left(-\frac{1}{g_E}\,\delta_{mb}\,\partial_0^2 + \frac{1}{g_M}\,\left[\mathcal{D}_i^{(\alpha)}\mathcal{D}_i^{(\alpha)}\,\delta_{mb} - \mathcal{D}_m^{(\alpha)}\,\mathcal{D}_b^{(\alpha)}\right] + \frac{k}{\pi}\,\epsilon_{mb}\partial_0\right) A_b = 0\,. \tag{4.48}$$

Note that we have assumed an arbitrary level k and re-wrote the K-matrix as $k\epsilon_{ab}$, where the ϵ_{ab} is a 4×4 anti-symmetric matrix with the elements in the upper triangle all equal to one. In momentum space, the equations of motion become

$$\left(\frac{1}{g_E}\,\delta_{mb}\,\omega^2 - \frac{1}{g_M}\,\left[\mathcal{P}^2\,\delta_{mb} - \mathcal{P}_m^{(\alpha)}\,\mathcal{P}_b^{(\alpha)}\right] + \frac{ik\,\omega}{\pi}\,\epsilon_{mb}\right) A_b = 0\,, \tag{4.49}$$

where we have defined

$$\mathcal{P}_m^{(\alpha)} \equiv \mathcal{T}_m^{(I,\alpha)}\,p_I\,, \quad p_I = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} (p_x^2 + p_y^2 + p_z^2)/2 \\ (p_x\,p_y + p_x\,p_z + p_y\,p_z) \end{pmatrix}. \tag{4.50}$$

The corresponding propagator obeys $\Delta_{mb}\,G_{bc} = \delta_{mc}$, i.e., $G_{bc} = \left(\Delta^{-1}\right)_{bc}$. The poles of G_{bc} then follow from $\det(G) = 0$, which in general is a very complicated function of ω, p_x, p_y, p_z. Since we are mostly interested in unveiling the gap of the

excitations, we can focus on the limit $p_x, p_y, p_z \to 0$, where the determinant $\det(G)$ simplifies dramatically. In this limit the poles of G can be obtained by solving the following quartic equation

$$\omega^4 - 6\left(\frac{k}{\pi}\right)^2 g_E^2\,\omega^2 + \left(\frac{k}{\pi}\right)^4 g_E^4 = 0\,, \tag{4.51}$$

which leads to the following mass gaps

$$\omega_1 = \frac{k g_E}{\pi}\left(1+\sqrt{2}\right) \quad \text{and} \quad \omega_2 = \frac{k g_E}{\pi}\left(-1+\sqrt{2}\right). \tag{4.52}$$

In the limit where we have the pure Chern-Simons-like action, $g_E, g_M \to \infty$, the gap becomes infinitely large, which shows that the effective action (4.26) is fully gapped.

It is interesting to trace back the physical origin of the two independent excitations we have found. This comes from the number of physical components of the gauge fields, namely, we have two pairs of fields, with each pair giving rise to an independent excitation. This is a direct reflex of the number of degrees of freedom per site of the lattice model, where the local Hilbert space accommodates two spin-1/2 degrees of freedom.

To further discuss this point, it is instructive to pursue a quantum mechanical analogy. The action (4.46) can be transformed into a simple quantum mechanical problem by studying field configurations depending only on time, $A_a \to \frac{1}{V^{1/2}} A_a(t)$, with V being the spatial volume of the system. With this, the action (4.46) reduces to

$$S = \int dt\, \left[\frac{1}{2g_E}\,\dot{A}_a^2 + \frac{k}{2\pi}\,\epsilon_{ab}\,A_a\,\dot{A}_b\right]. \tag{4.53}$$

With a simple change of basis we can split the action into two independent problems, with each one involving a pair of fields. This is achieved through an orthogonal transformation $A \to Q\,A$, where the matrix Q is given by

$$Q = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1+\sqrt{2}}{2\sqrt{3+2\sqrt{2}}} & -\frac{1}{2} & \frac{1-\sqrt{2}}{2\sqrt{3-2\sqrt{2}}} & -\frac{1}{2} \\ \frac{2+\sqrt{2}}{2\sqrt{3+2\sqrt{2}}} & 0 & \frac{2-\sqrt{2}}{2\sqrt{3-2\sqrt{2}}} & 0 \\ \frac{1+\sqrt{2}}{2\sqrt{3+2\sqrt{2}}} & \frac{1}{2} & \frac{1-\sqrt{2}}{2\sqrt{3-2\sqrt{2}}} & \frac{1}{2} \end{pmatrix}, \tag{4.54}$$

satisfying $QQ^\top = 1$ and $\det Q = 1$. The effect of this transformation in the action (4.53) is to leave the matrix ϵ in a block-diagonal form

$$Q\,\epsilon\,Q^\top = \begin{pmatrix} 0 & 1+\sqrt{2} & 0 & 0 \\ -(1+\sqrt{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & -1+\sqrt{2} \\ 0 & 0 & -(-1+\sqrt{2}) & 0 \end{pmatrix}. \tag{4.55}$$

In the new basis, the action (4.53) splits into two independent Landau problems with different magnetic fields,

$$S = S_1 + S_2\,, \tag{4.56}$$

where

$$S_1 \equiv \int dt \sum_{i=1}^{2} \left[\frac{1}{2g_E} \dot{A}_i^2 + \frac{k(1+\sqrt{2})}{2\pi} \epsilon_{ij} A_i \dot{A}_j \right] \tag{4.57}$$

and

$$S_2 \equiv \int dt \sum_{i=1}^{2} \left[\frac{1}{2g_E} \dot{A}_i^2 + \frac{k(-1+\sqrt{2})}{2\pi} \epsilon_{ij} A_i \dot{A}_j \right], \tag{4.58}$$

with ϵ_{ij} being the two-dimensional antisymmetric symbol defined by $\epsilon_{12} \equiv 1$. The gap between the Landau levels is of the form $\omega = B/m$, where B is the magnetic field and m is the mass of the particle. From (4.57) and (4.58) we can identify $m_1 = m_2 \equiv g_E^{-1}$, $B_1 \equiv (1+\sqrt{2})k/\pi$ and $B_2 \equiv (-1+\sqrt{2})k/\pi$, so that the corresponding gaps between the Landau levels are

$$\omega_1 = \frac{g_E\, k}{\pi} \left(1+\sqrt{2}\right), \qquad \omega_2 = \frac{g_E\, k}{\pi} \left(-1+\sqrt{2}\right), \tag{4.59}$$

matching precisely the result obtained from the poles of the propagator.

5 Final Remarks

5.1 Final Remarks

In this thesis we focused on building effective descriptions of fracton theories, we constructed a prescription such that given a lattice fracton model one is able to identify a consistent continuum description in terms of Chern-Simons-like theories. The construction is generic in that it applies to any system whose microscopic Hamiltonian is a sum of commuting projectors built from tensor products of spin-1/2 operators. Instead of working directly with tensor products of Pauli operators that represent the local variables, we utilize the Dirac representation of Clifford algebras. This representation makes a connection between the lattice model and the field theory simple. Our formalism can, in principle, be used to analyze other lattice models, such as those that exhibit subsystem symmetry protected topological (SSPT) phases (48, 21) or type II fracton phases (5). As we have done in the second half of the work.

In the field theory, the algebraic structure of the Dirac matrices is encoded in an anti-symmetric matrix K. The details about an specific lattice model enter via this matrix K (whose dimension depends on the size of the representation), the charge vectors T (that specify the operators that are placed on the sites), as well as the lattice vector positions of the sites themselves. Given these data, one can follow the prescription here presented and derive an effective field theory for any type of Clifford-like fracton, such as the 3D Chamon (with a 2×2 Dirac representation) or the 3D Haah (with a 4×4 Dirac representation) codes. As a concrete example, we built fracton theories in odd D spatial dimensional spaces. We discussed the properties of the resulting Chern-Simons-like theory, such as their currents, which are conserved in sub-manifolds, and the topological degeneracy of the ground states, which formally depends on the Pfaffian of the matrix K and, as usual in fracton systems, on the linear size of the system.

For the Haah code, we have been able to derive a low-energy effective field theory directly from the lattice model, by means of the same prescription as before. The effective theory is a 3+1 dimensional quadratic gauge theory of, once again, the Chern-Simons-type and hence fully gapped. For this case, the prescription we described is not enough to ensure the consistency of the continuum theory. The action must be supplemented with an extra condition, which is responsible for selecting physical states. This extra constraint emerges from the representation of the identity in the continuum, which is not automatic when the lattice operators are represented in terms of fields via the exponential map. Outside the physical subspace the theory contains

mobile excitations and thus corresponds to a type-I fractonic theory. However, inside the physical subspace, all the excitations are completely immobile due to the conservation of infinitely many charges that effectively takes place in this subspace. Hence, we think of the type-II fractonic character as embedded into a type-I fractonic theory.

The physical properties inside the constrained subspace are dictated by the differential operator $d_2 = \sum_{ijk} \left|\epsilon^{ijk}\right| \partial_j \partial_k$. In particular, the infinite set of conserved charges constructed out of functions that are annihilated by d_2 is an essential ingredient to ensure that all excitations are completely immobile. In this way, it is natural to ask whether it is possible to construct a consistent effective theory involving a single derivative operator like d_2, without the need of any extra condition for selecting physical states. The action would be of the form (4.26) with $\mathcal{T}_a^{(1,\alpha)} = 0$ in (4.24) and $\mathcal{T}_a^{(2,\alpha)}$ satisfying $\mathcal{T}_a^{(2,\alpha)} K_{ab} \mathcal{T}_b^{(2,\beta)} = 0$ (for gauge invariance). This would be a pure type-II fracton theory, in the sense that it is not embedded into a type-I one. Reversing the steps to obtain the microscopic model from the effective theory is possible, although it does not lead to a consistent lattice model since the dictionary between $\Gamma^{(I,\alpha)}$ and $T^{(I,\alpha)}$ is not one-to-one, i.e., two distinct $T^{(I,\alpha)}$ are mapped into the same $\Gamma^{(I,\alpha)}$. Thus, it is possible to have a type-II fracton theory in the continuum with no obvious lattice model. The details can be found in the Appendix E . Whether there exists a lattice model for this pure type-II fracton continuum theory remains an open question.

Properties such as the mutual statistics of the quasiparticles where not explored in the present work. The restricted mobility of fractons makes it unnatural to speak of standard braiding. However, the authors in (70) were able to develop a theory of fusion and statistical processes that incorporates the mobility restrictions common in fracton models. An interesting question for future exploration is how our formalism could incorporate their notion of statistics.

For readers familiar with the K-matrices and charge vectors T appearing in the description of Abelian fractional quantum Hall states (54), as well as their quantum wire constructions (71, 72, 73, 74), it is tempting to expect that the description here presented – for "integer" fractons given our K and T's – could possibly lend itself to the analysis of fractional fractons. This is an intriguing possibility that merits further investigation, but keeping the following points in mind.

Another interesting route for the future is due to the resemblance of the ideas here presented and the quantum wire constructions of topological phases. Here, instead of wires, we deploy $(0+1)$-dimensional degrees of freedom, i.e., ours is a "quantum dot" construction. Like in the wire constructions, we identify families of commuting operators that can be simultaneously pinned and gap the system. In the wire systems, fractionalization already takes place in the $(1+1)$-dimensional building blocks, and it is carried over to higher dimensions by coupling the wires, notably using only integer charge transfer operators. However, there is no fractionalization

in the quantum dots of the construction of this paper. Of course, one may generalize the construction presented here to start with wires instead of dots, in which case fractionalization may appear more easily.

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Appendices

APÊNDICE A – Quantum Gauge Theories

In this appendix we want to give a brief of review of some aspects of lattice quantum gauge theories that are used in the main text. This discussion is based on (3).

A.1 Simple Gauge Theories

The machinery of gauge theories is probably one of the most powerful tools we have to describe nature. Gauge theories is a name for a theory that has redundancies in his description, in the language of quantum mechanics, it is the same to say that a given quantum state can be labeled with the same set of parameters as another different quantum state. If two states $|a\rangle$ and $|b\rangle$ are related by a gauge transformation, we say that they have a gauge symmetry and the theory between them is a gauge theory.

We shall avoid the terminoly 'gauge symmetry' in what follows, since the 'gauge symmetry' is not a symmetry in the canonical way. If two states are related by a gauge transformation, they are the same state simply with different labels, thus, being two representations of the same state, it is not a symmetry as we are used to think. For this reason, we shall use the term *gauge structure* every time we can relate two states by a gauge transformation. To give a first taste of how we can define a gauge theory, we shall start with a simple example in which we have a particle propagating along a one dimensional periodic potential. The Hamiltonian is written as

$$H = \frac{1}{2m}p^2 + V\cos\left(2\pi x/a\right), \tag{A.1}$$

and the Hilbert space of this theory is given by the set of states which are labeled by the position of the particle

$$\mathcal{H} = \{|x\rangle\}. \tag{A.2}$$

It is necessary to stress that the Hamiltonian alone is not sufficient to specify the system, instead, we need to specify as well, the Hilbert space. In other words, the system is characterized by the pair $(H, \mathcal{H})$. Note, however, that the Hamiltonian (A.1) have a translation symmetry

$$T_a^\dagger H T_a = H, \quad T_a|x\rangle = |x+a\rangle. \tag{A.3}$$

We can modify the initial theory given by (A.1) and (A.2), by including the symmetry (A.3) into the gauge structure. The resulting theory is given by

$$H = \frac{1}{2m}p^2 + V\cos\left(2\pi x/a\right), \qquad \mathcal{H}_a = \left\{|\psi\rangle\,,\, T_a\,|\psi\rangle = |\psi\rangle\right\}. \tag{A.4}$$

The new theory $(H, \mathcal{H}_a)$ is our gauge theory. Note that, in order to define the gauge theory we modified the Hilbert space while we kept the same Hamiltonian. The new Hilbert space is a subspace of the original Hilbert space, which is invariant under the gauge transformation. Since the Hamiltonian possess the symmetry, it also within the invariant subspace. We refer to T_a as the *gauge transformation* and the states $|\psi\rangle$ in the new Hilbert space are the *gauge invariant states*. The gauge invariant theory (A.3) describes the motion of a particle along a circle.

Note that on a circle, the states $|x\rangle$ and $|x+a\rangle$ are the same state. For our purposes, x and $x+a$ are just labels, which are labeling the same quantum state. A gauge transformation is just an operation that relates these two labels which are describing the same state. In this sense, a gauge transformation is just a redundancy in the description, and thus, only gauge invariant quantities can carry physical information. If a state is not gauge invariant, then it does not belong to the physical Hilbert space. In our simple example of the particle moving in the circle, an unphysical wavefunction would be any function $\psi(x)$ that is not periodic. All observables, i.e., physical operators, must also be gauge invariant $O = T_a^\dagger O T_a$, if they were not, the action of such operators would take us outside the physical Hilbert space as well. We stress again, only gauge invariant quantities are meaningful in a physical sense.

Although this example gave us a brief idea of what is the procedure to build a gauge theory, in the following we shall analyze more interesting cases which are widely used in modern research, and also, a necessary background for the discussions in this work.

A.1.1 $\mathbb{Z}_2$ lattice gauge theory

Now we want to introduce a more interesting theory, but still, a simple one. We want to consider a theory with a $\mathbb{Z}_2$ gauge field. The $\mathbb{Z}_2$ gauge theory is interesting because is the simplest case where we can obtain a topological theory. To start the discussion, let us define the Hilbert space of the theory, and to do that we will consider the square presented in Fig.9. The square is labeled **i**, on every link we assign a spin variable $s_{ij} = \pm 1$. The states in the Hilbert space are labeled by the configurations of s_{ij} and we are going to write the state as $|s_{ij}\rangle$. The difference of this model to a Ising model is that the labelings s_{ij} are not one-to-one, instead, they are two-to-one in the case at hand. There are two gauge equivalent configurations that labels the same

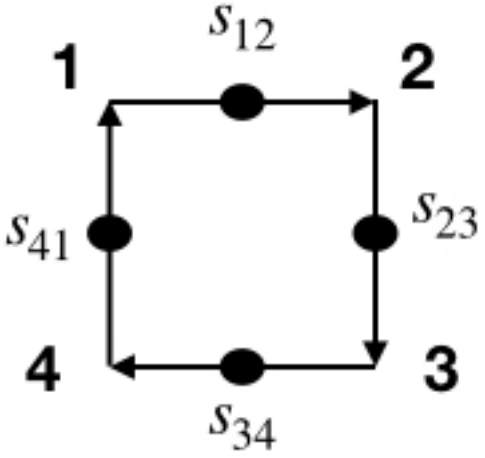

Figura 9 – A $\mathbb{Z}_2$ gauge theory on a square.

state. A $\mathbb{Z}_2$ gauge transformation can be represented by a function W_i which assumes values ± 1, then, two configurations s_{ij} and $\tilde{s}_{ij}$ are equivalent if they are related by

$$\tilde{s}_{ij} = W_i s_{ij} W_j^{-1}. \tag{A.5}$$

Thus, from (A.5) we can argue that the gauge transformation defines a equivalence relation. We can group all the equivalent configurations to form classes, such classes will be denominated *gauge equivalent classes*. Then, although the original Hilbert space had a two-to-one labeling of states, when we restrict ourselves to lay into the equivalent classes of the model, the Hilbert space acquire a one-to-one correspondence. As an example, consider the gauge theory defined in Fig.9. There are 4 links, each link has a binary variable which give us $2^4 = 16$ possible configurations $|s_{ij}\rangle$. Since there 4 sites, we also have $2^4 = 16$ possible choices for the $\mathbb{Z}_2$ gauge transformations. If the number of different configurations equals the number of the allowed gauge transformations, then there exists only one gauge equivalent class and therefore these states have a one-to-one correspondence. However, note that there two transformations that are inocuous

$$W_i = 1, \quad W_i = -1. \tag{A.6}$$

The two gauge transformations in (A.6) form a group, which is called *Invariant Gauge Group*, which in this case have two elements. Thus, (A.6) reduces the number of gauge transformation by 2 on each link, or, the number of gauge equivalent configurations is reduced by a factor which is the number of elements in the Invariant Gauge Group. The net result is that we are left with only 8 gauge transformations from the initial 16 we had. Thus, we only have two gauge equivalent classes, and hence the Hilbert space for the $\mathbb{Z}_2$ theory have dimension 2. The next step now, is that we need to find a way to explicitly label each of these states. In order to do this, we need gauge invariant

quantities, it is not hard to convince itself that such an operator $U(C)$ if written as

$$U(C) = s_{ij}s_{jk}\dots s_{li}, \tag{A.7}$$

is gauge invariant. The $i, j, k, l \dots$ represents the sites where the loop passes. $U(C)$ is commonly referred to as $\mathbb{Z}_2$ flux operator. Since $U(C)$ is defined in terms of the spins, it also can only assume values ± 1, and this tell us that the two states on the $\mathbb{Z}_2$ gauge theory on the square are labeled by these two values of $U(C)$. Is not difficult to extend this analisys to a finite lattice. On a finite square lattice with N sites and has periodic boundary conditions in both directions, we have $2N$ nearest neighbour links, which tell us that there 2^{2N} possible configurations of s_{ij}, with N sites we can build 2^N different gauge transformations, and since the transformation is $\mathbb{Z}_2$ the number of non innocuous gauge transformations is in fact $2^N/2$. Putting all of this together, we find that there is $2 \times 2^{2N}/2^N = 2 \times 2^N$ different gauge equivalent classes, which correspond to 2×2^N different states. A good attempt to label all these states is to consider the $\mathbb{Z}_2$ flux through a plaquette, which can be written as

$$F_i = s_{i,i+x}s_{i+x,i+x+y}s_{i+x+y,i+y}s_{i+y,i}. \tag{A.8}$$

Again, these flux operators are binary, $F_i = \pm 1$, which naively make us think that they would be able to give 2^N labels, but note that they obey the constraint

$$\prod_i F_i = 1. \tag{A.9}$$

Which reduces the number of labels by a factor of 2. Thus, the fluxes in (A.8) only give us 2^{N-1} labels for the 2×2^N states. Each flux configuration will correspond to four different states. The constraint (A.9) is valid for any finite lattice with periodic conditions, we can see that independent of the lattice being even-even, even-odd or odd-odd, each spin appear twice in a product like (A.9), which guarantees the equality in (A.9).

We still have to figure it out how to distinguish the four configurations that the operators (A.8) can not label. To do this, let us consider a configuration s^0_{ij} from which we can write

$$s^{(m,n)}_{ij} = f^m_x(ij)f^n_y(ij)s^0_i j, \quad m, n = 0, 1. \tag{A.10}$$

The functions $f_{x,y}(ij)$ take values ± 1, $f_{x,y}(ij) = -1$ if the link (ij) crosses the $x(\text{or}y)$ line, and it will be equal to 1 otherwise. See Fig.10 for a pictorial representation. The four configurations give rise to the same $\mathbb{Z}_2$ flux on every plaquette, since the x or y lines will cross each plaquette two times, which give us a $(-1)^2$ contribution to the flux. But, they are not gauge equivalent, because they correspond to the insertion of a $\mathbb{Z}_2$ flux around eachof the two cycle in the torus. Thus, the configurations (A.10) generates four gauge equivalent classes. Thus, the 2^{N-1} labels coming from the flux

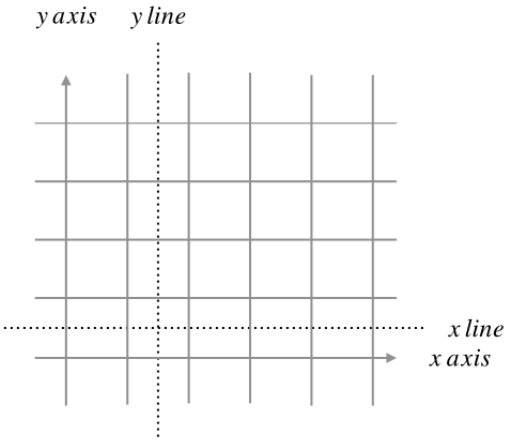

Figura 10 – Links crossing the x and/or y lines may receive an additional minus sign.

configurations (A.8) plus the four fold degeneracy give us all the labels necessary to describe the 2×2^N states in the $\mathbb{Z}_2$ gauge theory. We shall give an explicitly example later in this appendix to make all these ideas clearer. In the following we want to build the Hamiltonian for this kind of theory. The Hamiltonian of the model need a few properties. First of all, it has to be a local operator and has to be gauge invariant, since s_{ij} is a many-to-one label of the system, we need to make sure that each equivalent label gives rise to the same energy. The gauge invariance recquirement make our life much easier, because we already have these operators, the fluxes in (A.8). But they are not the only operators we could pick, for example, consider a operator σ^1_{ij} that flips the sign of the spin s_{ij}, i.e., $\sigma^1_{ij} : s_{ij} \to -s_{ij}$, this operator is also gauge invariant. To see this, consider two gauge equivalent configurations that are related as $W\left|\{s_{ij}\}\right\rangle = \left|\{\tilde{s}_{ij}\}\right\rangle$, now consider the gauge transformation of the configuration in which we flip the spin, i.e., $W\left(\sigma^1\left|\{s_{ij}\}\right\rangle\right) = -W\left|\{s_{ij}\}\right\rangle = -\left|\{\tilde{s}_{ij}\}\right\rangle$. This state can be written as

$$W\sigma^1 W^{-1}\left|\{\tilde{s}_{ij}\}\right\rangle = -\left|\{\tilde{s}_{ij}\}\right\rangle \quad \Rightarrow \quad W\sigma^1 W^{-1} = \sigma^1. \tag{A.11}$$

Thus, we see that in order for (A.11) to be consistent, the σ^1 operator must also be gauge invariant. This lead us to propose the following local Hamiltonian for the $\mathbb{Z}_2$ gauge theory

$$H_{\mathbb{Z}_2} = -g\sum_i F_i - t\sum_{\langle ij\rangle}\sigma^1_{ij}, \tag{A.12}$$

where $\langle ij\rangle$ means that we are summing over first neighbours. To analyze a few properties of the Hamiltonian (A.12) let us first consider the case where $t = 0$ and $g > 0$, in this situation the ground state of $H_{\mathbb{Z}_2}$ corresponds to $F_i = +1$ and it is four fold degenerate. The excitations of the model corresponds to flipping the sign of the F_i's, these excitations behaves like local particles and are called $\mathbb{Z}_2$ vortices. The constraint (A.9) forces the $\mathbb{Z}_2$ vortices to be created in pairs and the energy gap of these vortices is of order g. It's also possible to show that these vortices behave like bosons.

As a last discussion for this appendix, we would like to point out the topological degeneracy and the phases of the model. The topological degeneracy is, by definition, a kind of degeneracy that can not be lifted by any local perturbations. To understand the robustness of the model, let us treat the t term as a perturbation and analyze if this term can lift the four fold degenerate ground states of the $t = 0$, $\mathbb{Z}_2$ model. The four degenerate ground states are given by (A.10) with $s_{ij}^0 = 1$, the only way to change one state $s_{ij}^{(m,n)}$ into a different state $s_{ij}^{(m',n')}$ is to flip the signs of s_{ij}^0 on a line all the way around the torus. If the torus is is formed by a $L \times L$ lattice, then in order to change to a different ground state we need at least L σ^1 operators, and this is only accomplished at order L, at least, in perturbation theory. This means that the t term in the Hamiltonian can lift the four fold degeneracy, however, at this order in perturbation theory the energy splitting, ΔE is of order t^L/g^{L-1}. Since t/g is a small parameter, in the thermodynamic limit $L \to \infty$ means that $\Delta E \to 0$, and thus, the degeneracy is not lifted. Note that this discussion does not depend on any symmetry of the system, only on the topology of the manifold where the system lives. This is the robustness (assuming the thermodynamic limit) of the four fold degeneracy, and it can be used to classify the phase, in fact, it is one of the most important properties of the $\mathbb{Z}_2$ gauge theory and it is usually named as a *topological degeneracy*. The four fold degeneracy reflects the low energy dynamics of the $\mathbb{Z}_2$ flux. The $\mathbb{Z}_2$ gauge theory is a non local theory, and the non locality here means that the total Hilbert space of the theory can not be expressed as a direct product of local Hilbert spaces.

APÊNDICE B – Higher Moment Conservations

Consider the a general polynomial in three variables, (x, y, z), of degree d written in terms of a monomial basis $\mathbb{R}_d(x, y, z)$

$$P^{(d)} = \sum_{\alpha} p_\alpha \, X^\alpha \,, \tag{B.1}$$

where $p_\alpha = p_{abc}$ and $X^\alpha = x^a \, y^b \, z^c$ and α runs over all the elements that belongs to $\mathbb{R}_d(x, y, z)$. In the following we shall ommit that these are polynomials in (x, y, z), we shall write simply $\mathbb{R}_d$ and we will implicitly assume that these are polynomial in the three variables already mentioned. The basis $\mathbb{R}_d$ have dimension equal to

$$\kappa_d = dim(\mathbb{R}_d) = \frac{(d+2)!}{d!\,2!} \,. \tag{B.2}$$

In the main text we argued that there are infinitely many conserved charges, given that one find a polynomial P such that $d_2 \, P = 0$, with d_2 a linear differential operator that realize the mapping $d_2 \, : \, \mathbb{R}_d \, \to \, \mathbb{R}_{d-2}$. Then our problem amounts to find the kernel of the differential operator d_2. The dimension of the kernel can be easily found by counting the number of free coefficients after solving $d_2 \, P \, = \, 0$. Therefore, the dimension of the kernel is

$$dim \, (ker(d_2)) = \kappa_d - \kappa_{d-2} = \frac{(d+2)(d+1) - d(d-1)}{2} \,. \tag{B.3}$$

The elements of the monomial basis $\mathbb{R}_d$ can be represented as

$$\mathbb{R}_d = \left\{ x^d, \, x^{d-1} y, \, x^{d-1} \, z, \, x^{d-2} \, y^2, \, x^{d-2} \, y \, z, \, x^{d-2} \, z^2, \, \ldots y^d, \, y^{d-1} \, z, \, y^{d-2} \, z^2, \, \ldots, \, z^d \right\} \,, \tag{B.4}$$

and we associate a coefficient p_α for each of the elements above, i.e., p_1 is associated with the element in the first entry, a p_2 is associated with the second entry, and so on. It is trivial to note that any polynomial built from $\mathbb{R}_0$ and $\mathbb{R}_1$ trivially satisfies $d_2 \, P = 0$ and the dimension of the kernel is 1 and 3 respectively. The first non trivial case happens for polynomials built from $\mathbb{R}_2$ with a generic form

$$P^{(2)} = p_1 \, x^2 + p_2 \, x \, y + p_3 \, x \, z + p_4 \, y^2 + p_5 \, y \, z + p_6 \, z^2 \,, \tag{B.5}$$

solving $d_2 \, P = 0$ fixes $p_2 = -p_3 - p_5$, and therefore a solution is the polynomial

$$P^{(2)} = p_1 \, x^2 + p_3 \, (\, x \, z - \, x \, y) + p_4 \, y^2 + p_5 \, (\, y \, z - \, x \, y) + p_6 \, z^2 \,, \tag{B.6}$$

this polynomial is a solution for any p_1, p_3, p_4, p_5, $p6$ and will lead to the conservation of some component (or combination of components) of the quadrupole moment. The same analysis holds for higher degrees, in the following we write some solutions of higher orders,

$$P^{(3)} = \sum_{\alpha \notin \pi^{(3)}} p_\alpha \, X^\alpha + (p_6 - p_3 + p_9)\, x^2\, y + (p_6 - p_8 + p_9)\, x\, y^2 - (2p_6 + 2p_9)\, x\, y\, z\,, \tag{B.7}$$

$$\begin{aligned} P^{(4)} = \sum_{\alpha \notin \pi^{(4)}}^{15} p_\alpha \, X^\alpha &+ \left(\frac{2p_6}{3} - p_3 - p_{10} - p_{14}\right) x^3\, y + (p_6 - 3p_{10} + p_{13} - 3p_{14})\, x^2\, y^2 \\ &+ (3p_{10} + 3p_{14} - 2p_6)\, x^2\, y\, z + \left(\frac{2p_{13}}{3} - p_{10} - p_{12} - p_{14}\right) x\, y^3 \\ &+ (3p_{10} - 2p_{13} + 3P_{14})\, x\, y^2\, z - (3p_{10} - 3p_{14})\, x\, y\, z^2\,, \end{aligned} \tag{B.8}$$

$$\begin{aligned} P^{(5)} = \sum_{\alpha \notin \pi^{(5)}} p_\alpha X^\alpha &+ \left(\frac{p_6}{2} - p_3 - \frac{p_{10}}{2} + p_{15} + p_{20}\right) x^4\, y + (p_6 - 2p_{10} + 6p_{15} - p_{19} + 6p_{20})\, x^3\, y^2 \\ &+ (2p_{10} - 2p_6 - 4p_{15} - 4p_{20})\, x^3\, y\, z + (6p_{15} - p_{10} + p_{18} - 2p_{19} + 6p_{20})\, x^2\, y^3 \\ &+ (3p_{10} - 12p_{15} + 3p_{19} - 12p_{20})\, x^2\, y^2\, z + (6p_{15} - 3p_{10} + 6p_{20})\, x^2\, y\, z^2 \\ &+ \left(p_{15} - p_{17} + \frac{p_{18}}{2} - \frac{p_{19}}{2} + p_{20}\right) x\, y^4 + (2p_{19} - 4p_{15} - 2p_{18} - 4p_{20})\, x\, y^3\, z \\ &+ (6p_{15} - 3p_{19} + 6p_{20})\, x\, y^2\, z^2 - (4p_{15} + 4p_{20})\, x\, y\, z^3\,, \end{aligned} \tag{B.9}$$

$$\begin{aligned} P^{(6)} = \sum_{\alpha \notin \pi^{(6)}} p_\alpha \, X^\alpha &+ \left(\frac{2p_6}{5} - p_3 - \frac{3p_{10}}{10} + \frac{2p_{15}}{5} - p_{21} - p_{27}\right) x^5\, y \\ &+ \left(p_6 - \frac{3p_{10}}{2} + 3p_{15} - 10p_{21} + p_{26} - 10p_{27}\right) x^4\, y^2 + \left(\frac{3p_{10}}{2} - 2p_6 - 2p_{15} + 5p_{21} + 5p_{27}\right) x^4\, y\, z \\ &+ (4p_{15} - p_{10} - 20p_{21} - p_{25} + 4p_{26} - 20p_{27})\, x^3\, y^3 + (3p_{10} - 8p_{15} + 30p_{21} - 4p_{26} + 30p_{27})\, x^3\, y^2\, z \\ &+ (4p_{15} - 3p_{10} - 10p_{21} - 10p_{27})\, x^3\, y\, z^2 + \left(p_{15} - 10p_{21} + p_{24} - \frac{3p_{25}}{2} + 3p_{26} - 10p_{27}\right) x^2\, y^4 \\ &+ (30p_{21} - 4p_{15} + 3p_{25} - 8p_{26} + 30p_{27})\, x^2\, y^3\, z + (6p_{15} - 30p_{21} + 6p_{26} - 30p_{27})\, x^2\, y^2\, z^2 \\ &+ (10p_{21} - 4p_{15} + 10p_{27})\, x^2\, y\, z^3 + \left(\frac{2p_{24}}{5} - p_{21} - p_{23} - \frac{3p_{25}}{10} + \frac{2p_{26}}{5} - p_{27}\right) x\, y^5 \\ &+ \left(5p_{21} - 2p_{24} + \frac{3p_{25}}{2} - 2p_{26} + 5p_{27}\right) x\, y^4\, z + (4p_{26} - 10p_{21} - 3p_{25} - 10p_{27})\, x\, y^3\, z^2 \\ &+ (10p_{21} - 4p_{26} + 10p_{27})\, x\, y^2\, z^3 - (5p_{21} + 5p_{27})\, x\, y\, z^4\,, \end{aligned} \tag{B.10}$$

where $\pi^{(3)} = \{p_2,\, p_4,\, p_5\}$, $\pi^{(4)} = \pi^{(3)} \cup \{p_7,\, p_8,\, p_9\}$, $\pi^{(5)} = \pi^{(4)} \cup \{p_{11},\, p_{12},\, p_{13},\, p_{14}\}$, $\pi^{(6)} = \pi^{(5)} \cup \{p_{16},\, p_{17},\, p_{18},\, p_{19},\, p_{20}\}$. This construction continues to higher degrees polynomials, in each step the number of linearly dependent coefficients increases with the degree of the previous polynomial, i.e., for a polynomial of degree equal to seven

the coefficients contained in $\pi^{(7)} = \pi^{(6)} \cup \{p_{22}, p_{23}, p_{24}, p_{25}, p_{26}, p_{27}, \}$ will be given in terms of all others. In general, all linear dependent coefficients p_i will be contained in the set

$$\pi^{(d)} = \pi^{(d-1)} \cup \{p_{i+2}, p_{i+3}, \ldots, p_{i+d}\} \ , \quad d > 2 \, . \tag{B.11}$$

where p_i is the coefficient that sits in the last entry of $\pi^{(d-1)}$. We are assuming that one organizes this coefficients in increasing order. Note that the set represented by the curly brackets, contains $(d-1)$ elements.

The conclusion is that the effective theory for the Haah code contains infinitely many conserved charges, and these charges can be interpreted as components of higher moment multipoles (dipoles, quadrupoles, octupoles and so on) or combinations thereof.

APÊNDICE C – Gauge invariance to higher orders

In the main text we comment that the gauge invariance property of the effective theory emerges from the fact that the lattice cubic operators commute among themselves. This property holds at any order of the cubes expansion because it relies only on the t-vectors aspects. To see that, note that the differential operators can be written in general as

$$\mathcal{D}_a^{(\alpha)} = \sum_{I=0}^{7} t_a^{(I,\alpha)} \sum_j \frac{1}{j!} \left(\sum_{b=x,y,z} n_b^I \partial_b \right)^j , \tag{C.1}$$

the vectors $n_b^I = \left(n_x^I, n_y^I, n_z^I\right)$ corresponds to the corners of the cube, these positions are taken considering that the origin sits at the center of the cube, therefore

$$\begin{aligned} n_b^0 =& (-1,-1,-1)\,, \quad n_b^1 = (-1,+1,-1)\,, \quad n_b^2 = (+1,-1,-1)\,, \quad n_b^3 = (-1,-1,+1)\,, \\ n_b^4 =& (+1,+1,-1)\,, \quad n_b^5 = (-1,+1,+1)\,, \quad n_b^6 = (+1,-1,+1)\,, \quad n_b^7 = (+1,+1,+1)\,. \end{aligned} \tag{C.2}$$

This allow us to write the derivatives as

$$\begin{aligned} \mathcal{D}_a^{(1)} = \sum_j \frac{1}{j!} \Bigg[& t_a^{(4,1)} \left((-\partial_x + \partial_y - \partial_z)^j + (\partial_x - \partial_y - \partial_z)^j + (-\partial_x - \partial_y + \partial_z)^j \right) \\ & \left(t_a^{(2,1)} + t_a^{(3,1)} \right) \left((\partial_x + \partial_y - \partial_z)^j + (-\partial_x + \partial_y + \partial_z)^j + (\partial_x - \partial_y + \partial_z)^j \right) \\ & + \left(t_a^{(1,1)} + t_a^{(5,1)} \right) (-\partial_x - \partial_y - \partial_z)^j \Bigg] , \end{aligned} \tag{C.3}$$

$$\begin{aligned} \mathcal{D}_a^{(2)} = \sum_j \frac{1}{j!} \Bigg[& t_a^{(5,2)} \left((-\partial_x + \partial_y - \partial_z)^j + (\partial_x - \partial_y - \partial_z)^j + (-\partial_x - \partial_y + \partial_z)^j \right) \\ & \left(t_a^{(1,2)} + t_a^{(2,2)} \right) \left((\partial_x + \partial_y - \partial_z)^j + (-\partial_x + \partial_y + \partial_z)^j + (\partial_x - \partial_y + \partial_z)^j \right) \\ & + \left(t_a^{(3,2)} + t_a^{(4,2)} \right) (\partial_x + \partial_y + \partial_z)^j \Bigg] . \end{aligned} \tag{C.4}$$

We can put this into a more convenient form by using the identity

$$(x+y+z)^n = \sum_{k_1+k_2+k_3=n} \binom{n}{k_1,\, k_2,\, k_3} x^{k_1} y^{k_2} z^{k_3} , \tag{C.5}$$

Then the derivatives become

$$\mathcal{D}_a^{(1)} = \sum_j \sum_{k_1+k_2+k_3=j} \binom{j}{k_1,\, k_2,\, k_3} \frac{1}{j!} \Big[t_a^{(4,1)} \left((-1)^{k_1+k_2} + (-1)^{k_2+k_3} + (-1)^{k_1+k_3} \right) \left(t_a^{(2,1)} + t_a^{(3,1)} \right) \left((-1)^{k_1} + (-1)^{k_2} + (-1)^{k_3} \right) + \left(t_a^{(1,1)} + t_a^{(5,1)} \right) (-1)^{k_1+k_2+k_3} \Big] \partial_x^{k_1} \partial_y^{k_2} \partial_z^{k_3} , \tag{C.6}$$

$$\mathcal{D}_a^{(2)} = \sum_j \sum_{k_1+k_2+k_3=j} \binom{j}{k_1,\, k_2,\, k_3} \frac{1}{j!} \Big[t_a^{(5,2)} \left((-1)^{k_1+k_2} + (-1)^{k_2+k_3} + (-1)^{k_1+k_3} \right) \left(t_a^{(1,2)} + t_a^{(2,2)} \right) \left((-1)^{k_1} + (-1)^{k_2} + (-1)^{k_3} \right) + \left(t_a^{(3,2)} + t_a^{(4,2)} \right) \Big] \partial_x^{k_1} \partial_y^{k_2} \partial_z^{k_3} . \tag{C.7}$$

To proceed we have to analyze the product $K_{ab}\mathcal{D}_a^{(\alpha)}\mathcal{D}_b^{(\beta)}$. The product involving operators of the same type, i.e., $\alpha = \beta$, automatically vanishes due to the anti symmetry of the K matrix, the product that, in principle, can be non zero is the one involving $D_a^{(1)}$ and $D_b^{(2)}$, to analyze this product we can focus our attention only on the t-vector structure that arises from this product, which can be written as

$$\begin{aligned}
&\left(\sum_{m=1}^{3} (-1)^{k_m} \right) \left(\sum_{n=1}^{3} (-1)^{q_n} \right) \left(t_a^{(2,1)} + t_a^{(3,1)} \right) K_{ab} \left(t_b^{(1,2)} + t_b^{(2,2)} \right) \\
&+ \left(\sum_{m=1}^{3} \left| \epsilon^{mij} \right| (-1)^{k_i+k_j} \right) \left(\sum_{n=1}^{3} |\epsilon^{npr}| (-1)^{q_p+q_r} \right) t_a^{(4,1)} K_{ab} t_b^{(5,2)} \\
&+ \left(\sum_{m=1}^{3} (-1)^{q_m+\sum_i k_i} \right) \left(t_a^{(1,1)} + t_a^{(5,1)} \right) K_{ab} \left(t_b^{(1,2)} + t_b^{(2,2)} \right) \\
&+ \left(\sum_{m=1}^{3} \left| \epsilon^{mij} \right| (-1)^{q_i+q_j} \right) t_a^{(4,1)} K_{ab} \left(t_b^{(3,2)} + t_b^{(4,2)} \right) \\
&+ \left(\sum_{m=1}^{3} (-1)^{k_m} \right) \left(t_a^{(2,1)} + t_a^{(3,1)} \right) K_{ab} \left(t_b^{(3,2)} + t_b^{(4,2)} \right) \\
&+ \left(\sum_{m=1}^{3} \left| \epsilon^{mij} \right| (-1)^{q_i+q_j+\sum_a k_a} \right) \left(t_a^{(1,1)} + t_a^{(5,1)} \right) K_{ab} t_b^{(5,2)} .
\end{aligned} \tag{C.8}$$

Now we have to check the situations where we have products between even(odd) differential operators. The parity of $\mathcal{D}$ is encoded in the sums $\sum k$ and $\sum q$. When we have the situation where both $\mathcal{D}$'s are even(odd) it amounts to analyze the cases in which $\sum k$ = even(odd) and $\sum q$ = even(odd). Let us analyze the even-even case, this can be achivied in two different ways: (i) (k_1, k_2, k_3) = even, (ii) (k_i, k_j) = odd, $i \neq j$ and k_m = even, $m \neq i, j$ (Similarly for the q's). Therefore, the coefficients in (C.8) become such that the remaining combinations of products between the t-vectors can

be cancelled using the commutation relations among the cubes

$$\left(t_a^{(2,1)} + t_a^{(3,1)}\right) K_{ab} \left(t_b^{(1,2)} + t_b^{(2,2)}\right) + t_a^{(4,1)} K_{ab} t_b^{(5,2)} = 0\,, \tag{C.9}$$

$$\left(t_a^{(1,1)} + t_a^{(5,1)}\right) K_{ab} \left(t_b^{(1,2)} + t_b^{(2,2)}\right) + t_a^{(4,1)} K_{ab} \left(t_b^{(3,2)} + t_b^{(4,2)}\right) = 0\,, \tag{C.10}$$

$$\left(t_b^{(2,1)} + t_b^{(3,1)}\right) K_{ab} \left(t_a^{(3,2)} + t^{(4,2)}\right) + \left(t_a^{(1,1)} + t_a^{(5,1)}\right) K_{ab} t_b^{(5,2)} = 0\,. \tag{C.11}$$

By symmetry the odd-odd case is the same. Therefore, we obtain that for a theory with differential operators with the same parity the condition that ensures gauge invariance is given by

$$K_{ab}\, \mathcal{D}_a^{(\alpha)}\, \mathcal{D}_b^{(\beta)} = 0\,. \tag{C.12}$$

A similar relation can be obtained when the differential operators possess diferent parities, i.e., one derivative is odd and the other is even. The difference now is that the coefficients in (C.8) of the odd derivatives will acquire a relative sign, such that (C.9) can no longer be used to cancell all the terms in (C.8). This is because the gauge invariance for these theories with derivatives possessing different parities demands that we introduce the operator $\bar{\mathcal{D}}_a^{(\alpha)}$, which differs from $\mathcal{D}$ by a minus sign on every odd term. For these theories, the gauge invariance follows from the condition

$$K_{ab} \mathcal{D}_a^{(\alpha)} \bar{\mathcal{D}}_b^{(\beta)} = 0\,, \tag{C.13}$$

which is ensured by (C.9) once again.

APÊNDICE D – Determining the K-matrix

The matrix K is determined in three steps: i) we determine its dimensionality; ii) we find constraints on the possible values of the elements; iii) we fix the elements in order to match the physical properties of the lattice model. The steps i) and ii) follow directly from the algebra of the operators

$$\Gamma_{\mathbf{x}}^{(I,\alpha)} = \exp\left(i\, T_a^{(I,\alpha)}\, K_{ab}\, A_b(\mathbf{x})\right) , \tag{D.1}$$

which is essentially encoded into the following expressions

$$T_a^{(I,\alpha)}\, K_{ab}\, T_b^{(J,\alpha)} = \begin{cases} 1 \mod 2, & \text{if I}\neq\text{J} \\ 0 \mod 2, & \text{if I=J} \end{cases} \quad , \quad \left[A_a(\mathbf{x})\,, A_b(\mathbf{x}')\right] = i\,\pi\,\left(K^{-1}\right)_{ab}\,\delta(\mathbf{x}-\mathbf{x}')\,. \tag{D.2}$$

Step iii) is more subtle and follows from the algebra of the ground state holonomy gauge-invariant operators of the effective field theory.

Step (i): Dimensionality of K-matrix.

We have constructed an entire class of models in these notes, labeled by vectors T and the matrix K. In particular, the dimensionality of the "spin" operator acting at each site of the lattice is tied to the spatial dimensionality. Indeed,

$$\begin{aligned} D = 3 \quad &\Rightarrow \quad \sigma_{i_1} \Leftrightarrow \gamma_{2\times 2} \\ D = 5 \quad &\Rightarrow \quad \sigma_{i_1} \otimes \sigma_{i_2} \Leftrightarrow \gamma_{4\times 4} \\ D = 2n+1 \quad &\Rightarrow \quad \sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n} \Leftrightarrow \gamma_{2^n\times 2^n}. \end{aligned} \tag{D.3}$$

In this way, the algebra of the spin operators in D spatial dimensions can be written in terms of the Clifford algebra of Dirac matrices of dimensionality $2^n \times 2^n$. In this case, we have $2n$ Dirac matrices. Therefore, according to the representation (D.1), we need $2n$ distinct fields A to reproduce properly the algebra of the Dirac matrices. This fixes the dimensionality of the matrix K to be $2n \times 2n$.

Step (ii): Elements of the K-matrix.

The possible values of the elements of the K-matrix are determined from the algebra of operators at each site. Take for example the $D = 3$ case, and let the K-matrix

be given by

$$K = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \quad \text{and} \quad K^{-1} = \begin{pmatrix} 0 & -\frac{1}{k} \\ \frac{1}{k} & 0 \end{pmatrix}, \tag{D.4}$$

where k is, in principle, an arbitrary integer. In $D = 3$ dimensions we only have the two independent operators

$$\gamma^1 = e^{ikA_2} \quad \text{and} \quad \gamma^2 = e^{-ikA_1}. \tag{D.5}$$

They will anticommute if k is odd, this follows directly from the TKT condition in (D.2). For the $D = 2n + 1$ dimensional case, the reasoning is similar. For simplicity, we take advantage of the symmetry of the continuum theory

$$A \to W\,A\,, \quad T^{(I,\alpha)} \to W^{-1}\,T^{(I,\alpha)}\,, \quad K \to W^{\top}\,K\,W\,. \tag{D.6}$$

such that we choose an appropriate W that put the K-matrix into its block-diagonal form

$$W^{\top}\,K\,W = \text{Diag}\left\{ \begin{pmatrix} 0 & k_1 \\ -k_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & k_2 \\ -k_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & k_n \\ -k_n & 0 \end{pmatrix} \right\}. \tag{D.7}$$

In this representation, the condition that the operatos (D.1) anti commute translates into a TKT condition, like the one in (D.2), for each block. Therefore, the components of the K-matrix are fixed to be odd integers only. Note that this does not completely fix what are the allowed integers, to do that we have to analyze the size of the Hilbert space.

Step (iii): Lattice model and the K-matrix entries.

To completely fix the components of the K-matrix in (D.7), we need to consider the algebra of gauge-invariant operators. This is worked in detail in Sec.(3.3). The key equation is the algebra of the line operators

$$e^{i\,a_j^{(p)}}\; e^{i\,a_{j+1}^{p}} = e^{i\,\pi/k_j}\, e^{i\,a_{j+1}^{p}}\, e^{i\,a_j^{(p)}}\,, \tag{D.8}$$

where the $a_j^{(p)}$ are the fields describing the ground state configurations on each leaf p of the foliation along the sub-manifolds where charge is conserved.They are related to the A fields by

$$\tilde{A}_j^{(p)}(\mathbf{x},\, t) = \frac{1}{l^{(D-1)/2}}\, a_j^{(p)}(t)\,, \tag{D.9}$$

where the $\tilde{A}$ fields are the ones in the representation in which the K-matrix is block diagonal, as in (D.7). Following this analisys, we have showed in Sec.(3.3) that the ground state is degenerate, and in particular, each leaf p contributes to the degeneracy as

$$GSD(p) = \prod_{i=1}^{N} \left(\prod_{j=1}^{(D-1)/2} 2\,k_j \right) = \left(2^{\frac{D-1}{2}}\, Pf(K) \right)^N, \tag{D.10}$$

with the pfaffian being $Pf(K) = k_1 k_2 \dots k_n$. Therefore, the value of k_i determines the size of the representation of the ground state in a corresponding subdimensional manifold where charge is conserved. In other words, the ground state will possess a $\mathbb{Z}_{2k_i}$ symmetry for each pair of fields $a_i^{(p)}$, $a_{i+1}^{(p)}$. In these notes we have discussed only Z_2 symmetric spin liquids, which fixes the coefficients $k_i = 1 \; \forall \, i$. In this sense, the K-matrix is determined by the microscopic system since it carries information about the symmetries of the lattice model.

Note that the transformations (D.6), although not crucial for this discussion they simplify it dramatically. The important point to note, is that they allow us to define an equivalence class of continuum theories, i.e., two descriptions with matrices K and $\tilde{K}$ are equivalent if K and $\tilde{K}$ are related by a transformation like (D.6). This feature is similar to what happens in usual Chern-Simons theories, where two CS actions can lead to the same description with two different K-matrices. To make this point clear, consider a redefinition of the basis field in the effective action according to

$$A \to WA, \tag{D.11}$$

where W is a matrix with integer entries. This leads to a theory with new parameters

$$\tilde{K} = W^\top K W \quad \text{and} \quad \tilde{T}^{(I,\alpha)} = W^{-1} T^{(I,\alpha)}. \tag{D.12}$$

Thus, two effective theories with parameters $(K,\, T)$ and $(\tilde{K},\, \tilde{T})$ related through (D.12), with the matrix W possessing integer entries and $\det W = 1$, describes the same fracton system. Indeed, this implies

$$\mathrm{Pf}(\tilde{K}) = \mathrm{Pf}(W^\top K W) = \mathrm{Pf}(K) \tag{D.13}$$

and also leaves unchanged the quantization condition (the TKT relation) in (D.2), in particular for the principal basis it becomes

$$t_a^{(I)} \, (K^\top)_{ab} \; t_b^{(J)} = \tilde{t}_a^{(I)} \, (\tilde{K}^\top)_{ab} \; \tilde{t}_b^{(J)} = 2n^{(IJ)} + (1 - \delta_{IJ}) \,, \quad n^{(IJ)} \in \mathbb{Z} \,. \tag{D.14}$$

which is still an even integer if $I = J$ and an odd integer if $I \neq J$.

APÊNDICE E – Pure Type-II continuum fracton model

In this appendix we expand the short discussion in chapter 5 about constructing a type-II fracton field theory involving exclusively the d_2 differential operator. We emphasize that we will change the notation slightly. The corners of the cubes will be labeled from 0 to 7. As before, the product of Dirac operators in a given corner is determined by a vector $T_a^{(I,\alpha)}$, $I = 0, \ldots, 7$ according to (4.13). We will consider a model with two cubic operators (similarly to the Haah code), but with arbitrary Dirac operators in each of the corners. Upon expansion of these cubic operators (like the one that led to (4.18)) one can demand that the coefficients of the differential operators 1, ∂_i, ∂_i^2 vanish identically and that the coefficients of $\partial_i \partial_j$, $i \neq j$ are such that the resulting differential operator in the continuum is $\mathcal{D}_a^{(\alpha)} = \mathcal{T}_a^{(\alpha)} d_2$. These requirements give us a set of equations that allow us to determine the allowed T-vectors and hence the Dirac operators that constitutes each cube. Although, as we shall argue below, there are some ambiguities arising in returning from the continuum to the lattice which renders the lattice model inconsistent with the continuum one. Despite being able to construct a continuum theory compatible with the physics of a pure type-II fracton model it seems very difficult to obtain a corresponding lattice model within the framework discussed here.

To make this problem explicit, let us consider a generic cube operator

$$C_{\mathbf{x}}^{(\alpha)} = \exp\left(i \sum_{I=0}^{7} T_a^{(I,\alpha)} K_{ab} A_b \left(\mathbf{x} + \hat{r} \cdot \mathbf{n}_I\right) \right), \tag{E.1}$$

where $\hat{r} = (\hat{x}, \hat{y}, \hat{z})$ and $\mathbf{n}_I$ is a set of vectors specifying the positions of the corners of the cube relative to its center, explicitly given by

$$\begin{aligned} &\mathbf{n}_0 = -(1,1,1), \quad \mathbf{n}_1 = (-1,-1,1), \quad \mathbf{n}_2 = (-1,1,-1), \quad \mathbf{n}_3 = (-1,1,1), \\ &\mathbf{n}_4 = (1,-1,-1), \quad \mathbf{n}_5 = (1,-1,1), \quad \mathbf{n}_6 = (1,1,-1), \quad \mathbf{n}_7 = (1,1,1). \end{aligned}$$

We shall impose constraints on the combinations of T-vectors emerging from the expansion so that the continuum effective theory contains the single differential operator $d_2 = \sum_{ijk} \left|\epsilon^{ijk}\right| \partial_j \partial_k$. Such theory is a candidate for a pure type-II fracton. We shall determine the corresponding T-vectors and then try to reconstruct the lattice model.

Expanding the cube operators in (E.1) leads to a continuum theory. The resulting terms in the exponential can be organized according to the order of the derivatives they involve, namely, 1, ∂_i, ∂_i^2, and $\partial_i \partial_j$, with $i \neq j$. The coefficients of each one of

these differential operators are linear combinations of the T-vectors. We then demand that the coefficients of 1, ∂_i, and ∂_i^2 vanish. In addition, we impose that the coefficients of the operators $\partial_i\partial_j$, with $i \neq j$, are the same in order to identify the operator d_2. These conditions lead to a system of equations that can be solved. We find that only two T-vectors in each cube operator are independent, say, $T^{(6,\alpha)}$ and $T^{(7,\alpha)}$. The remaining ones can be written in terms of these two vectors:

$$\begin{aligned} T^{(0,\alpha)} &= 3T^{(6,\alpha)} + 2T^{(7,\alpha)}; \\ T^{(1,\alpha)} &= T^{(2,\alpha)} = T^{(4,\alpha)} = -2T^{(6,\alpha)} - T^{(7,\alpha)}; \\ T^{(3,\alpha)} &= T^{(5,\alpha)} = T^{(6,\alpha)}. \end{aligned} \tag{E.2}$$

The cube operators in (E.1) reduce to

$$C_{\mathbf{x}}^{(\alpha)} \sim \exp\left(iK_{ab}\mathcal{D}_a^{(\alpha)}A_b(\mathbf{x})\right), \tag{E.3}$$

where now $\mathcal{D}_a^{(\alpha)} \equiv \mathcal{T}_a^{(\alpha)}d_2$, with $\mathcal{T}^{(\alpha)} \equiv 4\left(T_a^{(6,\alpha)} + T_a^{(7,\alpha)}\right)$. We can immediately write down the corresponding effective action in the form (4.26), which is invariant under gauge transformations $A_a \to A_a + \mathcal{D}_a^{(\alpha)}\zeta^{(\alpha)}$, provided that $K_{ab}\mathcal{D}_a^{(\alpha)}\mathcal{D}_b^{(\beta)} = 0$. This condition is equivalent to $\mathcal{T}_a^{(\alpha)}K_{ab}\mathcal{T}_b^{(\beta)} = 0$. Therefore, gauge invariance of the continuum theory imposes restrictions only on $\mathcal{T}^{(\alpha)}$, but not on $T^{(6,\alpha)}$ and $T^{(7,\alpha)}$ individually. We refer to this condition as the *weak gauge condition*.

Let us try to reconstruct the lattice model. The main issue with the weak gauge condition is that it likely will lead to a lattice model that is not given in terms of commuting projectors, i.e., a lattice model where the cube operators are noncommuting. To obtain a lattice model of commuting projectors we need some lattice input. We import the commutation relations from the lattice, which correspond to restrictions on the vectors $T^{(6,\alpha)}$ and $T^{(7,\alpha)}$ individually, namely,

$$\begin{aligned} T_a^{(6,1)}\,K_{ab}\,T_b^{(7,1)} = 0\,, \quad T_a^{(6,2)}\,K_{ab}\,T_b^{(7,2)} &= 0\,, \\ T_a^{(6,1)}\,K_{ab}\,T_b^{(7,2)} = 0\,, \quad T_a^{(7,1)}\,K_{ab}\,T_b^{(6,2)} &= 0\,, \\ T_a^{(6,1)}\,K_{ab}\,T_b^{(6,2)} + T_a^{(7,1)}\,K_{ab}\,T_b^{(7,2)} &= 0. \end{aligned} \tag{E.4}$$

These conditions are stronger than the previous ones and, consequently, also ensure gauge invariance of the continuum theory. Moreover, they ensure that all the cube operators are simultaneously commuting. An explicit solution for this set is $T^{(6,1)} = (0,0,0,1)$, $T^{(7,1)} = (0,-1,1,0)$, $T^{(6,2)} = (1,1,-1,1)$, and $T^{(7,2)} = (1,-1,0,2)$. This enables us to identify the following spin operators

$$T^{(6,1)} \to \gamma^4, \quad T^{(7,1)} \to \gamma^2\gamma^3, \quad T^{(6,2)} \to \gamma^5, \quad T^{(7,2)} \to \gamma^1\gamma^2. \tag{E.5}$$

The corresponding cube operators are shown in Fig.(11). The lattice theory built from those cubes is *not* compatible with the effective theory constructed from the

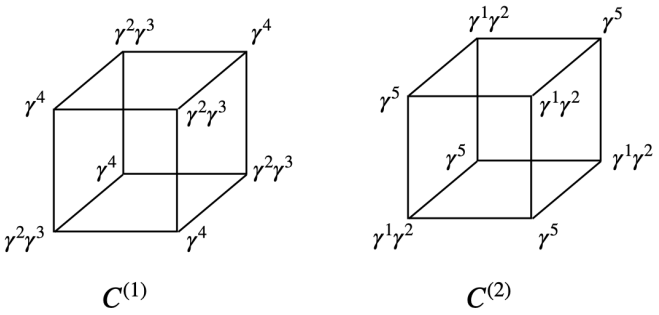

Figura 11 – Cubes obtained from the effective field theory.

operator (E.3). In fact, the lattice model defined in terms of the cubes of Fig. (11) supports *mobile excitations*, corresponding to a type-I fracton system. The mobility can be seen from the fact that one can act with a local operator that anti-commutes with all operators in $C^{(1)}$ (or $C^{(2)}$), and thus create eight defects, an octupole configuration. For example, take the action of a γ_2 operator on a single site, since γ_2 anti-commutes with all operators in $C^{(1)}$ (and $C^{(2)}$ as well in this case), it will create eight defects in the lattice (the sign of all eight neighboring cubes to that point will be flipped), this corresponds to the creation of an ocutpole excitation. The octupoles can then be used to move quadrupole excitations.

So what goes wrong? The subtle point is that there are ambiguities in the lattice which are not innocuous in the continuum theory. We have come across with this before: the identity operator in the lattice is not automatic implemented in the continuum theory. Let us consider another example, say, two operators characterized respectively by $T^{(6,1)}$ and $T^{(0,1)} = 3T^{(6,1)} + 2T^{(7,1)}$. While they are distinct from the continuum point of view, they lead to the same lattice operator γ^4, since the components of the T-vectors are defined mod 2. This violates the one-to-one map between $T^{(I,\alpha)}$ and the lattice operators, since two T-vectors are mapped to the same operator. Therefore, this lattice model is not a valid one. This can be made more explicitly if one starts with the lattice model defined by the cubes of Fig.(11) and applies the procedure in the main text to obtain the effective theory. The resulting theory corresponds to a fracton system with mobile quadrupole excitations, representing properly the lattice model, and not a type-II theory as the one that follows from (E.3). While the passage from the lattice to the continuum using the framework described here and in (55, 56) safely produces a *bona fide* effective description, the reverse is not true. On the other hand, the effective action that follows from (E.3) corresponds to a properly pure type-II continuum fractonic theory, but with no any obvious corresponding lattice model.