Universidade Estadual de Londrina

RODRIGO CORSO BAPTISTA DOS SANTOS

APLICAÇÕES DE TEORIA CONFORME EM FASES TOPOLÓGICAS DA MATÉRIA

LONDRINA

2023

RODRIGO CORSO BAPTISTA DOS SANTOS

APLICAÇÕES DE TEORIA CONFORME EM FASES TOPOLÓGICAS DA MATÉRIA

Dissertação apresentada ao Programa de Mestrado em Física da Universidade Estadual de Londrina para obtenção do título de Doutor em Física.

Orientador: Prof. Dr. Pedro Rogério Sergi Gomes

LONDRINA
2023

Ficha de identificação da obra elaborada pelo autor, através do Programa de Geração Automática do Sistema de Bibliotecas da UEL

Santos, Rodrigo Corso Baptista dos Santos.
Aplicações de Teoria Conforme em Fases Topológicas da Matéria / Rodrigo Corso Baptista dos Santos Santos. - Londrina, 2023.
86 f. : il.

Orientador: Pedro Rogério Sergi Gomes.
Tese (Doutorado em Física) - Universidade Estadual de Londrina, Centro de Ciências Exatas, Programa de Pós-Graduação em Física, 2023.
Inclui bibliografia.

1. Teoria de Conforme - Tese. 2. Fases Topológicas da Matéria - Tese. 3. Líquidos de Spin - Tese. 4. Supercondutor Topológico - Tese. I. Rogério Sergi Gomes, Pedro. II. Universidade Estadual de Londrina. Centro de Ciências Exatas. Programa de Pós-Graduação em Física. III. Título.

CDU 53

RODRIGO CORSO BAPTISTA DOS SANTOS

APLICAÇÕES DE TEORIA CONFORME EM FASES TOPOLÓGICAS DA MATÉRIA

Dissertação apresentada ao Programa de Mestrado em Física da Universidade Estadual de Londrina para obtenção do título de Doutor em Física.

BANCA EXAMINADORA

Orientador: Prof. Dr. Pedro Rogério Sergi Gomes
Universidade Estadual de Londrina

Dr. Rodrigo Arouca de Albuquerque
Universidade de Uppsala – UU

Prof. Dr. Carlos Hernaski
Universidade Tecnológica Federal do Paraná – UTFPR

Prof. Dr. Tobias Micklitz
Centro Brasileiro de Pesquisas Físicas – CBPF

Prof. Dr. Eduardo Cantera Marino
Universidade Federal do Rio de Janeiro – UFRJ

Londrina, 28 de março de 2023.

AGRADECIMENTOS

Throughout my academic journey I had to opportunity to encounter a series of remarkable people that heavily influenced my life, on a personal and a professional level. Here I would like to express my gratitude to them.

First and foremost, I would like to thank my advisor professor Dr. Pedro Gomes. Since the time we first met, when I was still an undergraduate student, Pedro's support, guidance and encouragement have been one of the cornerstones in my development as a physicist and a person. For that, I would like to express my deepest admiration and gratitude.

I would also like to thank my partner Lucas Valle Gaino, for making everyday a happy one, while being such a encouraging, kind and loving person with whom I have the pleasure of sharing my life with.

Furthermore, I would also like to thank my mother Luciana Corso and my sisters Isabela and Heloisa Corso. For all the dedication, care, attention and love given to me through my life, without which I would not be the person I am today, I would like to express my gratitude and love.

My gratitude also goes to the some professors that inspired me throughout these years. To Dr. Carlos Hernaski, for the helpful discussions that were paramount to my development and to the completion of this thesis. To Dr. Paula Bienzobas for the unassuming and light-hearted casual conversations. To Dr. Marcelo Gomes and Dr. Adilson da Silva for the contributions to my work. Lastly I would like to thank Dr. Eduardo Marino, Dr. Carlos Hernaski, Dr. Tobias Micklitz, Dr. Marcelo Gomes, Dr. Paula Bienzobas and Dr. Rodrigo Arouca for evaluating my thesis.

I would also like to thank some special friends. In special, I would like to thank Jonas Pinheiro, Eduardo Barroso, Weslei Fontana, Heitor Casasola, and Lara Petini for all the special moments we spent together, which I treasure at the bottom of my heart. To my friends from Switzerland, in special to Suvi Pachigolla, Lorenzo Amato, Edoardo Renaldin and Christian Vaccarin, for welcoming me with an open heart in Switzerland and for keeping me sane when I was far from home during the pandemic. I will forever treasure those times.

At last, I would like to thank Dr. Christopher Mudry, and the whole PSI team, for hosting me during my time in Switzerland. To CAPES for the financial support and to the Department of Physics of the State University of Londrina for the structural support.

CORSO B. SANTOS, R . **Aplicações de Teoria Conforme em Fases Topológicas da Matéria**. 2023. 85f. Dissertação (Mestrado em Ciência da Computação) – Universidade Estadual de Londrina, Londrina, 2023.

RESUMO

Segundo a **??**, 3.1-3.2, o resumo deve ressaltar o objetivo, o método, os resultados e as conclusões do documento. A ordem e a extensão destes itens dependem do tipo de resumo (informativo ou indicativo) e do tratamento que cada item recebe no documento original. O resumo deve ser precedido da referência do documento, com exceção do resumo inserido no próprio documento. (. . .) As palavras-chave devem figurar logo abaixo do resumo, antecedidas da expressão Palavras-chave:, separadas entre si por ponto e finalizadas também por ponto.

Palavras-chave: Teoria Conforme, Teoria de Campos, Fases Topológicas, Matéria Condensada, Líquidos de Spin

CORSO B. SANTOS, R . **Applications of Conformal Field Theory in Topological Phases of Matter**. 2023. 85p. Master's Thesis (Master in Science in Computer Science) – State University of Londrina, Londrina, 2023.

ABSTRACT

In this thesis we discuss two topological phases of matter and how conformal field theory can be applied to better understand it. Our first topic is the topological superconductor realized by a quantum anomalous Hall in proximity of a pairing potential. For this model, we firstly predict the existence of gapless excitations through a simple quantum wires argument. In spite of the nonrelativistic perturbation, we find the leading contribution to the effective action in a low-energy expansion to be a sum of Chern-Simons terms. At last, we deduce the boundary effective field theory by using the equivalence between the Chern-Simons and its induced rational conformal field theory. Furthermore, we discuss the $(2+1)$-dimensional quantum spin liquid through a wires approach. To do so, we use non-Abelian bosonization to obtain an effective action in terms of Wess-Zumino-Witten terms, from which the fixed point structure can be easily extracted. As a check, we calculate the one loop contribution to the C-function and compare it with the central charge at the fixed points.

Keywords: Quantum Field Theory, Conformal Field Theory, Condensed Matter, Topological Phases, Spin Liquids

SUMÁRIO

1 INTRODUCTION

Condensed matter physics is concerned with the behavior of particles at finite density and low temperatures, where external parameters like pressure and doping can force matter to reorganize itself producing different phases. Classically, we learn of phases such as liquid, crystal, solid and gas, but quantum physics holds much more fascinating phenomena such as charge density waves, Bose-Einstein condensates, superconductivity, (anti)-ferromagnetism and many others. All these phases can be classified by the underlying principle of symmetry breaking.

This symmetry breaking approach to describing quantum phases of matter is best encapsulated by the Landau-Ginzburg theory [1], where a *local* order parameter acquires a non-zero expected value in a phase, differentiating it from other phases [2, 3]. Furthermore, this parameter is generally some measurable physical observable. Consider a ferromagnet as an example, where by lowering its temperature, it undergoes a transition from a symmetric phase to a symmetry broken phase with non-zero magnetization [4].

The Landau-Ginzburg theory allows us to study the system close to criticality. Essentially, this is done by considering the free energy as an expansion in powers of the order parameter, which is small near phase transition by construction. Furthermore, the framework classifies the phase transitions as first, second or higher order depending on which coefficient of the expansion vanishes.

In short, the Landau-Ginzburg framework classifies phases according to their symmetries. More recently, a series of new phases of matter came to light which did not fit this paradigm. As an example, we call attention to the chiral spin state introduced in [5, 6] to study high temperature superconductivity. Shortly after their introduction, it was realized that many different chiral spin states share the same symmetry [7]. Another example is the fractional quantum Hall (FQH) [8, 9].

In this way, the symmetries were not sufficient for classifying different phases anymore and a new *topologial order* was made necessary to completely describe the chiral spin state [10]. For classifying these new phases new quantum numbers such as the ground state degeneracy [7, 9] and the non-Abelian geometric phase of the degenerate ground states[11, 10].

For these *topological* phases of matter, we are able to define an order parameter. But in the end, we find that this order parameter is generally non-local, which is incompatible with the Landau-Ginzburg theory. From a more fundamental point of view, this incompatibility stems from the nature of the interactions at hand [12]. In a topological phase, the long-range correlation of the electrons produces a ground state wave function that is topologically sensitive. In turn, this allows us to classify these phases by the degeneracy of the ground state when in a topologically non-trivial manifold and the non-Abelian geometric phase of such degenerate ground states [13].

Furthermore, the long-range entanglement of the electrons leads to some interesting phenomena that are shared among some topological phases. One such characteristic is the presence of quasiparticle excitations, which can carry fractional charge and statistics. This property has several applications on topological quantum memory and computation [13]. It also provides us with a path for experimentally detecting topological orders. Another property of some phases is the presence of emergent gauge fields and topologically protected gapless boundary excitations, which can also lead to some device application.

This thesis focuses on topological phases of matter and its uses of conformal field theory. In fact, here we discuss two topological phases of matter. In Chapters 2 and 3 we consider a quantum Hall phase realized in terms of free fermions with nonrelativistic dispersion relation, possessing a global $U(1)$ symmetry. Then, we couple this symmetry to a background gauge field and compute the effective action by integrating out the gapped fermions. Surprisingly, the non-relativistic case differs from the ordinary Hall by having an effective action described by an integer level Chern-Simons term, instead of the usual half-integer for the relativistic case [14, 15].

The proximity to a conventional superconductor induces a pairing potential in the quantum Hall state, favoring the formation of Cooper pairs [16]. When the pairing is strong enough, it drives the system to a topological superconducting phase, hosting Majorana fermions. Even though the continuum $U(1)$ symmetry is broken down to a $\mathbb{Z}_2$ one, we can forge fictitious $U(1)$ symmetries that enable us to derive the effective action for the topological superconducting phase, also given by a Chern-Simons theory. To eliminate spurious states coming from the artificial symmetry enlargement, we demand that the fields in the effective action are $O(2)$ instead of $U(1)$ gauge fields. In the $O(2)$ case we have to sum over the $\mathbb{Z}_2$ bundles in the partition function, which projects out the states that are not $\mathbb{Z}_2$ invariants. The corresponding edge theory is the $U(1)/\mathbb{Z}_2$ orbifold, which contains Majorana fermions in its operator content.

In Chapter 4, we do a brief detour to review the non-Abelian bosonization of the chiral Gross-Neveu model as a warm up for Chapter 5. More specifically, we use a quantum wires approach to discuss how we can introduce interactions in the action that effectively realize a (2+1)-dimensional phase. Then, we show that in the deep infrared these operators realize a new conformal field theory with a smaller central charge. Lastly, we discuss that this IR CFT is not achievable by an RG flow starting at the free fermion fixed point.

Chapter 5 uses a lot of the mechanisms we learned in the non-Abelian bosonization of the chiral Gross-Neveu model to discuss the fixed point structure of the a QSL phase. We consider a closed version of the model proposed in [17, 18, 19] as to study the gap opening of the bulk degrees of freedom in the IR limit. Among the proposed interactions that realize the QSL phase some are non-commuting, meaning they may compete to realize different phases in the low-energy limit. Fortunately, we are able to broaden the methods of Chapter 4 by introducing a variable change that accommodates for the non-commuting nature of the problem and allows us to safely perform the fermion integration process.

The resulting action has rich fixed point structure. Besides the expected free fermion fixed point, we find one partially gapped fixed point and two fully gapped fixed points, one competing and the other not. Then, motivated by our discussions of Chapter 4, we compute the one-loop C-function and come to the same conclusion. The fixed points are isolated from the rest of the RG flow, but the physics close to it is essentially gapped.

Publications

- Gomes, M., Pedro RS Gomes, K. Raimundo, Rodrigo Corso B. Santos, and A. J. da Silva. *"Topological superconductor from the quantum Hall phase: Effective field theory description."* Physical Review B 106, no. 19 (2022): 195111.
- Santos, Rodrigo Corso B., Pedro RS Gomes, and Carlos A. Hernaski. *"Bosonization of the Thirring model in 2+ 1 dimensions."*Physical Review D 101, no. 7 (2020): 076010.

Unpublished

- Santos, Rodrigo Corso B., Pedro RS Gomes, and Carlos A. Hernaski. *"A Quantum Wires Approach to Non-Abelian Spin Liquids".*

2 TOPOLOGICAL SUPERCONDUCTOR

2.1 Introduction

One remarkable accomplishment of the field of topological phases of matter is the provision of a concrete platform for the realization of systems exhibiting the elusive Majorana fermions, namely, the topological superconductors (TSs). In recent years, many efforts have been devoted to the study of such systems [20, 21, 22, 16, 23, 24, 25, 26, 27, 28]. In addition to their intrinsic interest, the Majorana fermions are generally believed to play a crucial role in quantum computing due to their non-Abelian braiding properties [29, 30, 31] (for a recent review, see [32]).

A simple setting for the realization of a topological superconductor was proposed in [16], where such phase arises from a quantum anomalous Hall (QAH) phase in proximity to a pairing potential inducing the formation of Cooper pairs (similar models were discussed in [20, 27] and possible experimental signatures were reported in [28]). The specific model of [16], which we shall review in the next section, consists of spinful noninteracting electrons with a *nonrelativistic* dispersion relation $E^2 = b_1^2\vec{p}^2 + (\Delta + m_0 + b_2\vec{p}^2)^2$, where b_1 and b_2 are positive parameters, m_0 is the insulator gap that can be positive or negative, and Δ is the strength of the pairing potential which is chosen to be positive. The relation between m_0 and Δ dictates in which phase the system is. For weak coupling, $\Delta < |m_0|$, the system is either in a quantum Hall ($m_0 < 0$) or in a trivial phase ($m_0 > 0$), but for strong enough coupling, $\Delta > |m_0|$, an s-wave topological superconducting phase emerges between them, hosting Majorana fermions both in the vortices and at the edges.

In this chapter we revisit this system from the perspective of *effective field theory*. Our main objective is to derive a low-energy action incorporating all the phases discussed above. Some interesting aspects concerning the effective field theory are the following. First, in the pure quantum Hall phase, $\Delta = 0$, the effective field theory is expected to be a simple Chern-Simons (CS) action with integer level. Starting with fermions possessing a global $U(1)$ symmetry associated with charge conservation, we follow the usual procedure of introducing a background gauge field for this global symmetry and then integrate out the fermions. This produces a fermionic determinant, which can be computed in the large mass (gap) limit. In relativistic fermionic theories (which correspond to $b_2 = 0$), the leading contribution in the effective action for the background field is a Chern-Simons term with a half-integer level [15, 14][1]. As our discussion sets

[1] In addition to the parity anomaly, this half-integer coefficient would lead to a gauge anomaly, but it can be removed by a local counterterm.

itself apart by being nonrelativistic, it is not clear whether a Chern-Simons term can arise from the corresponding fermionic determinant. We carry this computation in this work and show that in fact a Chern-Simons term with a properly quantized level arises from a highly nontrivial combination of several Feynman diagrams, so that we do not need any counterterm to cope with gauge anomaly.

Another interesting aspect of the effective field theory is evident when we consider the proximity effect to a superconductor. The introduction of a pairing potential in the system breaks the $U(1)$ charge conservation symmetry down to a $\mathbb{Z}_2$ symmetry, allowing the formation of Cooper pairs, so that charge is conserved only mod 2. In this case, we cannot introduce naively continuum gauge fields, since there is no longer a continuous symmetry. We proceed within the Bogoliubov–de Gennes (BdG) formalism (see, for example, [33, 34]), where the fermion operators are accommodated into the Nambu spinors, which are spinors whose components are not independent. This leads to an artificial particle-hole symmetry, such that there is a duplication of the degrees of freedom of the theory. If in addition we work with unconstrained (independent) components in the Nambu spinor, we end up with an enlarged theory possessing fictitious $U(1)$ symmetries [35]. The physical Hilbert space is then recovered by retaining in the spectrum only states that are properly $\mathbb{Z}_2$ invariants.

We can take advantage of the $U(1)$ fictitious symmetries in that they can be coupled to background fields, such that we can compute the low-energy effective action again simply by integrating out the fermions. The effective theory is given in terms of Chern-Simons theories, which are locally identical to the one derived in the case of the pure QAH phase. However, the projection onto the physical space amounts to considering the background fields as $O(2) = U(1) \rtimes \mathbb{Z}_2$ gauge fields, instead of $U(1)$. In the $O(2)$ path integral, the sum over nontrivial bundles of $O(2)$ projects out all the states that are not $\mathbb{Z}_2$ invariant. The corresponding edge theory is thus the $U(1)/\mathbb{Z}_2$ orbifold theory, which contains the Majorana fermions.

This chapter is organized as follows. In Sec. 2.2, we review the microscopic models that describe the QAH phase and the TS phase obtained by introducing a pairing potential in the QAH system. In Sec. 2.3, we study the edge states of both phases from the point of view of the quantum wires description, where one of the spatial dimensions of the system is discretized. Section 3.1 is dedicated to the derivation of the low-energy effective actions for the QAH and TS phases. In Sec. 3.2 we study the orbifold edge theory of the TS phase that follows from the effective field theory via bulk-edge correspondence. Complementary discussions are presented in Appendices A and B.

2.2 The Models

In this section we briefly review the model introduced in [16] describing the transition from the QAH to a TS. When a QAH state is coupled to a conventional s-wave superconductor through the proximity effect, the transition between the phases with trivial and nontrivial Hall conductance is, in general, split into two transitions, among which appears a chiral TS phase.

2.2.1 Quantum Anomalous Hall Hamiltonian

The QAH system can be conceived in terms of spinful electrons with a quadratic Hamiltonian,

$$H_{QAH} \equiv \sum_{\vec{p}} \psi^{\dagger}_{\vec{p}} h_{QAH}(\vec{p}) \psi_{\vec{p}}, \tag{2.1}$$

possessing a $U(1)$ global symmetry corresponding to the charge conservation. The spinor $\psi_{\vec{p}}$ is defined as $\psi_{\vec{p}} \equiv \begin{pmatrix} c_{\vec{p}\uparrow} & c_{\vec{p}\downarrow} \end{pmatrix}^T$. The specific form of the single-particle Hamiltonian h_{QAH} giving rise to the QAH phase is [16]

$$h_{QAH}(\vec{p}) \equiv \vec{b}(\vec{p}) \cdot \vec{\sigma} = \begin{pmatrix} m(\vec{p}^2) & b_1(p_x - ip_y) \\ b_1(p_x + ip_y) & -m(\vec{p}^2) \end{pmatrix}, \tag{2.2}$$

with $m(\vec{p}^2) \equiv m_0 + b_2 \vec{p}^{\,2}$. The parameters b_1 and b_2 are taken to be positive, whereas m_0 is allowed to change sign, with each one corresponding to a distinct phase.

To see this, we note that the vector $\vec{b}(\vec{p}) = (b_1 p_x, b_1 p_y, m(\vec{p}^2))$ allows us to define the unit vector

$$\vec{n} \equiv \frac{\vec{b}(\vec{p})}{|\vec{b}(\vec{p})|}, \tag{2.3}$$

which in turn corresponds to a map from the momentum region to the unit sphere S^2 parametrized by $\vec{n}$. Note that for $|\vec{p}| \to \infty$, the map (2.3) implies $\vec{n} = (0, 0, 1)$. Therefore, we can think of the momenta $\vec{p}$ as taking values over $\mathbb{R}^2 \cup \{\infty\}$, which is topologically equivalent to a sphere S^2. In this way, the relation (2.3) defines a class of maps from S^2 to S^2 classified by the homotopy group $\Pi_2(S^2) = \mathbb{Z}$, with winding number

$$\mathcal{W}(S) = \frac{1}{8\pi} \int_{S^2} d^2 p \epsilon^{ij} \vec{n} \cdot (\partial_{p_i} \vec{n} \times \partial_{p_j} \vec{n}). \tag{2.4}$$

Using the explicit form of the map (2.3), the winding number is

$$\mathcal{W}(S) = \begin{cases} 1 & m_0 < 0 \\ 0 & m_0 > 0. \end{cases} \tag{2.5}$$

As different values of $\mathcal{W}(S)$ correspond to topologically distinct situations, the system undergoes a phase transition as a function of m_0. In this way, the point $m_0 = 0$

is a quantum critical point between a topologically ordered phase when $m_0 < 0$ (QAH) and a trivial one with $m_0 > 0$ (trivial insulator). The Hall conductivity σ_{xy} is given in terms of the winding number as $\sigma_{xy} = \frac{1}{2\pi}\mathcal{W}$.

A hallmark of a topological phase is the presence of gapless edge states when a physical boundary is introduced in the system. This can be incorporated in the Hamiltonian description by promoting $m_0 \to m_0(y)$, with $m_0(y) < 0$ if $y < 0$ and $m_0(y) > 0$ if $y > 0$, so that $y = 0$ corresponds to an interface between two distinct phases. Hence, there must be edge states at this region. Because of the absence of time-reversal invariance, the edge states are chiral (one-way propagating) and the number of them is related to the bulk topological number (2.5) through [36, 37]

$$\mathcal{W} = N_R - N_L, \tag{2.6}$$

where $N_{R/L}$ are the number of chiral right/left propagating edge modes. According to (2.5), we see that there is a single stable chiral edge mode in the QAH phase.

An elegant way to capture the physics of the edge states is through the bulk-edge correspondence using Chern-Simons effective field theory [38]. However, the Lagrangian following from the Hamiltonian (2.2) is *nonrelativistic*, and the usual procedure of integrating out massive fermions in the presence of a background field is not guaranteed to generate a Chern-Simons term in the effective action. One of the purposes of this work is to show that a Chern-Simons term does emerge in a highly nontrivial way in this nonrelativistic setting. We shall carry out this computation in Sec. 3.1.

2.2.2 Proximity Effect to a Topological Superconductor

Next we consider the system in proximity to an s-wave superconductor. In this case, a finite pairing amplitude can be induced through the potential $\Delta c^\dagger_{\vec{p}\uparrow} c^\dagger_{-\vec{p}\downarrow} + \Delta^* c_{-\vec{p}\downarrow} c_{\vec{p}\uparrow}$, which breaks the $U(1)$ symmetry down to $\mathbb{Z}_2$. The QAH system in the presence of the pairing potential is more conveniently described in terms of the Bogoliubov–de Gennes (BdG) Hamiltonian

$$H_{BdG} = \sum_{\vec{p}} \Psi^\dagger_{\vec{p}} h_{BdG}(\vec{p}) \Psi_{\vec{p}}, \tag{2.7}$$

with the doubled Nambu spinor $\Psi_{\vec{p}} = \begin{pmatrix} c_{\vec{p}\uparrow} & c_{\vec{p}\downarrow} & c^\dagger_{-\vec{p}\uparrow} & c^\dagger_{-\vec{p}\downarrow} \end{pmatrix}^T$ and

$$h_{BdG}(\vec{p}) = \frac{1}{2} \begin{pmatrix} h_{QAH}(\vec{p}) - \mu & i\Delta\sigma_y \\ -i\Delta^*\sigma_y & -h^*_{QAH}(-\vec{p}) + \mu \end{pmatrix}, \tag{2.8}$$

where μ is a chemical potential and Δ is the superconducting gap. If we set $\Delta = 0$, this Hamiltonian reduces to (2.1) (with $\mu = 0$). In this basis, the spinor $\Psi_{\vec{p}}$ satisfies the constraint

$$\Psi_{\vec{p}} = (\sigma_x \otimes \mathbb{1}) \Psi^*_{-\vec{p}}. \tag{2.9}$$

To proceed, we simply ignore this constraint in the intermediate steps, but shall impose it in the end in order to retain the physical space.

In the absence of the chemical potential μ, we can easily block-diagonalize the Hamiltonian, namely,

$$H_{BdG} = \frac{1}{2}\sum_{\vec{p}} \tilde{\Psi}^{\dagger}_{\vec{p}} \begin{pmatrix} h_+(\vec{p}) & 0 \\ 0 & h_-(\vec{p}) \end{pmatrix} \tilde{\Psi}_{\vec{p}}, \tag{2.10}$$

where

$$h_\pm(\vec{p}) = \begin{pmatrix} m(\vec{p}^2) \pm \Delta & b_1(p_x - ip_y) \\ b_1(p_x + ip_y) & -(m(\vec{p}^2) \pm \Delta) \end{pmatrix}, \tag{2.11}$$

and $\tilde{\Psi}_{\vec{p}} = \begin{pmatrix} a_{+,\vec{p}} & a^{\dagger}_{+,-\vec{p}} & a_{-,\vec{p}} & -a^{\dagger}_{-,-\vec{p}} \end{pmatrix}^T$, with

$$a_{\pm,\vec{p}} \equiv \frac{1}{\sqrt{2}}\left(c_{\vec{p}\uparrow} \pm c^{\dagger}_{-\vec{p}\downarrow}\right). \tag{2.12}$$

In this basis, the constraint (2.9) acts on each block individually,

$$\tilde{\Psi}_{\vec{p}} = (\sigma_z \otimes \sigma_x)\tilde{\Psi}^*_{-\vec{p}}. \tag{2.13}$$

The operators $a_{\pm,\vec{p}}$ do not have a well-defined transformation property under $U(1)$ symmetry. They transform properly only under the subgroup of $U(1)$ transformations $e^{i\alpha}$ for $\alpha = 0, \pi$, i.e., under the $\mathbb{Z}_2$ group. Consequently, the excitations created upon application of these operators do not have well-defined electric charge. These considerations are important to correctly identify the physical excitations and, in particular, are extremely useful when we are describing this system in terms of the effective field theory, as it will be done in Sec. 3.1.2.

Note that the block Hamiltonians in (2.11) are equivalent to two copies of the Hamiltonian (2.2), but with a slight modification of the parameters, namely, $m_0 \to m_0 + \Delta$ for the block h_+ and $m_0 \to m_0 - \Delta$ for the block h_-. Thus, the winding number for the proximity effect can be obtained in the same way as for the QAH system simply as $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}_+ + \widetilde{\mathcal{W}}_-$, resulting in

$$\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}_+ + \widetilde{\mathcal{W}}_- = \begin{cases} 2 & |m_0| > \Delta, \text{ with } m_0 < 0 \\ 1 & |m_0| < \Delta \\ 0 & |m_0| > \Delta, \text{ with } m_0 > 0. \end{cases} \tag{2.14}$$

The corresponding phase diagram is shown in Fig. 1.

Some comments are in order. In the weak-coupling regime, $|m_0| > \Delta$, with $m_0 > 0$, the phase is trivial. For $m_0 < 0$, the system is a nontrivial topological phase which is adiabatically connected to the QAH phase in the limit $\Delta \to 0$. However, it is

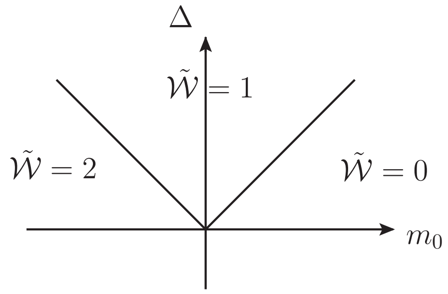

Figura 1 – Phase diagram of QAH in proximity to a superconducting pairing potential Δ.

important to notice that the winding number here corresponds to the number of gapless chiral fermions in the Majorana basis, because of the particle-hole symmetry inherent to the BdG formalism. So we interpret the two chiral gapless Majorana fermions as glued together to form a complex fermion, which corresponds to the chiral complex mode of the QAH. When the pairing becomes stronger, in the region $|m_0| < \Delta$, one of the Majorana fermions merges with the bulk states, and we are left with a single Majorana fermion in the boundary. This is the topological superconducting phase. We will see later how these patterns of phase transitions come up in terms of the effective field theory.

2.3 Quantum Wires Formulation and Edge States

In this section we discuss a simple way to derive the edge states associated with the QAH and TS phases. The idea is to transform the system into a system of quantum wires by discretizing one of the spatial directions. The edge states emerge in a quite natural way in this approach.

2.3.1 Quantum Anomalous Hall

The first step is to write a Lagrangian form for the QAH system. The action can be obtained from the Hamiltonian (2.1),

$$S_{QAH}[\psi, \bar{\psi}] = \int d^3x\, \bar{\psi}(i\gamma^0\partial_0 + ib_1\gamma^i\partial_i + b_2(i\gamma^i\partial_i)^2 - m_0)\psi, \tag{2.15}$$

with the following representation for the Dirac matrices,

$$\gamma^0 = \sigma^3\ ,\ \ \gamma^1 = i\sigma^2 \ \text{ and } \ \gamma^2 = -i\sigma^1, \tag{2.16}$$

where σ^1, σ^2, and σ^3 are the Pauli matrices. The spinor $\psi^T = (\psi_\uparrow\ \psi_\downarrow)$ is the coordinate counterpart of the fermion operator $\psi_{\vec{p}}$, and the spinor $\bar{\psi}$ is defined in the usual way as $\bar{\psi} \equiv \psi^\dagger\gamma^0$.

With the goal of discretizing the y direction, it is convenient to rotate to a new representation through the unitary operator

$$U = e^{\frac{i\pi}{4}\sigma^2}. \tag{2.17}$$

This converts the Dirac matrices to the set

$$\gamma^0 = -\sigma^1\ ,\ \ \gamma^1 = i\sigma^2 \ \text{ and } \ \gamma^2 = -i\sigma^3, \tag{2.18}$$

with the new spinor components

$$\psi_R \equiv \frac{1}{\sqrt{2}}(\psi_\uparrow + \psi_\downarrow) \ \text{ and } \ \psi_L \equiv \frac{1}{\sqrt{2}}(-\psi_\uparrow + \psi_\downarrow). \tag{2.19}$$

In this basis, the action in (2.15) becomes

$$\begin{aligned} S &= \int d^3x \left[i\psi_R^\dagger \partial_+ \psi_R + i\psi_L^\dagger \partial_- \psi_L - b_2 \psi_L^\dagger \partial_x^2 \psi_R - b_2 \psi_R^\dagger \partial_x^2 \psi_L \right. \\ &+ \left. m_0 \psi_L^\dagger \left(1 - m_0^{-1} b_1 \partial_y - m_0^{-1} b_2 \partial_y^2\right) \psi_R + m_0 \psi_R^\dagger \left(1 + m_0^{-1} b_1 \partial_y - m_0^{-1} b_2 \partial_y^2\right) \psi_L \right], \end{aligned} \tag{2.20}$$

where we have defined $\partial_\pm \equiv \partial_0 \pm b_1 \partial_1$. We are assuming that the mass m_0 is nonvanishing, so that the bulk remains always gapped. We discuss first the QAH phase, where the mass m_0 is negative. In this case, let us choose a representative point in the phase diagram corresponding to the QAH phase, where the edge states appear in a quite direct way. This is achieved by choosing the parameters so that

$$b_2 = -\frac{b_1^2}{2m_0}. \tag{2.21}$$

Notice that, as b_1 and b_2 are positive, this condition can only be satisfied for negative m_0, i.e., in the QAH phase. With this choice, the action can be written as

$$\begin{aligned} S &= \int d^3x \left[i\psi_R^\dagger \partial_+ \psi_R + i\psi_L^\dagger \partial_- \psi_L - \frac{b_1^2}{2|m_0|} \psi_L^\dagger \partial_x^2 \psi_R - \frac{b_1^2}{2|m_0|} \psi_R^\dagger \partial_x^2 \psi_L \right. \\ &- \left. |m_0| \psi_L^\dagger \left(1 + a\partial_y + \frac{1}{2} a^2 \partial_y^2\right) \psi_R - |m_0| \psi_R^\dagger \left(1 - a\partial_y + \frac{1}{2} a^2 \partial_y^2\right) \psi_L \right], \end{aligned} \tag{2.22}$$

where we have introduced the "wire spacing" $a \equiv \frac{b_1}{|m_0|}$. For a large gap $|m_0|$, the wire spacing a is small and the terms in the second line of this expression can be identified as second-order Taylor expansions, so that (2.22) is approximated by

$$\begin{aligned} S &\approx \int d^3x \left[i\psi_R^\dagger \partial_+ \psi_R + i\psi_L^\dagger \partial_- \psi_L - \frac{ab_1}{2} \psi_L^\dagger \partial_x^2 \psi_R - \frac{ab_1}{2} \psi_R^\dagger \partial_x^2 \psi_L \right. \\ &- \left. \frac{b_1}{a} \psi_L^\dagger(t,x,y) \psi_R(t,x,y+a) - \frac{b_1}{a} \psi_R^\dagger(t,x,y+a) \psi_L(t,x,y) \right]. \end{aligned} \tag{2.23}$$

This form leads naturally to the discretization of the y direction. To this, we make the following prescriptions:

$$\psi_{R/L}(t,x,y) \to \frac{1}{\sqrt{a}} \psi^j_{R/L}(t,x) \qquad \text{and} \qquad \int dy \to a \sum_j. \tag{2.24}$$

Then, by considering the system with open boundary conditions in the y direction and using (2.24), the action (2.23) becomes

$$S \approx \int d^2x \left[\sum_{j=1}^{N} \left(i\psi_R^{j\dagger}\partial_+\psi_R^j + i\psi_L^{j\dagger}\partial_-\psi_L^j \right) - \frac{b_1}{a} \sum_{j=1}^{N-1} \left(\psi_L^{j\dagger}\psi_R^{j+1} + \psi_R^{j+1\dagger}\psi_L^j \right) + \cdots \right] \quad (2.25)$$

where we have discarded the irrelevant terms of order a. We notice that the chiral modes ψ_R^1 and ψ_L^N, associated with the first and the last wires, respectively, are decoupled from the remaining modes in this action. In fact, they are governed exclusively by kinetic terms

$$S_{\text{edge}} = \int d^2x \left(i\psi_R^{1\dagger}\partial_+\psi_R^1 + i\psi_L^{N\dagger}\partial_-\psi_L^N \right), \quad (2.26)$$

and thus are identified as the gapless edge states of the QAH phase. It is worth emphasizing that these modes are spatially far apart from each other and, due to the locality of the interactions, they cannot be gapped through backscattering. Furthermore, deviations from (2.21) will generically introduce interactions between the gapless edge modes and gapped modes both at the same wire and at the neighboring wires. However, as long as the interaction strength is small compared to the gap, the edge states will remain gapless. These features imply that they are stable. In sum, we have found a complex one-way propagating fermion in each one of the boundaries; i.e., the edge theory of the QAH phase is a conformal field theory with chiral central charge $c = 1$.

Now we discuss the system in the trivial phase $m_0 > 0$. In this case, we cannot proceed as in the previous one, since the terms in the second line of (2.20), namely,

$$\psi_L^\dagger \left(1 - m_0^{-1} b_1 \partial_y - m_0^{-1} b_2 \partial_y^2 \right) \psi_R + \text{H. c.}, \quad (2.27)$$

do not correspond to a second-order Taylor expansion for any choice of the parameters and, consequently, after discretization no mode will be left decoupled. To see this, we define again the "wire spacing"$a \equiv \frac{b_1}{m_0}$ so that, for large gap, the above expression can be written as

$$\begin{aligned} &\psi_L^\dagger(t,x,y)\psi_R(t,x,y-a) - \frac{b_2}{ab_1}\psi_L^\dagger(t,x,y)\left[\psi_R(t,x,y+a)\right. \\ - \; & \left. 2\psi_R(t,x,y) + \psi_R(t,x,y-a)\right] + \text{H. c.}. \end{aligned} \quad (2.28)$$

We then proceed with the discretization of these terms according to the prescriptions in (2.24), taking into account open boundary conditions in the y direction. This produces in the action terms proportional to

$$\begin{aligned} - \; & \frac{b_2}{ab_1} \sum_{j=1}^{N-1} \psi_L^{j\dagger}(t,x)\psi_R^{j+1}(t,x) + \frac{2b_2}{ab_1} \sum_{j=1}^{N} \psi_L^{j\dagger}(t,x)\psi_R^j(t,x) \\ + \; & \left(1 - \frac{b_2}{ab_1}\right) \sum_{j=2}^{N} \psi_L^{j\dagger}(t,x)\psi_R^{j-1}(t,x) + \text{H. c.}, \end{aligned} \quad (2.29)$$

which shows that all modes partake in the interaction (for any choice of the parameters) and then turn out to be gapped. There are no gapless edge states left behind, as expected for the topologically trivial phase.

2.3.2 Superconducting Phase

Now we consider the system in the presence of the superconducting pairing potential and move to the BdG formalism. The action corresponding to the Hamiltonian (2.10) is

$$\begin{aligned} S_{QAH-TS} &\equiv \frac{1}{2}\int d^3x \left[\mathcal{L}_+(\psi_+,\bar{\psi}_+) + \mathcal{L}_-(\psi_-,\bar{\psi}_-)\right] \\ &= \frac{1}{2}\int d^3x \Big[\bar{\psi}_+(i\gamma^0\partial_0 + ib_1\gamma^i\partial_i + b_2(i\gamma^i\partial_i)^2 - (m_0+\Delta))\psi_+ \\ &+ \bar{\psi}_-(i\gamma^0\partial_0 + ib_1\gamma^i\partial_i + b_2(i\gamma^i\partial_i)^2 - (m_0-\Delta))\psi_-\Big], \end{aligned} \tag{2.30}$$

where the doubled spinors are $\psi^T_\pm = \begin{pmatrix} a_\pm & a^\dagger_\pm \end{pmatrix}$. We can then follow a strategy similar to the previous case for discretizing this action.

The first step is to rotate according to the unitary operator in (2.17), under which the spinors transform as

$$\psi_\pm \;\to\; \tilde{\psi}_\pm \equiv U\psi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} a_\pm + a^\dagger_\pm & a^\dagger_\pm - a_\pm \end{pmatrix}^T. \tag{2.31}$$

The transformed spinors naturally lead to the introduction of the Majorana operators

$$\chi^\pm_R \equiv \frac{1}{\sqrt{2}}\left(a_\pm + a^\dagger_\pm\right) \quad \text{and} \quad \chi^\pm_L \equiv \frac{i}{\sqrt{2}}\left(a_\pm - a^\dagger_\pm\right), \tag{2.32}$$

so that

$$\tilde{\psi}_\pm = (\chi^\pm_R \; i\chi^\pm_L)^T. \tag{2.33}$$

In terms of the Majorana fermions, the corresponding Lagrangians become

$$\begin{aligned} \mathcal{L}_\pm &= i\chi^\pm_R\partial_+\chi^\pm_R + i\chi^\pm_L\partial_-\chi^\pm_L + 2ib_2\chi^\pm_L\partial^2_x\chi^\pm_R \\ &- 2im_\pm\chi^\pm_L\left(1 - m^{-1}_\pm b_1\partial_y - m^{-1}_\pm b_2\partial^2_y\right)\chi^\pm_R, \end{aligned} \tag{2.34}$$

where we have defined the masses $m_\pm \equiv m_0 \pm \Delta$.

We are mostly interested in studying the edge states of the superconducting phase, where $m_+ > 0$ and $m_- < 0$. In this situation, the contribution of $\mathcal{L}_+$ works precisely as the Lagrangian of the trivial phase discussed previously, and consequently does not lead to any massless edge state. On the other hand, the contribution of $\mathcal{L}_-$ works like the Lagrangian of the QAH phase, but with the difference that now the edge states are given in terms of Majorana fermions. In fact, we see that the term

$$-2im_-\chi^-_L\left(1 - m^{-1}_- b_1\partial_y - m^{-1}_- b_2\partial^2_y\right)\chi^-_R \tag{2.35}$$

of $\mathcal{L}_-$ can be identified as a second-order Taylor expansion provided we choose the representative point

$$b_2 = -\frac{b_1^2}{2m_-}. \tag{2.36}$$

With this choice and identifying the wire spacing as $a \equiv \frac{b_1}{|m_-|}$, the expression (2.35) can be approximated by

$$\frac{2ib_1}{a}\chi_L^-(t,x,y)\chi_R^-(t,x,y+a), \tag{2.37}$$

in the limit of large gap $|m_-|$. After the discretization using the prescriptions in (2.24), this leads to a term in the action proportional to

$$\sum_{j=1}^{N-1} \chi_{L,j}^-(t,x)\chi_{R,j+1}^-(t,x), \tag{2.38}$$

which implies that the Majorana modes $\chi_{R,1}^-$ and $\chi_{L,N}^-$, associated with the first and the last wires, respectively, are decoupled and then remain gapless. Therefore, the edge theory of the superconducting phase is given by

$$S_{\text{edge}} = \int d^2x \left(i\chi_{R,1}^-\partial_+\chi_{R,1}^- + i\chi_{L,N}^-\partial_-\chi_{L,N}^- \right), \tag{2.39}$$

which corresponds to a conformal field theory with chiral central charge $c = \frac{1}{2}$.

We shall return to the edge theory later on, when discussing how the edge states can be recovered from the effective field theory through the bulk-edge correspondence.

3 TOPOLOGICAL SUPERCONDUCTOR EFFECTIVE ACTION AND SPECTRUM

3.1 Effective Field Theory

The main goal of this chapter is to derive the low-energy effective field theory for the QAH system in proximity to an s-wave superconductor. Given the nonrelativistic character of the fermion Lagrangian, it is not clear whether a Chern-Simons term can arise from the corresponding fermionic determinant.

3.1.1 Topological Effective Field Theory for the QAH Phase

The nontrivial part of the computation of the effective action is already present in the case of the pure QAH system, and so we will concentrate first on this case. The strategy is to introduce a background gauge field A for the global $U(1)$ symmetry of the QAH, and then integrate out the gapped fermions to obtain $S_{eff}[A]$.

The corresponding action is given in (2.15), which we repeat here for convenience,

$$S_{QAH}[\psi,\bar{\psi}] = \int d^3x\, \bar{\psi}(i\gamma^0\partial_0 + ib_1\gamma^i\partial_i + b_2(i\gamma^i\partial_i)^2 - m_0)\psi. \tag{3.1}$$

Then we introduce a background gauge field A for the global $U(1)$ symmetry[1],

$$S_{QAH}[\psi,\bar{\psi};A] = \int d^3x\, \bar{\psi}(i\gamma^0 D_0 + ib_1\gamma^i D_i + b_2(i\gamma^i D_i)^2 - m_0)\psi, \tag{3.2}$$

where $D_\mu \equiv \partial_\mu - iA_\mu$, $\mu = 0,1,2$. Before proceeding, it seems that there is certain ambiguity in this process. Indeed, if we write the term with coefficient b_2 in (3.1) as $b_2(i\gamma^i\partial_i)^2 = b_2\vec{\nabla}^2$, then the gauging of this term is simply $b_2\vec{D}^2$, which is different from the term $b_2(i\gamma^i D_i)^2$ of (3.2), namely,

$$\begin{aligned}(i\gamma^i D_i)^2 &= \vec{D}^2 + \frac{i}{4}[\gamma^i,\gamma^j]F_{ij} \\ &= \vec{D}^2 - \frac{1}{2}\gamma^0\epsilon^{0ij}F_{ij},\end{aligned} \tag{3.3}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$ and we have used $[\gamma^\mu,\gamma^\nu] = -2i\epsilon^{\mu\nu\rho}\gamma_\rho$. Therefore, these two ways of treating the higher-derivative term produce theories differing by an operator proportional to $\bar{\psi}\gamma^0\epsilon^{0ij}F_{ij}\psi$. In general, such a UV operator is expected simply to renormalize the parameters of the low-energy effective theory. However, as we shall discuss later, it does not affect the induced topological Chern-Simons term since it amounts just to a redefinition of the current. In the following we proceed with the form (3.2).

[1] Some aspects of this model with dynamical gauge field and positive mass have been analyzed previously in [39].

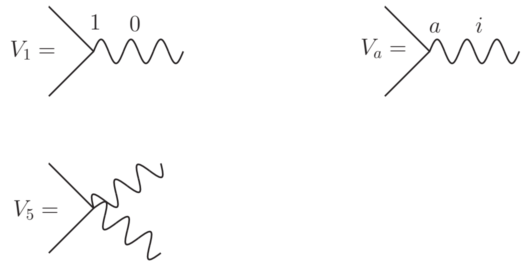

Figura 2 – Interaction vertices. The index 0 in the diagram V_1 means that the photon line involves the component A_0. The diagram V_a represents generically the three vertices with $a = 2, 3, 4$, and the index i means that the photon line involves the component A_i.

The strategy is to compute the effective action for the background gauge field A by integrating out the fermions,

$$e^{iS_{eff}[A]} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS_{QAH}[\psi,\bar{\psi}]}. \tag{3.4}$$

The effective action is then organized in a local expansion in powers of the external field,

$$\begin{aligned} iS_{eff}[A] &= \int d^3x \frac{1}{2} \left(A_0 \Pi_{00} A_0 + A_0 \Pi_{0i} A_i + A_i \Pi_{i0} A_0 + A_i \Pi_{ij} A_j + \cdots \right) \\ &= \int d^3x \frac{1}{2} \left(A_0 \Pi_{00} A_0 + 2 A_0 \Pi_{0i} A_i + A_i \Pi_{ij} A_j + \cdots \right), \end{aligned} \tag{3.5}$$

where we have used the fact that the two terms involving A_0 and A_i give the same contribution up to integration by parts. The operators $\Pi_{\mu\nu}$ can be computed from the relevant Feynman diagrams with fermions in the internal lines and the background field in the external ones.

From (3.2) we can identify immediately the interaction vertices,

$$V_1 \equiv \bar{\psi}\gamma^0 A_0 \psi, \quad V_2 \equiv b_1 \bar{\psi}\gamma^i A_i \psi, \tag{3.6}$$

and

$$V_3 \equiv -ib_2 (\partial_i \bar{\psi}) \gamma^i \gamma^j A_j \psi, \quad V_4 \equiv ib_2 \bar{\psi} \gamma^i \gamma^j A_i \partial_j \psi, \quad V_5 \equiv b_2 \bar{\psi} \left(\gamma^i A_i \right)^2 \psi. \tag{3.7}$$

They are shown in Fig. 2. The fermion propagator is

$$\begin{aligned} S(k) &= \frac{i}{\gamma^0 k_0 - b_1 \vec{\gamma} \cdot \vec{k} - b_2 \vec{k}^2 - m_0 + i\epsilon} \\ &= \frac{i(k_0 \gamma^0 - b_1 \vec{k} \cdot \vec{\gamma} + b_2 \vec{k}^2 + m_0)}{k_0^2 - b_1^2 \vec{k}^2 - (b_2 \vec{k}^2 + m_0)^2 + i\epsilon}. \end{aligned} \tag{3.8}$$

The one-loop contributions to the two-point functions of the background field are shown in Figs. 3, 4, and 5.

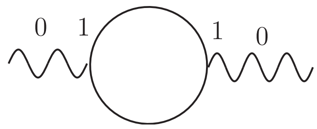

Figura 3 – One-loop contribution for the two-point function $\langle A_0 A_0 \rangle$, which contributes for Π_{00} in the effective action (3.5).

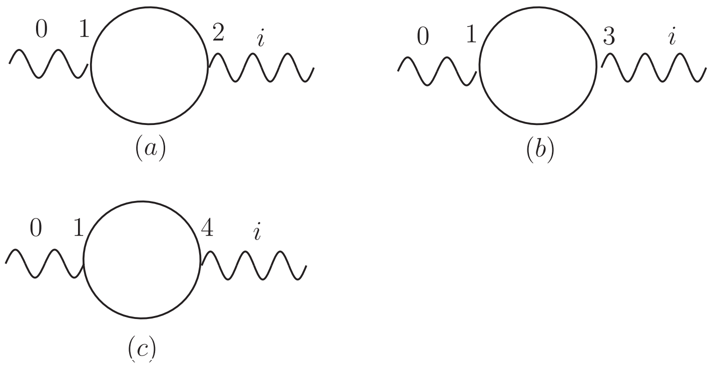

Figura 4 – One-loop contributions for the two-point function $\langle A_0 A_i \rangle$, which contribute for Π_{0i} in the effective action (3.5).

The one-loop contributions are organized in a derivative expansion or, equivalently, in a momentum expansion. There are no zero-th-order contributions to the two-point function $\langle A_\mu A_\nu \rangle$.

We then consider the first-order derivative contributions to the two-point function $\langle A_0 A_i \rangle$. They are contained in the diagrams of Fig. 4, and furnish

$$\Pi_{0i}^{(a)}(p) = \left[\Theta(m_0)\left(\frac{-b_1^2}{4\pi(b_1^2+4b_2m_0)}\right) + \Theta(-m_0)\left(\frac{1}{4\pi}\right)\right]\epsilon_{0ij}p_j + \cdots , \tag{3.9}$$

$$\Pi_{0i}^{(b)}(p) = \Pi_{0i}^{(c)}(p) = \left[\Theta(m_0)\left(\frac{b_1^2}{8\pi(b_1^2+4b_2m_0)}\right) + \Theta(-m_0)\left(\frac{1}{8\pi}\right)\right]\epsilon_{0ij}p_j + \cdots , \tag{3.10}$$

where Θ is the Heaviside step function. Adding up the three pieces, we get

$$\Pi_{0i}(p) = \Pi_{0i}^{(a)} + \Pi_{0i}^{(b)} + \Pi_{0i}^{(c)} = \Theta(-m_0)\frac{1}{2\pi}\epsilon_{0ij}p_j + \cdots . \tag{3.11}$$

Finally, we consider the two-point function $\langle A_i A_j \rangle$. Only the diagrams (a), (b), and (c) of Fig. 5 contribute to the first-order derivative term. The corresponding contributions are

$$\Pi_{ij}^{(a)}(p) = \Theta(m_0)\left(\frac{-b_1^2}{2\pi(b_1^2+4b_2m_0)}\right)\epsilon_{0ij}p_0 + \cdots , \tag{3.12}$$

$$\Pi_{ij}^{(b)}(p) = \Pi_{ij}^{(c)}(p) = \left[\Theta(m_0)\left(\frac{b_1^2}{4\pi(b_1^2+4b_2m_0)}\right) + \Theta(-m_0)\left(\frac{1}{4\pi}\right)\right]\epsilon_{0ij}p_0 + \cdots . \tag{3.13}$$

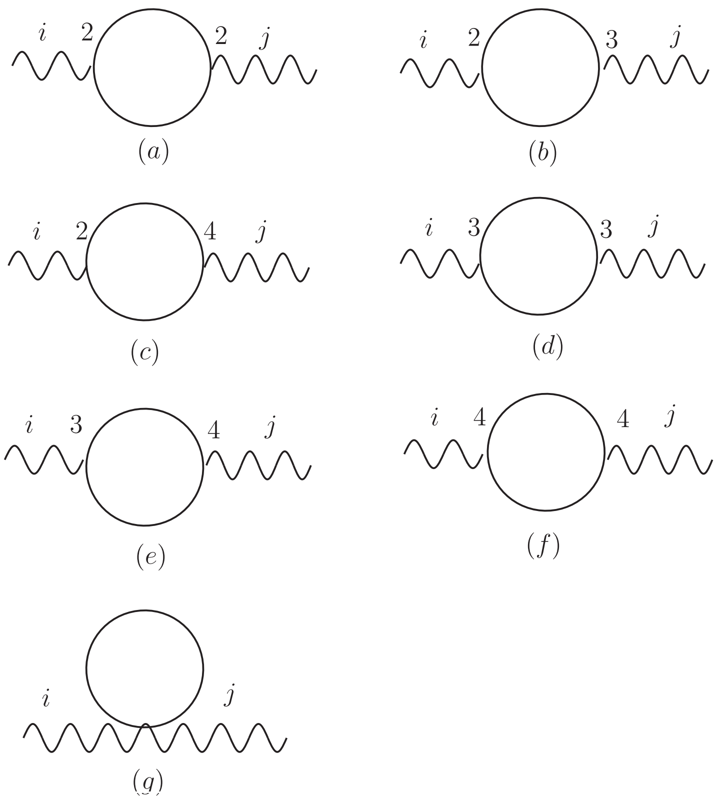

Figura 5 – One-loop contributions for the two-point function $\langle A_i A_j \rangle$, which contribute for Π_{ij} in the effective action (3.5).

Adding the three terms, we obtain

$$\Pi_{ij}^{(a)} + \Pi_{ij}^{(b)} + \Pi_{ij}^{(c)} = \Theta(-m_0)\frac{1}{2\pi}\epsilon_{0ij}p_0 + \cdots . \quad (3.14)$$

The diagrams (d), (e), and (f) contribute at least with two derivatives, while the diagram (g) does not contribute at all.

Including all the first-order contributions to the low-energy effective action (3.5), we obtain

$$\begin{aligned} iS_{eff}[A] &= \int \frac{d^3p}{(2\pi)^3}\left[\Theta(-m_0)A_0(p)\left(\frac{1}{2\pi}\epsilon_{0ij}p_j\right)A_i(-p) + \Theta(-m_0)A_i(p)\left(\frac{1}{4\pi}\epsilon_{0ij}p_0\right)A_j(-p) + \right. \\ &= \int \frac{d^3p}{(2\pi)^3}\left[-\Theta(-m_0)\frac{1}{4\pi}\epsilon^{\mu\nu\rho}A_\mu(p)p_\nu A_\rho(-p) + \cdots\right]. \end{aligned}$$

In the coordinate space this reads

$$S_{eff}[A] = \int d^3x \left[\Theta(-m_0)\frac{1}{4\pi}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho + \cdots\right]. \quad (3.16)$$

It is quite remarkable that even though the fermion dynamics is nonrelativistic, a usual Chern-Simons term is recovered in the topological sector of the effective theory. We can further appreciate this result recalling that in any gapped system of charged fermions the Hall conductivity, σ_{xy}, is quantized, and it is given in terms of the winding number of the momentum space Hamiltonian [40]. The Hall responses $J^i = \sigma_{xy}\epsilon^{ij}E^j$ and $\sigma_{xy} = \frac{\partial J^0}{\partial B}$ (Streda formula) to the application of electric and magnetic fields in the system, can be combined in a covariant way as $J^\mu = \sigma_{xy}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho$. Then, these responses can be obtained from the coupling $A_\mu J^\mu$, which in terms of an effective action corresponds to $\frac{\sigma_{xy}}{2}\int d^3x\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$. This is an indication that, even though the fermion system is nonrelativistic, the field theory has to manage to deliver a usual CS term, whose coefficient is related to the Hall conductivity.

In the present case, the Chern-Simons term arises from a nontrivial combination of several one-loop diagrams, whereas in the usual (relativistic) case the Chern-Simons term comes from a single one-loop diagram. The higher-order derivative corrections represented by the dots are nonrelativistic. In addition, the CS coefficient in (3.16) is properly quantized and does not break invariance under large gauge transformations, in contrast to the usual case of the relativistic fermionic determinant. The Hall conductivity σ_{xy} can be read from the Chern-Simons coefficient $\frac{\sigma_{xy}}{2}\int AdA$, which implies that $\sigma_{xy} = \frac{\Theta(-m_0)}{2\pi}$. This recovers precisely the results of the previous section for the Hall conductivity $\sigma_{xy} = \frac{1}{2\pi}\mathcal{W}$, with $\mathcal{W}$ given in (2.5). In Appendix A, we discuss that the CS term is protected against radiative corrections when the gauge field is dynamical.

Now we discuss that the introduction of a chemical potential μ in the system does not affect the topological sector of the effective field theory. The introduction of the chemical potential amounts to replacing $h_{QAH}(\vec{p}) \to h_{QAH}(\vec{p}) - \mu$ in the Hamiltonian (2.1). This, in turn, corresponds to the presence of the term $\mu\bar{\psi}\gamma^0\psi$ in the Lagrangian (3.1). We can absorb this factor in the time derivative, $\tilde{\partial}_0 \equiv \partial_0 - i\mu$, or equivalently, in the zero component of the momentum, $\tilde{p}_0 \equiv p_0 + \mu$. Therefore, it is immediate to see that the chemical potential does not affect the Chern-Simons contribution,

$$\int d^3p A_i(p)\epsilon_{0ij}\tilde{p}_0 A_j(-p) = \int d^3p A_i(p)\epsilon_{0ij}p_0 A_j(-p), \tag{3.17}$$

since $\mu\int d^3p A_i(p)\epsilon_{0ij}A_j(-p)$ vanishes by symmetry.

Before move on, we discuss the effect of considering $\vec{D}^2$ instead of $(i\gamma^i D_i)^2$ in the definition of the QAH system in the presence of an external field. According to Eq. (3.3), this amounts to including in the previous computation a new interaction vertex that we denote as

$$V_6 \equiv \frac{b_2}{2}\bar{\psi}\gamma^0\epsilon_{0ij}F_{ij}\psi. \tag{3.18}$$

This new vertex has the potential to generate a Chern-Simons term in combination with the vertices V_1, V_2, V_3, and V_4. However, we can check explicitly that they do not generate any Chern-Simons contribution, so that the topological sector of the theory is not sensitive to the presence of the above operator.

To understand the underlying reason for this, we consider the $U(1)$ current following from (3.2). It can be easily constructed by considering the generalization of the Noether theorem for higher derivative theories (see for example [41]). In particular, for an internal symmetry the current for a second-order derivative theory reads

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_I}\delta\phi_I + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi_I}\partial_\nu \delta\phi_I - \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi_I}\right)\delta\phi_I, \tag{3.19}$$

where ϕ_I stands for a generic set of fields. Using this expression for the global $U(1)$ symmetry of (3.2), we obtain the components

$$J^0 = \bar\psi\gamma^0\psi \quad \text{and} \quad J^i = b_1\bar\psi\gamma^i\psi + ib_2(\bar\psi\gamma^i\gamma^j\partial_j\psi - \partial_j\bar\psi\gamma^j\gamma^i\psi) + 2b_2\bar\psi\psi A^i. \tag{3.20}$$

A conserved current is not uniquely defined, since we can always redefine it as

$$J'^\mu = J^\mu + \partial_\nu\Omega^{\mu\nu}, \quad \text{with} \quad \Omega^{\mu\nu} = -\Omega^{\nu\mu}. \tag{3.21}$$

The new current J'^μ is as good as the initial one, once it is also conserved and gives rise to the same charge $\int_{\text{space}} J^0$.

Next we consider the current in the theory with the operator $b_2\vec{D}^2$ instead of $b_2(i\gamma^i D_i)^2$. In this case, the current is given by

$$J'^0 = J^0 \quad \text{and} \quad J'^i = J^i - b_2\partial_j\left(\bar\psi\epsilon^{ij0}\gamma_0\psi\right). \tag{3.22}$$

We see that the currents J^μ and J'^μ differ precisely by a term of the form (3.21), with $\Omega^{0i} = 0$ and $\Omega^{ij} = -b_2\bar\psi\epsilon^{ij0}\gamma_0\psi$.

Next, we consider the generic relation

$$\frac{\delta S_{eff}[A]}{\delta A_\mu} = \langle J^\mu \rangle. \tag{3.23}$$

Denoting by $S'_{eff}[A]$ the effective action coming from the UV theory involving $b_2\vec{D}^2$ instead of $b_2(i\gamma^i D_i)^2$, we have likewise

$$\frac{\delta S'_{eff}[A]}{\delta A_\mu} = \langle J'^\mu \rangle. \tag{3.24}$$

Now consider the density in the relations (3.23) and (3.24). As $J^0 = J'^0$, it follows that

$$\frac{\delta}{\delta A_0}\left(S'_{eff}[A] - S_{eff}[A]\right) = 0, \tag{3.25}$$

namely, the corresponding effective actions differ only by gauge-invariant terms independent of A_0,

$$S'_{eff}[A] = S_{eff}[A] + \int d^3x\, O(A_i). \tag{3.26}$$

Therefore, the CS contribution must be exactly the same in both effective actions.

3.1.2 EFT for the QAH Phase in Proximity to a TS

After the derivation of the effective field theory for the QAH system, we are ready to study the effective field theory in the presence of the superconducting pairing potential. As discussed previously, in this case the $U(1)$ global symmetry is broken down to the discrete $\mathbb{Z}_2$ symmetry. However, as is usual in treating superconductor systems, it is quite useful to insist on keeping a *fictitious* $U(1)$ symmetry. To conceive this, we simply ignore the constraint (2.13) and work with unconstrained spinors. This means that for each block in (2.10) we have a fictitious $U(1)$ symmetry.

In quantizing the theory with the fictitious $U(1)$ global symmetries we find that the states in this enlarged Hilbert space are in the representation of the fictitious $U(1)$. Among all these states, we have to project out all the states which are not invariant under the action of $\mathbb{Z}_2$, so that the physical Hilbert space contains only states that are properly $\mathbb{Z}_2$ invariant.

We can take advantage of the fictitious $U(1)$ symmetries to introduce background fields for them. In this way, we can follow the strategy of the previous section and integrate out the fermions to derive the (local) effective action for these background fields. The above projection onto the physical Hilbert space can be implemented concretely by considering the background fields as actually $O(2) = U(1) \rtimes \mathbb{Z}_2$ gauge fields, instead of $U(1)$. The local action is precisely the action of a $U(1) \times U(1)$ theory but the corresponding path integrals are different. In the $O(2)$ case we have to sum over the $\mathbb{Z}_2$ bundles, which is equivalent to computing the path integral with the insertion of projection operators that select only the $\mathbb{Z}_2$-invariant states.

We start considering again the action (2.30),

$$\begin{aligned} S_{QAH-TS} &= \frac{1}{2}\int d^3x \Big[\bar{\psi}_+(i\gamma^0\partial_0 + ib_1\gamma^i\partial_i + b_2(i\gamma^i\partial_i)^2 - (m_0+\Delta))\psi_+ \\ &+ \bar{\psi}_-(i\gamma^0\partial_0 + ib_1\gamma^i\partial_i + b_2(i\gamma^i\partial_i)^2 - (m_0-\Delta))\psi_-\Big], \end{aligned} \tag{3.27}$$

where $\psi_\pm$ are the unconstrained two-component spinors associated with the two blocks of (2.10). Upon the constraint (2.13), we get in the momentum space simply

$$\psi_+^T(\vec{p}) = (a_{+,\vec{p}}, a^\dagger_{+,-\vec{p}}) \quad \text{and} \quad \psi_-^T(\vec{p}) = (a_{-,\vec{p}}, -a^\dagger_{-,-\vec{p}}). \tag{3.28}$$

Proceeding with the unconstrained spinors, we can then introduce the corresponding background fields $A^{(\pm)}$ and compute the low-energy effective theory by integrating out the fermions, exactly as we did in the previous section. One subtle point that arises in this procedure is that there is no natural way to assign fictitious $U(1)$ charges for $\psi_\pm$. Therefore, we shall keep them unfixed momentarily, which amounts to considering covariant derivatives $D_\mu = \partial_\mu - iqA^{(\pm)}_\mu$, for a generic integer charge q. We will fix the charge posteriorly on physical grounds, so that the resulting effective theory describes properly the underlying physics.

The effective action in this case can be immediately read out from the result (3.29), with the appropriate adjustment of the parameters, as well as a corresponding rescaling of gauge fields by a factor of q,

$$S_{eff}[A^{(+)},A^{(-)}] = \int d^3x \left[\Theta(-m_0-\Delta)\frac{q^2}{4\pi}\epsilon^{\mu\nu\rho}A_\mu^{(+)}\partial_\nu A_\rho^{(+)} + \Theta(-m_0+\Delta)\frac{q^2}{4\pi}\epsilon^{\mu\nu\rho}A_\mu^{(-)}\partial_\nu A_\rho^{(-)} + \cdots\right]. \quad (3.29)$$

As anticipated, this local theory is the same as the one of a $U(1)\times U(1)$ Chern-Simons theory. However, these fields are $O(2) = U(1)\rtimes\mathbb{Z}_2$ gauge fields and we shall explore the consequences in the next section. It is quite interesting to see how the pattern of phases described by Eq. (2.14) manifests here. In the trivial case, when $|m_0| > \Delta,$ with $m_0 > 0$, both Chern-Simons coefficients vanish, which is expected for a topologically trivial phase. In the superconducting phase, when $|m_0| < \Delta$, only one of the Chern-Simons terms contributes, namely, the one associated with $A^{(-)}$. Finally, in the QAH phase, for $|m_0| > \Delta,$ with $m_0 < 0$, both Chern-Simons are nonvanishing.

3.2 Superconducting Phase and $U(1)/\mathbb{Z}_2$ Orbifold

We shall focus on the superconducting phase, $|m_0| < \Delta$, whose low-energy effective theory is then described by a single $O(2)$ Chern-Simons theory at the level q^2,

$$S_{eff}[A^{(-)}] = \int d^3x\, \frac{q^2}{4\pi}\epsilon^{\mu\nu\rho}A_\mu^{(-)}\partial_\nu A_\rho^{(-)}. \quad (3.30)$$

We can in principle proceed by studying the $O(2)$ CS theory itself (see for example [42, 43, 44]). However, an insightful way to unveil the spectrum of the $O(2)$ Chern-Simons gauge theory is through its relation with rational conformal field theory (RCFT), which goes back to the classic work by Moore and Seiberg [45]. The relationship between CS and RCFT is specially suitable for describing topological phases of matter, because of the presence of edge states described by conformal field theories whenever the system is defined on a manifold with a physical boundary. We can consider, for example, the spatial manifold as a disk D with a boundary S^1. If the time coordinate is also a circle S^1, then the resulting conformal field theory will be defined on a torus $T^2 = S^1\times S^1$.

In the $O(2)$ case, the one-to-one correspondence between Chern-Simons and RCFT occurs when q^2 is of the form $q^2 \equiv 2N$, with N being a positive integer. We see that only for even values of the charge q there is a solution with both q and N integers. The simplest case is $N = 2$, which corresponds to $q = 2$, and we shall discuss that this case describes precisely the physical properties of the superconducting phase.

3.2.1 Extension of the Chiral Algebra and the Orbifold

One of the key results of Moore and Seiberg [45] is that the edge theory associated with $O(2)$ CS at the level $2N$ is the $U(1)/\mathbb{Z}_2$ orbifold theory at the level $2N$, which contains $N+7$ primary fields. To construct the orbifold theory we firstly consider the theory of a compact free scalar field $\varphi \sim \varphi + 2\pi R$, where R is the compactification radius (it is identified as $R = \sqrt{2N}$ through the bulk-edge correspondence), with a $U(1)$ current $j(z) = \partial_z \varphi$. When the square of the compactification radius is a rational number, some vertex operators become purely holomorphic and then can be added to the chiral algebra to produce an extended algebra, denoted by $\mathcal{A}_N$. Specifically, the maximal extension is obtained when the square of the compactification radius is of the form $\frac{R^2}{2} = \frac{p}{p'}$, where p and p' are coprimes (their greatest common divisor is 1). In this case, the basic vertex operators

$$e^{\pm i\sqrt{2N}\varphi}, \quad \text{with } \; N = pp', \tag{3.31}$$

become purely chiral and can be used to extend the algebra. The representations of the extended algebra are given by the remaining vertex operators that have local OPE (trivial monodromy) with all the generators of the extended algebra, namely,

$$e^{i\frac{n}{\sqrt{2N}}\varphi}, \quad n \in \mathbb{Z}. \tag{3.32}$$

However, only those operators for which n belongs to the interval $n = 0, 1, ..., 2N-1$ are indeed primary fields. Integer values of n outside this interval amounts to the insertion of one of the generators $e^{\pm i\sqrt{2N}\varphi}$ and consequently correspond to descendant operators. This introduces equivalence classes among states, organized in a $\mathbb{Z}_{2N}$ structure. Therefore, the number of representations is truncated and become finite, so that the extended algebra ends up having $2N$ representations.

The next step is to project out the states that are noninvariant under $\mathbb{Z}_2$ symmetry, to obtain the orbifold $U(1)/\mathbb{Z}_2$. To this, we first note that under the action of the $\mathbb{Z}_2$ symmetry $\varphi \to -\varphi$, the vertex operators $e^{i\frac{n}{\sqrt{2N}}\varphi}$ are mapped onto $e^{-i\frac{n}{\sqrt{2N}}\varphi}$. Then we use the $\mathbb{Z}_{2N}$ equivalence to bring it back to the interval $0, 1, ..., 2N-1$. In sum, the effect of the $\mathbb{Z}_2$ symmetry is $|n\rangle \to |-n+2N\rangle$. With this in mind, we can determine the fate of the states under the action of the $\mathbb{Z}_2$ symmetry,

$$\begin{aligned}
|0\rangle &\to |0\rangle \\
|1\rangle &\to |-1+2N\rangle \\
|2\rangle &\to |-2+2N\rangle \\
&\vdots \\
|N\rangle &\to |N\rangle \\
&\vdots \\
|2N-1\rangle &\to |1\rangle\,.
\end{aligned} \tag{3.33}$$

We see that the states $|0\rangle$ and $|N\rangle$ are invariant. From the remaining $2N-2$ states, we can form $(2N-2)/2$ invariant linear combinations

$$\begin{array}{rcl} |1\rangle & + & |-1+2N\rangle \\ |2\rangle & + & |-2+2N\rangle \\ & \vdots & \\ |N-1\rangle & + & |N+1\rangle\,. \end{array} \tag{3.34}$$

The primary operators creating such states are

$$\cos\left(\frac{n}{\sqrt{2N}}\varphi\right), \quad n = 1, ..., N-1. \tag{3.35}$$

In addition, we have the operators $\mathbb{1}$ and $e^{i\sqrt{\frac{N}{2}}\varphi}$, where the latter one can be split in two independent parts

$$\sin\left(\sqrt{\frac{N}{2}}\varphi\right) \quad \text{and} \quad \cos\left(\sqrt{\frac{N}{2}}\varphi\right). \tag{3.36}$$

In the orbifold theory, operators that have OPE with the current that are local only after the action of the group $\mathbb{Z}_2$ are allowed. These are the so-called twist or order-disorder operators, and are defined through the following OPE with the current,

$$j(z)\sigma(w) \sim \frac{\tau(w)}{(z-w)^{\frac{1}{2}}}. \tag{3.37}$$

Notice that when z goes around w through a 2π rotation, the OPE picks up a minus sign. In the context of the superconducting phase, the operator $\sigma(w)$ represents a vortex located at w. The above relation implies the following relation between conformal dimensions of σ and τ: $h_\tau = h_\sigma + 1/2$. From the OPE of the twist fields with the energy-momentum tensor $T(z) \sim j(z)j(z)$ we can compute the conformal dimensions of the twist fields: $h_\sigma = 1/16$ and $h_\tau = 9/16$. Twist operators always come up in pairs. In the orbifold case, we have two pairs of twist fields satisfying the OPE (3.37), namely, σ_1, τ_1 and σ_2, τ_2. They correspond to the trivial and the nontrivial representations of $\mathbb{Z}_2$ [46]. Therefore, we end up with 4 new operators in the theory.

Summarizing, the field content of orbifold theory, with the respective conformal dimensions, is the following:

$$\underbrace{\mathbb{1}}_{0},\ \underbrace{j(z)}_{1},\ \underbrace{\cos\left(\sqrt{\frac{N}{2}}\varphi\right)}_{\frac{N}{4}},\ \underbrace{\sin\left(\sqrt{\frac{N}{2}}\varphi\right)}_{\frac{N}{4}},\ \underbrace{\cos\left(\frac{n}{\sqrt{2N}}\varphi\right)}_{\frac{n^2}{4N}},\ \underbrace{\sigma_a}_{\frac{1}{16}},\ \underbrace{\tau_a}_{\frac{9}{16}}, \tag{3.38}$$

with $n = 1, ..., N-1$ and $a = 1, 2$, in a total of $N+7$ fields. The respective fusion rules were derived in [46]. For the convenience of the reader we have included a derivation of them in the Appendix B, along with additional discussions of the algebra extension and the orbifold.

3.3 $N = 2$ Orbifold = Ising $\times$ Ising

We are particularly interested in the values of N compatible with solutions of the relation $q^2 = 2N$, with both q and N integers. The simplest case corresponds to $q = N = 2$. In this case, the spectrum is quite simple and, in particular, the operators $\cos\left(\sqrt{\frac{N}{2}}\varphi\right)$ and $\sin\left(\sqrt{\frac{N}{2}}\varphi\right)$ become fermion operators $\Upsilon_1 \equiv \cos\varphi$ and $\Upsilon_2 \equiv \sin\varphi$ of conformal dimension $\frac{1}{2}$, with no electric charge. Therefore, they are natural candidates to be identified with the Majorana fermions appearing at the boundary of the topological superconducting phase. This is also compatible with the results of [47].

In addition to containing the Majorana fermions, another remarkable property of the case $N = 2$, and which is crucial for its identification as the proper edge theory of the superconducting case, is that it corresponds precisely to two copies of the Ising CFT. To see this, we consider the corresponding fusion rules [46]:

$$\Phi \times \Phi = \mathbb{1} + j, \;\; j \times \Phi = \Phi, \;\; j \times j = \mathbb{1}, \;\; \Upsilon_a \times \Upsilon_a = \mathbb{1}, \;\; \Upsilon_1 \times \Upsilon_2 = j, \tag{3.39}$$

$$\sigma_a \times \sigma_a = \mathbb{1} + \Upsilon_a, \;\; \sigma_1 \times \sigma_2 = \Phi, \;\; j \times \sigma_a = \tau_a, \;\; \Phi \times \sigma_a = \sigma_a, \;\; \Upsilon_a \times \sigma_a = \sigma_a, \tag{3.40}$$

where we have defined $\Phi \equiv \cos\left(\frac{1}{2}\varphi\right)$. The above operator content can be decomposed into two sets,

$$\mathbb{1}: \; (h = 0), \;\; \Upsilon_1: \; \left(h = \frac{1}{2}\right), \;\; \sigma_1: \; \left(h = \frac{1}{16}\right), \tag{3.41}$$

and

$$\mathbb{1}: \; (h = 0), \;\; \Upsilon_2: \; \left(h = \frac{1}{2}\right), \;\; \sigma_2: \; \left(h = \frac{1}{16}\right). \tag{3.42}$$

From the fusion rules shown in (3.39) and (3.40), we see that each one of the above sets of operators generates an independent Ising CFT with central charge $c = 1/2$, namely,

$$\sigma_a \times \sigma_a = \mathbb{1} + \Upsilon_a, \;\; \Upsilon_a \times \sigma_a = \sigma_a, \;\; \Upsilon_a \times \Upsilon_a = \mathbb{1}. \tag{3.43}$$

Then, taking the tensor product of the two sets of fusion rules, we recover the full set of fusion rules of the orbifold model for $N = 2$.

Recalling the discussion of Sec. 2.3, the edge theory of the superconducting phase corresponds to a single Ising CFT with chiral central charge $c = 1/2$. The doubling of degrees of freedom in the orbifold theory with $N = 2$ is a direct reflection of the duplication of the degrees of freedom inherent to the BdG formalism used to derive the CS effective action (3.29), which in turn gives rise to the orbifold theory through the bulk-edge correspondence. Therefore, coming from the orbifold theory, in order to properly account for the physical degrees of freedom, we need to halve the theory keeping thus only one of the Ising CFT's. In terms of central charge, this means that from $c = 1 = 1/2 + 1/2$, we are left with only a half of the central charge, $c = 1/2$, which is expected for the superconducting phase.

After the discussion of the edge theory associated with a single $O(2)$ CS theory with level 4, it is immediate to see that the edge theory of the QAH phase also follows from the effective action (3.29) via bulk-edge correspondence. It emerges when $|m_0| > \Delta,$ with $m_0 < 0$, where both Chern-Simons theories contribute. In this case, we obtain an orbifold CFT with $N = 2$ for each of the CS theories. The resulting total central charge is $c = 2 = 1 + 1$, which reduces to $c = 1$ after we take into account the halving mechanism for eliminating duplicated degrees of freedom.

4 NON-ABELIAN BOSONIZATION

This chapter is meant as a warm up for quantum spin liquid (QSL) we discuss in the next chapter. Here, we will review a series of aspects of non-Abelian bosonization and conformal field theory which are relevant to our future discussions. More specifically, we use a quantum wires approach to discuss how we can introduce operators in the action that effectively realize a (2+1)-dimensional phase. Then, we show that in the deep infrared these operators realize a new conformal field theory with smaller central charge. Then, we calculate the first order of the C-function in a loop expansion and show that it agrees with the central charge at the fixed points.

4.1 The Model

Our starting point is the free fermion action in 1+1 dimensional Minkowski spacetime

$$S_0 = \int \mathrm{d}^2x \; i\bar{\Psi}_{i\sigma}\partial\!\!\!/\Psi_{i\sigma}, \tag{4.1}$$

where $\Psi_{i\sigma}$ is a two component spinor,

$$\Psi_{i\sigma} = \begin{pmatrix} \tilde{\psi}_{Ri\sigma} \\ \tilde{\psi}_{Li\sigma} \end{pmatrix}, \tag{4.2}$$

$\bar{\Psi} = \Psi^\dagger\gamma^0$ and γ^μ are the 1+1 dimensional Dirac matrices obeying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. We set the gamma matrices to

$$\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \gamma^1 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.3}$$

In this basis the free fermion action separates into right and left moving sectors

$$S_0 = \int \mathrm{d}^2x \; \left[i\tilde{\psi}^\dagger_{Ri\sigma}\left(\partial_0 - \partial_1\right)\tilde{\psi}_{Ri\sigma} + i\tilde{\psi}^\dagger_{Li\sigma}\left(\partial_0 + \partial_1\right)\tilde{\psi}_{Li\sigma}\right]. \tag{4.4}$$

Through this work we will use some concepts of conformal field theory. To make comparisons with the current literature easier, we perform a transformation from Minkowski to Euclidean spacetime. This can be achieved by the variable change $x_0 \to i\tau$, such that the action is rewritten as

$$S_0 = \int \mathrm{d}^2x \; \left[\tilde{\psi}^\dagger_{Ri\sigma}\partial\tilde{\psi}_{Ri\sigma} + \tilde{\psi}^\dagger_{Li\sigma}\bar{\partial}\tilde{\psi}_{Li\sigma}\right], \tag{4.5}$$

where we defined $\partial = \partial_z \equiv \partial_\tau - i\partial_1$ and $\bar{\partial} = \partial_{\bar{z}} \equiv \partial_\tau + i\partial_1$. For the light-cone Euclidean variables the metric and inverse metric tensors are given by

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}, \qquad \text{and} \qquad \eta^{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}. \tag{4.6}$$

In the free fermion action, the index $i = 1, \cdots, N_c$. From the quantum wires formalism, this index can be seen as the wire index specifying which wire the electron is. On the other hand, the index $\sigma = 1, \cdots, N_f$ specifies any other internal degree of freedom the fermion may have. In this way, the free fermion action is invariant under the symmetry group $U_R(N) \times U_L(N)$, with $N \equiv N_c N_f$, with the associated Lie algebra

$$u(N_c N_f)_1 \supset u(1) \oplus su(N_c)_{N_f} \oplus su(N_f)_{N_c}. \tag{4.7}$$

According to the Sugawara construction, the corresponding energy momentum tensor can be decomposed as

$$T[u(N)_1] = T[u(1)] + T[su(N_c)_{N_f}] + T[su(N_f)_{N_c}], \tag{4.8}$$

which can be easily checked with the central charge

$$c_0 = \underbrace{N_c N_f}_{u(N)} = \underbrace{1}_{u(1)} + \underbrace{\frac{N_c(N_f^2 - 1)}{N_c + N_f}}_{su(N_f)_{N_c}} + \underbrace{\frac{N_f(N_c^2 - 1)}{N_c + N_f}}_{su(N_c)_{N_f}}. \tag{4.9}$$

Then, it is natural to assume that we can introduce operators into the action that, when flowed into the deep infrared, open a gap in one of the sub-algebras of $u(N)_1$ and thus subtract the corresponding central charge from (4.9). For the case at hand, we want to open a gap in $su(N_c)_{N_f}$, the wires sector of our theory. This gapping interaction is important as, by allowing electrons to tunnel between different wires, it effectively realizes a (2+1)-dimensional gapped phase. Furthermore, in the next chapter we want to discuss the quantum spin liquid system, where (in contrast to the quantum Hall effect) there is no charge carrier and thus we also need to open a gap in the $u(1)$ sector.

By perturbatively calculating the beta functions, it has been shown the relevant interactions for openning a gap in the $U(1)$ and $SU(N_c)$ sectors are given by [19]

$$\mathcal{L}_{Thirring} = -\lambda_t J^0_\mu J^{0\mu} \tag{4.10}$$

$$\mathcal{L}_{Umklapp} = -\tilde{\lambda}_{U(1)} \left(\prod_{i=1}^{N_c} \prod_{\sigma=1}^{N_f} \tilde{\psi}^\dagger_{Ri\sigma} \right) \left(\prod_{i=N_c}^{1} \prod_{\sigma=N_f}^{1} \tilde{\psi}_{Li\sigma} \right) + (R \leftrightarrow L). \tag{4.11}$$

$$\mathcal{L}_{SU(N_c)} = -\frac{\tilde{\lambda}}{2} J^{\mu a} J^a_\mu = 2\tilde{\lambda} J^a_R J^a_L, \tag{4.12}$$

such that the new interacting Lagrangian is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{SU(N_c)} + \mathcal{L}_{Thirring} + \mathcal{L}_{Umklapp}. \tag{4.13}$$

The Thirring term does not open a gap, as it is marginal along the RG flow. Nonetheless, it is an allowed term by the symmetries of the theory, thus we add it to the action. The currents are the usual $SU(N_c)$ and $U(1)$ currents, $J^a_{R/L} = \tilde{\psi}^\dagger_{R/L;i\sigma} t^a_{ij} \psi_{R/L;j\sigma}$, $J^0_{R/L} = \tilde{\psi}^\dagger_{R/L;i\sigma} \tilde{\psi}_{R/L;i\sigma}$ and a is the $SU(N_c)$ generator t^a_{ij} index. We also note our $SU(N_c)$ conventions

$$\left[t^a, t^b\right] = i f^{abc} t^c, \quad \mathrm{tr}\left(t^a t^b\right) = \tfrac{1}{2}\delta^{ab} \quad \text{and} \quad f^{abc} f^{a'bc} = N_c \delta^{aa'}. \tag{4.14}$$

In order to find possible non-perturbative fixed points it is convenient to recast the fermion action into a bosonic form. This has the advantage of enabling the study of the RG flow non-perturbatively, which is essential to find strong coupling fixed points. The first step in this process is to introduce an auxiliary vector field, such that we eliminate the current-current interaction in favor of quadratic interactions

$$\begin{aligned} \mathcal{L} = {} & \tilde{\psi}^\dagger_{Ri\sigma}\left(\delta_{ij}\partial_z + i\delta_{ij}B_z + iA^a_z t^a_{ij}\right)\tilde{\psi}_{Rj\sigma} + \tilde{\psi}^\dagger_{Li\sigma}\left(\delta_{ij}\partial_{\bar{z}} + i\delta_{ij}B_{\bar{z}} + iA^a_{\bar{z}} t^a_{ij}\right)\tilde{\psi}_{Lj\sigma} \\ & + \frac{1}{\tilde{\lambda}}\,\mathrm{tr}\, A_z A_{\bar{z}} + \frac{1}{\lambda_t} B_z B_{\bar{z}} + \mathcal{L}_{Umklapp}. \end{aligned} \tag{4.15}$$

In this way, by integrating over the new vector fields we return to our original interacting action, equation (4.13).

Now, we can decouple the $U(1)$ and the $SU(N_c)$ degrees of freedom from the action. This process is quite similar to the abelian bosonization, see [48, 17, 49], but here this can be thought as a field redefinition. For the fermions we set

$$\tilde{\psi}_{Ri\sigma} = e^{i(\eta+\theta)}\psi_{Ri\sigma} \qquad \text{and} \qquad \tilde{\psi}_{Li\sigma} = e^{i(\eta-\theta)}\psi_{Li\sigma}, \tag{4.16}$$

and for the vector fields

$$B_\pm = -i\partial_\pm\left(\eta \mp \theta\right)\mathbb{I}. \tag{4.17}$$

This allows us to separate the scalar dependence from the rest of the partition function

$$Z = Z_{U(1)} Z_{SU(N)}, \tag{4.18}$$

with

$$\begin{aligned} Z_{U(1)} \;=\; & \int \mathcal{D}\eta\mathcal{D}\theta\,\det\left(-\partial^2\right)\exp\left[-\int d^2x\left(\frac{1}{\lambda_t} + a\right)\left(\partial_\mu\theta\partial^\mu\theta - \partial_\mu\eta\partial^\mu\eta\right)\right. \\ & \left. - \;\frac{N}{2\pi}\partial_\mu\theta\partial^\mu\theta - \lambda_{U(1)}\cos 2N\theta\right]. \end{aligned} \tag{4.19}$$

The factor $\det\left(-\partial^2\right)$ is the Jacobian for the vector field reparametrization, which can be rewritten as a ghost [50]. The last kinetic and the a dependent terms are originated form the fermion Jacobian associated with the change of variables (4.16), where a is a regularization dependent parameter, usually set to zero by gauge invariance[51]. The cosine term comes from the Umklapp interaction, we also absorbed any mass scale and numerical factors generated by the bosonization procedure into the coupling constant [52].

Now, let us analyze the central charge of the $U(1)$ sector. There are two fields that remain scale invariant along the RG flow, the free boson η with unit central charge, and the ghosts with $c = -2$. The cosine term breaks conformal invariance of the theory, completely gapping the θ boson. In the free fermion limit θ is free, with unit central charge. On the other hand, it is completely gapped in the low-energy limit, thus adding no central charge to the $U(1)$ action. At last, we combine the contributions for the ghosts and the two scalar bosons to find the total $U(1)$ contribution to the central charge. At high energies we find $c_{U(1)} = 0$ and at low energies $c_{U(1)} = -1$.

This negative central charge at the low-energy limit is not problematic as the complete partition function also depends on $Z_{SU(N)}$. This sector guarantees that the total central charge is non-negative. As an example, let us consider the $U(1)$ fermion, $N_c = N_f = 1$. In this case the $SU(N)$ partition function becomes that of a single $U(1)$. Thus, the total central charge of the theory is unit in the UV and zero in the IR limit.

4.2 Non-Abelian Bosonization

After separating the scalar part we are left with the $SU(N_c)$ partition function with action

$$S_{SU(N_c)} = \int \mathrm{d}^2x\ \psi^\dagger_{Ri\sigma}\left(\delta_{ij}\partial_z + iA^a_z t^a_{ij}\right)\psi_{Rj\sigma} + \psi^\dagger_{Li\sigma}\left(\delta_{ij}\partial_{\bar{z}} + iA^a_{\bar{z}} t^a_{ij}\right)\psi_{Lj\sigma} + \frac{1}{\bar{\lambda}}\,\mathrm{tr}\,A_{\bar{z}}A_z. \tag{4.20}$$

The gauge fields can be decoupled from the fermions by introducing the arbitrary complex matrix with unit determinant M through the variable change

$$\psi_R = M\chi_R, \qquad \psi^\dagger_R = \chi^\dagger_R M^{-1}, \qquad \psi_L = M^{\dagger-1}\chi_L, \qquad \psi^\dagger_L = \chi^\dagger_L M^\dagger, \tag{4.21}$$

$$A_z = i\partial_z M M^{-1} \qquad \text{and} \qquad A_{\bar{z}} = -iM^{\dagger-1}\partial_{\bar{z}}M^\dagger. \tag{4.22}$$

This produces the $SU(N_c)$ partition function

$$Z_{SU(N_c)} = \int \mathcal{D}M\mathcal{D}M^\dagger\mathcal{D}\psi\mathcal{D}\bar{\psi}J_B J_F e^{-S_0 - \frac{1}{\bar{\lambda}}\int \mathrm{d}^2x\,\mathrm{tr}\,M^{\dagger-1}\partial_{\bar{z}}M^\dagger\partial_z M M^{-1}} \tag{4.23}$$

where S_0 is the free fermion action. The Jacobians J_f and J_B, are related to the variables change (4.21) and (4.22) respectively. Their explicit form is given by [53]

$$J_F = \frac{\det\left(D_z D_{\bar{z}}\right)}{\det\left(\partial_z\partial_{\bar{z}}\right)} = e^{N_f W[M^\dagger M] + \frac{b}{4\pi}\int d^2x\,\mathrm{tr}\,M^{\dagger-1}\partial_{\bar{z}}M^\dagger\partial_z M M^{-1}} \tag{4.24}$$

$$J_B = \det\left(D_z D_{\bar{z}}\right)_{adj} = \det\left(\partial_z\partial_{\bar{z}}\right)_{adj} e^{2N_c W[M^\dagger M] + \frac{c}{4\pi}\int d^2x\,\mathrm{tr}\,M^{\dagger-1}\partial_{\bar{z}}M^\dagger\partial_z M M^{-1}}, \tag{4.25}$$

where $W[M]$ is the Wess-Zumino-Witten action

$$W[M] = \frac{1}{8\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\,\mathrm{tr}\,\partial_\mu M\partial^\mu M^{-1} + \frac{i}{12\pi}\int_{\mathcal{M}} \mathrm{d}^3x\,\epsilon^{\mu\nu\sigma}\,\mathrm{tr}\,M^{-1}\partial_\mu M M^{-1}\partial_\nu M M^{-1}\partial_\sigma M. \tag{4.26}$$

The determinants of the covariant derivatives in (4.24) are calculated in the N_c-dimensional representation of the $SU(N_f N_c)$ group that the fermions belong. The index of this representation is N_f, which gives the level of the WZW action on the right. The subscript *adj* stands for the adjoint representation of the $SU(N_c)$ group that the vector fields belong, and the level in the WZW action in the expression (4.25) is related to the Casimir invariant of $SU(N_c)$, which is given by $C_A = N_c$. The parameters b and c again parametrize the regularization ambiguities in the determinant evaluations. In terms of the new variables, the partition function can be put into the form

$$\begin{aligned} Z_{SU(N_c)} = \int \mathcal{D}\mu \exp - \Big[-kW[M] - kW[M^\dagger] + \frac{k\lambda}{4\pi}\int \mathrm{d}^2x\,\mathrm{tr}\,M^{\dagger-1}\partial_{\bar{z}}M^\dagger\partial_z M M^{-1} \\ + S_{fermion} + S_{ghost}\Big], \end{aligned} \tag{4.27}$$

where

$$\mathcal{D}\mu = \mathcal{D}M\mathcal{D}M^\dagger\mathcal{D}\chi\mathcal{D}\bar{\chi}\mathcal{D}b\mathcal{D}\bar{b}\mathcal{D}c\mathcal{D}\bar{c}$$

$$S_{fermion} = \int \mathrm{d}^2x\,\bar{\chi}_{i\sigma}\partial\!\!\!/\chi_{i\sigma} \qquad \text{and} \qquad S_{ghost} = \int \mathrm{d}^2x \sum_{i=1}^{N_c^2-1}\left[b^i\partial c^i + \bar{b}^i\bar{\partial}\bar{c}^i\right] \tag{4.28}$$

and the ghosts originate from the remaining adjoint representation determinant in the boson Jacobian. We also have defined $k \equiv 2N_c + N_f$ and

$$\lambda \equiv \frac{4\pi}{k\tilde{\lambda}} - \alpha + 1, \tag{4.29}$$

which plays the role of the effective coupling constant of the model and α is some combination of the previous regularization parameters b and c.

With the complete partition function at hand, we are able to compute the total central charge of the theory. In the free fermion sector each field gives a unit contribution, totaling $c = N$. Furthermore, the ghost action subtracts a central charge $-2\left(N_c^2 - 1\right)$. In general, the terms for g and h are not conformally invariant, except for three specific fixed points. The first one is found at $\lambda = 1$, where we can use the Polyakov-Wiegmann identity,

$$\begin{aligned} W(M^\dagger M) &= W(M) + W(M^\dagger) - \frac{1}{4\pi}\int \mathrm{d}^2x\,\mathrm{Tr}\left(M^{\dagger-1}\partial_{\bar{z}}M^\dagger\partial_z M M^{-1}\right) \\ &\equiv W(M) + W(M^\dagger) + P(M^\dagger, M), \end{aligned} \tag{4.30}$$

to rewrite the $SU(N_c)$ action as

$$-kW[M^\dagger M] + S_{fermion} + S_{ghost}. \tag{4.31}$$

This describes a low-energy conformal field theory with central charge

$$c_{IR} = N - 1 - 2(N_c^2 - 1) - \frac{k(N_c^2-1)}{N_c - k} = \frac{N_c(N_f^2-1)}{N_c + N_f}. \tag{4.32}$$

This is the $su(N_f)_{N_c}$ WZW central charge, indicating that we completely gapped the $U(1)$ and the $SU(N_c)$ sectors of the theory. An interesting check is the limit where we eliminate the $SU(N_f)$ group from the theory, that is, consider $N_f \to 1$ and $N_c \to N$. In this limit, the IR central charge goes to zero as is expected for a completely gapped theory.

At this fixed point, the action is invariant under the group transformation

$$M^\dagger \to M^{\dagger\prime} = \Omega(z) M^\dagger \Lambda^{-1}(z,\bar{z}) \qquad \text{and} \qquad M \to M' = \Lambda(z,\bar{z}) M \bar{\Omega}^{-1}(\bar{z}). \tag{4.33}$$

Where the Λ transformation corresponds to the emergent gauge invariance of the action (4.20) in the low-energy limit. In fact, under Λ transformations only, the vector fields transform according to a standard non-Abelian gauge transformation

$$A_\mu \to A'_\mu = \Lambda A_\mu \Lambda^{-1} - i\Lambda \partial_\mu \Lambda^{-1} \tag{4.34}$$

for $\mu = z, \bar{z}$. As the fields acquire this new gauge invariance at this fixed point, the action is written in terms of only $U \equiv M^\dagger M$. In this way, one of the path integrals in the partition function decouples. Even tough this seems harmless at this point, it will lead to a series of difficulties in the RG flow. On the other hand, the global version of the Ω and $\bar{\Omega}$ transformations generates the $SU(N_c)$ Noether currents

$$J = kU^{-1}\partial U \qquad \text{and} \qquad \bar{J} = -k\bar{\partial}UU^{-1}. \tag{4.35}$$

Another fixed point is found at $\lambda = 0$, where the $SU(N_c)$ action reads

$$-kW[M^\dagger] - kW[M] + S_{fermion} + S_{ghost}, \tag{4.36}$$

which describes a theory with central charge

$$c_{UV} = N - 2(N_c^2 - 1) - 2\frac{k(N_c^2-1)}{N_c - k} = N + 2\frac{N_c(N_c^2-1)}{N_c + N_f}. \tag{4.37}$$

This leads us to deduce the group content upon which the UV theory is supported

$$U(1) \times SU(N_f)_{Nc} \times SU(N_c)_{-k}. \tag{4.38}$$

Now, let us consider the action of the full symmetry group of the UV fixed point action

$$M^\dagger \to \Omega(z) M^\dagger \bar{\Omega}^{-1}(\bar{z}) \qquad \text{and} \qquad M \to \Omega'(z) M \bar{\Omega}'^{-1}(\bar{z}). \tag{4.39}$$

Following the standard WZW theory, this symmetry group leads us to two pairs of conserved currents, one for M and one for $M^\dagger$. On the other hand, the transformation of the group valued fields leads us to the transformation of the vector fields

$$\begin{aligned} A_z &\to \Omega' A_z \Omega'^{-1} - i\Omega' \partial \Omega'^{-1}, \\ A_{\bar{z}} &\to \bar{\Omega} A_{\bar{z}} \bar{\Omega}^{-1} - i\bar{\Omega} \bar{\partial} \bar{\Omega}^{-1}. \end{aligned} \tag{4.40}$$

This corresponds to a symmetry of the fermion action (4.20) at the UV fixed point, given by

$$\lambda = 0 \leftrightarrow \tilde{\lambda} = \frac{4\pi}{k(\alpha - 1)}. \tag{4.41}$$

Notice that, in terms of the fermion degrees of freedom the two previous fixed points can be described by the same action given an appropriate choice of α. Nonetheless, they differ by their symmetries. The IR fixed point is invariant under a standard $U(1)$ gauge symmetry. On the other hand, the UV fixed point is invariant under the combined action of two chiral $U(1)$ symmetries.

By fixing a gauge invariant regularization, $\alpha = 0$, the IR fixed point is realized in the strong coupling limit, $\tilde{\lambda} \to \infty$. On the other hand, the UV fixed point is realized for a negative value of the coupling constant, $\tilde{\lambda} = -4\pi/k$. For such values, the integration of the vector fields in Eq. (4.20) does not converge to the current-current interaction of the initial Lagrangian (4.13). For this reason, we consider this UV fixed point to be an artifact of the bosonization process and thus not physical.

The last fixed point is the free fermion fixed point, achieved in the limit $\lambda \to \pm\infty$. From a first analysis of the action (4.27), it seems that this fixed point introduces a divergence. As a matching condition, the free fermion limit the fields M and $M^\dagger$ go to the identity. This is necessary, as in the limit $\tilde{\lambda} = 0$ we turn off an interaction, and thus its associated vector field should simplify to the identity. Furthermore, the ghost contribution to the central charge can then be eliminated by inverting the variable change (4.22) and returning the path integral to the vector field A_μ. Moreover, the $U(1)$ sector also contributes with zero central charge. Thus, the only non-zero contribution to the free fermion limit is the free fermion action in (4.27) and the total central charge in this limit reads

$$c_{\text{free fermion}} = N. \tag{4.42}$$

At last we have the fixed point structure of our theory. Notice that the two central charges c_{UV} and c_{IR} are higher and lower than the free fermion central charge respectively. Thus, starting from the free fermion fermion fixed point, the C-theorem guarantees that we can only reach the IR fixed point.

4.3 Loop Expansion and C-function

The Zamolodchikov C Theorem [54] is one of the most important aspects of conformal field theory. The basic idea is that, along the RG flow from high energy to low-energy theories we must loose some information. This loss is something we must be able to encapsulate in some precise way. In fact, the theorem states that there exists a C-function of the coupling constants that is monotonically decreasing along the RG flow and stable only at the fixed points of the theory. Its value at these fixed points is equal to the central charge of the corresponding conformal field theory. The C-function is defined as

$$C = (2\pi)^2 \left[2x^4 \langle T(x)T(0) \rangle - x^3 \bar{x} \langle T(x)\Theta(0) \rangle - \frac{3}{8} x^2 \bar{x}^2 \langle \Theta(x)\Theta(0) \rangle \right]_{x=x_0}, \tag{4.43}$$

where $T(x) \equiv T_{zz}(x)$, $\Theta(x) = \eta^{\mu\nu} T_{\mu\nu}(x)$ are combinations of the components of the energy momentum tensor and x_0 is an arbitrary scale.

As C must be a function of λ, its only non-trivial dependence must originate from terms in the action dependent on the $SU(N_c)$ fields. This allows us to consider only the action for M and $M^\dagger$ in determining the C-function. In this manner, we only consider the first line of the partition function (4.27).

Furthermore, we consider the C-function in a small $\hbar$ approximation. In doing so, our analysis becomes independent of the value of the coupling constant λ and thus of the energy scale. To this end, we define the background field expansion

$$M^\dagger = M_0^\dagger e^{i\sigma} \quad \text{and} \quad M = M_0 e^{-i\rho}, \tag{4.44}$$

where M_0 and $M_0^\dagger$ are constant matrices, σ and ρ are also matrices.

To first order of approximation in $\hbar$ we find the effective action

$$e^{-S^{(1)}_{eff}} = \int \mathcal{D}\sigma \mathcal{D}\rho \exp -\frac{k}{8\pi} \int d^2x \, \eta^{\mu\nu} \operatorname{tr} \left[-\partial_\mu \sigma \partial_\nu \sigma - \partial_\mu \rho \partial_\nu \rho + 2\lambda \partial_\mu \sigma \partial_\nu \rho \right]. \tag{4.45}$$

It is convenient define fields

$$\phi_\pm \equiv \frac{\rho \mp \sigma}{\sqrt{2}}, \tag{4.46}$$

which diagonalize the one loop effective action

$$\begin{aligned} S^{(1)} &= -\int d^2x \frac{k}{8\pi} \eta^{\mu\nu} \operatorname{tr} \left[(1+\lambda) \partial_\mu \phi_+ \partial_\nu \phi_+ + (1-\lambda) \partial_\mu \phi_- \partial_\nu \phi_- \right] \\ &= \int d^2x \frac{k}{16\pi} \left[(1+\lambda)\, \phi_+^a \nabla \phi_+^a + (1-\lambda)\, \phi_-^a \nabla \phi_-^a \right], \end{aligned} \tag{4.47}$$

where $\nabla \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. The corresponding propagators are

$$\left\langle \phi^a_\pm(x)\phi^b_\pm(y)\right\rangle = \frac{8\pi}{k}\frac{\delta^{ab}}{1\pm\lambda}G(x,y), \tag{4.48}$$

where $\nabla G(x,y) = \delta^{(2)}(x-y)$.

Up to one loop the action is just a pair of free scalar fields, apparently with central charge $c = 2(N_c^2-1)$. However, the Green function $G(x,y)$ is not well defined in the low-energy limit, which could affect the RG flow. Nonetheless, we eliminate these divergences by adding a regularization to the Lagrangian

$$\mathcal{L}_{ct} \propto m^2\phi^a_+\phi^a_+ + m^2\phi^a_-\phi^a_-. \tag{4.49}$$

where m^2 is a small positive mass, which should be taken to zero at the end of the calculations.

For ϕ_+ this has no impact on the RG flow. It remains scale invariant and its contribution to the C-function is simply its central charge

$$C^{(1)}_{\phi_+} = N_c^2 - 1. \tag{4.50}$$

On the other hand, ϕ_- has a pole at $\lambda = 1$, the IR fixed point. For this case we need to consider the C-function more carefully. Let us define the regularized propagator

$$\left\langle \phi^a_-(x)\phi^b_-(y)\right\rangle_{reg} = \frac{8\pi}{k}\delta_{a,b}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{1}{-(1-\lambda)p^2+m^2}e^{ip\cdot r} = \frac{4\delta_{a,b}}{kl}K_0\left(rm/\sqrt{l}\right), \tag{4.51}$$

where we assumed the mass to be positive, we also defined $l \equiv -(1-\lambda) = 4\pi/k\tilde{\lambda} > 0$ and $r \equiv \sqrt{(x-y)(\bar{x}-\bar{y})}$.

As we are still working at one loop, the action is quadratic on the fields and thus the ϕ_- contribution to the C-function can be directly calculated from its definition (4.43) using Noether theorem. At the end of this procedure we obtain the one loop C-function

$$C^{(1)}(\lambda) = C^{(0)} + \left(N_c^2-1\right)\left[1 + \frac{m^4r_0^4}{4l^2}\left(K_2(mr_0/\sqrt{l})^2 + 2K_1(mr_0/\sqrt{l})^2 - 3K_0(mr_0/\sqrt{l})^2\right)\right], \tag{4.52}$$

which has fixed points at $l = 0$ and $l \to \infty$. $K_n(x)$ is the modified Bessel function of the second kind, we have also set the renormalization scale at $r = r_0$ and $C^{(0)}$ has the contribution from the $U(1)$ sector, the free fermion and the ghosts. Now that we have a closed expression we can take the limit $m \to 0$ to obtain the regularized one loop C-function

$$C^{(1)}(\lambda) = C^{(0)} + \left(N_c^2-1\right)\left[2-\delta_{l,0}\right], \tag{4.53}$$

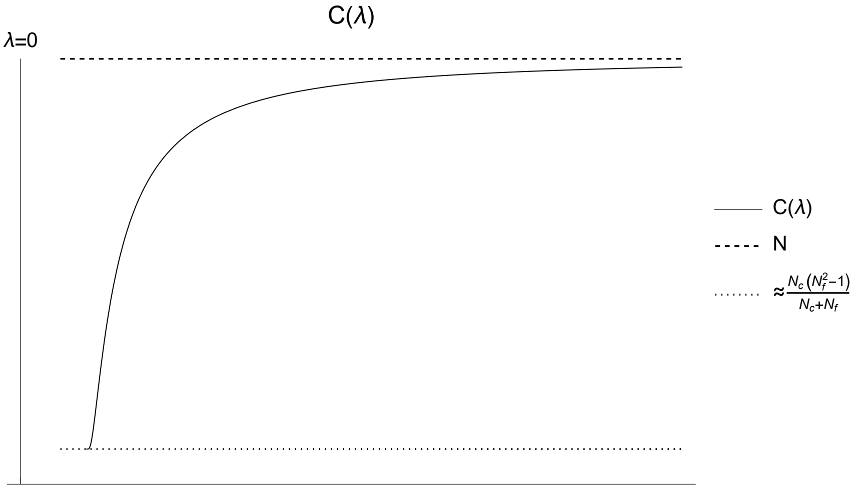

Figura 6 – Flow of the one loop C-function.

where $\delta_{l,0}$ is the continuum analogue of the Kronecker delta, which can be defined by

$$\delta_{l,0} \equiv \lim_{\epsilon \to 0^+} \Theta(\epsilon - l), \tag{4.54}$$

where theta is the Heaviside step function. For a plot of the one loop C-function before we take the mass to zero see Figure 6.

As expected from the fixed points structure of the theory, the C-function has fixed points at $\lambda = \infty$ and $\lambda = 1$. Furthermore, at the fixed points

$$C^{(1)}(\lambda = 1) = N - 2(N_c^2 - 1) - 1 + (N_c^2 - 1), \tag{4.55}$$

$$C^{(1)}(\lambda \to \infty) = N - \underbrace{2(N_c^2 - 1)}_{ghost} + \underbrace{0}_{U(1)} + \underbrace{2(N_c^2 - 1)}_{WZW}. \tag{4.56}$$

Notice that at the fixed points the C-function is independent of the regulator mass, thus the limit $m \to 0$ is trivial.

As we are performing a loop expansion of the action, the C-function is expected to predict the central charge in a small $\hbar$ approximation. For the WZW action, this is equivalent to a large k approximation. In this way, the first order contribution to the $SU(N_c)$ C-function at the IR fixed point, given by the last term of (4.55), must match large k expansion of the first order WZW central charge at $\lambda = 1$,

$$-\frac{k(N_c^2 - 1)}{N_c - k} = (N_c^2 - 1)\left[1 + \frac{N_c}{k}\right] + \mathcal{O}(N_c/k)^2, \tag{4.57}$$

where the higher orders in N_c/k must be matched by further terms in the loop expansion. In the free fermion limit, the one loop cancels out the ghost contribution to the central charge, leaving only the free fermion. Thus, any higher loop contribution must vanish in this limit.

Now that we have closed the one loop discussion we proceed to the next order of corrections in the loop expansion. As these will involve products of a higher number of fields, we cannot easily proceed like we did for the two fields action. Instead, consider a classical action in a curved background $S[\gamma]$, its classical energy-momentum can be computed as

$$T_{\mu\nu} = -\frac{2}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{\mu\nu}}. \tag{4.58}$$

Then, from the effective action

$$e^{-S_{eff}} \equiv \int \mathcal{D}M\mathcal{D}M^{\dagger} e^{-S[\gamma,M,M^{\dagger}]}, \tag{4.59}$$

we obtain the correlation functions of the energy-momentum tensor by taking functional derivatives with respect to the metric

$$\begin{aligned} -\frac{2}{\sqrt{\gamma}(x)}\frac{2}{\sqrt{\gamma}(y)}\frac{\delta^2 S_{eff}}{\delta\gamma^{\mu\nu}(x)\delta\gamma^{\rho\sigma}(y)} &= \langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle - \langle T_{\mu\nu}(x)\rangle\langle T_{\rho\sigma}(y)\rangle \\ &- \frac{2}{\sqrt{\gamma}(x)}\frac{2}{\sqrt{\gamma}(y)}\left\langle \frac{\delta^2 S}{\delta\gamma^{\mu\nu}(x)\delta\gamma^{\rho\sigma}(y)}\right\rangle. \end{aligned} \tag{4.60}$$

At last, we take the flat spacetime limit to define the tensor

$$\begin{aligned} B_{\mu\nu\rho\sigma} &\equiv -\frac{2}{\sqrt{\gamma(x)}}\frac{2}{\sqrt{\gamma(y)}}\frac{\delta^{(2)}S_{\text{eff}}[\gamma]}{\delta\gamma^{\mu\nu}(x)\delta\gamma^{\rho\sigma}(y)}\bigg|_{\gamma^{\mu\nu}=\eta^{\mu\nu}} \\ &= \langle T_{\mu\upsilon}(x)T_{\rho\sigma}(y)\rangle - \langle T_{\mu\nu}(x)\rangle\langle T_{\rho\sigma}(y)\rangle - \frac{2}{\sqrt{\gamma(x)}}\frac{2}{\sqrt{\gamma(y)}}\left\langle\frac{\delta^{(2)}S[\gamma]}{\delta\gamma^{\mu\nu}(x)\delta\gamma^{\rho\sigma}(y)}\right\rangle\bigg|_{\gamma^{\mu\nu}=\eta^{\mu\nu}}. \end{aligned} \tag{4.61}$$

from which we can extract the C-function.

Proceeding with the loop expansion, we get to the two loops correction to the effective action

$$S^{(2)}_{eff} = \langle S_4\rangle - \frac{1}{2}\langle S_3S_3\rangle\,, \tag{4.62}$$

where S_3 and S_4 are the three and four field terms of the action given the expansion (4.44). Their explicit form is given by

$$\begin{aligned} S_3 = &-\frac{k}{48\pi\sqrt{2}}\int \mathrm{d}^2x\, f_{abc}\Big[i\epsilon_{\mu\nu}\Big(3(1-\lambda)\chi^a\partial_\mu\varphi^b\partial_\nu\varphi^c + (1+3\lambda)\chi^a\partial_\mu\chi^b\partial_\nu\chi^c\Big) \\ &+ 3\eta^{\mu\nu}(1-\lambda)\chi^a\partial_\mu\chi^b\partial_\nu\varphi^c\Big] \end{aligned} \tag{4.63}$$

$$\begin{aligned} S_4 = &-\frac{k}{384\pi}\int \mathrm{d}^2x\, f_{abe}f_{cde}\eta^{\mu\nu}\Big[(1-\lambda)\varphi^a\partial_\mu\varphi^b\partial_\nu\varphi^c\varphi^d + 6(1-\lambda)\varphi^a\partial_\mu\varphi^b\partial_\nu\chi_c\chi^d \\ &+(1+7\lambda)\chi^a\partial_\mu\chi^b\partial_\nu\chi^c\chi^d\Big]. \end{aligned} \tag{4.64}$$

Where the fields φ and χ are defined in terms of $\phi_\pm$, such that the product $M^\dagger M$ is written in terms of only χ. The explicit form of the transformation can be obtained from the Baker-Campbell-Hausdorff (BCH) formula

$$\phi_- = \varphi + \frac{1}{24}[\chi, [\varphi, \chi]] + \cdots, \tag{4.65}$$

$$\phi_+ = -\chi + \frac{i}{2\sqrt{2}}[\varphi, \chi] + \frac{1}{24}[\varphi, [\varphi, \chi]]. \tag{4.66}$$

In this way, at the IR fixed point ($\lambda = 1$) the action can be written in terms of only χ to all orders of the expansion in the background fields.

We note that there are more terms in the four field action S_4 of the type $\epsilon^{\mu\nu}\chi\partial_\mu\chi\partial_\nu\varphi\chi$. As there is no correlation $\langle\varphi\chi\rangle$ these terms can be dropped off. If we were to calculate higher corrections to the effective action, contributions proportional to $\langle S_4 S_4\rangle$ would make these terms important.

Taking the expected values we obtain the two loop contribution to the effective action

$$\begin{aligned} S^{(2)}_{eff} = &\frac{\pi N_c(N_c^2-1)}{3k}\frac{(3\lambda^2-4\lambda-1)}{(\lambda+1)^2(1-\lambda)}\int \mathrm{d}^2x\,\sqrt{\gamma}\gamma^{\mu\nu}\left[\partial^x_\mu G\partial^y_\nu G - G\partial^x_\mu\partial^y_\nu G\right]_{x=y} \\ &- \frac{4\pi N_c(N_c^2-1)}{3k}\frac{(3\lambda^2+3\lambda+1)}{(1+\lambda)^3}\int \mathrm{d}^2x\,\mathrm{d}^2y\,\epsilon^{\mu\nu}\epsilon^{\sigma\rho}G\partial^x_\mu\partial^y_\sigma G\partial^x_\nu\partial^y_\rho G \\ &+ \frac{\pi\,(1-\lambda)\,N_c(N_c^2-1)}{2k(1+\lambda)^2}\int \mathrm{d}^2x\,\mathrm{d}^2y\,\gamma^{\frac{1}{2}}(x)\gamma^{\frac{1}{2}}(y)\gamma^{\mu\nu}(x)\gamma^{\sigma\rho}(y)\left[G\partial^x_\mu\partial^y_\sigma G\partial^x_\nu\partial^y_\rho G - \partial^x_\mu G\partial^y_\sigma G\partial^x_\nu\partial^y_\rho G\right] \end{aligned} \tag{4.67}$$

where G is the propagator defined in the equation (4.48).

The metric dependence of the two loops effective action is not trivial, as it is encoded inside the propagators. A way to extract the energy momentum tensor is to consider conformal metric transformations $\gamma_{\mu\nu} = e^{\omega}\eta_{\mu\nu}$, such that the metric dependence is evident in the conformal field ω [55]. Furthermore, these integrals are divergent in both the IR and the UV limit and need to be carefully regularized. The IR divergence can be tamed by the addition of a small mass. On the other hand, the UV divergence requires a more meticulous analysis, which is reviewed in the Appendix C. The resulting effective action reads

$$S^{(2)}_{eff} = \frac{N_c(N_c^2-1)}{k}f(\lambda)\Gamma[\gamma], \tag{4.68}$$

where

$$f(\lambda) = 2\frac{2\lambda+1}{(1-\lambda)(1+\lambda)^3} \tag{4.69}$$

and $\Gamma[\gamma]$ is the Polyakov action. Note that, as promised, the second order correction to the effective action is of order $\frac{N_c}{k}$.

Now that we have the two loops effective action in terms of the Polyakov action,

$$\Gamma[\gamma] \equiv \frac{1}{96\pi} \int \mathrm{d}^2x\, \mathrm{d}^2y\, \sqrt{\gamma(x)}\sqrt{\gamma(y)} R(x) R(y) G(x,y), \tag{4.70}$$

we take the metric derivatives and then take the flat spacetime limit. We find the tensor defined by the equation (4.61)

$$B_{\mu\nu\sigma\rho} = -\frac{1}{12\pi}\left[\frac{1}{2\pi}\partial_\mu\partial_\nu\partial_\sigma\partial_\rho \ln|x-y| - \left(\eta_{\mu\nu}\partial_\sigma\partial_\rho + \eta_{\sigma\rho}\partial_\mu\partial_\nu\right)\delta^2(x-y) + \eta_{\mu\nu}\eta_{\sigma\rho}\partial^2\delta^2(x-y)\right]. \tag{4.71}$$

With the explicit form of $B_{\mu\nu\sigma\rho}$ at hand we notice that there is no term of the form $f(x)f(y)$, comparing with its definition, equation (4.61), we conclude that $\langle T(x)\rangle = 0$. Furthermore, the last term in the definifiton of $B_{\mu\nu\sigma\rho}$ is proportional to contact terms and does not contribute to our calculation. Now, we just need to choose the corresponding indexes to find

$$\begin{aligned}
\langle T(x)T(0)\rangle &= \frac{N_c(N_c^2-1)}{8\pi^2 k} f(\lambda)\frac{1}{x^4},\\
\langle T(x)\Theta(0)\rangle &= 0,\\
\langle \Theta(x)\Theta(0)\rangle &= 0.
\end{aligned} \tag{4.72}$$

At last, we find the C-function

$$C^{(2)}(\lambda) = \left(N_c^2 - 1\right)\frac{N_c}{k} f(\lambda). \tag{4.73}$$

At the free fermion fixed point $f(\lambda\to\infty) = 0$, thus the one-loop central charge cancels the ghost contribution $c_{ghost} = -2(N_c^2-1)$ resulting in the correct free fermion central charge $c = N$. On the other hand, the C-function is not stable at the IR fixed point, $(\partial_\lambda C(\lambda = 1) \neq 0)$.

In fact, its divergence at this point is a consequence of the ϕ_- propagator and the fact that, at this point, the action (4.27) is written in terms of only $M^\dagger M$ instead of the independent pair $M^\dagger$ and M. This abrupt change in the integration procedure, and its consequent divergence, is manifest in the partition function. Consequently, we expect to find divergences or incompatibilities to all orders in the large k expansion in the IR limit.

In order to better understand the RG flow to the IR fixed point we introduce a deformation to our theory, such that we modify the action in (4.27) to

$$\begin{aligned}
S_{SU(N_c)} = {}& -k_1 W[M^\dagger] - k_2 W[M] + \frac{\lambda\sqrt{k_1k_2}}{4\pi}\int \mathrm{d}^2x\, \mathrm{tr}\, M^{\dagger-1}\partial_{\bar z} M^\dagger \partial_z M M^{-1}\\
&+ S_{fermion} + S_{ghost}.
\end{aligned} \tag{4.74}$$

Firstly, we note that this model has a very interesting T-duality [56], defined by the combined exchange

$$k_{1,2} \to -k_{2,1} \qquad \text{and} \qquad \lambda \to 1/\lambda. \tag{4.75}$$

With that in mind, we follow the discussions of [57, 58, 59] and study the RG flow under this symmetry. Henceforth, we start at the $G_{k_1} \times G_{k_2}$ WZW action, which now represents the free fermion fixed point at $\lambda = 0$, and flow to a $G_{k_1} \times G_{k_2-k_1}$ WZW action at the IR fixed point $\lambda = \lambda_0 \equiv \sqrt{k_1/k_2}$. In fact, it is possible to check that the two CFTs are indeed fixed points by calculating the β function for the model [59]

$$\begin{aligned}\beta(\lambda) = &-\frac{N_c}{2\sqrt{k_1k_2}}\frac{\lambda^2(\lambda-\lambda_0)(\lambda-\lambda_0^{-1})}{(1-\lambda^2)^2}\\ &+\frac{N_c^2}{4k_1k_2}\frac{\lambda^4(\lambda-\lambda_0)\left(\lambda-\lambda_0^{-1}\right)\left(\left(\lambda_0+\lambda_0^{-1}\right)(1+5\lambda^2)-8\lambda-4\lambda^3\right)}{(1-\lambda^2)^5}+\mathcal{O}(k_{1,2}^{-3}).\end{aligned} \tag{4.76}$$

Then, by taking the limit $\lambda_0 \to 1$ we return to our original model. In doing so, the IR fixed point is abruptly described by a G_k algebra, instead of the $G_{k_1} \times G_{k_2-k_1}$ algebra for $\lambda_0 \neq 1$. Furthermore, at this fixed point the model acquires a new gauge invariance, the Λ transformations of equations (4.33) and (4.34). Lastly, from the beta function in this limit,

$$\lim_{\lambda_0\to 1}\beta(\lambda) = -\frac{N_c}{2k}\frac{\lambda^2}{(1+\lambda)^2}+\frac{N_c^2}{2k^2}\frac{\lambda^4(1-2\lambda)}{(1-\lambda)(1+\lambda)^5}+\mathcal{O}(k_{1,2}^{-3}), \tag{4.77}$$

we infer that the sudden change of algebra and emergence of a gauge invariance remove the IR fixed point from the RG flow [56].

This idea is made stronger by considering the Zamolodchikov metric associated with the current-current perturbation (4.12) [54]. For such case, the Abelian part of the metric, its k independent part, was first calculated in [60], but a more recent discussion can also be found in the Appendix of [61]. The resulting metric reads

$$g_{\lambda,\lambda} = \frac{1}{2}\frac{N_c^2-1}{(1-\lambda^2)^2}. \tag{4.78}$$

With this in mind, we look at the geodesic distance induced in the parameter space,

$$\Delta s = \int \sqrt{g_{\lambda,\lambda}}\,\mathrm{d}\lambda = \sqrt{2\,(N_c^2-1)}\log\frac{1+\lambda}{1-\lambda} \tag{4.79}$$

and find that the distance between any point and the IR fixed point is infinite. In this way, the parameter space of the theory is separate in three parts, $0 \leq \lambda < 1$, $\lambda = 1$ and $\lambda > 1$. The $\lambda > 1$ and $\lambda < 1$ sectors are related by the T-duality. Furthermore, the self-dual $\lambda = 1$ fixed point is physical. Although it is not reachable by the RG flow, it is possible to construct it by fine tuning of parameters.

This discontinuity in the RG flow is reflected in the one loop C-function (4.53), where the discontinuity at $\lambda = 1$ represents the abrupt loss of one the boson fields in the first order in the loop expansion, see action (4.47). In turn, this is a consequence of the sudden change of integration fields, as in taking the limit $\lambda \to 1$ the action dependence jumps from the independent pair $M^\dagger$ and M to the product $M^\dagger M$.

This connection between the C-function and the separation of the RG flow is made clearer when we consider that at the fixed points the C-function equals the central charge of the corresponding CFT. Furthermore, it is known that the central charge is associated with the number of conformal degrees of freedom of the CFT [62]. Thus, it is not surprising that the abrupt change of degrees of freedom would result in a discontinuous C-function. In fact, the divergences in the metric, the C-function and the β-function and the discontinuity in the C-function are all tied up as these quantities are related by

$$\partial_\lambda C(\lambda) = 24 g_{\lambda,\lambda} \beta_\lambda. \tag{4.80}$$

Although the IR fixed point is not accessible to the RG flow, we can study points arbitrarily close to it by performing a zoom in limit [59], which ties the $\lambda \to 1$ and the large k limits together

$$\lambda = 1 - \frac{\kappa^2}{k}, \qquad M = M^{\dagger -1}(1 + i\frac{u_a t^a}{\sqrt{k}}). \tag{4.81}$$

In this limit the $SU(N_c)$ action looks like the PCM model [63],

$$S_{PCM} = \frac{\kappa^2}{4\pi} \int \mathrm{d}^2x \,\mathrm{tr}\, M^{\dagger -1} \partial M^\dagger M^{\dagger -1} \bar{\partial} M^\dagger + \frac{1}{8\pi} \int \mathrm{d}^2 x \,\mathrm{tr}\, \partial u \bar{\partial} u, \tag{4.82}$$

where the first term is the PCM model. The second term consists of $N_c^2 - 1$ gapless free bosons, which is consistent with the first term in a large k expansion of the WZW central charge in (4.32). A more in depth analysis can be found in [58, 56, 60, 63, 59, 61].

As a last comment, we highlight that even though the limit $\lambda \to 1$ leads to problems in the C-function, the β-function and the metric, the IR fixed point is nothing but a WZW theory. In this way, we are able to take $\lambda = 1$ in the actions (4.63) and (4.64) and then follow the same procedure for computing the C-function. We obtain the expected result for the large k expansion of the WZW central charge

$$C^{(2)}(\lambda = 1) = \frac{N_c}{k}. \tag{4.83}$$

5 SPIN LIQUID

5.1 Introduction

A quantum spin liquid (QSL) is a proposed theoretical system where the spins are highly correlated, yet they do not order even at very low-energies [64]. In contrast to magnetically ordered phases, where the wavefunction is a product state, QSL wave functions have long range entanglement between local degrees of freedom. In this way, these QSL wave functions cannot be smoothly deformed into a product state.

Here, we are interested in the class of gapped spin liquids, where the long range entanglement of the ground state wave function produces some distinctive phenomena. Contrary to magnetic orders, we are not able to discern different QSL phases by their symmetry structure. Instead, we are forced to differentiate them by the topological data of the ground state wave function. In this way, they are said to have a topological order, similar to fractional quantum Hall states. Loosely speaking, different spin liquids differ by their patterns of long range entanglement.

The non-perturbative nature of the QSL produces a series of remarkable phenomena. Firstly, we call attention to the presence of gapped quasiparticles. As these are topologically charges, they can only be crated in topologically neutral multiplets. Thus, a single quasiparticle represents a non-local perturbation of the ground state. In this way, quasiparticles interact non-locally to produce non-trivial statistics. More specifically, in two space dimensions the quasiparticles are anyons that pick up a phase as they circle around each other. This phase is associated with the braiding group as the world lines of these quasiparticles cross [65].

Another hallmark feature of 2+1 dimensional topological phases is the presence of gapless edge excitations. This leads us to expect that the low-energy limit is described by a conformal field theory (CFT) on the edge of the manifold. Then, by showing the equivalence between the current algebra and the Hilbert spaces, we are able to demonstrate the equivalence between the edge Wess-Zumino-Witten model (WZW) and a bulk Chern-Simons theory (CS) [66, 45]. In summary, this scheme provides us a method of obtaining low-energy effective actions for gapped topological phases of matter [67, 68, 69, 70, 71].

In parallel to this discussion, progress was also made in the description of topological phases of matter using a framework called quantum wires. This consists of dividing the 2+1 dimensional theory in a series of (1+1)-dimensional systems, called quantum wires, where electrons can move freely. Then, by introducing interactions between wires we effectively realize a (2+1) dimensional phase. The wire construction was shown to successfully reproduce the Fractional quantum Hall effect [72, 48, 73, 74] among other phases of matter [19, 75].

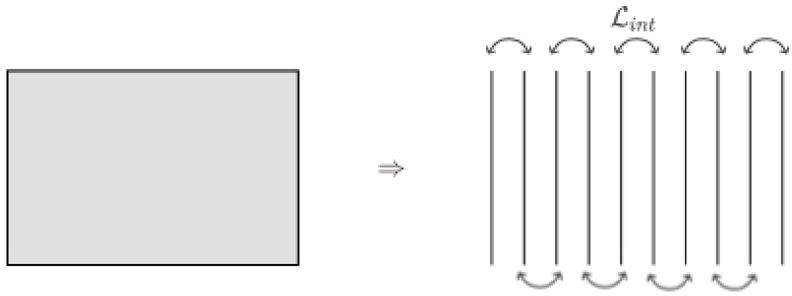

Figura 7 – Dimensional deconstruction of a two dimensional surface.

Concerning the QSL at hand, one such interaction must open a gap in the group structure representing the wires degrees of freedom. In this way, we guarantee that the electrons can tunnel between different wires reproducing a gapped (2+1)-dimensional phase, instead of a collection of self interacting (1+1)dimensional theories. Furthermore, the QSL has no charge carriers in its spectrum. Then, in order to reproduce this behavior we must introduce interactions that gap the $U(1)$ sector for all the wires.

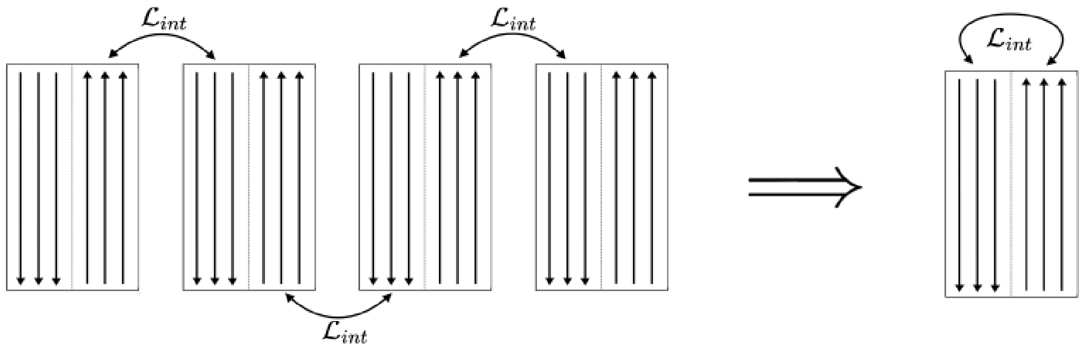

Figura 8 – Process of closing the manifold.

Following this approach, the study of the reference [19] proposed a construction of quantum wires to a class of non-Abelian spin liquids. To this end, the set of wires is divided into a series of interacting bundles as show in left side of the Figure 8. Here, as we are interested in the RG flow from the free fermion theory to a topological fixed point, we only need for the correct interactions to be defined on each wire. For this reason, we simplify the formalism presented in [19] to look at only of the bundles, thus effectively considering a closed manifold. Notice that we neglected the interactions defined between wires of the same bundle in the picture fr illustrative purposes. Nonetheless, they are present in the open and closed manifolds.

In this way, we start with a series of non interacting (1+1)-dimensional CFTs with a starting central charge c_0. Then, by turning the tunneling interactions $\mathcal{L}_{int}$ we may produce a new, nontrivial and interacting CFT, with central charge c. This process can be seen as an RG flow starting from the a free fermion fixed point and ending in a still unknown CFT. Then, by Zamolodchikov C-theorem there must exist a monotonically decreasing C-function that interpolates between the two central charges. In this way the central charge must decrease along this process. In fact we find that the remaining central charge is associated with the coset

$$\left(\frac{su(2)_k \oplus su(2)_{k'}}{su(2)_{k+k'}}\right)_R \quad \text{and} \quad \left(\frac{su(2)_k \oplus su(2)_{k'}}{su(2)_{k+k'}}\right)_L . \tag{5.1}$$

The indexes R and L indicate the propagation direction of the nonmassive states in the first and last blocks. These are represented by a conformal field theory with central charge

$$c(k,k') = 3\left(\frac{k}{k+2} + \frac{k'}{k'+2}\right) - 3\frac{k+k'}{k+k'+2}, \tag{5.2}$$

such that, for $k' = 1$, this reproduces the minimum model series. For $k' = 2$ this central charge corresponds to the superconformal minimal model series, which besides conformal invariance, exhibit supersymmetry [76].

In fact, this remaining central charge belongs to the wires which are left uncoupled at the edges by the wires formalism. These are immediately identified with the gapless edge states of the spin liquid, which are then described by the edge CFT. Further achievements were obtained by considering the strong coupling limit of the tunneling interactions [17, 18]. This allows us to use a constrained fermion approach to determine the chiral actions of the gapless fermions on the edge and the gap opening in the bulk. Then, by using the bulk-edge correspondence we are able to determine the corresponding bulk action. This is given in terms of a non-Abelian CS theory, which is know to be topologically sensitive.

The wires description has some attractive features. Firstly, it allows us to describe the topological properties of the system in terms of the original fermionic degrees of freedom. In this way, it can be seen as a more microscopic approach to the problem. Besides, from a more technical point of view, it transforms the (2+1) dimensional system into a set of coupled (1+1) dimensional systems. Thus, we are able to use a series of powerful non-perturbative methods developed for two dimensional theories, like conformal invariance [62, 76] and bosonization [52, 77].

An important question that was raised since the initial formulation in [19] is about the stability of the spin liquid phases constructed from quantum wires. The delicate point arises from the fact that not all the terms involved in the interactions between the blocks of wires

$$\sum_i \lambda_i \mathcal{L}^i_{int}, \tag{5.3}$$

commute with each other, that is $[\mathcal{L}^i_{int}, \mathcal{L}^j_{int}] \neq 0$, which prohibits the simultaneous localization of the currents due to quantum uncertainty. This means that the respective coupling constants, λ_i, will compete among themselves and the topological phase may be destabilized. Although perturbative calculations of the renormalization group indicate that the interactions are relevant in the low-energy limit. To demonstrate the existence of the phase is a difficult problem, as it is necessary to determine an effective action that is valid along the entire RG flow.

5.2 The Model

Our starting point is the free fermion action

$$S_0 = \int \mathrm{d}^2x \, \left[\tilde{\psi}^\dagger_{R,i\sigma} \partial \tilde{\psi}_{Ri\sigma} + \tilde{\psi}^\dagger_{L,i\sigma} \bar{\partial} \tilde{\psi}_{Li\sigma} + \tilde{\psi}'^\dagger_{R,i\sigma'} \partial \tilde{\psi}'_{Ri\sigma'} + \tilde{\psi}'^\dagger_{L,i\sigma'} \bar{\partial} \tilde{\psi}'_{Li\sigma'} \right], \tag{5.4}$$

where we are assuming that repeated indexes are summed. Latin ones run from one to N_c, non primed greek indexes run from one to N_f and the primed ones run from one to N'_f. The action is invariant under transformations of the symmetry groups $U_R(N) \times U_L(N) \times U_R(N') \times U_L(N')$, where $N = N_c N_f$ and $N' = N_c N'_f$.

The starting action has central charge $c = N + N'$. Our strategy is quite similar to our previous discussion. We want to introduce low-energy relevant operators that gap certain sectors of the theory, thus reducing its total central charge. In this section we will gap the $U(1)$, $SU(N_f)$ and $SU(N'_f)$ internal degrees of freedom. To this end, we start by adding the interactions that gap the abelian sectors of the theory

$$\mathcal{L}_{Umklapp} + \mathcal{L}'_{Umklapp} + \mathcal{L}_{Thirring} + \mathcal{L}'_{Thirring} \tag{5.5}$$

where Thirring and Umklapp interactions are given by the equations (4.11) and (4.10), the primed versions are trivially generalized by priming everything.

The Thirring interaction by itself does not open a gap, it is exactly marginal. Nonetheless, as we saw in our previous discussion, it is necessary to introduce the $U(1)$ vector fields. The Abelian bosonization closely follows our discussion in the previous section. In doing so, we change the fermions from $\tilde{\psi}_{R/Li\sigma}$ to $\psi_{R/Li\sigma}$ and find that the partition function can be rewritten as

$$Z = Z_{U(1)} Z'_{U(1)} Z_{non-Abelian}, \tag{5.6}$$

where the Abelian parts are given by the partition function (4.19) and its primed generalization. Concerning the central charge, the Abelian sectors give no contribution at high energies and in the infrared $c^{U(1)}_{IR} + c^{U(1)'}_{IR} = -2$.

Now we proceed to the non-Abelian sector. To gap the $SU(N_f)$ and $SU(N_f')$ we introduce the interactions

$$\mathcal{L}_{SU(N_f)} = 2\tilde{g} J_R^A J_L^A, \qquad \mathcal{L}_{SU(N_f')} = 2\tilde{g}' J_R^{A\prime} J_L^{A\prime}, \tag{5.7}$$

where the $SU(N_f)$ currents are given by $J^A_{R/L} = \psi^\dagger_{R/L;i\sigma} t^A_{\sigma\rho} \psi_{R/L;i\rho}$. The matrices t^A are the $SU(N_f)$ generators, with A running from one to the group dimension, $N_f^2 - 1$. We will follow the same conventions established in the previous section for $SU(N_c)$ in equation (4.14).

To gap the $SU(N_c)$ sectors we introduce the interactions

$$\mathcal{L}_{SU(N_c)} = 2\lambda J_R^a J_L^a, \quad \mathcal{L}'_{SU(N_c)} = 2\lambda' J_R'^a J_L'^a \quad \text{and} \quad \mathcal{L}_d = 2\lambda_d K_R^a K_L^a, \tag{5.8}$$

with the same conventions as before. The diagonal current is given by $K^a_{R/L} = J^a_{R/L} + J'^a_{R/L}$. As it is argued in [19], this interaction is conjectured to stabilize a strongly interacting fixed point by completely gapping the $SU(N_c)$ sectors. In that work, the authors used the quantum wires construction in such a away that the bulk is completely gapped, leaving massless degrees of freedom only at the edges of the theory. Here, we study whether the gap persists in the bulk in the presence of the competing interactions.

The first step to investigate the stability of the phase is to produce a bosonized version of the fermion action. This is achieved by introducing auxiliary fields, such that the current-current interactions become quadratic in the fermions,

$$\begin{aligned}\mathcal{L} = {} & \psi^\dagger_{R,i\sigma} D_z^{A+B+C} \psi_{Rj\rho} + \psi^\dagger_{L,i\sigma} D_{\bar{z}}^{A+B+C} \psi_{Lj\rho} + \psi'^\dagger_{R,i\sigma'} D_z^{A+B'+C'} \psi'_{Rj\rho'} + \psi'^\dagger_{L,i\sigma'} D_{\bar{z}}^{A+B'+C'} \psi'_{Lj\rho'} \\ & + \frac{1}{\lambda} \operatorname{tr} B_{\bar{z}} B_z + \frac{1}{\lambda'} \operatorname{tr} B'_{\bar{z}} B'_z + \frac{1}{\lambda_d} \operatorname{tr} A_{\bar{z}} A_z + \frac{1}{\tilde{g}} \operatorname{tr} C_{\bar{z}} C_z + \frac{1}{\tilde{g}'} \operatorname{tr} C'_{\bar{z}} C'_z,\end{aligned} \tag{5.9}$$

where we introduces the covariant derivative $D_\mu^{A+B+C} = \delta_{ij}\delta_{\sigma\rho}\partial_\mu + iA^a_\mu t^a_{ij}\delta_{\sigma\rho} + iB^a_\mu t^a_{ij}\delta_{\sigma\rho} + iC^A_\mu t^A_{\sigma\rho}\delta_{ij}$. The fields A, B and B' are $SU(N_c)$ non-Abelian vector fields, meanwhile C and C' are $SU(N_f)$ are $SU(N_f')$ non-Abelian vector fields respectively.

Now we are in a position to start the non-Abelian bosonization procedure. The two non-competing sectors can be integrated by making the field redefinition

$$C_{\bar{z}} = -iu^{\dagger-1}\bar{\partial}u^\dagger, \qquad C_z = i\partial u u^{-1} \tag{5.10}$$

$$C'_{\bar{z}} = -iu'^{\dagger-1}\bar{\partial}u'^\dagger, \qquad C_z = i\partial u' u'^{-1}. \tag{5.11}$$

The remaining $SU(N_c)$ sector presents a more challenging problem, which will be discussed later. Nonetheless, we are able to easily integrate the non-competing sectors and separate them from the non-Abelian partition function

$$Z_{non-Abelian} = Z_{SU(N_f)} Z_{SU(N_f')} \tilde{Z},. \tag{5.12}$$

Let us start with the first and second terms. As there is no competition in the $SU(N_f)$ and $SU(N'_f)$ sectors, the calculations proceed quite similar to the case where there is no competition, as in equations (4.20) to (4.27). The resulting partition functions reads

$$Z_{SU(N_f)} = \int \mathcal{D}u\mathcal{D}u^\dagger \exp - \left[-\kappa W(u) - \kappa W(u^\dagger) + g\kappa P(u^\dagger, u) + S_{ghost} \right]; \quad (5.13)$$

$$Z_{SU(N'_f)} = \int \mathcal{D}u'\mathcal{D}u'^\dagger \exp - \left[-\kappa' W(u') - \kappa' W(u'^\dagger) + g'\kappa' P(u'^\dagger, u') + S_{ghost} \right], \quad (5.14)$$

where $P(u^\dagger, u)$ is the last term in the Polyakov-Wiegmann identity, equation (4.30). We defined the renormalized coupling constants according to our previous discussion

$$g \equiv \frac{4\pi}{\kappa\tilde{g}} + 1 - \beta \qquad \text{and} \qquad g' \equiv \frac{4\pi}{\kappa'\tilde{g}'} + 1 - \beta', \quad (5.15)$$

we also have defined $\kappa \equiv 2N_f + N_c$ and $\kappa' \equiv 2N'_f + N_c$.

The partition function (5.13) has three fixed points, one for $g = 0$, where we have two decoupled WZW actions and another fixed point is found at $g = 1$, where we can use the Polyakov-Wiegmann identity to produce one WZW action for $u^\dagger u$. Lastly, there is a fixed point when we turn the interactions off, by setting $\tilde{g} = 0$, this leads to a null central charge for the sector. Using the same reasoning for the second partition function, equation (5.14), we find the combined central charge of these sectors

$$c_{SU(N_f)} + c_{SU(N'_f)} = \begin{cases} -2(N_f^2 - 1) - 2(N_f'^2 - 1) - 2\frac{\kappa\left(N_f^2 - 1\right)}{-\kappa + N_f} - 2\frac{\kappa'\left(N_f'^2 - 1\right)}{-\kappa' + N'_f} & \text{if} \quad g = g' = 0, \\ -2(N_f^2 - 1) - 2(N_f'^2 - 1) - \frac{\kappa\left(N_f^2 - 1\right)}{-\kappa + N_f} - \frac{\kappa'\left(N_f'^2 - 1\right)}{-\kappa' + N'_f} & \text{if} \quad g = g' = -1, \\ 0 & \text{if} \quad g, g' \to \pm\infty. \end{cases} \quad (5.16)$$

We will assume that all the coupling constants flow together from the UV fixed points to the IR fixed points. This way, a situation where g is in the UV limit and the g' is in the IR limit is forbidden. Just like in the noncompeting case, we set the regularization to produce the RG flow from the free fermion fixed point to the IR fixed point, $\beta = \beta' = 0$. In turn, this eliminates the UV fixed points $g = g' = 0$.

The loop expansion and the process of constructing the C-function of the $SU(N_f)$ and $SU(N'_f)$ sectors is analogous to the case without competition. To obtain the correct expressions one should make the exchange $k \to \kappa$, $N_c \to N_f$ and $\lambda \to g$ in the expressions of the previous section for the $SU(N_f)$ sector and similarly for the $SU(N'_f)$. We add the ghost and the WZW contribution to find the non-competing sectors C-

function

$$C_{SU(N_f)} = \left(N_f^2 - 1\right)\left[\frac{1}{4\bar{g}^2}\left(K_2(1/\sqrt{\bar{g}})^2 + 2K_1(1/\sqrt{\bar{g}})^2 - 3K_0(1/\sqrt{\bar{g}})^2\right) - 1\right]$$
$$C_{SU(N_f')} = \left(N_f^{2\prime} - 1\right)\left[\frac{1}{4\bar{g}^{2\prime}}\left(K_2(1/\sqrt{\bar{g}'})^2 + 2K_1(1/\sqrt{\bar{g}'})^2 - 3K_0(1/\sqrt{\bar{g}'})^2\right) - 1\right], \tag{5.17}$$

where we defined $\bar{g} \equiv -(1-g)$, $\bar{g}' \equiv -(1-g')$.

5.3 The $SU(N_c)$ Competing Sector

Now we discuss the remaining $SU(N_c)$ sector. After the integration of the non-competing sectors we are left with the action

$$\tilde{S} = \int d^2x \left[\psi^\dagger_{R,i\sigma} D_z^{A+B} \psi_{Rj\rho} + \psi^\dagger_{L,i\sigma} D_{\bar{z}}^{A+B} \psi_{Lj\rho} + \psi'^\dagger_{R,i\sigma'} D_z^{A+B'} \psi'_{Rj\rho'} + \psi'^\dagger_{L,i\sigma'} D_{\bar{z}}^{A+B'} \psi'_{Lj\rho'} + \frac{1}{\lambda}\,\mathrm{tr}\, B_{\bar{z}} B_z + \frac{1}{\lambda'}\,\mathrm{tr}\, B'_{\bar{z}} B'_z + \frac{1}{\lambda_d}\,\mathrm{tr}\, A_{\bar{z}} A_z\right], \tag{5.18}$$

which is associated with the partition function $\tilde{Z}$. The vector fields in the covariant derivatives do not commute with each other. This makes the fermion integration more complicated than the non-competing cases. Nonetheless, we can simplify the problem by making the field redefinitions

$$B_\mu \to \tilde{B}_\mu - A_\mu \qquad \text{and} \qquad B'_\mu \to \tilde{B}'_\mu - A_\mu. \tag{5.19}$$

This simplifies the action to

$$\tilde{S} = \int d^2x \left[\psi^\dagger_{R,i\sigma} D_z^{B} \psi_{Rj\rho} + \psi^\dagger_{L,i\sigma} D_{\bar{z}}^{B} \psi_{Lj\rho} + \psi'^\dagger_{R,i\sigma'} D_z^{B'} \psi'_{Rj\rho'} + \psi'^\dagger_{L,i\sigma'} D_{\bar{z}}^{B'} \psi'_{Lj\rho'} + \frac{1}{\lambda}\,\mathrm{tr}\left(\tilde{B}_{\bar{z}} - A_{\bar{z}}\right)\left(\tilde{B}_z - A_z\right) + \frac{1}{\lambda'}\,\mathrm{tr}\left(\tilde{B}'_{\bar{z}} - A_{\bar{z}}\right)\left(\tilde{B}'_z - A_z\right) + \frac{1}{\lambda_d}\,\mathrm{tr}\, A_{\bar{z}} A_z\right]. \tag{5.20}$$

In this form we are able to integrate the fermions. We first set the vector fields to

$$\tilde{B}_{\bar{z}} = -iH^{\dagger -1}\bar{\partial}H^\dagger, \quad \tilde{B}_z = i\partial H H^{-1}, \quad \tilde{B}'_{\bar{z}} = -iH'^{\dagger -1}\bar{\partial}H'^\dagger, \quad \tilde{B}'_z = i\partial H' H'^{-1}$$
$$A_{\bar{z}} = -ig^{\dagger -1}\bar{\partial}g^\dagger \qquad \text{and} \quad A_z = i\partial g g^{-1}. \tag{5.21}$$

The field redefinitions allows us to separate the $SU(N_c)$ sector from the fermions, leading us to the partition function

$$\tilde{Z} = Z_0 Z_{SU(N_c)}, \tag{5.22}$$

where Z_0 is the free fermion partition function, associated with the action (5.4). Furthermore, the $SU(N_c)$ sector partition function is given by

$$\begin{aligned}\tilde{Z} = Z_0 \int \mathcal{D}\mu J[H^\dagger, H] J[H'^\dagger, H'] J[g^\dagger, g] \times \exp - \Big[-N_f W(H^\dagger H) - N_f' W(H'^\dagger H') \\ + \frac{1}{\lambda} \int \operatorname{tr} \left(H^{\dagger -1} \bar{\partial} H^\dagger - g^{\dagger -1} \bar{\partial} g^\dagger \right) \left(\partial H H^{-1} - \partial g g^{-1} \right) + \frac{1}{\lambda_d} \int \operatorname{tr} g^{\dagger -1} \bar{\partial} g^\dagger \partial g g^{-1} \\ + \frac{1}{\lambda'} \int \operatorname{tr} \left(H'^\dagger \bar{\partial} H'^\dagger - g^\dagger \bar{\partial} g^\dagger \right) \left(\partial H' H_-'^{-1} - \partial g g^{-1} \right) \Big], \end{aligned} \tag{5.23}$$

where the result of the fermion integration is given by the first line of the partition function above and the integration measure is given by

$$\mathcal{D}\mu \equiv \mathcal{D}g \mathcal{D}g^\dagger \mathcal{D}H \mathcal{D}H^\dagger \mathcal{D}H' \mathcal{D}H'^\dagger \tag{5.24}$$

To obtain the form we presented here, we added counterterms to ensure that the resulting action remains gauge invariant in the strong coupling limit. This process is equivalent to setting $\alpha = 0$ for the non-competing action (4.24). Furthermore, we consider similar requirements to obtain the Jacobians for the field redefinitions (5.21)

$$J[H^\dagger, H] = [\det(\partial_+ \partial_-)]_{ADJ} \exp \left[2N_c W(H^\dagger H) \right], \tag{5.25}$$

where the determinant in the adjoint representation can be rewritten as reparametrization ghosts resulting in

$$\begin{aligned}\tilde{Z} = Z_0 \int \mathcal{D}\mu \exp - \Big[-(2N_c + N_f) W(H^\dagger H) - (2N_c + N_f') W(H'^\dagger H') - 2N_c W(g^\dagger g) \\ + \frac{1}{\lambda} \int \operatorname{tr} \left(H^{\dagger -1} \bar{\partial} H^\dagger - g^{\dagger -1} \bar{\partial} g^\dagger \right) \left(\partial H H^{-1} - \partial g g^{-1} \right) + \frac{1}{\lambda_d} \int \operatorname{tr} g^{\dagger -1} \bar{\partial} g^\dagger \partial g g^{-1} \\ + \frac{1}{\lambda'} \int \operatorname{tr} \left(H'^\dagger \bar{\partial} H'^\dagger - g^\dagger \bar{\partial} g^\dagger \right) \left(\partial H' H_-'^{-1} - \partial g g^{-1} \right) + S^{(3)}_{\text{ghost}} \Big]. \end{aligned} \tag{5.26}$$

As a last step to the fermion integration, it is convenient to rewrite the partition function in terms of the complex $SU(N_c)$ valued fields for the original vector fields B and B'. To this end, we perform one last field redefinition by setting

$$H = gh, \qquad H^\dagger = h^\dagger g^\dagger, \qquad H' = gh' \qquad \text{and} \qquad H'^\dagger = h'^\dagger g^\dagger, \tag{5.27}$$

while we keep the integration measure (5.24) in terms of the upper case fields. In terms of these new variables the $SU(N_c)$ partition function reads

$$\begin{aligned}\tilde{Z} = Z_0 \int \mathcal{D}\mu \exp - \Big[-N_f W(h^\dagger g^\dagger g h) - N_f' W(h'^\dagger g^\dagger g h') \\ - 2N_c \left[W(h^\dagger g^\dagger g h) + W(h'^\dagger g^\dagger g h') + W(g^\dagger g) \right] + S^{(3)}_{\text{ghost}} \\ + \frac{1}{\lambda} \int \operatorname{tr} h^{\dagger -1} g^{\dagger -1} \bar{\partial} h^\dagger g^\dagger g \partial h h^{-1} g^{-1} + \frac{1}{\lambda'} \int \operatorname{tr} h'^{\dagger -1} g^{\dagger -1} \bar{\partial} h'^\dagger g^\dagger g \partial h' h'^{-1} g^{-1} \\ + \frac{1}{\lambda_d} \int \operatorname{tr} g^{\dagger -1} \bar{\partial} g^\dagger \partial g g^{-1} \Big], \end{aligned} \tag{5.28}$$

where $S^{(n)}_{\text{ghost}}$ is the action for n copies of $SU(N_c)$ reparametrization ghosts, each with central charge $c = -2(N_c^2 - 1)$.

5.4 Fixed Points

Now we are in a position to discuss the fixed points structure of the theory. Firstly we look at the free fermion limit, $\lambda, \lambda', \lambda_d \to 0$, we call this the fixed point 1. At first glance, this introduces divergences in the action. However, these can be resolved by imposing that, by turning off an interaction, its associated $SU(N_c)$ field goes to the identity. In doing so, we invert the Jacobian and return its path integral to the original vector field. For example, in taking the limit $\lambda_d \to 0$ we should also take $g = g^\dagger = \mathbb{I}$. Likewise, in the free fermion limit we take all the $SU(N_c)$ fields to the identity. This procedure eliminates all the terms dependent on the $SU(N_c)$ fields from the action (5.28). The remaining ghosts are then eliminated by undoing the variable change (5.21). This process completely removes the gapping of the $SU(N_c)$ sector and reproduces the correct free fermion central charge. Note that this fixed point is independent of the choice of our regularization parameter α. This matches our expectation that the free fermions fixed point should be independent of the regularization choice.

There are three more IR fixed points we will discuss, for these we will assume that the $U(1)$, $SU(N_f)$ and $SU(N_f')$ sectors are fully gapped. Our first non-trivial fixed point is found at $\lambda = \lambda' = 0$ and $\lambda_d \to \infty$, we will call this the fixed point 2. In this limit we take $h = h^\dagger = h' = h'^\dagger = \mathbb{I}$ and return the path integral to the vector fields

$$\int \mathcal{D}(h^\dagger g^\dagger)\mathcal{D}(gh)\mathcal{D}(h'^\dagger g^\dagger)\mathcal{D}(gh') \exp\left(2N_c\left[W(h^\dagger g^\dagger gh) + W(h'^\dagger g^\dagger gh')\right] + S^{(2)}_{\text{ghost}}\right) = \int \mathcal{D}\tilde{B}_\mu \mathcal{D}\tilde{B}'_\mu. \tag{5.29}$$

Thus, at the fixed point 2 the $SU(N_c)$ action reads

$$S_{2,SU(N_c)} = -k_d W(g^\dagger g) + S^{(1)}_{ghost}, \tag{5.30}$$

where we defined $k_d \equiv 2N_c + N_f + N_f'$.

Adding the contribution from non-competing sectors we obtain the complete central charge for the IR fixed point 2

$$\begin{aligned} c_2 = {} & N + N' - 2 - 2(N_c^2 - 1) - \frac{k_d(N_c^2-1)}{N_c - k_d} - 2(N_f^2 - 1) \\ & - \frac{\kappa(N_f^2-1)}{N_f - \kappa} - 2(N_f'^2 - 1) - \frac{\kappa'(N_f'^2-1)}{N_f' - \kappa'} \\ = {} & \frac{k_d N_f N_f'(N_c^2-1)}{(N_c+N_f)(N_c+N_f')(N_c+N_f+N_f')}, \end{aligned} \tag{5.31}$$

where in the first line the first two terms correspond to the free fermion, the next to the $U(1)$ sector, the next two terms to the $SU(N_c)$ sector, and the remaining ones to the

$SU(N_f)$ and $SU(N'_f)$ sectors. This central charge is supported by the coset group

$$\frac{SU(N_c)_{N_f} \times SU(N_c)_{N'_f}}{SU(N_c)_{N_f+N'_f}}. \tag{5.32}$$

As there is a remaining central charge, the diagonal interaction does not fully gap the theory. In fact it leaves some degress of freedom gapless preserving a smaller conformal invariance.

A third fixed point can be found at $\lambda_d = 0$ and $\lambda, \lambda' \to \infty$. In this limit, we take $g = g^\dagger = \mathbb{I}$ and return its path integral to its original vector field A_μ alongside its respective ghost field. This leads us to the fixed point 3 action

$$S_{3,SU(N_c)} = -kW(h^\dagger h) - k'W(h'^\dagger h') + S^{(2)}_{ghost'}, \tag{5.33}$$

where we defined $k \equiv 2N_c + N_f$ and $k' \equiv 2N_c + N'_f$. Its total central charge is given by

$$\begin{aligned} c_3 = {} & N_c N_f + N_c N'_f - 2 - 4(N_c^2 - 1) - \frac{k(N_c^2-1)}{N_c - k} - \frac{k'(N_c^2-1)}{N_c - k'} \\ & - 2(N_f^2 - 1) - \frac{\kappa(N_f^2-1)}{N_f - \kappa} - 2(N_f'^2 - 1) - \frac{\kappa'(N_f'^2-1)}{N'_f - \kappa'} \\ = {} & 0. \end{aligned} \tag{5.34}$$

As expected this is the fully gapped fixed point.

Lastly, there is the fixed point 4 found at the competing strong coupling limit, $\lambda, \lambda', \lambda_d \to \infty$. A naive analysis might consider that, as the fixed point 3 is fully gapped, one cannot turn on a relevant interaction, as this would lead to an overgapping or a negative central charge. In fact, in this limit

$$S_{4,SU(N_c)} = -kW(H^\dagger H) - k'W(H'^\dagger H') - 2N_c W(g^\dagger g) + S^{(3)}_{ghost} \tag{5.35}$$

and its total central charge

$$\begin{aligned} c_4 = {} & N + N' - 2 - 6(N_c^2 - 1) - \frac{k(N_c^2-1)}{N_c - k} - \frac{k'(N_c^2-1)}{N_c - k'} - \frac{2N_c\left(N_c^2-1\right)}{N_c - 2N_c} \\ & - 2(N_f^2 - 1) - \frac{\kappa(N_f^2-1)}{N_f - \kappa} - 2(N_f'^2 - 1) - \frac{\kappa'(N_f'^2-1)}{N'_f - \kappa'} \\ = {} & 0. \end{aligned} \tag{5.36}$$

Thus the strong coupling competing limit is a fully gapped conformal field theory.

Note that this is not the complete fixed point structure of our theory. There are additional UV fixed points similar to the $\lambda = 0$ fixed point in the non competing case. As we are interested in building the C-function from the free fermion to the IR fixed points, we will not discuss these any further. At last, we summarize the fixed point structure of our theory and its central in the table

$$\begin{array}{c|cc} \text{central charge} & \lambda = \lambda' = 0 & \lambda, \lambda' \to \infty \\ \hline \lambda_d = 0 & N + N' & 0 \\ \lambda_d \to \infty & su(N_c)_{N_f} \times su(N_c)_{N'_f} / su(N_c)_{N_f + N'_f} & 0 \end{array} \tag{5.37}$$

5.5 Loop Expansion and C-function

Just like in the non-competing case in the last section, we must build the C-function to check that the IR fixed point structure we discussed is indeed realized. Our strategy closely follows our previous discussion. We take the background field expansion to find the up to one loop effective action. Afterwards, we calculate the up to one loop C-function by taking correlation functions of the energy-momentum tensor.

Here we will make an important simplification. As we noticed in our previous discussion, the $SU(N_c)$ IR fixed point occurs for $\alpha = 0$ and the free fermion fixed point is independent of the choice of the regulator parameter. Because of this peculiar characteristic of our theory, we will consider $\alpha = 0$. Note that, if we were interested in other RG flows this may not be valid.

We expand the $SU(N_c)$ fields as

$$H = H_0 e^{-i(\phi_1+\phi_2)/\sqrt{2}}, \quad H' = H'_0 e^{-i(\phi_3+\phi_4)/\sqrt{2}}, \quad g = g_0 e^{-i(\phi_5+\phi_6)/\sqrt{2}}, \tag{5.38}$$

$$H^\dagger = H_0^\dagger e^{i(\phi_1-\phi_2)/\sqrt{2}}, \quad H'^\dagger = H_0'^\dagger e^{i(\phi_3-\phi_4)/\sqrt{2}} \quad \text{and} \quad g^\dagger = g_0^\dagger e^{i(\phi_5-\phi_6)/\sqrt{2}}, \tag{5.39}$$

where H_0, H'_0 and g_0 are constant matrices. The fields ϕ_n, for $n = 1, \cdots, 6$, are background hermitian matrix fields. We collect the quadratic terms in this expansion to find the two fields action

$$S_2 = \frac{1}{2} \int \mathrm{d}^2 x \, \sqrt{\gamma} \gamma^{\mu\nu} \partial_\mu \phi_m^a \mathcal{O}_{mn} \partial_\nu \phi_n^a, \tag{5.40}$$

where we are implying sums in the indexes. The index a runs from 1 to $N_c^2 - 1$ and $\mathcal{O}$ is given by

$$\mathcal{O} = \frac{1}{8\pi} \begin{pmatrix} l & 0 & 0 & 0 & -l & 0 \\ 0 & -2k-l & 0 & 0 & 0 & l \\ 0 & 0 & l' & 0 & -l' & 0 \\ 0 & 0 & 0 & -2k'-l' & 0 & l' \\ -l & 0 & -l' & 0 & l+l'+l_d & 0 \\ 0 & l & 0 & l' & 0 & -4N_c - l - l' - l_d \end{pmatrix}, \tag{5.41}$$

where

$$l \equiv \frac{4\pi}{\lambda}, \quad l' \equiv \frac{4\pi}{\lambda'} \quad \text{and} \quad l_d \equiv \frac{4\pi}{\lambda_d}. \tag{5.42}$$

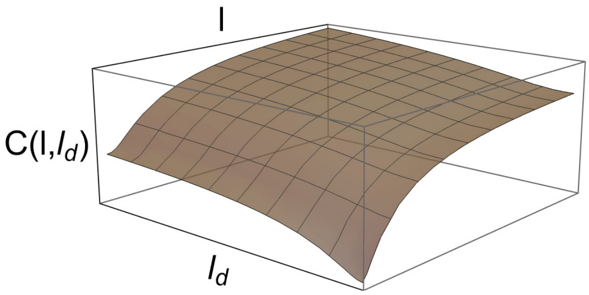

Figura 9 – 3D plot for $C(m^2r_0^2 l, m^2r_0^2 l_d)$

Adding sources to the fields and integrating the action we find the corresponding two point functions

$$\left\langle \phi_m^a(x)\phi_n^b(y)\right\rangle = \delta_{ab}\mathcal{O}_{mn}^{-1}G(x,y), \tag{5.43}$$

where $G(x,y)$ is the Green function to the Laplace-Beltrami operator $\nabla = \frac{1}{\sqrt{\gamma}}\partial_\mu\left(\gamma^{\mu\nu}\sqrt{\gamma}\partial_\nu\right)$.

Notice that, up to one loop, the competing problem can be separated into two non-interacting problems, one for the even and one for odd ϕ_n. The sector for even n remains gapless along the RG flow. On the other hand, the odd ϕ_n generates the one loop RG flow. Our strategy for doing so closely follows the non competing case. As the action (5.40) is quadratic, we simply find its energy-momentum tensor and take the desired expected values in the quadratic theory. We find that the up to one loop $SU(N_c)$ C-function reads

$$C^{(1)}_{SU(N_c)}(l,l_d) = -6(N_c^2-1) + (N_c^2-1)\left[C^{(s)}(l) + C^{(s)}\left(l_+\right) + C^{(s)}\left(l_-\right)\right], \tag{5.44}$$

where the first term is the contribution from the ghosts. We also have defined

$$l_\pm \equiv 2ll_d/\left(3l + ld \pm \sqrt{9l^2 + 2ll_d + l_d^2}\right) \tag{5.45}$$

and $C^{(s)}$ is the resulting one loop C-function for the non-competing case

$$C^{(s)}(l) = 1 + \frac{m^4r_0^4}{4l^2}\left[K_2(mr_0l^{-1/2})^2 + 2K_1(mr_0l^{-1/2})^2 - 3K_0(mr_0l^{-1/2})^2\right]. \tag{5.46}$$

At this point we can take the regulator mass to zero. The resulting one loop C-function can be summarized in the formula

$$C^{(1)}_{SU(N_c)} = -(N_c^2-1)\left[2\delta_{l,0} + \delta_{l_d,0}\right], \tag{5.47}$$

where $\delta_{l,0}$ is the continuum version of the Kronecker delta function given by (4.54).

As expected, the one loop C-function at its fixed points matches the first order expansion of the $SU(N_c)$ sector central charge for small $\hbar$.

	C-function	central charge expansion
free fermion	$C^{(1)}_{SU(N_c)} = 0$	0
$\lambda_d \to \infty, \lambda = 0$	$C^{(1)}_{SU(N_c)} = -(N_c^2-1)$	$-2(N_c^2-1) + (N_c^2-1)\left[1 + \frac{N_c}{k_d}\right]$
$\lambda_d = 0, \lambda \to \infty$	$C^{(1)}_{SU(N_c)} = -2(N_c^2-1)$	$-4(N_c^2-1) + (N_c^2-1)\left[2 + \frac{N_c}{k} + \frac{N_c}{k'}\right]$
$\lambda, \lambda_d \to \infty$	$C^{(1)}_{SU(N_c)} = -3(N_c^2-1)$	$-6(N_c^2-1) + (N_c^2-1)\left[3 + \frac{N_c}{k} + \frac{N_c}{k'} + \frac{N_c}{2N_c}\right]$

Concerning the free fermion fixed point, the one loop C-function matches to $SU(N_c)$ sector's central charge at this fixed point. Thus, further order corrections to the C-function must vanish at this point. On the other hand, we expect higher order corrections to the C-function at the IR fixed points. For a plot of the one loop C-function before the renormalization limit see Fig. 9.

In Chapter 4, we discussed how by fully gapping a sector of our theory we introduce divergences in the partition function originated in the sudden change in the number of independent fields. These properties lead a discontinuity and divergences in the C-function and divergences in the β-function and the Zamolodchikov metric. Along these lines, we concluded that the IR fixed point is infinitely far away, and is thus not accessible from the rest of the RG flow.

Here in the competing case something similar happens. By fully gapping some sector (taking λ and/or λ_d to strong coupling) the number of independent fields in the action of (5.26) suddenly changes. This can be easily seen by the sudden disappearance of some of the odd ϕ_n in (5.40) as we take $l \to 0$ and/or $l_d \to 0$, which in turn is manifested in the C-function (5.47) as discontinuities in these limits.

Motivated by our discussion in the non-competing case we propose that, with the exception of the free fermion, the fixed points shown in Table 5.37 are not reachable by a RG flow starting at the free fermion fixed point. In turn, if we create the system at one of the fixed points by fine tuning it is also not connected to the rest of the RG flow, in this way these fixed points are stable against perturbations. Nonetheless, generalizations of the zoom in limit (4.81) allows us to study the theory arbitrarily close to the IR fixed point. As we consider the gap opening to be continuous along the RG flow, the zoom in limit allows us to study a theory which is also close to the IR fixed point.

The methods developed in [58, 56, 60, 63, 59, 61] can be applied to the competing case. The fixed points 3 and 4 can be expanded in a zoom in limit similar to the non-competing case. For the fixed point 3, we can expand the action (5.26) in a zoom in limit

$$\frac{1}{\lambda} = \epsilon + \mathcal{O}(1/k), \qquad \frac{1}{\lambda'} = \epsilon' + \mathcal{O}(1/k'), \qquad \frac{1}{\lambda_d} \to \infty,$$
$$H = H^{\dagger -1}\left[\mathbb{I} + i\frac{u_a t^a}{\sqrt{k}}\right], \qquad H' = H'^{\dagger -1}\left[\mathbb{I} + i\frac{u'_a t^a}{\sqrt{k'}}\right], \qquad g^\dagger = g = \mathbb{I}, \tag{5.48}$$

for ϵ and ϵ' close to zero we obtain the IR limit of the fixed point 3

$$S_3 \approx \epsilon \int d^2x \,\mathrm{tr}\, H^{\dagger -1}\bar{\partial}H^\dagger \partial H^\dagger H^{\dagger -1} - \frac{1}{8\pi}\int d^2x\,\mathrm{tr}\,\partial_\mu u \partial^\mu u$$
$$+ \epsilon' \int d^2x \,\mathrm{tr}\, H'^{\dagger -1}\bar{\partial}H'^\dagger \partial H'^\dagger H'^{\dagger -1} - \frac{1}{8\pi}\int d^2x\,\mathrm{tr}\,\partial_\mu u' \partial^\mu u'. \tag{5.49}$$

Indeed, in the zoom in expansion we obtain a pair of decoupled free bosons, one for each sector, corresponding to the large k and k' expansion of the central charge for the fixed point c. Furthermore, we obtain a pair of PCM actions, which are not conformally invariant.

Likewise, we can expand the action in (5.26) around the fixed point 4

$$\frac{1}{\lambda} = \epsilon + \cdots, \qquad \frac{1}{\lambda'} = \epsilon' + \cdots, \qquad \frac{1}{\lambda_d} \to \epsilon_d + \cdots,$$
$$H = H^{\dagger -1}\left[\mathbb{I} + i\frac{u_a t^a}{\sqrt{k}}\right], \qquad H' = H'^{\dagger -1}\left[\mathbb{I} + i\frac{u'_a t^a}{\sqrt{k'}}\right], \qquad g = g^{\dagger -1}\left[\mathbb{I} + i\frac{v_a t^a}{\sqrt{2N_c}}\right], \tag{5.50}$$

for ϵ, ϵ' and ϵ_d close to zero we obtain

$$S_4 \approx \epsilon \int d^2x \,\mathrm{tr}\, H^{\dagger -1}\bar{\partial}H^\dagger \partial H^\dagger H^{\dagger -1} - \frac{1}{8\pi}\int d^2x\,\mathrm{tr}\,\partial_\mu u \partial^\mu u$$
$$+ \epsilon' \int d^2x \,\mathrm{tr}\, H'^{\dagger -1}\bar{\partial}H'^\dagger \partial H'^\dagger H'^{\dagger -1} - \frac{1}{8\pi}\int d^2x\,\mathrm{tr}\,\partial_\mu u' \partial^\mu u'$$
$$+ \epsilon_d \int d^2x \,\mathrm{tr}\, g^{\dagger -1}\bar{\partial}g^\dagger \partial g^\dagger g^{\dagger -1} - \frac{1}{8\pi}\int d^2x\,\mathrm{tr}\,\partial_\mu v \partial^\mu v.$$

Unfortunately, we are unable to compare the resulting beta functions as we do not have a method for determining the generalization of the T-duality (4.75) in the competing case, which is necessary to correctly relate our model to the ones discussed in the literature.

As we discussed previously, the fixed point 4 has a zero central charge and thus it is gapped. Even tough we are not able to access the fixed point by the RG flow, the zoom in limit (5.50) allows us to study the theory arbitrarily close to it. Furthermore, as the RG flow (and the gap opening) are continuous, the theory close to the competing fixed point is essentially gapped, and the QSL phase is stable.

6 FINAL REMARKS

Along this work we have studied three topological phases of matter. In the first part, we studied low-energy effective theories associated with the QAH phase and with a topological superconducting phase arising from the QAH system in proximity to a pairing potential. We first derive the edge theory of the corresponding phases by transforming the 2+1 dimensional systems into a set of 1+1 dimensional quantum wires, where the edge states appear in a quite transparent way. Next, the EFT for the QAH phase was derived directly from the microscopic model by computing the fermionic determinant at the leading order in the large gap limit. A new aspect involved in this computation is that the fermions are of nonrelativistic nature, so that the leading term in the effective action comes from several Feynman diagrams with nonrelativistic pieces that together conspire to produce a usual relativistic CS term. Of course, higher-order corrections (nontopological) are nonrelativistic. It is quite remarkable that the CS term arising from this computation does not suffer from the gauge anomaly, in contrast to the relativistic case that is plagued with a half-integer CS contribution.

Our computation of the fermionic determinant can be used even in the presence of the superconductor pairing potential that breaks the $U(1)$ charge conservation symmetry. To this, we follow the standard treatment of superconductors where we turn to the BdG formalism and work with unconstrained spinors. In effect, this leads to a duplication of the degrees of freedom and also introduces fictitious $U(1)$ symmetries, which can be coupled with background gauge fields. This enables us to compute the local effective action for the gauge fields in a similar way to the QAH case. The physical Hilbert space is recovered by considering the gauge fields as $O(2)$, instead of $U(1)$. In this way, the bulk-edge correspondence implies that the corresponding edge theory is the orbifold $U(1)/\mathbb{Z}_2$. The level of the CS, which is related to the compactification radius of the edge theory, is fixed under physical requirements of the superconducting phase. This leads to the $O(2)_4$ CS theory whose edge states are described by the $N = 2$ orbifold theory, which in turn corresponds to two copies of the Ising CFT. The doubling of the degrees of freedom is a direct consequence of the way the effective theory was constructed, i.e., employing the BdG formalism and working with unconstrained spinors.

A natural extension of this work is to consider the case of Laughlin states of the fractional quantum Hall phase. While this is not described in terms of free fermions, and consequently it is very difficult to integrate out the massive fermions to produce a fermionic determinant, it can be analyzed in the framework of quantum wires [48]. Recent works have shown how the effective theory containing a CS with level m, with m

being an integer odd, emerges directly from the quantum wires system through explicit identifications between quantum wires variables and gauge fields in the continuum [74, 78, 79]. In this way, we see this as a promising setting to study the proximity effect in the fractional case, which in principle drive the system to a fractional topological superconducting phase.

Furthermore, we also discussed the realization of topological phases of non-Abelian spin liquids. To do so, we used a quantum wires approach to discuss how we can introduce operators in the action that effectively realize a (2+1)-dimensional phase in the deep infrared. Then, we used non-Abelian bosonization to obtain a bosonic version of the action, from which we were able to extract the fixed point structure.

In our simplified, non-competing, model we obtained three fixed points. Firstly, we found that, in order for the free fermion fixed point to be realized, it was necessary to introduce consistency conditions to the gauge fields. This also led to nontrivial implications on the competing case. Furthermore, we found two more fixed points. One is an UV fixed point, which was discarded on physical grounds and the other is the strong coupling IR fixed point with the expected partially gapped central charge.

With the fixed point structure at hand we proceeded to study the RG flow in a loop expansion. In our first analysis we studied the C-function, where in taking the IR limit, we found a discontinuity at first order and a divergence at second order. Inspired by the connection between the central charge and the number of conformal degrees of freedom, we were able to trace these properties back to the sudden change of the number of independent fields in the action as we take the IR limit and realize the Polyakov-Wiegmann identity.

Some discussions in the literature corroborate the idea that the discontinuity and divergence in the C-function are a result of a separation of the RG flow into three regions. The first two are $\lambda < 1$ and $\lambda > 1$, which are equivalent by a T-duality. Furthermore, the IR fixed point, $\lambda = 1$, remains inaccessible from the rest of the RG flow [58, 56, 63, 59, 61, 60]. In turn, if we create the system at one of the fixed points by fine tuning it is also not connected to the rest of the RG flow, in this way these fixed points are stable against perturbations.

With our simplified model analysis finished, we proceed to the competing case, which realizes the QSL phase. We follow the discussions of the non-competing case with slight modifications to accommodate for the competing nature of the spin liquid. Such competition leads us perform the variable changes (5.19), (5.21) and (5.27), otherwise we would have found that some fixed points would be overgapped in the strong coupling limit, with a negative central charge.

With the bosonization process finished, we were able to derive the fixed point structure of the model. Firstly, by turning off the interactions we find the free fermion fixed point. Then by taking the strong coupling limit of each interaction selectively we found three more fixed points. One is realized by taking only the diagonal interaction to the strong coupling limit, this is the partially gapped, cosset fixed point associated with the boundary bundles of [17, 19]. Furthermore, we found a fully gapped fixed point by taking only the non-diagonal interaction to the strong coupling limit. At last, we found another fully gapped fixed point when all the interactions flow to strong coupling. This is the fixed point associated with the bulk bundles which is thought to realize the QSL phase [17, 19].

Finally, we follow our non-competing discussion to compute the first order contribution to the C-function in a loop expansion, the general picture is the same in both models. The competing C-function is discontinuous as we take each anyone of the coupling constants to strong coupling. This is nothing but a reflection of the reduction of the effective number of degrees of freedom in the action.

By comparing the competing C-function with our simplified model's discussion we propose that the strong coupling fixed points we discussed are isolated, and thus inaccessible, from the free fermion fixed point. Moreover, as the process of gapping the conformal degrees of freedom is thought to be continuous, the physics near the strong coupling fixed point should essentially the same. In this way, the theory near the fixed point 4 is essentially gapped, and the QSL phase is stable against fluctuations.

Furthermore, developing a generalization of the T-duality (4.75) to our model would allow us to use the methods of [58, 56, 60, 63, 59, 61] to compare the β-functions with our model, even though the strong coupling fixed point is not apparent in the β-function. As a last comment, we highlight that the methods we used throughout our discussions are generally not dependent on the specific interaction we chose or the group structure of the theory. By this reasoning, generalizations to different phases of matter should be achievable.

REFERÊNCIAS

[1] GINZBURG, V. L.; LANDAU, L. D. *On the theory of superconductivity*. [S.l.]: Springer, 2009.

[2] BERNEVIG, B. A. Topological insulators and topological superconductors. In: *Topological Insulators and Topological Superconductors*. [S.l.]: Princeton university press, 2013.

[3] LANDAU, L. D. On the theory of phase transitions. *Zh. Eksp. Teor. Fiz.*, v. 7, p. 19–32, 1937.

[4] MUSSARDO, G. *Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics*. OUP Oxford, 2010. (Oxford Graduate Texts). ISBN 9780199547586. Disponível em: <https://books.google.com.br/books?id=fakVDAAAQBAJ>.

[5] KALMEYER, V.; LAUGHLIN, R. Equivalence of the resonating-valence-bond and fractional quantum hall states. *Physical review letters*, APS, v. 59, n. 18, p. 2095, 1987.

[6] WEN, X.-G.; WILCZEK, F.; ZEE, A. Chiral spin states and superconductivity. *Physical Review B*, APS, v. 39, n. 16, p. 11413, 1989.

[7] WEN, X.-G. Vacuum degeneracy of chiral spin states in compactified space. *Physical Review B*, APS, v. 40, n. 10, p. 7387, 1989.

[8] TSUI, D. C.; STORMER, H. L.; GOSSARD, A. C. Two-dimensional magnetotransport in the extreme quantum limit. *Physical Review Letters*, APS, v. 48, n. 22, p. 1559, 1982.

[9] WEN, X.-G.; NIU, Q. Ground-state degeneracy of the fractional quantum hall states in the presence of a random potential and on high-genus riemann surfaces. *Physical Review B*, APS, v. 41, n. 13, p. 9377, 1990.

[10] WEN, X.-G. Topological orders in rigid states. *International Journal of Modern Physics B*, World Scientific, v. 4, n. 02, p. 239–271, 1990.

[11] KESKI-VAKKURI, E.; WEN, X.-G. The ground state structure and modular transformations of fractional quantum hall states on a torus. *International Journal of Modern Physics B*, World Scientific, v. 7, n. 25, p. 4227–4259, 1993.

[12] MARINO, E. C. *Quantum field theory approach to condensed matter physics*. [S.l.]: Cambridge University Press, 2017.

[13] ZENG, B. et al. *Quantum information meets quantum matter*. [S.l.]: Springer, 2019.

[14] REDLICH, A. N. Gauge noninvariance and parity nonconservation of three-dimensional fermions. *Physical Review Letters*, APS, v. 52, n. 1, p. 18, 1984.

[15] NIEMI, A. J.; SEMENOFF, G. W. Axial-anomaly-induced fermion fractionization and effective gauge-theory actions in odd-dimensional space-times. *Physical Review Letters*, APS, v. 51, n. 23, p. 2077, 1983.

[16] QI, X.-L.; HUGHES, T. L.; ZHANG, S.-C. Chiral topological superconductor from the quantum hall state. *Physical Review B*, APS, v. 82, n. 18, p. 184516, 2010.

[17] HERNASKI, C. A.; GOMES, P. R. Effective Theories for 2+1 Dimensional Non-Abelian Topological Spin Liquids. *JHEP*, v. 10, p. 021, 2017.

[18] HERNASKI, C. A.; GOMES, P. R. Chiral edge states in (2+ 1)-dimensional topological phases. *International Journal of Modern Physics A*, World Scientific, v. 34, n. 29, p. 1950171, 2019.

[19] HUANG, P.-H. et al. Non-Abelian topological spin liquids from arrays of quantum wires or spin chains. *Physical Review B*, v. 93, n. 20, p. 205123, maio 2016. ISSN 2469-9950, 2469-9969. ArXiv: 1601.01094.

[20] FU, L.; KANE, C. L. Superconducting proximity effect and majorana fermions at the surface of a topological insulator. *Physical review letters*, APS, v. 100, n. 9, p. 096407, 2008.

[21] LAW, K. T.; LEE, P. A.; NG, T. K. Majorana fermion induced resonant andreev reflection. *Physical review letters*, APS, v. 103, n. 23, p. 237001, 2009.

[22] TANAKA, Y.; YOKOYAMA, T.; NAGAOSA, N. Manipulation of the majorana fermion, andreev reflection, and josephson current on topological insulators. *Physical review letters*, APS, v. 103, n. 10, p. 107002, 2009.

[23] CHUNG, S. B. et al. Conductance and noise signatures of majorana backscattering. *Physical Review B*, APS, v. 83, n. 10, p. 100512, 2011.

[24] MOURIK, V. et al. Signatures of majorana fermions in hybrid superconductor-semiconductor nanowire devices. *Science*, American Association for the Advancement of Science, v. 336, n. 6084, p. 1003–1007, 2012.

[25] LIU, J. et al. Zero-bias peaks in the tunneling conductance of spin-orbit-coupled superconducting wires with and without majorana end-states. *Physical review letters*, APS, v. 109, n. 26, p. 267002, 2012.

[26] NADJ-PERGE, S. et al. Observation of majorana fermions in ferromagnetic atomic chains on a superconductor. *Science*, American Association for the Advancement of Science, v. 346, n. 6209, p. 602–607, 2014.

[27] WANG, J. et al. Chiral topological superconductor and half-integer conductance plateau from quantum anomalous hall plateau transition. *Physical Review B*, APS, v. 92, n. 6, p. 064520, 2015.

[28] HE, Q. L. et al. Chiral majorana edge state in a quantum anomalous hall insulator-superconductor structure. *arXiv preprint arXiv:1606.05712*, 2016.

[29] KITAEV, A. Anyons in an exactly solved model and beyond. *Annals of Physics*, Elsevier, v. 321, n. 1, p. 2–111, 2006.

[30] NAYAK, C. et al. Non-abelian anyons and topological quantum computation. *Reviews of Modern Physics*, APS, v. 80, n. 3, p. 1083, 2008.

[31] SARMA, S. D.; FREEDMAN, M.; NAYAK, C. Majorana zero modes and topological quantum computation. *npj Quantum Information*, Nature Publishing Group, v. 1, n. 1, p. 1–13, 2015.

[32] MARRA, P. Majorana nanowires for topological quantum computation. *Journal of Applied Physics*, AIP Publishing LLC, v. 132, n. 23, p. 231101, 2022.

[33] RYU, S. et al. Topological insulators and superconductors: tenfold way and dimensional hierarchy. *New Journal of Physics*, IOP Publishing, v. 12, n. 6, p. 065010, 2010.

[34] CHIU, C.-K. et al. Classification of topological quantum matter with symmetries. *Reviews of Modern Physics*, APS, v. 88, n. 3, p. 035005, 2016.

[35] CHAMON, C. et al. Quantizing majorana fermions in a superconductor. *Physical Review B*, APS, v. 81, n. 22, p. 224515, 2010.

[36] WITTEN, E. Three lectures on topological phases of matter. *La Rivista del Nuovo Cimento*, Springer, v. 39, n. 7, p. 313–370, 2016.

[37] CAYSSOL, J. Introduction to dirac materials and topological insulators. *Comptes Rendus Physique*, Elsevier, v. 14, n. 9-10, p. 760–778, 2013.

[38] WEN, X.-G. Topological orders and edge excitations in fractional quantum Hall states. *Advances in Physics*, Taylor & Francis, v. 44, n. 5, p. 405–473, out. 1995. ISSN 0001-8732.

[39] ALVES, V. S. et al. Chern-simons terms in lifshitz-like quantum electrodynamics. *Physical Review D*, APS, v. 88, n. 6, p. 067703, 2013.

[40] QI, X.-L.; WU, Y.-S.; ZHANG, S.-C. Topological quantization of the spin hall effect in two-dimensional paramagnetic semiconductors. *Physical Review B*, APS, v. 74, n. 8, p. 085308, 2006.

[41] GOMES, P. R.; GOMES, M. Ward identities in lifshitz-like field theories. *Physical Review D*, APS, v. 85, n. 6, p. 065010, 2012.

[42] AMANO, K.; SHIROKURA, H. O (2) chern–simons gauge theory and z 2 orbifolds. *International Journal of Modern Physics A*, World Scientific, v. 7, n. 05, p. 1007–1023, 1992.

[43] FRADKIN, E.; HUERTA, M.; ZEMBA, G. R. Effective chern–simons theories of pfaffian and parafermionic quantum hall states, and orbifold conformal field theories. *Nuclear Physics B*, Elsevier, v. 601, n. 3, p. 591–606, 2001.

[44] BARKESHLI, M.; WEN, X.-G. U (1)$\times u(1) \times z_2$chern-simons theory and z 4 parafermion fractional quantum hall states. *Physical Review B*, APS, v. 81, n. 4, p. 045323, 2010.

[45] MOORE, G. W.; SEIBERG, N. Taming the conformal zoo. *Phys. Lett. B*, v. 220, p. 422–430, 1989.

[46] DIJKGRAAF, R. et al. The operator algebra of orbifold models. *Communications in Mathematical Physics*, Springer, v. 123, n. 3, p. 485–526, 1989.

[47] VAEZI, A. Fractional topological superconductor with fractionalized majorana fermions. *Physical Review B*, APS, v. 87, n. 3, p. 035132, 2013.

[48] TEO, J. C. Y.; KANE, C. L. From luttinger liquid to non-abelian quantum hall states. *Physical Review B*, American Physical Society, v. 89, n. 8, p. 085101, fev. 2014.

[49] MORENO, E.; SCHAPOSNIK, F. Issues of symmetries in the Gross-Neveu model. *International Journal of Modern Physics A*, World Scientific Publishing Co., v. 04, n. 11, p. 2827–2835, jul. 1989. ISSN 0217-751X.

[50] FURUYA, K.; SARAVI, R. G.; SCHAPOSNIK, F. Path Integral Formulation of Chiral Invariant Fermion Models in Two-dimensions. *Nucl. Phys. B*, v. 208, p. 159–181, 1982.

[51] JACKIW, R.; RAJARAMAN, R. Vector meson mass generation by chiral anomalies. *Physical Review Letters*, American Physical Society, v. 54, n. 12, p. 1219–1221, mar. 1985.

[52] GOGOLIN, A. O.; NERSESYAN, A. A.; TSVELIK, A. M. *Bosonization and Strongly Correlated Systems*. [S.l.]: Cambridge University Press, 2004. ISBN 9780521617192.

[53] CABRA, D.; MORENO, E.; REICHENCACH, C. von. Conformally invariant constrained fermion models. *International Journal of Modern Physics A*, World Scientific Publishing Co., v. 05, n. 12, p. 2313–2330, jun. 1990. ISSN 0217-751X.

[54] ZOMOLODCHIKOV, A. B. "Irreversibility" of the flux of the renormalization group in a 2D field theory. *Soviet Journal of Experimental and Theoretical Physics Letters*, v. 43, p. 730, jun. 1986. ISSN 0021-3640.

[55] LEUTWYLER, H.; SHIFMAN, M. Perturbation theory in the Wess-Zumino-Novikov-Witten model. *International Journal of Modern Physics A*, World Scientific Publishing Co., v. 07, n. 04, p. 795–841, fev. 1992. ISSN 0217-751X.

[56] GEORGIOU, G.; SFETSOS, K. Integrable flows between exact cfts. *Journal of High Energy Physics*, Springer, v. 2017, n. 11, p. 1–22, 2017.

[57] LECLAIR, A. Chiral stabilization of the renormalization group for flavor and color anisotropic current interactions. *Physics Letters B*, Elsevier, v. 519, n. 1-2, p. 183–187, 2001.

[58] GEORGIOU, G.; SFETSOS, K.; SIAMPOS, K. λ-deformations of left–right asymmetric cfts. *Nuclear Physics B*, Elsevier, v. 914, p. 623–641, 2017.

[59] GEORGIOU, G. et al. An exact symmetry in λ-deformed cfts. *Journal of High Energy Physics*, Springer, v. 2020, n. 1, p. 1–21, 2020.

[60] KUTASOV, D. Duality off the critical point in two-dimensional systems with non abelian symmetries. *Physics Letters B*, Elsevier, v. 233, n. 3-4, p. 369–373, 1989.

[61] GEORGIOU, G.; SFETSOS, K.; SIAMPOS, K. All-loop anomalous dimensions in integrable λ-deformed σ-models. *Nuclear Physics B*, Elsevier, v. 901, p. 40–58, 2015.

[62] GINSPARG, P. H. APPLIED CONFORMAL FIELD THEORY. In: *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena*. [S.l.: s.n.], 1988.

[63] GEORGIOU, G.; SFETSOS, K.; SIAMPOS, K. Double and cyclic λ-deformations and their canonical equivalents. *Physics Letters B*, Elsevier, v. 771, p. 576–582, 2017.

[64] SAVARY, L.; BALENTS, L. Quantum spin liquids: a review. *Reports on Progress in Physics*, IOP Publishing, v. 80, n. 1, p. 016502, 2016.

[65] BROHOLM, C. et al. Quantum spin liquids. *Science*, American Association for the Advancement of Science, v. 367, n. 6475, p. eaay0668, 2020.

[66] WITTEN, E. Quantum field theory and the jones polynomial. *Commun. Math. Phys.*, v. 121, p. 351–399, 1989.

[67] WEN, X. G. Chiral luttinger liquid and the edge excitations in the fractional quantum hall states. *Phys. Rev. B*, v. 41, p. 12838–12844, 1990.

[68] WEN, X. G. Edge excitations in the fractional quantum hall states at general filling fractions. *Modern Physics Letters B*, v. 5, n. 1, p. 39–46, jan. 1991.

[69] FRADKIN, E. et al. A chern-simons effective field theory for the pfaffian quantum hall state. *Nuclear Physics B*, Elsevier, v. 516, n. 3, p. 704–718, 1998.

[70] CABRA, D. C. et al. Non-abelian fractional quantum hall states and chiral coset conformal field theories. *International Journal of Modern Physics A*, World Scientific, v. 15, n. 30, p. 4857–4870, 2000.

[71] BLASI, A. et al. Non-abelian bf theory for 2+ 1 dimensional topological states of matter. *New Journal of Physics*, IOP Publishing, v. 14, n. 1, p. 013060, 2012.

[72] KANE, C. L.; MUKHOPADHYAY, R.; LUBENSKY, T. C. Fractional Quantum Hall Effect in an Array of Quantum Wires. *Physical Review Letters*, American Physical Society, v. 88, n. 3, p. 036401, jan. 2002.

[73] KLINOVAJA, J.; TSERKOVNYAK, Y.; LOSS, D. Integer and fractional quantum anomalous Hall effect in a strip of stripes model. *Physical Review B*, American Physical Society, v. 91, n. 8, p. 085426, fev. 2015.

[74] FONTANA, W. B.; GOMES, P. R. S.; HERNASKI, C. A. From quantum wires to the Chern-Simons description of the fractional quantum Hall effect. *Physical Review B*, American Physical Society, v. 99, n. 20, p. 201113, maio 2019.

[75] NEUPERT, T. et al. Wire deconstructionism of two-dimensional topological phases. *Physical Review B*, American Physical Society, v. 90, n. 20, p. 205101, nov. 2014.

[76] FRANCESCO, P. D.; MATHIEU, P.; SÉNÉCHAL, D. *Conformal Field Theory*. Island Press, 1996. (Graduate texts in contemporary physics). ISBN 9781461222576. Disponível em: <https://books.google.com.br/books?id=mcMbswEACAAJ>.

[77] MUDRY, C. Lecture notes on field theory in condensed matter physics. 04 2014.

[78] IMAMURA, Y.; TOTSUKA, K.; HANSSON, T. H. From coupled-wire construction of quantum hall states to wave functions and hydrodynamics. *Physical Review B*, APS, v. 100, n. 12, p. 125148, 2019.

[79] TOLEDO, J. et al. Quantum wires, chern-simons, and dualities in the quantum hall system. *arXiv preprint arXiv:2205.08488*, 2022.

[80] COLEMAN, S.; HILL, B. No more corrections to the topological mass term in qed3. *Physics Letters B*, Elsevier, v. 159, n. 2-3, p. 184–188, 1985.

Apêndices

APÊNDICE A – COLEMAN-HILL THEOREM

One important question on the one-loop generation of the CS term we described in Sec. 3.1 concerns its stability. We discuss now that the topological CS term so obtained is protected against higher-order radiative corrections when the gauge field is dynamical. Actually, this result extends the Coleman-Hill theorem which asserts that in a Lorentz-invariant setting and in the absence of infrared singularities the CS term does not have corrections beyond the one-loop contribution [80]. To establish this result for our case, we consider the $U(1)$ conserved current following from (3.2). Its components are given in (3.20), which we repeat here for convenience,

$$J^0 = \bar{\psi}\gamma^0\psi \quad \text{and} \quad J^i = b_1\bar{\psi}\gamma^i\psi + ib_2(\bar{\psi}\gamma^i\gamma^j\partial_j\psi - \partial_j\bar{\psi}\gamma^j\gamma^i\psi) + 2b_2\bar{\psi}\psi A^i. \tag{A.1}$$

The corresponding Ward identities following from its conservation can be derived with the help of the algebraic relation

$$\begin{aligned} p_0\gamma^0 + b_1 p_i\gamma^i + b_2(p_ip^i + 2k_ip^i + 2p_iA^i) &= (p_0+k_0)\gamma^0 + b_1(p_i+k_i)\gamma^i - b_2(p+k)^2 - m_0 \\ &- (k_0\gamma^0 + b_1k_i\gamma^i + b_2k_ik^i - m_0) + 2b_2p_iA^i \\ &= iS^{-1}(k+p) - iS^{-1}(k) + 2b_2p_iA^i, \end{aligned} \tag{A.2}$$

which appears in the current vertex whenever it is contracted with the external momentum p_μ entering at that vertex. Indeed, by applying this identity to the set of closed fermionic loop graphs with $N > 2$ amputated external gauge field lines, denoted by $\Gamma_{\mu_1\ldots}(p_1, p_2, \ldots, p_{N-1})$, we get

$$p_1^{\mu_1}\Gamma_{\mu_1\ldots}(p_1, p_2, \ldots p_{N-1}) = 0. \tag{A.3}$$

Taking a derivative of this expression with respect to $p_1^{\mu_1}$ and then setting $p_1^{\mu_1} = 0$ leads to $\Gamma_{\mu_1\ldots}(0, p_2, \ldots) = 0$. This, in turn, implies that $\Gamma_{\mu_1\ldots}(p_1, p_2, \ldots, p_{N-1}) = O(p_1p_2\ldots p_{N-1})$.

We then proceed as in [80] by considering the case $N = 2$. If the two trilinear gauge vertices belong to different loops then the result is $O(p^2)$ and no CS term is generated. The remaining possibility is that the two external vertices belong to the same closed fermionic loop with some internal gauge field lines. Here, by cutting the internal lines, we can put this graph in correspondence with another graph without internal lines and with independent external momenta (up to the global momentum conservation). The original graph is obtained as a limit process by identifying gauge field lines and multiplying its analytical expression by the corresponding propagators. As before, we conclude that the sum of the graphs of this type is $O(p^2)$ so that no CS term is generated. Therefore, beyond the one-loop graphs with two external lines and without internal lines, there is no contribution to the CS term.

APÊNDICE B – COMPACT BOSON, CHIRAL ALGEBRA EXTENSION, AND ORBIFOLD

This appendix is meant to be a supplementary material to the text, covering topics that are well discussed in the CFT literature in a concise and unified way [76, 46, 62] (we follow mainly the conventions of [76]). We start by discussing the compact boson and its action. Then, by summing over all inequivalent topological configurations we obtain the compact boson partition function, from which we read the characters of the representations. Furthermore, we extend the algebra introducing a $\mathbb{Z}_{2N}$ structure. This process reorganizes the infinite families of the Virasoro algebra into a finite number of families in the extended algebra.

B.1 Chiral Algebra Extension

On the torus there are two winding directions, such that we consider the following compactification condition,

$$\varphi(z+n\omega_1+n'\omega_2)=\varphi(z)+2\pi R\left(nm+n'm'\right), \qquad n,n',m,m'\in\mathbb{Z}, \tag{B.1}$$

where R and ω_i are the compactification radius and directions of the torus in complex coordinates. The indices m and m' specify inequivalent topological classes of configurations. The boson integration may be done by separating the boson field into a topologically nontrivial part $\varphi_{m,m'}$ and a periodic part $\tilde{\varphi}$, i.e., $\varphi=\varphi_{m,m'}+\tilde{\varphi}$, where

$$\varphi_{m,m'}=2\pi R\left[\frac{z}{\omega_1}\frac{m\bar{\tau}-m'}{\bar{\tau}-\tau}-\frac{\bar{z}}{\omega_1^*}\frac{m\tau-m'}{\bar{\tau}-\tau}\right] \tag{B.2}$$

is compatible with (B.1) and $\tau\equiv\omega_2/\omega_1$ is the modular parameter. Since $\partial_z\partial_{\bar{z}}\varphi_{m,m'}=0$, the action decomposes into $S[\varphi]=S[\varphi_{m,m'}]+S[\tilde{\varphi}]$ and we can write the partition function for each topological class as

$$Z_{m,m'}(\tau)=Z_{per}e^{-S[\varphi_{m,m'}]}, \tag{B.3}$$

where Z_{per} is the result of the periodic boson integration and the remaining action is given by

$$\begin{aligned}S[\varphi_{m,m'}]&=\frac{1}{2\pi}\int dz d\bar{z}\partial_z\varphi_{m,m'}\partial_{\bar{z}}\varphi_{m,m'}\\&=\pi R^2\frac{|m\tau-m'|^2}{2\mathrm{Im}\tau}.\end{aligned} \tag{B.4}$$

Under the action of the generators of modular invariance,

$$\mathcal{T}:\tau\to\tau+1 \qquad \text{and} \qquad \mathcal{S}:\tau\to-\frac{1}{\tau}, \tag{B.5}$$

the partition functions (B.3) transform among themselves:

$$Z_{m,m'}(\tau+1) = Z_{m,m'-m}(\tau), \tag{B.6}$$

$$Z_{m,m'}(-1/\tau) = Z_{-m',m}. \tag{B.7}$$

As this only amounts to a redefinition of the indices, the complete partition function,

$$Z = Z_{per} \sum_{m,m'\in\mathbb{Z}} e^{-\pi R^2 \frac{|m\tau - m'|^2}{2\mathrm{Im}\tau}}, \tag{B.8}$$

is modular invariant.

In order to identify the Virasoro characters from the partition function, it is useful to recast it in a manner that better reflects its holomorphic separation. To this end, we apply the Poisson resummation formula, which states that for two sums over the integers

$$\sum_{m'\in\mathbb{Z}} f(m') = \sum_{n\in\mathbb{Z}} \tilde{f}_n, \tag{B.9}$$

where $\tilde{f}_k$ is the Fourier transform of $f(x)$,

$$\tilde{f}_n = \int_{-\infty}^{\infty} \mathrm{d}x\, f(x) e^{-2\pi i x n}. \tag{B.10}$$

Applied to the compact boson partition function, the resummation formula yields the familiar expression for the compact boson partition function

$$Z = \sum_{n,m\in\mathbb{Z}} \chi_{n,m}(q)\bar{\chi}_{n,m}(\bar{q}), \tag{B.11}$$

where the Virasoro characters

$$\chi_{n,m}(q) = \frac{1}{\eta(q)} q^{(n/R+mR/2)^2/2} \quad \text{and} \quad \bar{\chi}_{n,m}(\bar{q}) = \frac{1}{\bar{\eta}(\bar{q})} \bar{q}^{(n/R-mR/2)^2/2}, \tag{B.12}$$

with $q \equiv e^{2\pi i\tau}$ and $\bar{q} \equiv e^{-2\pi i\bar{\tau}}$, are associated with the conformal families of the theory. Each family contains an infinite number of conformal fields, which are generated by successive application of the positive modes of the energy-momentum tensor on the primary field of the family. As expected for a free scalar CFT, the primary fields are the vertex operators

$$V_{n,m} = e^{i(n/R+mR/2)\varphi} \qquad \text{and} \qquad \bar{V}_{n,m} = e^{i(n/R-mR/2)\bar{\varphi}}, \tag{B.13}$$

with conformal dimension

$$h_{n,m} = \frac{1}{2}(n/R+mR/2)^2 \qquad \text{and} \qquad \bar{h}_{n,m} = \frac{1}{2}(n/R-mR/2)^2. \tag{B.14}$$

At this point we can also infer the invariance of the partition function under $R \to 2/R$, as it amounts to the exchange $n \leftrightarrow m$.

So far the partition function in Eq. (B.11) embodies only the Virasoro algebra. In order to add the $\mathbb{Z}_{2N}$ algebra, we need to perform an extension of the Virasoro algebra. To do so, we restrict the compactification radius according to

$$\frac{R^2}{2} = \frac{p}{p'} \tag{B.15}$$

with p, p' natural coprimes and introduce the new indices

$$\begin{aligned} n &= 2pn' + r, \quad & 0 \leq r \leq 2p - 1, \quad & r, n' \in \mathbb{Z}; \\ m &= 2p'm' + s, \quad & 0 \leq s \leq 2p' - 1, \quad & s, m' \in \mathbb{Z}; \end{aligned} \tag{B.16}$$

such that now we sum over the integers

$$\begin{aligned} u &= n' + m', \quad & u \in \mathbb{Z}; \\ l &= p'r + ps, \quad & l \in \mathbb{Z}. \end{aligned} \tag{B.17}$$

In terms of the new indices the partition function reads

$$Z = \sum_{l,\bar{l}} \chi_l(q) \bar{\chi}_{\bar{l}}(\bar{q}), \tag{B.18}$$

where the limits of the sum over l are yet to be specified and the extended algebra characters are given by

$$\chi_l(q) = \frac{1}{\eta(q)} \sum_{u \in \mathbb{Z}} q^{N(u+l/2N)^2}, \tag{B.19}$$

where $N \equiv pp'$. This is the maximal extension of the algebra.

From the definition of the extended character (B.19) it is easy to derive that $\chi_l = \chi_{l+2N}$. As we associate each character to a primary field, there are $2N$ primary vertex fields, given by

$$V_l = e^{il\varphi/\sqrt{2N}}. \tag{B.20}$$

Consequently, two vertex fields, V_l and V_{l+2N}, must necessarily belong to the same Verma module and there must exist operators which connects them. This is analogous to the role played by the L_{-n} in the Virasoro algebra. We find the ladder operators to be

$$\Gamma_\pm = e^{\pm i\sqrt{2N}\varphi}, \tag{B.21}$$

with conformal dimensions $h_{\Gamma_\pm} = N$.

It might be useful to extend the notion of a primary field of the extended algebra $\mathcal{A}$ by requiring that it must be annihilated by all the positive modes of the currents that generate the corresponding algebra. In terms of OPE, this is equivalent to defining a primary field of $\mathcal{A}$ to have the OPE

$$\mathcal{J}(z)\Phi(0) = z^{-h_{\mathcal{J}}}\Phi(0) + \text{less singular}, \tag{B.22}$$

where $\mathcal{J}$ is an algebra generating current. This equation should be understood in the sense that the OPE of $\mathcal{J}$ with a primary field has a *maximum* allowed singularity; i.e., less singular OPEs are allowed. This is nothing else than the Virasoro primary field condition generalized to an extended algebra. In this more familiar case, the generators of the algebra are the holomorphic and antiholomorphic parts of the energy momentum tensor, $T(z) \equiv T_{zz}(z)$ and $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})$, with conformal dimensions $h = 2$ and $\bar{h} = 2$, respectively.

The maximal extension is equivalent to choosing the $U(1)$ current j, the ladder operators $\Gamma_{\pm}$, and the energy-momentum tensor $T = T_{z,z}(z)$ to be the set of algebra generating currents $\mathcal{J}$. Applying the primary field condition (B.22) to the ladder operators, it follows that

$$\begin{aligned} \Gamma_{\pm}(z)V_l(0) &= e^{\pm i\sqrt{2N}\varphi(z)}e^{il\varphi(0)/\sqrt{2N}} = e^{i(l\pm 2N)\varphi(0)/\sqrt{2N}}e^{\mp l\langle\phi(z)\phi(0)\rangle} \\ &= z^{\pm l}V_{l\pm 2N}(0). \end{aligned} \tag{B.23}$$

Comparing with the primary field OPE, we see that the spectrum of primary fields is given by the V_l with $-(N-1) \leq l \leq N$ and the $U(1)$ partition function reads

$$Z_{U(1)} = \sum_{l,\bar{l}=-(N-1)}^{N} \chi_l(q)\bar{\chi}_{\bar{l}}(\bar{q}). \tag{B.24}$$

It is standard in the literature to choose the range of l in the definition of the primaries, Eq. (B.20), to be $l = 0, 1, 2, \cdots, 2N-1$. We use this convention in the main body of this paper.

B.2 Orbifold

Now that we have considered the extension of the chiral algebra we are in a position to study the $U(1)/\mathbb{Z}_2$ orbifold. Such a theory is obtained by considering only field configurations of $U(1)$ that are invariant under the $\mathbb{Z}_2$ group action, $\varphi \to -\varphi$. To do so, we extend the scalar field compactification condition such that it admits twists when going around the torus

$$\varphi\left(z + n\omega_1 + n'\omega_2\right) = e^{2\pi i(nv+n'u)}\varphi(z), \tag{B.25}$$

where $v, u = 0, \frac{1}{2}$ for periodic and antiperiodic boundary conditions, respectively. At this point it is convenient to identify the windings to be oriented along one of the

Cartesian directions. We choose ω_1 to represent winding around the space direction and ω_2 around time.

To each pair v, u, we associate a partition function $Z_{v,u}$. The complete orbifold partition function is given by the sum of the $Z_{v,u}$. The untwisted sector, $Z_{0,0}$, corresponds to the $Z_{U(1)}$ of the previous discussion, so that we are left to calculate the partition function for the twisted sectors. For the antiperiodic condition along at least one of the torus directions, we separate the partition function into its holomorphic blocks

$$Z_{vu} = |f_{vu}|^2, \tag{B.26}$$

where the conformal blocks are given by the character

$$f_{vu} \equiv \mathrm{Tr}\, q^{L_0^{(v)} - 1/24}. \tag{B.27}$$

The superscript on $L_0^{(v)}$ means that L_0 should be taken with the corresponding boundary conditions along the space direction, specified by v.

The computation of the trace in (B.26) requires some care in the case of antiperiodic boundary conditions along the time, i.e., when $u = \frac{1}{2}$. To understand this, let us examine a correlation function with the time antiperiodic boundary condition

$$\langle T\varphi(z) X(\{z_i\}) \rangle\,, \tag{B.28}$$

where T is the time ordering and $X(\{z_i\})$ stands for the insertion of any number of boson operators at the positions z_i. Now, consider that we take φ along a continuous path from z to $z + \omega_2$. Because of the time ordering φ will pass over all the insertions on X in succession and return to the starting place. As we are dealing with bosons this operation does not pick a sign. On the other hand, under this process φ picks up a sign when $u = 1/2$. To reconcile this discrepancy, we can introduce an operator $\mathcal{G}$ on every correlation function of φ in the case of the antiperiodic boundary condition along time, where $\mathcal{G}$ is the operator that implements the $\mathbb{Z}_2$ symmetry, $\mathcal{G}\varphi\mathcal{G}^{-1} = -\varphi$ [1]. In particular, the traces with $u = \frac{1}{2}$ in the partition function (B.26) must also include the operator $\mathcal{G}$.

To illustrate the procedure, we consider the specific case of $f_{0,1/2}$. The above prescription leads us to write the holomorphic blocks of the partition function as

$$f_{0,1/2} = \mathrm{Tr}\, \mathcal{G} q^{L_0^{(0)} - 1/24} = q^{-1/24} \Pi_{n>0}\, \mathrm{tr}^{(n)}\, \mathcal{G}_n q^{a_{-n} a_n}, \tag{B.29}$$

where the trace on the right-hand side is defined to be taken over states with fixed n.

1 This discussion is similar to what happens on compact fermions, where in the time periodic case we introduce $\mathcal{G} = (-1)^F$, with F being the fermion number operator.

That is,

$$\begin{aligned}
\mathrm{tr}^{(n)}\, \mathcal{G}_n q^{a_{-n}a_n} &= \sum_{k=0}^{\infty} \langle k|\, \mathcal{G}_n q^{a_{-n}a_n}\, |k\rangle = \sum_{k=0}^{\infty} \langle 0|\, (a_n)^k\, \mathcal{G}_n q^{a_{-n}a_n}\, (a_{-n})^k\, |0\rangle \\
&= \sum_{k,m=0}^{\infty} \frac{(2\pi i\tau)^m}{m!} \langle k|\, \mathcal{G}_n\, (a_{-n}a_n)^m\, (a_{-n})^k\, |0\rangle \\
&= \sum_{k,m=0}^{\infty} \frac{(2\pi i\tau)^m}{m!} \langle k|\, \mathcal{G}_n\, (a_{-n}a_n)^{m-1}\, (nk)\, (a_{-n})^k\, |0\rangle \\
&= \sum_{k,m=0}^{\infty} \frac{(2\pi i\tau nk)^m}{m!} \langle k|\, \mathcal{G}_n\, |k\rangle = \sum_{k=0}^{\infty} (-1)^k q^{kn} \\
&= \frac{1}{1+q^n},
\end{aligned} \tag{B.30}$$

where we have used the commutation relation $\left[a_{-n}a_n, (a_{-n})^k\right] = nk\,(a_{-n})^k$ to get from the second to the third line. Returning to the conformal block, we obtain

$$f_{0,1/2} = q^{-1/24} \prod_{n=1}^{\infty} \frac{1}{1+q^n} = \frac{\sqrt{\theta_3\theta_4}}{\eta} \approx \frac{1}{\eta}\left(1 - 2q + 2q^4 - 2q^9 + 2q^{16} + \cdots\right), \tag{B.31}$$

where θ_i are the Jacobi theta functions, as defined in [76].

We calculate the next two conformal blocks in a similar way,

$$\begin{aligned}
f_{1/2,0} &= q^{1/48} \prod_{n\in\mathbb{N}+1/2} \mathrm{Tr}\, q^{a_{-n}a_n} = q^{1/48} \prod_{n\in\mathbb{N}+1/2} \sum_{N=0}^{\infty} q^{nN} = q^{1/48} \prod_{n\in\mathbb{N}+1/2} \frac{1}{1-q^n} \\
&= \frac{\sqrt{\theta_2\theta_3/2}}{\eta} \approx \frac{q^{1/16}}{\eta}\left(1 + q^{1/2} + q^{3/2} + q^3 + q^5 + q^{15/2} + \cdots\right),
\end{aligned} \tag{B.32}$$

and for the next one,

$$\begin{aligned}
f_{1/2,1/2} &= q^{1/48} \prod_{n\in\mathbb{N}+1/2} \mathrm{Tr}\, \mathcal{G}_n q^{a_{-n}a_n} = q^{1/48} \prod_{n\in\mathbb{N}+1/2} \sum_{N=0}^{\infty} (-1)^N q^{nN} = q^{1/48} \prod_{n\in\mathbb{N}+1/2} \frac{1}{1+q^n} \\
&= \frac{\sqrt{\theta_2\theta_4/2}}{\eta} \approx \frac{q^{1/16}}{\eta}\left(1 - q^{1/2} - q^{3/2} + q^3 + q^5 - q^{15/2} + \cdots\right).
\end{aligned} \tag{B.33}$$

The orbifold partition function then reads

$$Z_{orb} = \frac{1}{2}\left[Z_{U(1)} + |f_{0,1/2}|^2 + |f_{1/2,0}|^2 + |f_{1/2,1/2}|^2\right]. \tag{B.34}$$

It is important to note that these conformal blocks individually are not invariant under the modular transformations (B.5). Under $\mathcal{T}$ the conformal blocks $f_{v,u}$ transform as

$$\begin{aligned}
f_{0,1/2}(\tau+1) &= e^{-i\pi/12} f_{0,1/2}(\tau) \\
f_{1/2,0}(\tau+1) &= e^{i\pi/24} f_{1/2,1/2}(\tau) \\
f_{1/2,1/2}(\tau+1) &= e^{i\pi/24} f_{1/2,0}(\tau)
\end{aligned} \tag{B.35}$$

and under $\mathcal{S}$, $f_{v,u}(-1/\tau) = f_{u,v}(\tau)$. In this way, even though the conformal blocks themselves are not modular invariant, the total partition function is.

Notice that $f_{1/2,0}$ and $f_{1/2,1/2}$ mix under $\mathcal{T}$ and a similar thing happens with $f_{0,1/2}$ and χ_0 of the $U(1)$ partition function as given in (B.19). Furthermore, these are the only terms that can be written in the form $|f|^2$. Let us check their contribution to the partition function:

$$\begin{aligned}Z_{orb} &\supset \frac{1}{2}\left(\chi_0\bar{\chi}_0 + |f_{0,1/2}|^2 + |f_{1/2,0}|^2 + |f_{1/2,1/2}|^2\right) \\ &= \frac{1}{4}\left[(\chi_0 + f_{0,1/2})\left(\bar{\chi}_0 + \bar{f}_{0,1/2}\right) + (\chi_0 - f_{0,1/2})\left(\bar{\chi}_0 - \bar{f}_{0,1/2}\right) + \right. \\ &\left. + \left(f_{1/2,0} + f_{1/2,1/2}\right)\left(\bar{f}_{1/2,0} + \bar{f}_{1/2,1/2}\right) + \left(f_{1/2,0} - f_{1/2,1/2}\right)\left(\bar{f}_{1/2,0} - \bar{f}_{1/2,1/2}\right)\right].\end{aligned} \tag{B.36}$$

This motivates us to define the orbifold characters and their associated primary fields as

$$\begin{array}{lll} \mathbb{1} : & \chi_{\mathbb{1}}^{orb} \equiv (\chi_0 + f_{0,1/2})/2 \approx \eta^{-1}\left(1 - q + q^2 + q^4 + \cdots\right), & h_{\mathbb{1}} = 0; \\ j : & \chi_j^{orb} \equiv (\chi_0 - f_{0,1/2})/2 \approx \eta^{-1} q\left(1 + q - q^3 + q^7 + \cdots\right), & h_j = 1; \\ \sigma : & \chi_\sigma^{orb} \equiv \left(f_{1/2,0} + f_{1/2,0}\right)/2 \approx \eta^{-1} q^{\frac{1}{16}}\left(1 + q^3 + q^5 + q^{14} + \cdots\right), & h_\sigma = \frac{1}{16}; \\ \tau : & \chi_\tau^{orb} \equiv \left(f_{1/2,0} - f_{1/2,0}\right)/2 \approx \eta^{-1} q^{\frac{9}{16}}\left(1 + q + q^7 + q^{10} + \cdots\right), & h_\tau = \frac{9}{16}; \end{array} \tag{B.37}$$

where the expansions are given for $N = 2$ and the antiholomorphic sector characters are defined similarly. With the new characters, this contribution to the partition function can be rewritten as

$$Z_{orb} \supset \left|\chi_{\mathbb{1}}^{orb}\right|^2 + \left|\chi_j^{orb}\right|^2 + \left|\chi_\sigma^{orb}\right|^2 + \left|\chi_\tau^{orb}\right|^2. \tag{B.38}$$

The remaining terms in the orbifold partition function are the $U(1)$ characters, $\chi_{k\neq 0}$. Let us consider

$$Z_{orb} \supset \frac{1}{2}\sum_{k=-(N-1)}^{N-1}{}' \sum_{\bar{k}=-(N-1)}^{N-1}{}' \chi_k\bar{\chi}_{\bar{k}} = 2\sum_{k,\bar{k}=1}^{N-1} \chi_k\bar{\chi}_{\bar{k}} \equiv 2\sum_{k,\bar{k}=1}^{N-1} \chi_k^{orb}\bar{\chi}_{\bar{k}}^{orb}, \tag{B.39}$$

where the primes on the sums mean that the $k, \bar{k} = 0$ are excluded. We associate with these characters the $N-1$ primary fields

$$\phi_k = \cos\left(\frac{k}{\sqrt{2N}}\varphi\right), \qquad h_k = \frac{k^2}{4N} \qquad \text{for} \qquad k = 1, 2, \cdots, N-1. \tag{B.40}$$

For the remaining $k = N$ character, we define

$$\phi_N^i : \; \chi_N^{orb} \equiv \frac{1}{2}\chi_N \approx \eta^{-1} q^{1/2}\left(1 + q^4 + q^{12} + q^{24} + \cdots\right), \qquad h_{\phi_i^N} = \frac{N}{4}, \tag{B.41}$$

where the expansion is given for $N = 2$. Its contribution to the partition function reads

$$Z_{orb} \supset 2\chi_N^{orb}\bar{\chi}_N^{orb}. \qquad \text{(No index sum).} \tag{B.42}$$

Thus, the $U(1)/Z_2$ orbifold partition function reads

$$Z_{orb} = \left|\chi_{\mathbb{1}}^{orb}\right|^2 + \left|\chi_j^{orb}\right|^2 + \left|\chi_\sigma^{orb}\right|^2 + \left|\chi_\tau^{orb}\right|^2 + 2\sum_{k,\bar{k}=1}^{N} \chi_k^{orb}\bar{\chi}_k^{orb}. \tag{B.43}$$

Our final task is to determine the fusion rules of the model. One way to do so is from the Verlinde formula

$$N_{i,j}^k = \sum_n \frac{S_{i,n}S_{j,n}S_{k,n}}{S_{\mathbb{1},n}}, \tag{B.44}$$

where $N_{i,j}^k$ is the fusion coefficients [2]

$$\phi_i \times \phi_j = \sum_k N_{i,j}^k \phi_k \tag{B.45}$$

and $S_{i,j}$ determines how the characters behave under the modular transformation $\mathcal{S}$, namely,

$$\chi_i' = \chi_i(-\frac{1}{\tau}) = \sum_j S_{i,j}\chi_j(\tau). \tag{B.46}$$

We will consider the case of even N, which is relevant for the discussion of the Sec. 3.2 (the case of odd N can be similarly constructed [46]). Let us discuss the transformation of some characters starting with the identity

$$\chi_{\mathbb{1}}' = \frac{1}{\sqrt{8N}}\left[\chi_{\mathbb{1}} + \chi_j + 2\sum_{k=1}^{N-1}\chi_k + \chi_N^i + 2\sqrt{N}\chi_\sigma + 2\sqrt{N}\chi_\tau\right]. \tag{B.47}$$

The factor of two for the σ and τ representations should be understood as a reflection of the fact that there are two representations for σ and τ. Therefore, when building the S matrix, the contribution for the twists should read $\sqrt{N}\left[\chi_{\sigma^1} + \chi_{\sigma^2} + \chi_{\tau^1} + \chi_{\tau^2}\right]$, where the symmetry of the splitting between σ^1 and σ^2 is a consequence of the unitarity of the S matrix.

Let us proceed to the next one. A naive analysis of the modular transformation leads us to

$$\chi_N^{i\prime} = \frac{1}{\sqrt{8N}}\left[\chi_{\mathbb{1}} + \chi_j + 2\sum_{k=1}^{N-1}(-1)^k\chi_k + (-1)^N\chi_N\right], \tag{B.48}$$

[2] In general the fusion matrices as defined in (B.44) and (B.45) are not precisely the same, but are related by a raising and lowering matrix, which in the present case is diagonal [76].

but this is still not correct. This modular transformation, as is presented, hides the twist characters. In fact, it only tells us that the sum of the contributions for the characters should cancel out, similarly to the last calculation. In this manner, the contributions for χ_{σ_1} should cancel out the contribution for χ_{σ_2}, and similarly for χ_{τ_i}. We implement this by adding to the brackets above the term $x\sigma_{i,j}\left(\chi_{\sigma_j}+\chi_{\tau_j}\right)$, where x is determined by demanding that the S matrix must be unitary and $\sigma_{i,j}\equiv 2\delta_{i,j}-1$. Similar considerations are needed for the transformation of χ_k, χ_σ and χ_τ, after which we obtain the S matrix for even N [3]

	$\mathbb{1}$	j	ϕ_N^i	ϕ_k	σ_i	τ_i
$\mathbb{1}$	1	1	1	2	$\sqrt{N}$	$\sqrt{N}$
j	1	1	1	2	$-\sqrt{N}$	$-\sqrt{N}$
ϕ_N^j	1	1	1	$2(-1)^k$	$\sigma_{i,j}\sqrt{N}$	$\sigma_{i,j}\sqrt{N}$
$\phi_{k'}$	2	2	$2(-1)^{k'}$	$4\cos\left(\frac{\pi k k'}{N}\right)$	0	0
σ_j	$\sqrt{N}$	$-\sqrt{N}$	$\sigma_{i,j}\sqrt{N}$	0	$\sqrt{2N}\delta_{i,j}$	$-\sqrt{2N}\delta_{i,j}$
τ_j	$\sqrt{N}$	$-\sqrt{N}$	$\sigma_{i,j}\sqrt{N}$	0	$-\sqrt{2N}\delta_{i,j}$	$\sqrt{2N}\delta_{i,j}$

,

where we have factored out $(8N)^{-1/2}$.

With the S matrix at hand, we only need to run the Verlinde formula to find the fusion rules:

$$\begin{gathered} j\times j=1,\quad \phi_N^i\times\phi_N^i=1,\quad \phi_k\times\phi_{k'}=\phi_{k+k'}+\phi_{k-k'}, \\ j\times\phi_k=\phi_k,\quad j\times\sigma_i=\tau_i\ \text{ and }\ \phi_k\times\phi_k=1+j+\phi_{2k}. \end{gathered} \tag{B.49}$$

[3] We call attention to a typo in the $\phi_k,\phi_{k'}$ entry of this table in Ref. [46]. Nonetheless, the fusion rules are correct.

APÊNDICE C – EFFECTIVE ACTION REGULARIZATION

Here we want to discuss the procedure to regularize the effective action (4.67). This section is intended to familiarize the reader with the methods we used in a convenient way, and is largely based in previous works, namely [55, 76]. Before we do that, it is convenient to find an alternative expression for the free boson effective action

$$\Gamma = \frac{1}{2}\,\mathrm{tr}\ln -\nabla, \tag{C.1}$$

where $\nabla = \frac{1}{\sqrt{\gamma}}\partial_\nu\left(\sqrt{\gamma}\gamma^{\mu\nu}\partial_\nu\right)$ is the Laplace-Beltrami operator. We start by giving an integral representation to the logarithm,

$$\ln x = -\int_\epsilon^\infty \frac{\mathrm{d}t}{t}\left[e^{-xt} - e^{-t}\right], \tag{C.2}$$

where the limit $\epsilon \to 0$ must be taken in the end of the calculations. Under an infinitesimal scale transformation of the metric, $\gamma'_{\mu\nu} = e^{\delta\omega}\,\gamma_{\mu\nu}$, the Laplace-Beltrami operator transforms as $\nabla' = e^{-\delta\omega}\nabla$ and we can write the variation of the boson effective action as

$$\begin{aligned}\delta\Gamma &= \frac{1}{2}\,\mathrm{tr}\int_\epsilon^\infty \mathrm{d}t\,\delta\omega\nabla e^{t\nabla} = \frac{1}{2}\,\mathrm{tr}\int_\epsilon^\infty \mathrm{d}t\,\delta\omega\frac{d}{dt}e^{t\nabla} = -\frac{1}{2}\,\mathrm{tr}\,\delta\omega e^{\epsilon\nabla}\\ \delta\Gamma &= -\frac{1}{2}\int \mathrm{d}^2x\,\sqrt{\gamma}\delta\omega(x)K(x,x,\epsilon),\end{aligned} \tag{C.3}$$

where $K(x,x,\epsilon)$ is the heat kernel of the Laplace-Beltrami operator

$$K(x,y,t) = \langle x|e^{t\nabla}|y\rangle\,, \quad \text{for} \quad t \geq 0. \tag{C.4}$$

In the limit $\epsilon \to 0$, we can expand the heat kernel in powers of the infinitesimal time

$$\delta\Gamma = -\frac{1}{2}\int \mathrm{d}^2x\,\sqrt{\gamma}\delta\omega(x)\left(\frac{1}{4\pi\epsilon} + \frac{R(x)}{24\pi}\right), \tag{C.5}$$

where $R(x)$ is the scalar curvature. The divergent term inside the bracket can be traced back to our assumption that the manifold is finite, it has nothing to do with curvature. We can eliminate this divergence with the addition of a local, field independent, counterterm to the effective action. Now we integrate the effective action, as the metric changes $\gamma'_{\mu\nu} = e^{\omega}\gamma_{\mu\nu}$ so does the Ricci scalar $R' = e^{-\omega}\left(R - \nabla\omega\right)$

$$\Gamma = -\frac{1}{96\pi}\int \mathrm{d}^2x\,\sqrt{\gamma}\left(\partial_\mu\omega\partial^\mu\omega + 2R\omega\right). \tag{C.6}$$

Our previous expression for the boson effective action can be made compatible if we consider a conformal change that takes a general metric to the plane metric $\gamma'_{\mu\nu} = e^{\hat{\omega}}\eta_{\mu\nu}$, that is, we choose $\omega = \hat{\omega}$ and $\gamma_{\mu\nu} = \eta_{\mu\nu}$ in the expression above. Now we insert an identity with the propagator equation

$$\int d^2y \sqrt{\gamma(y)}\nabla G(x,y) = 1 \tag{C.7}$$

After some partial derivatives and noting that in flat spacetime $R' = -e^{-\hat{\omega}}\partial_\mu\partial^\mu\hat{\omega}$, we find

$$\Gamma = \frac{1}{96\pi}\int d^2x d^2y \sqrt{\gamma(x)}\sqrt{\gamma(y)}R(x)G(x,y)R(y). \tag{C.8}$$

Now that we have a more useful expression for the effective action of a free boson in curved spacetime, we can start to relate the integrals of the effective action (4.67) to the free boson effective action. We start with the first integral

$$I_1 = \int d^2x \sqrt{\gamma}\gamma^{\mu\nu}\left[\partial^x_\mu G\partial^y_\nu G - G\partial^x_\mu\partial^y_\nu G\right]_{x=y}, \tag{C.9}$$

Firstly we write te propagator as

$$\begin{aligned} \bar{G}(x,y) &\equiv -\frac{1}{2\pi}\ln s(x,y) + G(x,y); \\ \bar{G}(x) &\equiv \bar{G}(x,x), \end{aligned} \tag{C.10}$$

where $s(x,y)$ is the geodesic distance between the points x and y. The logarithm term completely accounts for the divergence that is present in the propagator, such that $\bar{G}(x,y)$ has no divergence. In the limit $x = y$ we regularize the divergence, in doing so $G(x,y) \to \bar{G}(x,y)$.

With the symmetry of the propagator $G(x,y) = G(y,x)$ it is possible to show that

$$\partial^x_\mu G(x,y) = \partial^y_\mu G(x,y) = \frac{1}{2}\partial^x_\mu\bar{G}(x), \tag{C.11}$$

this simplifies the first term in I_1. From a similar argument it is possible to show that

$$\nabla\bar{G}(x) = 2\left[\nabla G(x,y) + \partial^x_\mu\partial^{\mu,y}G(x,y)\right]_{x=y}. \tag{C.12}$$

To simplify the second term in I_1 we note that, under a conformal transformation, the finite part of the propagator changes as

$$\bar{G}'(x) = \bar{G}(x) - \frac{\omega(x)}{4\pi} + 2\Omega(x) - \Omega_0, \tag{C.13}$$

such that

$$\Omega(x) = \frac{1}{V}\int \mathrm{d}^2y\,\sqrt{\gamma}e^{\omega(y)}G(x,y);$$
$$\Omega_0 = \frac{1}{V}\int \mathrm{d}^2x\,\sqrt{\gamma}\Omega(x)e^{\omega(x)}. \tag{C.14}$$

Choosing $\omega = \hat{\omega}$ and acting with ∇ on the equation (C.13) we obtain

$$\nabla\bar{G}' = \frac{R'}{4\pi} - \frac{2}{V} \quad \text{and} \quad \partial^x_\mu\partial^{\mu,y}G(x,y)\Big|_{x=y} = \frac{1}{8\pi}R. \tag{C.15}$$

This simplifies the second term such that

$$I_1 = -\frac{3}{16\pi}\int \mathrm{d}^2x\,\sqrt{\gamma}\bar{G}R, \tag{C.16}$$

under a transformation of the metric we find

$$I_1 = \frac{3}{64\pi^2}\int \mathrm{d}^2x\,\sqrt{\gamma}\partial_\mu\omega\partial^\mu\omega + 2\omega R = -\frac{9}{2\pi}\Gamma. \tag{C.17}$$

The last integral in the effective action (4.67) can be related to the individual terms of the integral I_1 through partial integrations and the propagator equation. We start by considering

$$\begin{aligned} I_{3.2} &= \int \mathrm{d}^2x\,\mathrm{d}^2y\,\sqrt{\gamma(x)}\sqrt{\gamma(y)}\gamma^{\mu\nu}(x)\gamma^{\sigma\rho}(y)\partial^x_\mu G\partial^y_\sigma G\partial^x_\nu\partial^y_\rho G \\ &= -\int \mathrm{d}^2x\,\mathrm{d}^2y\,\sqrt{\gamma(x)}\sqrt{\gamma(y)}\gamma^{\mu\nu}(x)\gamma^{\sigma\rho}(y)\left[\nabla^x G\partial^y_\sigma G\partial^y_\rho G + \partial^x_\mu G\partial^x_\nu\partial^y_\sigma G\partial^y_\rho G\right] \end{aligned} \tag{C.18}$$

where we discarded a total derivative term. Then we notice that the second term can be brought to the same form as $I_{3.2}$ by the exchange $\sigma \leftrightarrow \rho$. The remaining calculation follows the discussions for I_1, in the end we obtain

$$I_3 = \int \mathrm{d}^2x\,\mathrm{d}^2y\,\sqrt{\gamma(x)}\sqrt{\gamma(y)}\gamma^{\mu\nu}(x)\gamma^{\sigma\rho}(y)\left[G\partial^x_\mu\partial^y_\sigma G\partial^x_\nu\partial^y_\rho G - \partial^x_\mu G\partial^y_\sigma G\partial^x_\nu\partial^y_\rho G\right] = -\frac{3}{\pi}\Gamma \tag{C.19}$$

The remaining term involves a much more complicated and lengthy calculation, which is beyond the scope of this work [55]. The result of this calculation yields

$$I_2 = \int \mathrm{d}^2x\,\mathrm{d}^2y\,\epsilon^{\mu\nu}\epsilon^{\sigma\rho}G(x,y)\partial^x_\mu\partial^y_\sigma G(x,y)\partial^x_\nu\partial^y_\rho G(x,y) = \frac{3}{4\pi}\Gamma. \tag{C.20}$$