Universidade Estadual de Londrina

William Araújo de Souza

Spectral distortions on CMB due acoustic waves in bounce models

Silk Damping effect

Londrina

2024

William Araújo de Souza

Spectral distortions on CMB due acoustic waves in bounce models

Silk Damping effect

Dissertação apresentada ao Programa de Pós-graduação do Departamento de Física da Universidade Estadual de Londrina, como requisito parcial à obtenção do título de Mestre em Física.
Orientador: Prof. Dr. Sandro Dias Pinto Vitenti

Londrina
2024

Ficha de identificação da obra elaborada pelo autor, através do Programa de Geração Automática do Sistema de Bibliotecas da UEL

A663s Souza, William.

Spectral distortions on CMB due acoustic waves in bounce models : Silk Damping effect / William Souza. - Londrina, 2024.
127 f.

Orientador: Sandro Vitenti.
Dissertação (Mestrado em Física) - Universidade Estadual de Londrina, Centro de Ciências Exatas, Programa de Pós-Graduação em Física, 2024.
Inclui bibliografia.

1. Spectral distortions - Tese. 2. Bouce models - Tese. 3. Cosmic Microwave Background - Tese. I. Vitenti, Sandro. II. Universidade Estadual de Londrina. Centro de Ciências Exatas. Programa de Pós-Graduação em Física. III. Título.

CDU 53

William Araújo de Souza

Spectral distortions on CMB due acoustic waves in bounce models

Silk Damping Effect

Dissertação apresentada ao Programa de Pós-graduação do Departamento de Física da Universidade Estadual de Londrina, como requisito parcial à obtenção do título de Mestre em Física. Orientador: Prof. Dr. Sandro Dias Pinto Vitenti

BANCA EXAMINADORA

Orientador: Prof. Dr. Sandro Dias Pinto Vitenti
Universidade Estadual de Londrina - UEL

Thiago dos Santos Pereira
Universidade Estadual de Londrina - UEL

Rodrigo von Marttens
Universidade Federal da Bahia - UFBA

Londrina, 09 de Setembro de 2024.

À memória de minha mãe.

Agradecimentos

Agradeço ao apoio da minha família, em especial ao meu pai, meus tios e amigos próximos, que me apoiaram no momento mais difícil da minha vida, que foi lidar com o luto pela morte da minha mãe. À sua memória dedico este e futuros trabalhos. Agradeço ao meu orientador, Sandro Dias Pinto Vitenti, pelos ensinamentos. À CAPES, pela bolsa de pesquisa, e ao departamento de Física da UEL, pelo custeio de viagens para eventos científicos. Agradeço também pelas amizades desenvolvidas no departamento em especial o Bruno Araújo, Ricardo, Luiz Demétrio e Wesley Cruz, pelas conversas ao longo desses dois anos. Gostaria de agradecer em particular ao Luiz Demétrio pela ajuda no desenvolvimento da parte numérica do capítulo 4.

"Queria apenas tentar viver aquilo que brotava espontaneamente de mim. Por que isso me era tão difícil?"

(Hermann Hesse)

Souza, W. A. **Spectral distortions on CMB due acoustic waves in bounce models** 2024. (127)fls. Dissertação de Mestrado (Programa de Pós-Graduação em Física).– Universidade Estadual de Londrina, Londrina, 2024.

Resumo

Na era da cosmologia de precisão, missões são projetadas para coletar dados cosmológicos e testar modelos teóricos, com a radiação do Fundo Cósmico de Micro-ondas (CMB) sendo uma fonte chave de informação. Análises detalhadas da distribuição de temperatura dos fótons, polarização e frequência aprofundaram nossa compreensão da evolução e da história térmica do universo. Este trabalho foca nas pequenas flutuações no espectro de radiação do CMB, conhecidas como distorções espectrais, especificamente as distorções do tipo-μ que surgem antes da recombinação devido ao efeito de atenuação de Silk. Essas distorções podem distinguir entre diferentes modelos cosmológicos, já que a energia trocada entre o plasma primordial e os fótons do CMB é sensível ao espectro de potência das perturbações de curvatura. Nosso objetivo é estender a análise das distorções espectrais do tipo-μ para um modelo de salto quântico, que aborda questões de singularidade cosmológica e os mesmos problemas que a inflação, mas com condições iniciais menos rigorosas.

Palavras-chave: Radiação Cósmica de Fundo, distorções espectrais, oscilação acústica, ricochete quântico.

Souza, W. A **Spectral distortions on CMB due acoustic waves in bounce models** 2024. 127 p. Master Deegre Thesis – Universidade Estadual de Londrina, Londrina, 2024.

Abstract

In the era of precision cosmology, missions are designed to collect cosmological data to test theoretical models, with the Cosmic Microwave Background (CMB) radiation being a key source of information. Detailed analyses of photon temperature distribution, polarization, and frequency have deepened our understanding of the universe's evolution and thermal history. This work focuses on small fluctuations in the CMB's blackbody radiation spectrum, known as spectral distortions, specifically type-μ distortions arising before recombination due to the Silk damping effect. These distortions can distinguish between different cosmological models, as the energy exchanged between the primordial plasma and CMB photons is sensitive to the power spectrum of curvature perturbations. We aim to extend the analysis of μ spectral distortions to a quantum bounce model, which addresses cosmological singularity issues and the same problems as inflation but with less stringent initial conditions.

Keywords: Cosmic Microwave Background, spectral distortions, Silk damping, quantum bounce.

List of Figures

List of Tables

Table of Contents

1 Introduction

Cosmology is a precision science, with detailed theoretical models that provide a good fit for the cosmological observables. Among many cosmological data sets, the CMB temperature provides one of the most robust constraints to theoretical models. It allows us to determine key parameters of cosmological models, and to explore the early universe physics [1].

Besides the temperature fluctuations, the frequency spectrum of CMB encodes rich information. Departures of the frequency spectrum from a blackbody defined as spectral distortions, carry information about the thermal history of the early universe.

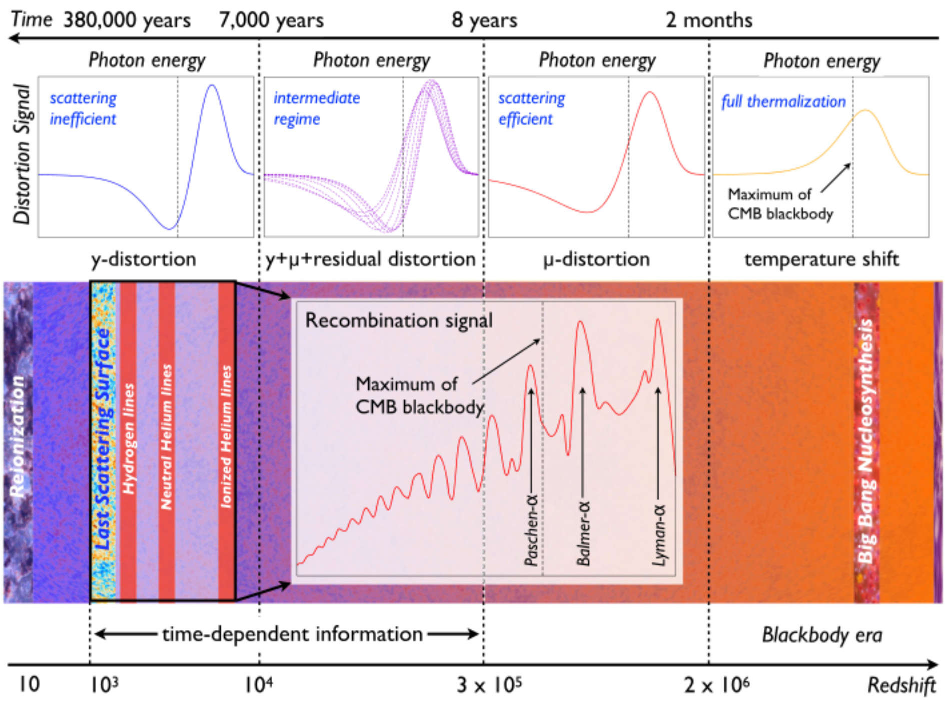


Figure 1 – Time evolution of spectral distortions [1].

There is a time dependence on the shape of the distortions, allowing us to explore the μ-distortion generated in the early universe by physical processes of that time. Among many physical effects in the early universe[1], we will pay attention to the diffusion damping (Silk damping) of acoustic waves in the photon-baryon fluid [2].

It was shown [3] that this effect leads to a spectrum distortion on CMB which depends on the spectral index n_S of the power spectrum of curvature perturbations, therefore providing constraints on inflationary models [4]. Moreover, recently proposed

[1] This effect is allowed by ΛCDM model, but one may study the energy release by effects such primordial black holes evaporation, annihilating dark matter, or decaying relict particles which have the potential to generate spectral distortions on CMB.

CMB experiments (e.g. PIXIE [5], PRISM [6]) are expected to have sensitivity to constrain $|\mu| \lesssim 5 \times 10^{-8}$ with 5σ of confidence.

The literature has shown that distortions generated by acoustic waves can constrain various inflationary scenarios that fall below the sensitivity of the PIXIE experiment. In this work, we aim to extend these estimations to a quantum bounce model for a barotropic fluid, seeking to identify distinguishing signatures between the bounce and inflationary models.

In this dissertation, we will develop the theory to understand the subject of the spectral distortions in the CMB in an isotropic and homogeneous universe. The first chapter, is an overview of Cosmology, commenting on some fundamental problems that motivated the inflationary model. A bounce model is also summarized, showing that it solves the same problems. In the second chapter, we explore the cosmological perturbations, obtaining the linear Einstein's equations and solving for the scalar equation to get the evolution of the scalar perturbation.

In the third chapter, we made a study of kinetic theory in curved spacetime, applying the concepts to the Friedmann-Lemaître-Robertson-Walker (FLRW) universe obtaining the temperature evolution of photons and baryons, and analyzing the perturbed Boltzmann equation to get the evolution of the temperature fluctuation of CMB, and the hydrodynamics equations of the photon gas.

In the last chapter, we estimated the μ-spectral distortion resulting from the Silk damping effect on small scales in the primordial plasma of baryons and photons, considering the power spectrum of a quantum bounce model for a single barotropic fluid [7]. Additionally, we estimated a stationary baryon temperature solution for the case of sound wave dissipation in the plasma.

2 Standard model of Cosmology

Modern cosmology can be constructed from two principles:

- Cosmological Principle: This principle postulates that the Universe on large scales[1] is approximately homogeneous and isotropic, meaning that from any point in the universe, there will be no preferred direction, and consequently, no preferred position. We use this hypothesis for the spatial sections Σ_t of the manifold M^4.

- Spacetime Decomposition Principle: The topology of the universe can be expressed as a disjoint product $M^4 = \Sigma_t \times \mathbb{R}$. Here, the spatial sections Σ_t can have any topology and are described by an evolution parameter t.

The foliation of M^4 guaranteed by the second principle will be essential to restrict the Cosmological Principle to the spatial sections, ensuring a lesser constraint on the functional form of the material content of the universe that we will describe[2].

Using the assumptions of spatial homogeneity and isotropy, we can determine the spacetime metric, which will be given by[3].

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + f_K(r)^2 d\Omega^2\right), \tag{2.1}$$

where

$$f_K(r) = \begin{cases} f_1(r) = \sin(r), \\ f_0(r) = r, \\ f_{-1}(r) = \sinh r, \end{cases} \tag{2.2}$$

and $d\Omega^2 \equiv \sin^2\theta d\phi^2 + d\theta^2$ is the solid angle element. The factor $a(t)$ is a dimensionless arbitrary function of proper time only, as we are imposing spatial homogeneity on the spatial hypersurfaces, and t is the proper time measured by isotropic observers.

2.1 Simultaneity hypersurfaces

Let us decompose the spacetime M^4 into disjoint submanifolds Σ_t, with $\Sigma_{t_1} \cap \Sigma_{t_2} = \emptyset$ and $\sqcup_t \Sigma_t = M^4$. One way to section the manifold is by defining a normalized field $n_\mu = -c\nabla_\mu t$ orthogonal to these spatial hypersurfaces.

[1] Approximately 100 Mpc, where 1 pc = 3.26 light-years.

[2] If the hypothesis of homogeneity and isotropy applied to M^4, we would have a restriction on the material content of the Universe, as will be discussed later.

[3] We obtain the metric explicitly in Appendix B.

We can define the projector $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$[4], which will map any tensor in M^4 to the tangents to the spatial hypersurfaces Σ_t, that is, it will collect only the spatial part of the tensors. We can raise one of the indices of the projector in order to act on the tensors

$$h^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} + n^\mu n_\nu. \tag{2.3}$$

We will denote the action on the tensors as

$$h[T^{\mu_1 \dots \mu_n}_{\ \ \nu_1 \dots \nu_l}] \equiv h^{\mu_1}_{\ \lambda_1} \dots h^{\beta_l}_{\ \nu_1} \dots T^{\lambda_1 \dots}_{\ \ \beta_1 \dots}. \tag{2.4}$$

If we project a time-like field we will see that the projection will be naturally zero

$$h_{\mu\nu} n^\nu = g_{\mu\nu} n^\nu + n_\nu n^\nu n_\mu = n_\mu - n_\mu = 0. \tag{2.5}$$

If we act the projector on the metric

$$h[g_{\mu\nu}] = h^{\alpha}_{\ \mu} h^{\beta}_{\ \nu} g_{\alpha\beta} = (\delta^{\alpha}_{\ \mu} + n^\alpha n_\mu)(\delta^{\beta}_{\ \nu} + n^\beta n_\nu) g_{\alpha\beta} = h_{\mu\nu}. \tag{2.6}$$

Therefore, the tensor $h_{\mu\nu}$ can be interpreted as the metric projected on the slices, being the fundamental dynamic quantity for our physical model.

Regarding the slicing of the manifold, there is an ambiguity in how to do it, since we can choose any covector field n_μ to slice it, as long as this field is normal to the spatial hypersurfaces. For this, it is enough that there exists a globally defined function t on the manifold such that $n_\mu = -c\nabla_\mu t$.

However, in the context of Cosmology following the Cosmological Principle, we are interested in a specific field that will define isotropic observers, that is, a field that is geodesic and normal to the hypersurfaces. It can be shown that these free-falling observers define a unique frame in which the spatial slices are homogeneous and isotropic [8].

Assuming there exists a hypersurface Σ_{t_0} such that there exists a time-like vector n_μ normal to this hypersurface, i.e., $n^\mu n_\mu = -1$ and $n_\mu s^\mu = 0$, where s^μ are spatial vectors tangent to the slice Σ_{t_0}. And that we have isotropic observers following geodesic curves $\lambda(t)$ in M^4 with tangent vector $l^\mu(t)$. We impose that

$$l^\mu(t_0) = n^\mu \Rightarrow l^\mu(t_0) l_\mu(t_0) = -1. \tag{2.7}$$

We can verify that the field l^μ will remain time-like, i.e., the norm is preserved along the integral curves of the free-falling observers.

$$\mathcal{L}_l(l^\mu l_\mu) = l^\nu \nabla_\nu (l^\mu l_\mu) = l_\mu l^\nu \nabla_\nu l^\mu + l^\mu l^\nu \nabla_\nu l_\mu = 0, \tag{2.8}$$

4 $g_{\mu\nu}$ are the components of ds^2, i.e., $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

where we defined the Lie derivative in appendice C, and used the assumption that l^μ is geodesic. Thus, we see that $l^\mu(t)l_\mu(t) = -1$ for all Σ_t with $t \in \mathbb{R}$. At any point $p \in \Sigma_{t_0}$, we can define a Gaussian coordinate system, in which we construct the spatial-like fields s^μ. With this, we will have a set of coordinate fields $\{l^\mu, s^\mu\}$. We know that the field of observers is orthogonal to the spatial sections in Σ_{t_0}. Let us check if they remain normal for any time interval.

$$\mathcal{L}_l(l^\mu s_\mu) = l^\nu\nabla_\nu(l^\mu s_\mu) = s_\mu l^\nu \nabla_\nu l^\mu + l^\mu l^\nu \nabla_\nu s_\mu \tag{2.9}$$

$$= l^\mu s^\nu \nabla_\nu l_\mu = \frac{1}{2} s^\nu \nabla_\nu \underbrace{(l^\mu l_\mu)}_{-1} = 0, \tag{2.10}$$

where we used the fact that the fields l^μ and s^ν are coordinate fields, i.e.,

$$\mathcal{L}_l s = [l, s] = 0 \Rightarrow \tag{2.11}$$

$$l(s) - s(l) = l^\nu \nabla_\nu s^\mu - s^\nu \nabla_\nu l^\mu = 0, \tag{2.12}$$

$$l^\nu \nabla_\nu s^\mu = s^\nu \nabla_\nu l^\mu. \tag{2.13}$$

If we project the covariant derivative onto the spatial sections, we will have an operator defined only on the spatial hypersurfaces

$$h[\nabla_\alpha T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l}] \equiv D_\alpha T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l}, \tag{2.14}$$

this operator will be compatible with the metric projected on the hypersurface

$$D_\alpha h_{\mu\nu} = h[\nabla_\alpha h_{\mu\nu}] = h[\nabla_\alpha (g_{\mu\nu} + n_\mu n_\nu)] \tag{2.15}$$

$$= h[n_\mu \nabla_\alpha n_\nu + n_\nu \nabla_\alpha n_\mu] = 0. \tag{2.16}$$

With this covariant derivative on the spatial section, we can define the spatial Riemann tensor analogously as we do in M^4.[5]

$$\mathcal{R}_{\mu\nu\sigma}{}^{\beta} v_\beta = [D_\mu, D_\nu] v_\sigma, \tag{2.17}$$

the commutator of the derivatives has been defined in apendice C, and for any 1-form $v_\sigma \in T_p^* M^4$. If we raise one of the indices of the Riemann tensor with the spatial metric, we can define it as a linear functional that acts on the antisymmetric space of 2-forms Λ_p^2 and maps to the same space, given a point $p \in \Sigma_t$, i.e.,

$$\mathcal{R} : \Lambda_p^2 \rightarrow \Lambda_p^2 \tag{2.18}$$

$$\omega \rightarrow \mathcal{R}(\omega), \tag{2.19}$$

[5] We will use calligraphic symbols to represent the tensors projected onto the three-dimensional hypersurface.

explicitly

$$\mathcal{R}_{\mu\nu}{}^{\sigma\beta}\omega_{\sigma\beta} = \bar{\omega}_{\mu\nu}. \tag{2.20}$$

The 2-form ω encodes the notion of an area at point p. Given a local chart $(\phi, U) \subset \Sigma_t$, we have $\omega = \omega_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$. Assuming spatial sections are isotropic, spatial vectors must transform as $s^\mu \to \bar{s}^\mu = \delta^\mu{}_\nu s^\nu$. Otherwise, we would have a specific field different from others on the hypersurface, indicating a privileged direction on spatial sections, contradicting the isotropy hypothesis. the components of the new transformed 2-form will be

$$\bar{\omega}_{[\alpha\beta]} = (\bar{s}_\alpha \wedge \bar{s}_\beta)(\omega) = \omega_{\mu\nu}\, \delta^\sigma{}_\alpha \delta^\gamma{}_\beta\, s_\sigma \wedge s_\gamma (dx^\mu \wedge dx^\nu) = \omega_{[\alpha\beta]}, \tag{2.21}$$

thus, due to the isotropy hypothesis, there is no privileged notion of area. In this sense, the Riemann operator in equation (2.20) must lead to the same 2-form, thus being a map proportional to the identity.

$$\mathcal{R}_{\mu\nu}{}^{\sigma\beta} = K\left(\delta^\alpha_{\ \mu}\delta^\beta_{\ \nu} - \delta^\alpha_{\ \nu}\delta^\beta_{\ \mu}\right), \tag{2.22}$$

lowering the indices σ and β on both sides with the spatial metric.

$$\mathcal{R}_{\mu\nu\sigma\beta} = K\left(h_{\mu\sigma}h_{\nu\beta} - h_{\sigma\nu}h_{\mu\beta}\right). \tag{2.23}$$

We can contract the indices ν and β, in order to obtain the spatial Ricci tensor $\mathcal{R}_{\mu\sigma}$, and another contraction μ and σ to obtain the spatial Ricci scalar $\mathcal{R}$.

$$\mathcal{R}_{\mu\sigma} = K[3h_{\mu\sigma} - h_{\nu\sigma}h_\mu^{\ \nu}] = 2Kh_{\mu\sigma}, \tag{2.24}$$

$$\mathcal{R} = 2Kh_\mu^{\ \mu} = 6K, \tag{2.25}$$

where K is the eigenvalue of the intrinsic curvature of space. The above result will be valid for any maximally symmetric space, i.e., possessing the 6 symmetries associated with rotation and translation in Σ_t, or in other words, being homogeneous and isotropic. The extrinsic curvature in M^4 can be expressed in terms of the irreducible decomposition (reference the Greeks),

$$\mathcal{K}_{\mu\nu} \equiv h[\nabla_\mu n_\nu] = \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu} + \omega_{\mu\nu} + a_\mu n_\nu, \tag{2.26}$$

such that, $\sigma_{\mu\nu} = \mathcal{K}^T_{(\mu\nu)}$ is the shear tensor[6], symmetric and traceless, it has the property of modifying the shape of the flow of curves orthogonal to the spatial sections, $\theta = \mathcal{K}_\mu^{\ \mu} = \nabla_\mu n^\mu$ is the expansion scalar, associated with the stretching of curves for each time interval, $\omega_{\mu\nu} = \mathcal{K}_{[\mu\nu]}$ is the vorticity tensor, associated with twists in the integral curves, allowing them to undergo deformations tangent to the spatial sections. Finally,

[6] The letter T in the tensor symbolizes that the tensor is traceless

a_μ is the four-acceleration of observers, indicating modifications in the integral curves due to external forces.

We are interested in geodesic observers, that is, those who follow geodesic curves, thus $a_\mu = 0$. Since n_μ is orthogonal to the spatial hypersurfaces, and using the Frobenius theorem [8], we can show that $\omega_{\mu\nu} = 0$, as follows:

$$\nabla_{[\mu}n_{\nu]} = n_{[\mu}v_{\nu]}, \tag{2.27}$$

given $v_\nu \in T_p^*M^4$ is any 1-form, then

$$\begin{aligned} n^\nu K_{[\mu\nu]} &= n^\nu \nabla_{[\mu}n_{\nu]} && (2.28)\\ &= \frac{1}{2}(n^\nu \nabla_\mu n_\nu - n^\nu \nabla_\nu n_\mu) && (2.29)\\ &= -\frac{1}{2}n^\nu \nabla_\mu n_\nu + \nabla_\mu (n^\nu n_\nu) = 0, && (2.30)\end{aligned}$$

then

$$\begin{aligned} n^\nu n_{[\mu}v_{\nu]} &= 0, && (2.31)\\ n^\nu n_\mu v_\nu + v_\mu &= 0, && (2.32)\\ v_\mu &= -n^\nu n_\mu v_\nu. && (2.33)\end{aligned}$$

Therefore the vorticity as defined earlier will be

$$\omega_{\mu\nu} = \mathcal{K}_{[\mu\nu]} = -n^\sigma v_\sigma n_{[\mu}n_{\nu]} = 0, \tag{2.34}$$

then the extrinsic curvature

$$\mathcal{K}_{\mu\nu} = \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu}. \tag{2.35}$$

This tensor is purely spatial,

$$\begin{aligned} h[\mathcal{K}_{\mu\nu}] = h_\mu^{\ \alpha} h_\nu^{\ \beta} \nabla_\alpha n_\beta &= (\delta_\mu^{\ \alpha} + n_\mu n^\alpha)(\delta_\nu^{\ \beta} + n_\nu n^\beta)\nabla_\alpha n_\beta = && (2.36)\\ &= \nabla_\mu n_\nu = \mathcal{K}_{\mu\nu}. && (2.37)\end{aligned}$$

We can verify how the spatial metric $h_{\mu\nu}$ evolves in time, i.e., how the metric changes when it is dragged along the geodesic curves generated by the timelike field n_μ, by calculating the Lie derivative of the metric with respect to this field, we obtain

$$\begin{aligned} \mathcal{L}_n h_{\mu\nu} \doteq \partial_t h_{\mu\nu} &= n^\alpha \nabla_\alpha h_{\mu\nu} + h_{\mu\alpha}\nabla_\nu n^\alpha + h_{\nu\alpha}\nabla_\mu n^\alpha = && (2.38)\\ &= h_\mu^{\ \alpha}\nabla_\nu n^\alpha + h_\nu^{\ \alpha}\nabla_\mu n^\alpha = (\delta_\mu^{\ \alpha} + n^\alpha n_\mu)\nabla_\nu n_\alpha + (\delta_\nu^{\ \alpha} + n^\alpha n_\nu)\nabla_\mu n_\alpha = && (2.39)\\ &= \nabla_\nu n_\mu + \nabla_\mu n_\nu = 2\nabla_{(\mu}n_{\nu)} = 2\mathcal{K}_{(\mu\nu)}, && (2.40)\end{aligned}$$

where in the second line we used that $n^\alpha \nabla_\alpha h_{\mu\nu} = n^\alpha \nabla_\alpha (g_{\mu\nu} + n_\mu n_\nu) = 0$.

To obtain the geometric equations of motion describing the spatial part of the Universe arising from symmetry assumptions, we need to project tensors from M^4 onto Σ_t. In particular, we are interested in the Ricci tensor, the Ricci scalar, and the energy-momentum tensor, as they appear in the Einstein equations. To obtain this relationship, it is valid to use the definition of spatial curvature in terms of three-dimensional covariant derivatives (2.17).

$$R_{\alpha n} = h^{\mu}_{\ \alpha}\nabla_\nu \mathcal{K}_{\mu}^{\ \nu} - D_\alpha\theta + n_\alpha(\mathcal{K}_{\nu\mu}\mathcal{K}^{\mu\nu} + \dot{\theta}). \tag{2.41}$$

From (2.41), we can calculate the projections of the Ricci tensor

$$R_{nn} = n^\alpha R_{\alpha n} = n^\alpha h[\nabla_\nu \mathcal{K}_{\mu}^{\ \nu}] - n^\alpha D_\alpha\theta + n^\alpha n_\alpha(\mathcal{K}_{\nu\mu}\mathcal{K}^{\mu\nu} + \dot{\theta}) \tag{2.42}$$

$$R_{nn} = -(\mathcal{K}_{\nu\mu}\mathcal{K}^{\mu\nu} + \dot{\theta}), \tag{2.43}$$

and the space-time projection

$$h[R_{\alpha n}] = h_{\alpha}^{\ \mu}\nabla_\nu \mathcal{K}_{\mu}^{\ \nu} \tag{2.44}$$

where the last term of (2.41) was canceled, and we used the idempotent property of the projector.

Decomposing the Ricci tensor as

$$R_{\mu\nu} = n_\mu n_\nu R_{nn} + h[R_{\mu\nu}] + n_\mu h[R_{\nu n}]. \tag{2.45}$$

And taking the trace of this tensor, we obtain a relationship between the four-dimensional Ricci scalar and its three-dimensional version projected onto the leaves.

$$R = R_{\mu}^{\ \mu} = -R_{nn} + h[R], \tag{2.46}$$

$$R = \mathcal{R} + 2\dot{\theta} + \theta^2 + \mathcal{K}_{\nu\mu}\mathcal{K}^{\mu\nu}. \tag{2.47}$$

Similarly, we aim to find a relationship between the Ricci tensor on the manifold and on the leaves. Starting from the definition of the Riemann tensor (2.17), we can perform the most general possible decomposition and utilize the tensor's symmetries, yielding

$$R_{\mu\nu\alpha\beta} = h[R_{\mu\nu\alpha\beta}] + 2\left(h[R_{\mu\nu n[\alpha}n_{\beta]}] + h[R_{\alpha\beta n[\mu}n_{\nu]}]\right). \tag{2.48}$$

Using the previously obtained projections, we have

$$\begin{aligned} R_{\mu\nu\alpha\beta} = \mathcal{R}_{\mu\nu\alpha\beta} + \mathcal{K}_{\mu\alpha}\mathcal{K}_{\nu\beta} - \mathcal{K}_{\nu\alpha}\mathcal{K}_{\mu\beta} - 4(D_{[\mu}\mathcal{K}_{\nu][\alpha}n_{\beta]} + D_{[\alpha}\mathcal{K}_{\beta][\mu}n_{\nu]}) + \\ +4(n_{[\mu}\mathcal{K}_{\nu]}^{\ \lambda}\mathcal{K}_{\lambda[\alpha}n_{\beta]} + \nabla_n(n_{[\mu}\mathcal{K}_{\nu][\alpha}n_{\beta]}). \end{aligned} \tag{2.49}$$

Contracting the second and last index, we obtain the expression for the Ricci tensor[7].

$$R_{\mu\alpha} = \mathcal{R}_{\mu\alpha} + (\theta + \nabla_n)\mathcal{K}_{\mu\alpha} + 2(D_{(\mu}\theta n_{\alpha)} - D_\sigma\mathcal{K}^{\sigma}_{\ (\mu}n_{\alpha)}) - n_\mu n_\alpha(\dot{\theta} + \mathcal{K}_{\sigma\lambda}\mathcal{K}^{\sigma\lambda}). \tag{2.50}$$

[7] Explicit calculations are provided in the appendix.

The relationships (2.50), (2.46) will be sufficient to express the geometric part of the Einstein equations. We are interested in expressing the most general energy-momentum tensor given an arbitrary field ${n_\mu}$[8]. The extrinsic curvature for this field is given by (2.26). The projection of the energy-momentum tensor in this case is

$$T_{\mu\nu} = T_{nn}n_\mu n_\nu + 2n_{(\mu}h[T_{\nu)n}] + h[T_{\mu\nu}]. \tag{2.51}$$

Defining $T_{nn} \equiv \rho$, $h[T_{\nu n}] \equiv q_\nu$, $h[T_{\mu\nu}] \equiv \Pi_{\mu\nu} + \frac{1}{3}Th_{\mu\nu}$, where ρ is the energy density of the fluid, q_ν is the energy flux in the spatial sections, $\Pi_{\mu\nu}$ is the anisotropic pressure, and $T = 3p$ is the trace of the spatial part, with p being the isotropic pressure of the fluid, note that we separate the last term of the decomposition into a traceless part $\Pi_\mu^{\ \mu} = 0$ and the trace part, then

$$T_{\mu\nu} = \rho n_\mu n_\nu + 2q_{(\nu}n_{\mu)} + ph_{\mu\nu} + \Pi_{\mu\nu}, \tag{2.52}$$

this is the energy-momentum tensor given an arbitrary field n_μ, it is a symmetric tensor with 10 degrees of freedom. Continuing, we are able to verify how the hypotheses of homogeneity and isotropy will simplify the tensors obtained above, in order to express the Einstein equations.

2.2 Friedmann equations

2.2.1 Homogeneity and isotropy

As discussed in Appendix B, in homogeneous and isotropic spaces, one can define a basis $\{e^a{}_\mu\}$, $a = 1, 2, 3$, such that the spatial metric can be written in sections as $h_{\mu\nu} = a(t)^2\delta_{ab}e^a_\mu e^b_\nu = a^2(t)\tilde{h}_{\mu\nu}$, thus

$$\begin{aligned} \partial_t(h_{\mu\nu}) &= \partial_t(a^2\delta_{ab}e^a{}_\mu e^b{}_\nu) = 2a\dot{a}\delta_{ab}e^a_\mu e^b{}_\nu, \\ 2\mathcal{K}_{\mu\nu} &= 2a\dot{a}\tilde{h}_{\mu\nu}, \\ \mathcal{K}_{\mu\nu} &= H(t)a^2\tilde{h}_{\mu\nu} = H(t)h_{\mu\nu}, \end{aligned} \tag{2.53}$$

where in the last passage we multiplied the right-hand side by a/a, and defined the Hubble function $H(t) \equiv \frac{\dot{a}}{a}$. From these relations, it follows that

$$\theta = h[\nabla_\mu n^\mu] = \mathcal{K}^\mu_{\ \mu} = 3H, \tag{2.54}$$

$$\mathcal{K}_{\mu\nu} = \frac{\theta}{3}h_{\mu\nu} \tag{2.55}$$

we see that naturally

$$D_\alpha\mathcal{K}_{\mu\nu} = H(t)D_\alpha h_{\mu\nu} = 0. \tag{2.56}$$

8 In the general case, this field will not necessarily be orthogonal to the hypersurfaces, nor geodesic.

The extrinsic curvature above will always be proportional to the metric projected onto the sections for any homogeneous and isotropic Universe and due to homogeneity and the above result, it will be the same at any point on the spatial hypersurface. From the definition of the Hubble function, we can express

$$\int_{a_0}^{a} \frac{da'}{a'} = \int_{t_0}^{t} H(t')dt',$$

$$a(t) = a_0 e^{\int_{t_0}^{t} H(t')dt'}. \tag{2.57}$$

We can apply the simplification (2.55) in equation (B.35), to obtain a relation between spatial curvature and the scale factor.

$$\dot{K} + \frac{2K\theta}{3} = 0 \tag{2.58}$$

$$\frac{\dot{K}}{K} = -\frac{2\theta}{3} = -2H(t)$$

$$\int_{K_0}^{K} \frac{dK'}{K'} = -2\int_{0}^{t} H(t')dt'$$

$$K(t) = \frac{\tilde{K}}{a^2}, \tag{2.59}$$

where $\tilde{K} = -1, 0, 1$ are the possible eigenvalues of the intrinsic curvature of spatial hypersurfaces, expressing hyperbolic, flat, or spherical geometries respectively, from here we will implicitly denote temporal dependence by K.

Let us return to the tensors obtained previously and use (2.2.1), (2.54), and (2.56) to simplify them.

$$R_{\mu\nu\alpha\beta} = 2(K + H^2)h_{\mu[\alpha}h_{\nu]\beta} + 4(\dot{H} + H^2)n_{[\mu}h_{\nu][\alpha}n_{\beta]}, \tag{2.60}$$

$$R_{\mu\alpha} = (K + H^2)(3h_{\mu\alpha} - h_{\mu\alpha}) + (\dot{H} + H^2)(h_{\mu\alpha} - 3n_\mu n_\alpha)$$

$$= (2K + 3H^2 + \dot{H})h_{\mu\alpha} - 3(\dot{H} + H^2)n_\mu n_\alpha. \tag{2.61}$$

The Ricci scalar can be obtained using (2.46)

$$R = 6K + 9\dot{H} + 12H^2 \tag{2.62}$$

$$= 6(K + \dot{H} + 2H^2), \tag{2.63}$$

thus the spatial and temporal components of the Einstein tensor will be

$$G_{nn} = n^\mu n^\nu G_{\mu\nu} = R_{nn} + R/2 \tag{2.64}$$

$$= -3(\dot{H} + H^2) + 3(K + \dot{H} + 2H^2) \tag{2.65}$$

$$= 3(H^2 + K), \tag{2.66}$$

$$h[G_{\mu\nu}] = h[R_{\mu\nu}] - \frac{1}{2}h[g_{\mu\nu}R] \tag{2.67}$$

$$= [2K + 3H^2 + \dot{H} - 3(K + \dot{H} + 2H^2)]h_{\mu\nu} \tag{2.68}$$

$$= -h_{\mu\nu}(K + 3H^2 + 2\dot{H}). \tag{2.69}$$

Let us see how the stress-energy tensor will simplify given the FLRW metric, the most general energy-momentum tensor is

$$T_{\mu\nu} = T^{(n)} n_\mu n_\nu + 2q_{(\mu} n_{\nu)} + T^{(h)} h_{\mu\nu} + \Pi_{\mu\nu}, \tag{2.70}$$

in which $T^{(n)} \equiv n^\mu n^\nu T_{\mu\nu} = \rho$ is the energy density, $q_\mu \equiv h[T_{\mu n}]$ is the heat flux, $T^{(h)} \equiv \dfrac{1}{3} h^{\mu\nu} T_{\mu\nu} = p$ is the isotropic pressure and $\Pi_{\mu\nu} = 0$ is the anisotropic pressure, with $\Pi_\mu^{\ \mu}$=0, since we are considering n_μ defining isotropic observers we will not have diagonal terms on the tensor, and the anisotropic pressure will be zero on the spatial sections. Furthermore, the energy-momentum tensor of a perfect fluid is

$$T_{\mu\nu} = \rho n_\mu n_\nu + p h_{\mu\nu}. \tag{2.71}$$

Finally, Einstein's equations are as follows:

$$\begin{aligned} G_{\mu\nu} &= \kappa T_{\mu\nu} \\ H^2 + K &= \frac{\kappa\rho}{3} \\ H^2 &= \frac{\kappa\rho}{3} - \frac{\tilde{K}}{a^2}, \end{aligned} \tag{2.72}$$

$$\begin{aligned} h[G_{\mu\nu}] &= \kappa h[T_{\mu\nu}] \\ -h_{\mu\nu}(K + 3H^2 + 2\dot{H}) &= \kappa p h_{\mu\nu} \\ 3H^2 + 2\dot{H} &= -\left(\kappa p + \frac{\tilde{K}}{a^2}\right), \end{aligned} \tag{2.73}$$

where

$$\begin{aligned} \dot{H} &= \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - H^2 \\ 2\frac{\ddot{a}}{a} + H^2 &= -\left(\kappa p + \frac{\tilde{K}}{a^2}\right) \end{aligned} \tag{2.74}$$

we can substitute (2.72) into (2.74)

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p). \tag{2.75}$$

Equations (2.72) and (2.75) are the Friedmann equations, equation (2.72) represents a constraint on the geometric and material content variables that must be satisfied regardless of the Universe's state, while (2.75) represents the model's dynamics, describing how the scale factor changes over time. We can take the time derivative of equation (2.72) and use (2.74), so that

$$\begin{aligned} 2H\dot{H} &= \frac{\kappa\dot{\rho}}{3} - \dot{K}, \\ \frac{\kappa\dot{\rho}}{3} - (\dot{K} + 2HK) &= -\kappa H(\rho + p), \\ \dot{\rho} &= -3H(\rho + p), \end{aligned} \tag{2.76}$$

we used (2.72), (2.74) in the first passage, and in the last (2.58). Equation (2.76) is called the continuity equation, it is a universal equation, that is, regardless of the nature of $T_{\mu\nu}$[9], this equation will be valid. This equation is not an independent equation from the Friedmann equations since we used them to derive this latter, however, it is interesting to note that we could have derived this expression using $\nabla_\mu T^{\mu 0} = 0$.

Note that we have two equations for three variables $(\rho, a(t), p)$ to have a closed system of equations it is necessary to impose one more equation that relates these variables, for this, we define an equation of state for the fluid that composes the system, let us suppose that the fluid is barotropic $p = p(\rho)$.[10]

$$p = w\rho, \tag{2.77}$$

then

$$\begin{aligned}
\dot{\rho} + 3H(1+w)\rho &= 0, \\
a\dot{\rho} + 3(1+w)\dot{a}\rho &= 0, \\
a^{3(1+w)}\dot{\rho} + 3(1+w)a^{3(1+w)-1}\dot{a}\rho &= 0, \\
\int_{t_0}^{t} \frac{d}{dt'}(a^{3(1+w)}\rho) &= 0, \\
\rho(a) &= \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}.
\end{aligned} \tag{2.78}$$

In the second passage, we use the definition of the Hubble function, and multiply both sides of the equation by $a^{3(1+w)-1}$. Note that the equation of state (2.77) does not presuppose perfect thermodynamic equilibrium, we will see in chapter 4 that from kinetic theory we can obtain this relation without the fluid being in perfect equilibrium, as we will see is the case for any system in an FLRW universe.

We will deal with fluids with $w \geq -1$, that is, the energy density will always decrease for any component, remaining constant only for the case $w = -1$, let's check some notable examples in table 1. In which $C_i = \rho_{0i} a_0^{3(1+w_i)}$. It is interesting to note

Table 1 – Energy density for different components of the universe

Component	w	$\rho(a)$
Dust	0	$C_p a^{-3}$
Radiation	$\frac{1}{3}$	$C_r a^{-4}$
Curvature	$-\frac{1}{3}$	$C_k a^{-2}$
Cosmological constant	-1	C_Λ

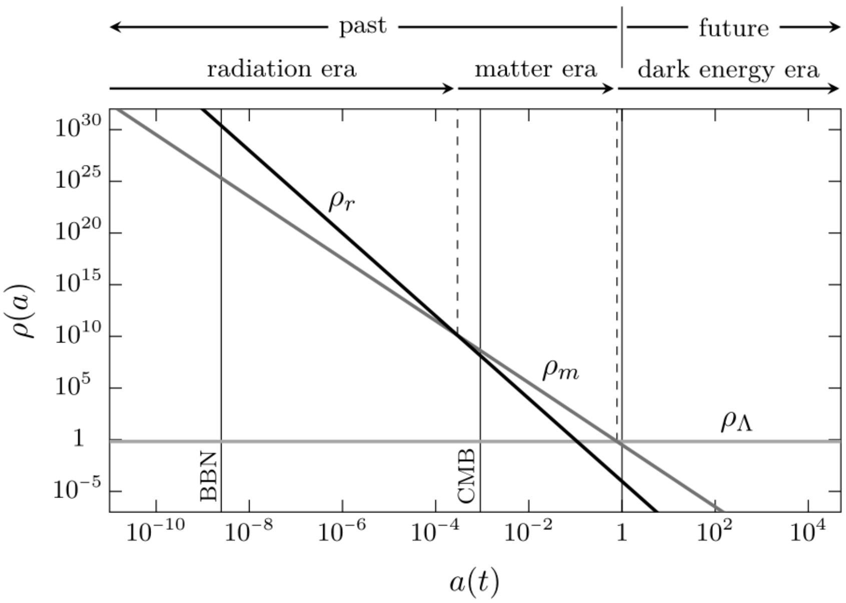
that the last two components of the table are not considered proper fluids. Curvature

[9] For example, if it is an energy-momentum tensor associated with a field, or probability, thermodynamic fluid, etc.

[10] Barotropic fluids, with constant equation of state parameter, agree with cosmological observations.

density is a component arising from the curvature of spatial sections, in which we can associate pressure and energy density, analogously to the cosmological constant (Λ), which is merely a form of energy in the Universe with negative pressure.[11] From the Einstein equations, we have that in the vacuum of matter $G_{\mu\nu} = T_{\mu\nu(\Lambda)} = -\Lambda g_{\mu\nu}$, if we consider a perfect barotropic fluid $ph_{\mu\nu} + \rho n_\mu n_\nu = -\Lambda(h_{\mu\nu} - n_\mu n_\nu)$, then $p = -\Lambda$ and $\rho = \Lambda \Rightarrow p = -\rho$.

Figure 2 – Evolution of energy density in the universe, a notable transition occurs from the radiation-dominated era to the matter-dominated era around $a(t) \approx 10^{-5}$, and from the matter era to the dark energy-dominated era around $a(t) \approx 1$. In certain calculations, it is relevant to account for two distinct components near these transitional epochs.



Source: Cosmology [10].

Furthermore, from the relationships obtained in Table 1, it can be affirmed that in the past where $a(t) \ll a_0$, the universe was dominated by radiation; shortly after, it was dominated by matter, and presently it is dominated by the cosmological constant (Figure 2).

From Equation (2.75), it can be seen that the Universe will accelerate ($\ddot{a} > 0$) if, and only if, $3p + \rho < 0$, i.e., when $w < -\dfrac{1}{3}$. This is the case, for example, of the cosmological constant. Observing the Universe expanding acceleratingly in the present days [11], Λ becomes a fundamental ingredient in the model. We can express the total

[11] There are alternative interpretations for dark energy that define it as a fluid with equation of state $w < -\frac{1}{3}$, for further details see [9].

energy-momentum tensor as

$$T_{\mu\nu} = \sum_i T^{(i)}_{\mu\nu}, \tag{2.79}$$

$$\rho = \sum_i \rho_i, \tag{2.80}$$

$$p = \sum_i p_i, \tag{2.81}$$

for each component (i) of the system. Thus, the Friedmann equation (2.72) can be defined as

$$H^2 = \frac{\kappa}{3}\sum_i \rho_i - \frac{\tilde{K}}{a^2}. \tag{2.82}$$

Finally, to analyze some aspects of the Universe's evolution, it is interesting to parameterize the above Friedmann equation in terms of the critical density $\rho_c \equiv \frac{3H^2}{\kappa}$. If we divide both sides of (2.82) by ρ_c,

$$\sum_i \frac{\rho_i}{\rho_c} - \frac{\tilde{K}}{a^2H^2} = 1, \tag{2.83}$$

defining $\Omega_i \equiv \frac{\rho_i}{\rho_c}$ and $\Omega_k \equiv \frac{\tilde{K}}{a^2H^2}$, then

$$\sum_i \Omega_i + \Omega_k = 1, \tag{2.84}$$

where Ω_i is the energy density of component (i) in the Universe, while Ω_k is the curvature density. Equation (2.84) allows us to estimate the curvature of the Universe on large scales (and consequently, its geometry) based on the energy density of the components constituting the Universe.

For example, if $\sum \rho_i < \rho_c \Rightarrow \sum \Omega_i < 1 \Rightarrow \Omega_k > 0 \Rightarrow \tilde{K} < 0$ (spherical geometry). On the other hand, if $\sum \rho_i \geq \rho_c \Rightarrow \sum \Omega_i \geq 1 \Rightarrow \Omega_k \leq 0 \Rightarrow \tilde{K} \geq 0$ (flat or hyperbolic geometry). It is interesting to express the above equation in terms of the critical densities measured today. For this, we define the normalized Hubble function $E \equiv \frac{H}{H_0}$ such that $E(t_0) = 1$, and $E^2\Omega_i = E^2\frac{\rho_i}{\rho_c} = \frac{\rho_i}{\rho_{c0}}$

$$\sum_i E^2\Omega_i + E^2\Omega_k = E^2,$$

$$\sum_i \Omega_{i0}x^{3(1+w_i)} + \Omega_{k0}x^2 = E^2,$$

$$\Omega_{r0}x^4 + \Omega_{m0}x^3 + \Omega_{k0}x^2 + \Omega_{\Lambda 0} = E^2, \tag{2.85}$$

where $x(t) = \frac{a_0}{a(t)}$, and $\Omega_{i0} = \frac{\rho_i}{\rho_{c0}}$. For $t = t_0$

$$\Omega_{\gamma 0} + \Omega_{\nu 0} + \Omega_{b0} + \Omega_{CDM0} + \Omega_{k0} + \Omega_{\Lambda 0} = 1, \tag{2.86}$$

where $\Omega_{m0} = \Omega_{b0} + \Omega_{CDM0}$ refers to the density of baryonic matter and cold dark matter respectively, and $\Omega_{r0} = \Omega_{\gamma 0} + \Omega_{\nu 0}$ refers to the density of photons and neutrinos[12] respectively. Therefore, considering H_0, we have 6 free parameters in the theory (since one of these parameters can be constrained by (2.86)).

Assuming a flat Universe ($\Omega_{k0} \approx 0$), with $T_{CMB} \approx 2.7255$ K, if we constrain $\Omega_{\Lambda 0}$ by Equation (2.86), we will have only 3 free parameters in the model, namely, H_0, Ω_{b0}, and Ω_{CDM0}. Recent observations [12] indicate that the abundances in the Universe today are approximately $\Omega_{b0} \approx 0.05$, $\Omega_{CDM0} \approx 0.26$, $\Omega_{k0} \approx 0$, $\Omega_{\Lambda 0} \approx 0.7$, with $H_0 \approx 67.4$ km/s/Mpc. In Chapter 4, we will see that we can estimate $\Omega_{\gamma 0}$, $\Omega_{\nu 0}$ from the temperature of CMB radiation today.

In summary, the Standard Model of Cosmology (ΛCDM) is the theory with material content (2.86), whose dynamics are determined by General Relativity (GR), satisfying the Cosmological Principle.

2.2.2 Comoving coordinates and evolution of one fluid

The comoving coordinate system is widely employed in Cosmology, offering the advantage that observers defining this coordinate system move with the expansion of the Universe in a comoving manner. This implies that the coordinates remain constant over time. Moreover, as we will see later, the scale relation between variables in this coordinate system becomes more evident in certain cases. The coordinates defining the observers are $(\tilde{r}, \theta, \phi)$, where $\tilde{r} = \frac{r(t)}{a(t)}$, with $r(t)$ being the physical coordinate, and $\tilde{r} = \frac{r(t_1)}{a(t_1)} = \frac{r(t_2)}{a(t_2)}$. The conformal time is defined as $d\eta \equiv \frac{dt}{a(t)}$, hence

$$\eta(t) - \eta(t_0) = \int_{t_0}^{t} \frac{dt}{a(t)}. \tag{2.87}$$

Geometrically, we are changing the foliation of the manifold M^4 using the function η as the global time, so that the tangent vector to comoving observers is $\tilde{n}_\mu = -\nabla_\mu \eta = \dot{\eta} n_\mu$, where $n_\mu = -c\nabla_\mu t$, hence $n^\mu \tilde{n}_\mu = \frac{1}{a(t)}$. We will use the notation $\mathcal{L}_{\tilde{n}} = \frac{\partial}{\partial \eta} \equiv \ '$ to denote the temporal derivative with respect to conformal time. Finally, we can express some

[12] We are considering relativistic neutrinos.

relevant quantities in the conformal coordinate system.

$$H = \frac{\dot{a}}{a} = \frac{a'}{a^2} = \frac{\mathcal{H}}{a} \tag{2.88}$$

$$\mathcal{H}^2 = a^2 \frac{\kappa\rho}{3} \tag{2.89}$$

$$\theta = \frac{3\mathcal{H}}{a} \tag{2.90}$$

$$\rho' = -3\mathcal{H}(\rho + p) \tag{2.91}$$

where $\mathcal{H} \equiv \frac{a'}{a}$ is the conformal Hubble function. Note that the last expression retains the same functional form under the choice of conformal time, which is expected since any monotonic time transformation $\left(\frac{d\tilde{t}}{dt} > 0\right)$ will preserve the form (2.91), hence preserving dynamics.

Another variable of interest, which measures the scale of causal interactions relative to the expansion of the universe and will be relevant when analyzing the horizon problem, is the Hubble radius, defined as

$$R_H = \frac{c}{H(t)} = \frac{1}{H(t)}, \tag{2.92}$$

its conformal version is

$$\tilde{R}_H = \frac{1}{\mathcal{H}}. \tag{2.93}$$

The Friedmann equation (2.85) normalized using the conformal Hubble function becomes

$$\tilde{R}_H^2 \mathcal{H}^2 = \frac{E^2}{x^2} = \Omega_{r0} x^2 + \Omega_{m0} x + \Omega_{k0} + \Omega_{\Lambda 0} x^{-2}. \tag{2.94}$$

We define $E = H/H_0$ before, in which $E(t_0) = 1$. From the above equation, we can express the evolution of the scale factor analytically with respect to conformal time. For this, we will use the fluid approximation, i.e., at different time scales x, one fluid dominates over the other $\Omega_{w_i} \gg \Omega_{w_j}$, hence we can consider only the contribution of the dominant fluid in the solution.

$$\frac{E}{x} = \sqrt{\Omega_{w0}}\, x^{\frac{1+3w}{2}} = \tilde{R}_H \mathcal{H} \tag{2.95}$$

$$x = \left[\frac{\tilde{R}_H \mathcal{H}}{\sqrt{\Omega_{w0}}} \right]^{\beta}, \tag{2.96}$$

where $\beta = \frac{2}{1+3w}$. From equation (2.78) in (2.94), we have $a \propto |\eta|^{\beta}$, thus $\mathcal{H} = \frac{\beta}{|\eta|}$, hence

$$x(\eta) = \left[\frac{\beta \tilde{R}_H}{\sqrt{\Omega_{w0}}\, |\eta|} \right]^{\beta}. \tag{2.97}$$

We thus have the evolution of the scale factor of the fluid with sound velocity ω_i as a function of conformal time. For the expansion phase $\eta > 0$.

2.2.3 Cosmological distances

Due to the expansion of the Universe, photons originating from some source in the Cosmos undergo the Doppler effect, whereby the frequency is modified. We observe this frequency shifting towards the red end of the electromagnetic spectrum, a phenomenon known as redshift. Let's deduce a relationship between the frequency of these photons and the scale factor.

Consider two isotropic observers, O and O', where one of these observers emits a photon following a light-like geodesic. The four-velocity for both observers is n^α, and the four-momentum of the photon is k_α. The frequency at the initial instant measured by one of the observers is

$$\omega = -k_\alpha n^\alpha, \tag{2.98}$$

differentiating the above expression with respect to an affine parameter λ of the light-like curve,

$$\begin{aligned} \frac{d\omega}{d\lambda} &= -k^\alpha \frac{dn_\alpha}{d\lambda} && (2.99) \\ &= -k_\alpha\, k^\beta \nabla_\beta u_\alpha && (2.100) \\ &= -k^\alpha\, k^\beta\, H\, h_{\beta\alpha} && (2.101) \\ &= -k^\alpha\, k_\alpha\, \frac{\dot{a}}{a} && (2.102) \\ &= -\frac{\dot{a}}{a}\, \omega^2 && (2.103) \end{aligned}$$

where we used (2.55), and $k^\alpha k_\alpha = \omega^2$. Solving explicitly for ω the above differential equation, we obtain

$$\frac{\omega(t)}{\omega_0} = \frac{a_0}{a(t)}, \tag{2.104}$$

where we used the method of separation of variables, and the relation $d\lambda = \frac{1}{\omega}\, dt$, obtained from (2.98) for normalized n^α. Equation (2.104) informs us that the frequency of the photon is inversely proportional to the scale factor; as previously stated, if the scale factor increases, then the frequency of the photon will decrease, consequently, the wavelength of the photon will increase, leading to the redshift effect[13].

We then see that the wavelength $\lambda \approx a(t)$, which justifies the energy density of the photon decreasing with a^{-4}, as besides the decrease in energy due to the increase in the volume of the three-dimensional spatial sections, we will have the Doppler effect decreasing its energy. The redshift variable is defined as

$$z \equiv \frac{\lambda_r - \lambda_e}{\lambda_e}, \tag{2.105}$$

[13] Note that in the calculation, we consider geodesic observers under the effect solely of the expansion of the Universe; if we consider some peculiar velocity due to some external effect, we may have the case of blueshift, that is, the wavelength of the photon will decrease, leading to a blue shift.

where λ_e, λ_r are the wavelength of the emitted and received photon respectively; we can then rewrite,

$$z = \frac{\omega_e}{\omega_r} - 1 = \frac{a(t_r)}{a(t_e)} - 1, \tag{2.106}$$

taking into account only the expansion of the Universe, we will always have $a(t_r) > a(t_e)$, hence $z > 0$. The redshift will be useful in some cases as a distance/time parameter. Note that if the time interval between the emission and reception of the photon is small, that is, $t_r - t_e = r$, where r is the physical distance between the two observers, then

$$\begin{aligned} a(t_r) &\approx a(t_e) + \dot{a}(t_e) r \Rightarrow \\ z &= \left.\frac{\dot{a}}{a}\right|_{t_e} r, \\ z &= H\, r. \end{aligned} \tag{2.107}$$

Equation (2.107) is the Hubble law, valid only for low redshifts, for example, for galaxies close to Earth.

2.2.4 Comoving and physical distances

To discuss certain cosmological issues, it is necessary to define cosmological distances. In particular, we are interested in the comoving distance on the hypersurface $\Sigma(t_0)$, which coincides with the current cosmological time. The distance that a photon travels from O (emission) to O′ (reception) is the length of the null geodesic described by the photon. We know that the metric is given by

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + d\Omega^2\right), \tag{2.108}$$

where, without loss of generality, we consider the photon's radial travel from observer O′ to observer O in the corresponding spatial section, hence $d\Omega^2 = 0$. Furthermore, the photon's geodesic is described by $ds^2 = 0$, thus yielding

$$\frac{dr}{\sqrt{1-Kr^2}} = \frac{dt}{a}. \tag{2.109}$$

Integrating both sides, we get

$$\int_{r_0}^{r} \frac{dr}{\sqrt{1-Kr^2}} = \int_{t_0}^{t} \frac{dt'}{a}. \tag{2.110}$$

We will solve the above integral for $K = -1, 0, 1$, with $r_0 = 0$, corresponding to the radial coordinate today:

$$\begin{cases} K = 0 & \Rightarrow r = \int_{t_0}^{t} \frac{dt'}{a} \equiv d_c \\ K = 1 & \Rightarrow r = \frac{\sin(\sqrt{K} d_c)}{\sqrt{K}} \\ K = -1 & \Rightarrow r = \frac{\sinh(\sqrt{K} d_c)}{\sqrt{K}} \end{cases} \tag{2.111}$$

where in the last case, we use the property $\sin(ix) = i\sinh(x)$. Let's focus on the distance in a Universe with $K = 0$, as our universe seems to satisfy this geometry on large scales. Note that we have calculated the comoving distance in a spatial section for a specific time ($\Sigma(t_0)$). We could calculate it in any section, as the notion of distance will change due to the expansion of the universe. Therefore, the physical distance is defined as

$$d = a(t)d_c, \tag{2.112}$$

thus allowing us to determine the physical distance in any section, i.e., at any cosmic time, by simply multiplying by the corresponding scale factor of the spatial section.

The ΛCDM model aligns with current observations on large scales, hence it serves as a suitable model. However, when we extrapolate its dynamics to the past ($z > 10^3$), we encounter some inconsistencies in the model.

2.3 Problems of ΛCDM model

2.3.1 Flatness problem

Given that $E^2(t)\Omega_i = \Omega_{i0}x^{3(1+w_i)}$ for a type of matter, we can normalize the matter density at the time of Nucleosynthesis ($z \approx 10^{11}$),

$$\Omega_{iN}x_N^{3(1+w_i)} = E_N^2(t)\Omega_i, \tag{2.113}$$

where $x_N = \dfrac{a_N}{a}$ and $E_N = \dfrac{H}{H_N}$. If we use the same normalized equation today for a fluid $\Omega_{i0}x^{3(1+w_i)} = E^2(t_0)\Omega_i$, we can compare the densities of the two epochs,

$$E_N^2(t_0)\Omega_{i0}x^{3(1+w_i)} = \Omega_{iN}x_N^{3(1+w_i)} \tag{2.114}$$

$$E_N^2(t_0)\Omega_{i0} = \Omega_{iN}\left(\frac{a_N}{a_0}\right)^{3(1+w_i)} = \frac{\Omega_{iN}}{(1+z_N)^{3(1+w_i)}}. \tag{2.115}$$

Similarly, for the curvature component, we have

$$E_N^2(t_0)\Omega_{k0} = \frac{\Omega_{kN}}{(1+z_N)^2}. \tag{2.116}$$

Recent observational measurements indicate that $\Omega_{k0} \ll \Omega_{i0}$, regardless of the matter content, thus

$$\frac{\Omega_{kN}}{(1+z_N)^2} \ll \frac{\Omega_{iN}}{(1+z_N)^{3(1+w_i)}} \tag{2.117}$$

$$\Omega_{kN} \ll \frac{\Omega_{iN}}{(1+z_N)^{1+3w_i}}. \tag{2.118}$$

Knowing that during Nucleosynthesis $z_N \approx 10^{11}$, the Universe was radiation-dominated $(w_r = \frac{1}{3})$ and $\Omega_{rN} \approx 1$, therefore

$$\Omega_{kN} \ll 10^{-22} \tag{2.119}$$

This implies that to justify the spatial curvature we measure today using the ΛCDM model, we conclude that the Universe at the time of Nucleosynthesis must have had a curvature much closer to zero[14]. This is the flatness problem, as we need to assume a specific initial condition for the curvature in the past to justify the current observational measurements of curvature.

2.3.2 Horizon problem

Modern Cosmology is built upon the pillars of homogeneity and isotropy of space. Homogeneity is directly observed in the CMB. However, the ΛCDM model does not account for this observation without an inflationary mechanism. This issue arises due to the presence of a particle horizon in the model, which is the maximum distance photons can travel, leading to causally disconnected regions. We will calculate the comoving horizon distance $d_{h,c}$, which is the distance a photon travels from the beginning of the Universe until the recombination period when photons decoupled from the primordial plasma [13]. Assuming $\Lambda = 0$ and $\Omega_{k0} = 0$, we have

$$d_{h,c} = \int_0^{t_r} \frac{dt}{a} = \int_{z_r}^{\infty} \frac{dz}{H(z)}, \tag{2.120}$$

where we use $dz = -\frac{H}{a}dt$. Let us approximate this as a fluid, specifically cold matter, where $H(z) = H_0\sqrt{\Omega_{m0}}(z+1)^{3/2}$,

$$d_{h,c} = \frac{1}{H_0\sqrt{\Omega_{m0}}} \int_{z_r}^{\infty} \frac{dz}{(z+1)^{3/2}} = \frac{2}{H_0\sqrt{\Omega_{m0}}} \left.\frac{1}{\sqrt{1+z}}\right|_{\infty}^{z_r} \approx \frac{2}{H_0\sqrt{\Omega_{m0}(1+z_r)}}, \tag{2.121}$$

then the physical distance at the recombination epoch ($z_r \approx 10^3$) will be,

$$\begin{aligned} d_{h,f} = a(z_r)d_{h,c} &= \frac{d_{h,c}}{1+z_r} \\ &= \frac{2}{H_0\sqrt{\Omega_{m0}}}\frac{1}{(1+z_r)^{3/2}} \approx \frac{2}{H_0\sqrt{\Omega_{m0}}10^{9/2}} \approx 0.02 \text{ rad} \approx 1.14°. \end{aligned} \tag{2.122}$$

where we used $\Omega_{m0} \approx 0.31$ and $H_0 = 67.4$ km/s/Mpc as discussed in section 2.2. We see that the angular distance of approximately 1.14° corresponds to the maximum distance two points in space could be causally connected at the time of recombination when observed today. However, what we observe in the CMB is homogeneity corresponding to an angular distance larger than this, meaning two distant points in the CMB are causally associated, even though the ΛCDM model indicates this is not possible. This is known as the horizon problem.

[14] In the calculation, we normalized at the Nucleosynthesis epoch, but if we assume a period with a redshift $z > 10^{11}$, we arrive at an even flatter curvature than obtained.

2.3.3 Large scale structures problem

When we observe the Universe today, we measure a correlation in the distribution of large-scale structures in different regions of space. To explain this correlation, we employ the theory of cosmological perturbations, specifically analyzing the dynamics of the scalar field and its evolution in the universe with a particular interest in the primordial universe. We will detail this construction in Chapter 3.

There is a range in Cosmology that we impose to associate causal interactions between different entities in an expanding universe, defined as sub-hubble. Below this limit, the field can be decomposed into Fourier modes with an associated wavelength, which must be smaller than the Hubble radius R_H, characterizing the limit. We will discuss this in more detail in the next chapter. For now, it is sufficient to say that these modes will have the form of a plane wave $\phi(x) \propto e^{i\omega\cdot x}$, where $\lambda = \frac{a}{|\omega|}$. Thus, to account for the observed correlation today, it is important to estimate the wavelength of this field in the primordial universe, as it determines the interaction scale.

The temporal evolution of the Hubble function for a fluid is given by

$$H = H_0 \sqrt{\Omega_{0i}}(z+1)^{\frac{3}{2}(1+w)}, \tag{2.123}$$

hence the Hubble radius

$$R_H(z) = \frac{R_{H0}}{(z+1)^{\frac{3}{2}(1+w)}}, \tag{2.124}$$

with $R_{H0} = \frac{1}{H_0\sqrt{\Omega_{0i}}} \approx \frac{1}{H_0}$ being the Hubble radius today. We will see in more detail that the correlation function is inversely proportional to the wavelength of the modes, and since $\lambda(t) = \lambda_0 a(t) = \frac{a_0\lambda_0}{1+z}$, taking $a_0 = 1$, we will verify the evolution of the causal scale of the modes for a fluid

$$\frac{\lambda}{R_H} = \frac{\lambda_0}{R_{H0}}(z+1)^{\frac{1+3w}{2}}, \tag{2.125}$$

note that if the modes are entering the Hubble radius today $\lambda \approx R_{H0}$ then $\lambda \gg R_H$ in the past. We refer to this limit as super-hubble, where different modes do not interact causally, with dynamics determined solely by the expansion of the Universe. We say that the modes are frozen in this limit.

If we assume that the modes are already super-hubble today $\lambda_0 \gg R_{H0}$, then in the past they will remain super-hubble. The same will occur if the modes are sub-hubble today $\lambda_0 \ll R_{H0}$.

Thus, the ΛCDM model is not sufficient to explain the correlation observed today, as it does not obtain the sub-hubble wavelength scale for the modes in the past, and therefore does not account for the causal interaction of the field with the material content.

2.3.4 Inflation

The Horizon problem can be visualized as follows: the comoving horizon distance of photons in the early universe (considering the instant of last scattering during recombination) is smaller than the comoving horizon distance of photons today, i.e., $d_{h,c}(t_{rec}) < d_{h,c}(t_0)$ (figure 3). Therefore, to solve this causal problem, we must have the condition of the inverted inequality, i.e., the comoving horizon of photons at recombination being greater than or equal to the comoving horizon of photons today: $d_{h,c}(t_{rec}) \geq d_{h,c}(t_0)$. From equation (2.120), we can rewrite

$$d_{h,c} = \int_0^{t_0} \frac{dt}{a} = \int_0^{t_0} \frac{1}{aH}\, dlna = \int_0^{t_0} \frac{1}{\mathcal{H}}\, dlna = \int_0^{t_0} R_{\mathcal{H}}\, dlna = \int_0^{t_0} \frac{1}{\dot{a}}\, dlna, \qquad (2.126)$$

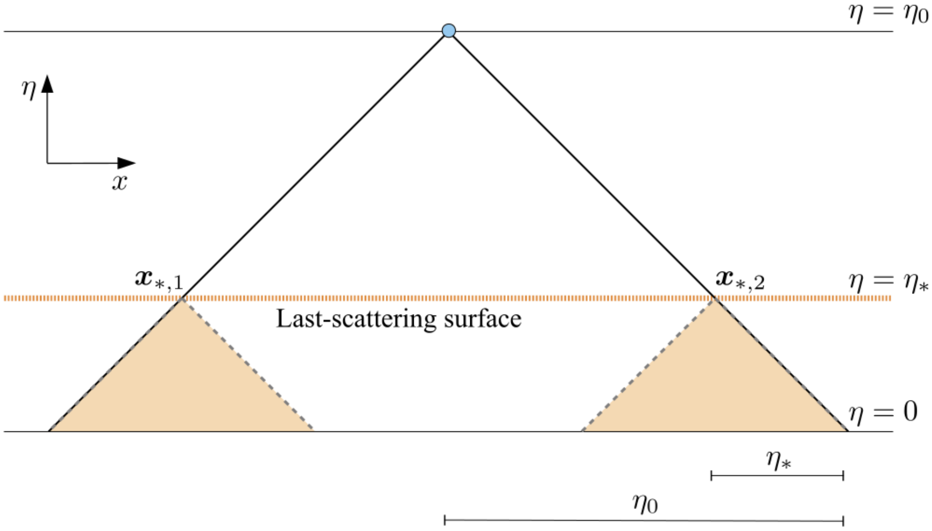
where we define the comoving Hubble radius as $R_{\mathcal{H}} = \dfrac{1}{\mathcal{H}}$. Assuming a universe predominantly composed of radiation and/or matter, equation (2.126) indicates that the largest contribution to the horizon size will be in the recent epochs of the universe $(a \approx a_0)$. This allows us to propose that the comoving horizon in the past was much larger than the comoving horizon today, and due to some physical mechanism, it drastically decreased, gradually increasing again until reaching the size we calculate in the present time.

For the comoving horizon given by the above expression to decrease, we must have $\dot{a}_{(t_1)} < \dot{a}_{(t_2)}$, with $t_1 < t_2$, thus, $\ddot{a} > 0$, i.e., the universe must undergo a phase of acceleration, which we refer to as inflation. One possible solution that allows for this accelerated expansion is given by [14]

$$a(t) = a_e\, e^{H_{inf}(t_e - t)}, \qquad (2.127)$$

with $t < t_e$, where t_e is the time corresponding to the end of inflation and $H_{inf} = \text{const}$ is the Hubble function during inflation.

Figure 3 – Diagram $x - \eta$ illustrating the horizon problem. Only signals from within the shaded regions below each point on the last-scattering surface could have influenced the CMB photons emitted from x_{*1} and x_{*2}. Since these regions do not overlap, no form of causal physics could have allowed them to adjust to the same temperature if they started from different temperatures. This is because the comoving horizon $d_{h,c}(rec) = \eta_*$ at the time the CMB was emitted is much smaller than our comoving horizon now, $d_{h,c}(t_0) = \eta_0$.



Source: Modern Cosmology [14].

Let us estimate how much the universe must have expanded so that we observe causally connected regions today. As previously mentioned, the condition for this to occur is $\dfrac{d_{h,c}(t_0)}{d_{h,c}(t_{rec})} < 1$. The distance in the numerator $d_{h,c}(t_0)$ was calculated in (2.121), thus, we only need to calculate the comoving horizon distance from the beginning of the Universe to recombination $d_{h,c}(t_{rec})$, using (2.120).

$$d_{h,c}(t_{rec}) = d_i + d_{i,e} + d_{e,rec}, \tag{2.128}$$

where d_i is the comoving distance of the photon from the beginning of the Universe to the start of inflation, $d_{i,e}$ is the distance of the photon during inflation, and $d_{e,rec}$ is the distance from the end of inflation to recombination. Assuming the Hubble function $H^2_{(x)}$ is continuous over each integration interval and that the Universe was radiation-dominated, using the variable $x = 1 + z$, we have

$$H^2 = \begin{cases} \Omega_{r0} H_0^4 \left(\dfrac{x_i}{x_e}\right)^4 x^4, & \infty > x \geq x_i \\ \Omega_{r0} H_0^4 x_e^4, & x_i \geq x \geq x_e \\ \Omega_{r0} H_0^4 x^4, & x_e \geq x \geq x_{rec}. \end{cases} \tag{2.129}$$

thus the distances are

$$d_i = \frac{1}{\sqrt{\Omega_{r0}}H_0}\left(\frac{x_i}{x_e}\right)^2 \int_{x_i}^{\infty} x^{-2}\, dx = \frac{1}{\sqrt{\Omega_{r0}}H_0}\frac{x_i}{x_e}^2, \tag{2.130}$$

$$d_{i,e} = \frac{x_e^{-2}}{\sqrt{\Omega_{r0}}H_0}\int_{x_e}^{x_i} dx = \left(\frac{x_i - x_e}{x_e^2}\right)\frac{1}{\sqrt{\Omega_{r0}}H_0}, \tag{2.131}$$

$$d_{e,rec} = \frac{1}{\sqrt{\Omega_{r0}}H_0}\int_{x_{rec}}^{x_e} x^{-2}\, dx = \frac{1}{\sqrt{\Omega_{r0}}H_0}\left(\frac{1}{x_{rec}} - \frac{1}{x_e}\right). \tag{2.132}$$

using $x_i \gg x_e \gg x_{rec}$

$$d_{h,c}(x_{rec}) \approx \frac{2}{\sqrt{\Omega_{r0}}H_0}\frac{x_i}{x_e^2}. \tag{2.133}$$

thus the ratio

$$\frac{d_{h,c}(t_0)}{d_{h,c}(t_{rec})} = \sqrt{\frac{\Omega_{r0}}{\Omega_{m0}}}\frac{x_e^2}{x_i} < 1, \tag{2.134}$$

using (2.127), we have $x_i = e^N x_e$, with $N = H_{inf}(t_i - t_e)$ representing the number of e-folds of accelerated expansion

$$\sqrt{\frac{\Omega_{r0}}{\Omega_{m0}}}\frac{x_e}{e^N} < 1 \Rightarrow N > \ln\left(\sqrt{\frac{\Omega_{r0}}{\Omega_{m0}}}x_e\right) \Rightarrow N > 21, \tag{2.135}$$

using $\sqrt{\frac{\Omega_{r0}}{\Omega_{m0}}} \approx 10^{-2}$, and considering that the end of inflation is approximately at Nucleosynthesis: $x_e \approx 10^{11}$. We see that the further back we place the end of inflation, the greater the number of e-folds N of accelerated expansion, thereby resolving the horizon problem due to the freedom in choosing the initial condition x_e, leading to an appropriate estimate of the number of e-folds.

Similarly, we can address the causal problem and the flatness problem. Let us consider the first case, given by (2.125), considering the Hubble radius during the inflation phase

$$\frac{\lambda}{R_{\mathcal{H}}} = \frac{\frac{\lambda_0}{x}}{\frac{1}{\mathcal{H}_0\sqrt{\Omega_{r0}}x_e^2}} = \frac{\lambda_0}{\tilde{R}_{\mathcal{H}}}\left(\sqrt{\Omega_{r0}}\frac{x_e^2}{x}\right), \tag{2.136}$$

Assuming the modes are entering the horizon today: $\lambda_0 \approx \tilde{R}_{\mathcal{H}}$, let us see how the modes behaved during the beginning of inflation

$$\frac{\lambda(x_i)}{R_{\mathcal{H}}} = \sqrt{\Omega_{r0}}\frac{x_e^2}{x_i} = \sqrt{\Omega_{r0}}x_e e^{-N}, \tag{2.137}$$

where $x_i = e^N x_e$, so $x_i \gg x_e$, therefore if we choose $N \approx 30$, at the beginning of inflation the modes will be subhorizon, thus having causal contact with other matter fields. Even if we consider the modes superhorizon today, they will have been subhorizon in the past by choosing an appropriate number of e-folds.

We can estimate the spatial curvature density at the beginning of inflation to address the flatness problem. Let us consider equation (2.113)

$$E_N^2(t_i)\Omega_{ki} = \frac{\Omega_{kN}}{x_N^2} = \Omega_{ke}\left(\frac{x_i}{x_e}\right)^2, \tag{2.138}$$

we are considering that the end of inflation is approximately at Nucleosynthesis, that is, $t_e \approx t_N$, hence $x_N(t_i) = \dfrac{a_e}{a_i}$ knowing that $E_N^2 = \Omega_{rN}x_e^4$, then

$$\Omega_{ki} = \frac{\Omega_{ke}}{\Omega_{rN}}\frac{x_i^2}{x_e^6} = \frac{\Omega_{ke}}{\Omega_{rN}}\frac{e^{2N}}{x_e^4}, \tag{2.139}$$

substituting the values $\Omega_{rN} \approx 1$, $x_e \approx 10^{11}$, we obtain

$$\Omega_{ki} \approx 10^{-44}e^{2N}\Omega_{kN} \approx 10^{16}\Omega_{kN} \gg \Omega_{kN}. \tag{2.140}$$

In which we chose $N = 30$, we thus see that the spatial curvature density at the beginning of inflation can be very large, but at the end of the process the Universe will have a spatial curvature close to zero, we can intuitively think that inflation leads to the homogeneity and isotropy observed in the primordial universe.

We see that inflation is a process that shifts the initial conditions of the ΛCDM model to the number of e-folds N, giving us the freedom to choose this parameter to resolve the above problems. However, for this, we need to introduce an additional entity in the model to justify this accelerated expansion process, which could be, for example, a scalar field. Let us explore another mechanism that will address the aforementioned problems without adding something new to the model, using only the material content we know in the Universe today.

It is important to emphasize that inflation does not completely resolve the cited problems but mitigates them by transferring the fine-tuning problem of the initial conditions to the initial conditions of inflation.

2.3.5 Bounce

In addition to the cosmological issues mentioned in the previous section, there is another problem with the model: the cosmological singularity. This is an obstacle to General Relativity and is not a specific characteristic of Cosmology. Einstein's equations are not valid at $t = 0$, where t is cosmic time. This means the dynamic law we use to describe the universe fails at its beginning. From the Friedmann equations (2.78), we can see that for $\omega_i > -1$, when $a \to 0$, we have $\rho \to \infty$.

We will not explore the bounce mechanism in detail here; for a review on this topic, see [15, 16]. The central idea of this mechanism is that the universe begins with a given size with scale factor $a(t_i) = a_i$ and material content that we observe today in the standard model, and goes through a contraction phase until it reaches a minimum size $a(t_b) = a_b < a_i$, where it undergoes a bounce and enters the expansion phase[15]. Two classes of bounces are known in the literature: quantum bounce and classical bounce. We will be more interested in the former, which arises due to quantum gravitational effects.

We will not focus on how this mechanism is generated but rather describe the classical evolution of the universe in the contraction and expansion regimes to address the aforementioned problems. Let the normalized Hubble function be

$$E = \begin{cases} -\sqrt{\Omega_{r0}}x^2, & x_i > x > x_b, \\ \sqrt{\Omega_{r0}}x^2, & x_b > x \end{cases} \tag{2.141}$$

where $x_b = x(t_b)$, with t_b being the time when the bounce occurs, and we are considering a radiation-dominated universe near the bounce. The distance traveled by a photon during the contraction period is

$$d_p = \int_{t_i}^{t_b} \frac{dt_1}{a(t_1)} = \int_{x_b}^{x_i} \frac{dx_1}{H(x_1)} \tag{2.142}$$

$$= \frac{1}{H_0\sqrt{\Omega_{r0}}}\left(\frac{1}{x_i} - \frac{1}{x_b}\right) \approx \frac{1}{H_0\sqrt{\Omega_{r0}}}\frac{1}{x_i}. \tag{2.143}$$

Imposing that $x_b \gg x_i$, i.e., the scale factor at the bounce is much smaller than the initial scale factor, note that if we assume a very large universe, the distance traveled by photons becomes infinite: $\lim_{x_i \to 0} d_p = \infty$.

Using this condition, the ratio (2.134) will be approximately[16]

$$\frac{d_{h,c}(t_0)}{d_{h,c}(t_{rec})} \approx x_i = \frac{a_0}{a_i}, \tag{2.144}$$

we just need to assume that $a_i > a_0$, that is, the universe started with a size greater than what we observe today, to have a ratio less than 1 and thus solve the horizon problem.

Similarly, from equation (2.136), it is possible to solve the large-scale structure problem. At the beginning of the contraction phase, we have $\lambda/\tilde{R}_{\mathcal{H}} = \frac{\lambda_0}{\tilde{R}_{\mathcal{H}}}\sqrt{\Omega_{r0}}x_i$, so the modes will be in the sub-Hubble limit just by imposing that $a_i > a_0$.

Assuming that at the beginning of the contraction phase the curvature is of the order of $\Omega_{k0}x_i^2 \approx 1$, we verify from the equation (2.138) normalized today that $\Omega_{k0} \approx x_i^2$, hence $a_i \gg a_0$ would lead to a very small curvature today, solving the flatness problem.

Therefore, the bounce model for the primordial universe appears to be reasonable. Besides solving some of the mentioned cosmological problems, the model does not

[15] At the moment of the bounce, the universe has a finite volume, thus avoiding the cosmological singularity

[16] In the limit where $a_i \to \infty$, we should use the dust approximation for a complete description, but the conclusion would be the same.

invoke additional content in a specific phase of the universe as in the case of inflation (inflaton).

However, up to the present moment, the quantum bounce model has the theoretical difficulty of predicting a power spectrum suitable for observations, making it a potential topic for future research [7].

3 Cosmological Perturbation theory

3.1 Gauge transformations

The universe on large scales satisfying the FLRW metric will be isotropic and homogeneous, but this result comes from a theoretical hypothesis that is an ideal case with limitations to describe some observations of the universe today.

As two examples, it is well known in the recombination epoch the signatures of anisotropies on CMB can not be explained only with the ΛCDM model, in addition, one needs to explain the large-scale structure formation in the universe, and how the inhomogeneities evolve in time leading to the inhomogeneous small scales structures that we observe nowadays, such as galaxies, and cluster of galaxies [17].

Furthermore, a perturbative approach to Einstein's equations is necessary to get a complete understanding of these topics. In this chapter our objective is to write the equations of motions linearized of these perturbations, in particular, we will focus on the scalar perturbation since its dynamic will describe the Silk-damping effect on the primordial plasma, which includes the photons of CMB, we will explore it with more details in chapter 5. We will follow the formalism of cosmological perturbation theory developed in [18], the major results can be found there, here we will develop the calculus and interpret them.

A direct way to study these a perturbation on cosmological scales is by performing a perturbation around an FLRW universe. We will choose a pseudo-Riemannian background manifold (M, g) as the isotropic and homogeneous universe, where g is the FLRW metric, and a physical manifold $(\widehat{M}, \hat{g})$, in which $\hat{g}$ is the physical metric or the perturbated metric.

The manifold $\widehat{M}$ is where the physical geometric objects will be constructed. One may associate quantities between M and $\widehat{M}$ by a diffeomorphism $\Omega : M \to \widehat{M}$, then a background quantity $Q \in X(M)$ can be taken to $\widehat{M}$ using the pull-back of $\Omega^{-1} : \widehat{M} \to M$ [1]

$$\widehat{Q} = (\Omega^{-1})^{*}Q, \tag{3.1}$$

we define the perturbation of Q as

$$\delta Q = \widehat{Q} - Q \tag{3.2}$$

where $Q \in \mathcal{X}(M)$ is the physical quantity defined on the physical manifold. The central hypothesis is that the perturbation above is "small", there is a problem with the validity

[1] See the appendix A for the definitions of the mathematical objects used in this chapter.

of this hypothesis in some gauges choices, see [19] for more details. However, in this work, we will assume that it is valid in the situations of our interest.

Let's assume a vector field $V^\mu \in T_pM$, we can define a flux as a one-parameter diffeomorphism group $\phi_t : M \to M$ generated by this field, in such a way that V^μ is tangent to the curves $\lambda(t) : \mathbb{R} \to M$

$$\Omega_t \equiv \phi_t(\Omega(p)), \tag{3.3}$$

with $p \in M$, doing a diffeomorphism on the metrics

$$g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_V g_{\mu\nu} = g^{(t)}_{\mu\nu}, \tag{3.4}$$

$$\hat{g}_{\mu\nu} \to \hat{g}_{\mu\nu} \tag{3.5}$$

then the perturbation

$$\delta g^{(t)}_{\mu\nu} = \hat{g}^{(t)}_{\mu\nu} - g^{(t)}_{\mu\nu} = \hat{g}_{\mu\nu} - g_{\mu\nu} - \mathcal{L}_V g_{\mu\nu} \tag{3.6}$$

$$\delta g^{(t)}_{\mu\nu} = \delta g_{\mu\nu} - \mathcal{L}_V g_{\mu\nu} \tag{3.7}$$

Calculating the lie derivative of the metric above

$$\begin{aligned} \mathcal{L}_V g_{\mu\nu} &= V^\sigma \nabla_\sigma g_{\mu\nu} + g_{\mu\sigma}\nabla_\nu V^\sigma + g_{\sigma\nu}\nabla_\mu V^\sigma \\ &= \nabla_\nu V^\mu + \nabla_\mu V_\nu = 2\nabla_{(\mu}V_{\nu)}, \end{aligned} \tag{3.8}$$

where we used Cartan's formula on the first line, and the second line we used $\nabla_\sigma g_{\mu\nu} = 0$ and lowered the indices of the vector fields with the metric. Therefore, if we want the perturbation not depend on the arbitrary vector field V^μ, one needs to impose that $\mathcal{L}_V g_{\mu\nu}$ is very small, or V^μ be a killing vector field, i.e., $\mathcal{L}_V g_{\mu\nu} = -2\nabla_{(\mu}V_{\nu)} = 0$.[2]

In what follows sometimes we will refer to V^μ as a gauge, hence the perturbation is gauge invariant when the conditions above are satisfied. The metric on M does not change on that diffeomorphism, we can construct a general transformation, thinking in a coordinate transformation on $\widehat{M}$ generated by a vector field $B^\mu \in \widehat{M}$, and combining the diffeomorphism with the coordinate change

$$g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_B g_{\mu\nu}, \tag{3.9}$$

$$\hat{g}_{\mu\nu} \to \hat{g}_{\mu\nu} + \mathcal{L}_B \hat{g}_{\mu\nu} = \hat{g}_{\mu\nu} + \mathcal{L}_B g_{\mu\nu}, \tag{3.10}$$

on the second transformation, we ignore the term $\mathcal{L}_B \delta g_{\mu\nu}$ assuming B^μ as the same order of the perturbation. Performing a combination of the transformations: coordinate change + diffeomorphism:

$$g^{(t)}_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_B g_{\mu\nu} + \mathcal{L}_V(g_{\mu\nu} + \mathcal{L}_B g_{\mu\nu}) = g_{\mu\nu} + \mathcal{L}_{(B+V)} g_{\mu\nu}; \tag{3.11}$$

$$\hat{g}^{(t)}_{\mu\nu} \to \hat{g}_{\mu\nu} + \mathcal{L}_B g_{\mu\nu}, \tag{3.12}$$

[2] Henceforward we will use the background metric to rise and lower indices of the tensors in M.

if we set the field $B^{\mu} = -V^{\mu}$, then

$$g_{\mu\nu} \to g_{\mu\nu}; \tag{3.13}$$

$$\hat{g}_{\mu\nu} \to \hat{g}_{\mu\nu} - \mathcal{L}_V g_{\mu\nu}. \tag{3.14}$$

We define the transformations (3.13) and (3.14) as gauge transformations on the metrics, thus in general, for any quantity $\widehat{Q} \in \widehat{M}$, and $Q \in M$, a gauge transformation will be

$$Q^{\mu_1\mu_2\ldots\mu_n}_{\quad\nu_1\nu_2\ldots\nu_l} \to Q^{\mu_1\mu_2\ldots\mu_n}_{\quad\nu_1\nu_2\ldots\nu_l}; \tag{3.15}$$

$$\widehat{Q}^{\mu_1\mu_2\ldots\mu_n}_{\quad\nu_1\nu_2\ldots\nu_l} \to \widehat{Q}^{\mu_1\mu_2\ldots\mu_n}_{\quad\nu_1\nu_2\ldots\nu_l} - \mathcal{L}_V \widehat{Q}^{\mu_1\mu_2\ldots\mu_n}_{\quad\nu_1\nu_2\ldots\nu_l}, \tag{3.16}$$

the background quantity Q will stay fixed by a gauge transformation, while the physical quantity $\hat{Q}$ will change. As we can see, the gauge transformation (3.16) depends on the vector field V^{μ}, we say this transformation is gauge dependent. The metric perturbation after a gauge transformation $\widetilde{\delta g}_{\mu\nu}$ will be

$$\begin{aligned} \widetilde{\delta g}_{\mu\nu} = \tilde{\hat{g}}_{\mu\nu} - \tilde{g}_{\mu\nu} &= \hat{g}_{\mu\nu} - \mathcal{L}_V g_{\mu\nu} - g_{\mu\nu} \\ &= \delta g_{\mu\nu} - \mathcal{L}_V g_{\mu\nu}, \end{aligned} \tag{3.17}$$

defining $\Delta(\delta g_{\mu\nu}) \equiv \widetilde{\delta g}_{\mu\nu} - \delta g_{\mu\nu}$, then

$$\Delta(\delta g_{\mu\nu}) = -\mathcal{L}_V g_{\mu\nu}. \tag{3.18}$$

Furthermore, the perturbation $\delta g_{\mu\nu}$ will be gauge invariant (GI), if the vector field V is a killing vector field, or the background metric satisfying

$$\mathcal{L}_V g_{\mu\nu} = 0 \Leftrightarrow \begin{cases} g_{\mu\nu} = 0; \\ g_{\mu\nu} \propto k\,\delta_{\mu\nu}, \quad k \in \mathbb{R}. \end{cases} \tag{3.19}$$

The result (3.19) is called the Stewart-Walker(SW) lemma applied to the metric tensor,[3] a necessary condition for a GI tensor [20]. Although was shown in [21], that this lemma is valid only for pure tensors, and not for general ones, we will comment on these tensors more ahead in this section.

Assuming the existence of an inverse $\hat{g}^{\mu\nu}$, such that

$$\begin{aligned} \hat{g}^{\mu\sigma}\hat{g}_{\sigma\nu} &= \delta^{\mu}_{\ \nu}, \\ (g^{\mu\sigma} + \delta g^{\mu\sigma})(g_{\nu\sigma} + \delta g_{\nu\sigma}) &= \delta^{\mu}_{\ \nu}, \\ g_{\nu\sigma}\delta g^{\mu\sigma} &= -g^{\mu\sigma}\delta g_{\nu\sigma}, \\ \delta g^{\mu\beta} &= -g^{\nu\beta}g^{\mu\sigma}\delta g_{\nu\sigma}, \end{aligned} \tag{3.20}$$

on the third line, we exclude the second-order terms of the perturbations, and on the last line, we multiply both sides by $g^{\nu\beta}$. The equation (3.20) shows us that the metric

[3] This result is valid for any pure tensors defined in $\widehat{M}$.

perturbation gets a sign when one rises or lowers its indices, it will be important because all the modes we get from the metric perturbation follow this property[4].

The most general decomposition of the physical metric on M is

$$ds^2 = \hat{g}_{\mu\nu}dx^\mu dx^\nu = a^2(\eta)\left[-(1-2\phi)d\eta^2 + 2B_i dx^i d\eta + (h_{ij} + 2C_{ij})dx^i dx^j\right], \tag{3.21}$$

we are using the conformal time parametrization, $A(\eta, \vec{x}), B_i(\eta, \vec{x}), C_{ij}(\eta, \vec{x})$ are tensors in $\widehat{M}$, with $1+3+6$ degrees of freedom respectively, totalizing 10 degrees of freedom as expected, sometimes we will refer to these tensors as perturbative modes. The perturbation on the metric in a matrix representation is

$$[\hat{g}_{\mu\nu}] = \begin{bmatrix} -1+2\phi & B_i \\ B_i & h_{ij}+2C_{ij} \end{bmatrix}, \tag{3.22}$$

$$[g_{\mu\nu}] = \begin{bmatrix} -1 & 0 \\ 0 & h_{ij} \end{bmatrix}, \tag{3.23}$$

$$[\delta g_{\mu\nu}] = \begin{bmatrix} 2\phi & B_i \\ B_i & 2C_{ij} \end{bmatrix}, \tag{3.24}$$

in which we are using a geodesic sectioning on M with the normalized covetorial field $n_\mu = -\nabla_\mu t$ as we did in chapter 2. And the gaussian coordinate system given by $e^{\mu}_{\ i}$, with $i = 1, 2, 3, \mu = 0, 1, 2, 3$ such that $h[\mathcal{L}_n e^{\mu}_{\ i}] = \dot{e}^{\mu}_{\ i} = 0$. The metric perturbation may be decomposed using a background covector field $\bar{n}_\mu$

$$\delta g_{\mu\nu} = 2\phi n_\mu n_\nu + 2B_{(\mu}\bar{n}_{\nu)} + 2C_{\mu\nu} \tag{3.25}$$

wherein these quantities are defined as

$$\phi \equiv \frac{1}{2}\delta g_{nn}, \tag{3.26}$$

$$B_\mu \equiv h[\delta g_{\mu n}], \tag{3.27}$$

$$C_{\mu\nu} \equiv \frac{1}{2}h[\delta g_{\mu\nu}], \tag{3.28}$$

in which $h_{\mu\nu} \equiv n_\mu n_\nu + g_{\mu\nu}$.

To conclude this section, in the context of perturbations, it is valid to define the two classes of tensors: pure tensors and mixed tensors as the authors of [21] did. The pure tensors are defined only with physical tensors in $\widehat{M}$, while the mixed tensors are defined as a combination of tensors of $\widehat{M}$ and M.

This classification will be important to understand some perturbations' nature and apply the SW lemma if it is the case. In [21], the authors define the following

4 Note that $\delta g^{\mu\beta}$ is not the inverse of $\delta g_{\mu\beta}$, because (3.20) is only valid on first order.

tensors:

$$\widehat{\mathcal{P}} \equiv \frac{\hat{g}_{\mu\nu}n^{\mu}n^{\nu}}{2} \to \mathcal{P} = \frac{g_{\mu\nu}n^{\mu}n^{\nu}}{2} = -\frac{1}{2}; \tag{3.29}$$

$$\widehat{\mathcal{P}}_{\mu} \equiv h[\hat{g}_{\mu n}] \to \mathcal{P}_{\mu} = h[g_{\mu n}] = 0; \tag{3.30}$$

$$\widehat{\mathcal{P}}_{\mu\nu} \equiv \frac{h[\hat{g}_{\mu\nu}]}{2} \to \mathcal{P}_{\mu\nu} = \frac{h[g_{\mu\nu}]}{2} = \frac{h_{\mu\nu}}{2}. \tag{3.31}$$

Wherein on the right sides of the arrows are the background versions of the quantities, one could define these background versions by changing the physical metric $\hat{g}_{\mu\nu}$ to the background metric $g_{\mu\nu}$. Furthermore, the perturbations (3.26),(3.27) and (3.28) may defined as

$$\phi \equiv \widehat{\mathcal{P}} - \mathcal{P}, \tag{3.32}$$

$$B_{\mu} \equiv \widehat{\mathcal{P}}_{\mu} - \mathcal{P}_{\mu}, \tag{3.33}$$

$$C_{\mu\nu} \equiv \widehat{\mathcal{P}}_{\mu\nu} - \mathcal{P}_{\mu\nu}. \tag{3.34}$$

Analogously a simple perturbation is defined as a perturbation similarly to (3.17), furthermore, the physical tensor is defined by physical geometric objects in $\widehat{M}$ merely. Otherwise, if the physical tensor on the perturbation is constructed as combinations of physical tensors in $\widehat{M}$ and background tensors in M then we say it is a mixed perturbation, thus the SW lemma is not valid for these tensors.

For example, the perturbations defined in (3.32), (3.33), and (3.34), are all mixed perturbations since they mixed the background covector field n_{μ} defined in T_pM with the physical metric $\hat{g}_{\mu\nu}$ defined on $\widehat{M}$.

3.2 Gauge invariant variables

The perturbation approach to general relativity has an intrinsic problem associated with the gauge transformations of metric and matter variables. As we will see in this section the perturbations will depend on the gauge explicitly. This feature comes from the fact that GR is a gauge theory, the gauge group being the diffeomorphism group Diff(M) of the spacetime manifold M [22]. Thus a possible way to contour this difficulty is to construct gauge invariant quantities and use them in the linear Einstein's equations. In order to interpret the perturbations and get predictions of the theory we need to choose a specific gauge. [5]

An advantage of working with the FLRW spacetime as a background manifold is the possibility of performing the scalar, vectorial, and tensorial (SVT) decomposition of the modes of the physical metric (3.21).

[5] The GI quantities will continue being GI after a specific gauge choice.

This decomposition was made for the first time by Bardeen, J. in 1980 [23], and will allow us to explore the dynamics of these modes separately. One can decompose the vectorial mode B_μ as

$$B_\mu = D_\mu \mathcal{B} + b_\mu, \tag{3.35}$$

where $D^\mu b_\mu = 0$, i.e., b_μ is divergence free, and $\mathcal{B}$ is an arbitrary scalar function. Taking the divergence of (3.35) we get

$$D^\mu B_\mu = D^2 \mathcal{B}, \tag{3.36}$$

$$\mathcal{B} = D^{-2} D^\mu B_\mu, \tag{3.37}$$

in which D^2 is the Laplace operator on the spatial sections, and D^{-2} its inverse associated with the Green's function.

At first, the mode $\mathcal{B}$ is not unique, because we can sum an arbitrary homogeneous function $u(x,\eta)$, performing $\mathcal{B} \to \mathcal{B} + u$, with $D^2 u = 0$. However, suppose the spatial sections are a compact manifold equipped with a Riemann metric, or there a boundary conditions in which the function u is zero. In that case, one can affirm that $\mathcal{B}$ is uniquely determined [24].

Consequently, the covector field b_μ=$B_\mu - D_\mu D^{-2} D^\nu B_\nu$ will be a divergent free field uniquely determined. So this guarantees the unicity of the decomposition (3.35).

In the same way, we can decompose the tensor $C_{\mu\nu}$ in SVT modes, its decomposition is more complicated, so it will be useful to write this tensor as

$$C_{\mu\nu} = C^t_{\mu\nu} + C\frac{h_{\mu\nu}}{3}, \tag{3.38}$$

in which $C^t_{\mu\nu}$ and C is the traceless part and the trace of $C_{\mu\nu}$ respectively. Now we can decompose $C^t_{\mu\nu}$ in its general form

$$C^t_{\mu\nu} = \left(D_\mu D_\nu - \frac{h_{\mu\nu} D^2}{3} \right) \mathcal{E} + D_{(\mu} F_{\nu)} + W_{\mu\nu}, \tag{3.39}$$

where $D^\mu F_\mu = D^\mu W_{\mu\nu} = W_\mu^{\ \mu} = 0$. Performing the divergence two times on (3.39) in the same way we did for the vector field, one can check that $\mathcal{E}$, F_ν and $W_{\mu\nu}$ are uniquely determined, see [18] for more details. Therefore, we can write

$$C_{\mu\nu} = \psi h_{\mu\nu} - D_\mu D_\nu \mathcal{E} + D_{(\mu} F_{\nu)} + W_{\mu\nu}, \tag{3.40}$$

where $\psi \equiv \frac{1}{3}(C + D^2 \mathcal{E})$.

These decompositions will simplify the linear differential equations involving the perturbations of (3.21), once the modes are uniquely determined we still have the 10 degrees of freedom on the modes:

$$\textbf{Scalars: } \psi,\ \mathcal{E},\ C,\ \mathcal{B}; \tag{3.41}$$

$$\textbf{Vectors: } F_\mu,\ b_\mu; \tag{3.42}$$

$$\textbf{Tensors: } W_{\mu\nu}. \tag{3.43}$$

For vectors and tensor modes, there is no time degree of freedom, even though μ,ν are running to 0,1,2,3. The vector modes have 2 degrees of freedom each, because they have null divergence, and the tensor $W_{\mu\nu}$ has 2 degrees of freedom because it has null divergence and is traceless.

We are interested in finding the gauge transformations of the modes above, before that, we should calculate the algebra of the operators D_μ, D^2, and ∂_t when acting on tensors[6]. Performing the first relation between the time derivative operator and the covariant spatial derivative on scalars, we get

$$D_\mu \partial_t \phi = h[\nabla_\mu \partial_t \phi] = h[\nabla_\mu (n^\sigma \nabla_\sigma \phi)] \tag{3.44}$$

$$= h[\mathcal{K}_\mu{}^\sigma \nabla_\sigma \phi + n^\sigma \nabla_\sigma \nabla_\mu \phi] = \mathcal{K}_\mu{}^\sigma D_\sigma \phi + \nabla_n D_\mu \phi \tag{3.45}$$

$$[D_\mu, \partial_t]\phi = 0, \tag{3.46}$$

in which we commuted the spatial derivatives when acting on ϕ, and $\mathcal{K}_\mu{}^\sigma D_\sigma \phi = 0$. The algebra between the contravariant spatial derivative with the time derivative on scalars will be

$$\partial_t D^\mu \phi = \partial_t (h^{\mu\sigma} D_\sigma \phi) = \dot{h}^{\mu\sigma} D_\sigma \phi + h^{\mu\sigma} \partial_t (D_\sigma \phi) \tag{3.47}$$

$$= 2\mathcal{K}^{\mu\sigma} D_\sigma \phi + D^\mu \dot{\phi} \tag{3.48}$$

$$[D^\mu, \partial_t]\phi = -\frac{2}{3}\theta D^\mu \phi. \tag{3.49}$$

Wherein we use (3.46), and (2.55), using this result we can calculate

$$\partial_t D^2 T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l} = \partial_t (h_{\mu\nu} D^\mu D^\nu T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l}) \tag{3.50}$$

$$= \dot{h}_{\mu\nu} D^\mu D^\nu T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l} + h_{\mu\nu} \partial_t (D^\mu D^\nu T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l}) \tag{3.51}$$

$$= \frac{2}{3}\theta D^2 T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l} + h_{\mu\nu} \left[D^\mu D^\nu \partial_t T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l} - \frac{2}{3}\theta D^2 T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l} \right] \tag{3.52}$$

$$= D^2 \partial_t T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l} - \frac{2}{3}\theta D^2 T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l} \tag{3.53}$$

$$[\partial_t, D^2] T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l} = -\frac{2}{3}\theta D^2 T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_l}. \tag{3.54}$$

On the third line, we used (3.49) to commute the operators two times. Thus the last convenient bracket to calculate is between D_μ and D^2 on scalars

$$[D_\mu, D^2]\phi = [D_\mu, D^\sigma D_\sigma]\phi = [D_\mu, D_\sigma] D^\sigma \phi - D_\sigma [D_\mu, D^\sigma]\phi \tag{3.55}$$

$$= \mathcal{R}_{\mu\sigma}{}^{\sigma\nu} D_\nu \phi = -\mathcal{R}_{\sigma\mu}{}^{\sigma\nu} D_\nu \phi = -\mathcal{R}_\mu{}^\nu D_\nu \phi = -2K D_\mu \phi \tag{3.56}$$

[6] We will not present the action of these algebras in general tensors, only the algebra that we will use further in this work

in which in the first line the the second term of the right side of the equation is null since we are taking the spatial derivative of the Riemann tensor in homogeneous and isotropic spatial sections, on the second line we used the antisymmetry property of Riemann tensor, and the definition of the Ricci tensor in FLRW geometry.

A useful result that we will use more ahead, is the double divergence of $C_{\mu\nu}$, defined in (3.40)

$$D^\mu D^\nu C_{\mu\nu} = D^2\psi - D^\mu D^2 D_\mu \mathcal{E} + \frac{1}{2}\left(D^\mu D^\nu D_\mu F_\nu + D^\mu D^2 F_\mu\right) \tag{3.57}$$

$$= D^2\psi - D^\mu(2KD_\mu\mathcal{E} + D_\mu D^2\mathcal{E}) \tag{3.58}$$

$$= D^2\psi - D^2(D^2 + 2K)\mathcal{E}, \tag{3.59}$$

where on the first line the third term is zero since

$$D^\mu D^\nu D_\mu F_\nu = D^2 D^\nu F_\nu + D^\mu \mathcal{R}_{\mu\nu}{}^{\nu\alpha} F_\alpha = 0, \tag{3.60}$$

$$D^\mu D^2 F_\mu = -2KD^\mu F_\mu + D^2 D^\mu F_\mu = 0. \tag{3.61}$$

We should use (3.8) to perform the gauge transformation of the perturbative modes. Is interesting to make a decomposition of the gauge V^μ, to analyze how the gauge acts in different modes,

$$V_\mu = V^{\parallel} n_\mu + h[V_\mu] = V^{\parallel} n_\mu + D_\mu V^\perp + V^\perp_\mu, \tag{3.62}$$

where $V^{\parallel} = -n^\mu V_\mu$, and we did the decomposition of the spatial part as in (3.35), with $D^\mu V^\perp_\mu = 0$.

Is straightforward to calculate the components $\widetilde{\delta g_{nn}}$, $\widetilde{\delta g_{\mu n}}$, and $\widetilde{\delta g_{\mu\nu}}$ using (3.8) and (3.62):

$$\begin{aligned}\widetilde{\delta g_{nn}} &= \delta g_{nn} - 2\nabla_n V_n \\ &= 2\phi + 2\dot{V}^{\parallel};\end{aligned} \tag{3.63}$$

$$\begin{aligned}h[\widetilde{\delta g_{\mu n}}] &= h[\delta g_{\mu n}] - 2h[\nabla_{(\mu} V_{n)}] \\ &= -B_\mu + h_\mu^\alpha[\nabla_\alpha V_n + \nabla_n V_\alpha] \\ &= -B_\mu - \partial_t(D_\mu V^\perp + V^\perp_\mu) + D_\mu V^{\parallel};\end{aligned} \tag{3.64}$$

$$\begin{aligned}h[\widetilde{\delta g_{\mu\nu}}] &= h[\delta g_{\mu\nu}] - 2h[\nabla_{(\mu} V_{\nu)}] \\ &= 2C_{\mu\nu} - h_\mu^\alpha h_\nu^\beta(\nabla_\alpha V_\beta + \nabla_\beta V_\alpha) \\ &= 2C_{\mu\nu} - 2V^{\parallel}\mathcal{K}_{\mu\nu} - 2D_{(\mu}D_{\nu)}V^\perp - 2D_{(\mu}V^\perp_{\nu)},\end{aligned} \tag{3.65}$$

we may rewrite the transformations of ϕ, B_μ and $C_{\mu\nu}$ explicitly

$$\phi \longrightarrow \phi + \dot{V}^{\parallel}; \tag{3.66}$$

$$B_\mu \longrightarrow B_\mu + \partial_t(D_\mu V^\perp + V^\perp_\mu) - D_\mu V^{\parallel}; \tag{3.67}$$

$$C_{\mu\nu} \longrightarrow C_{\mu\nu} - V^{\parallel}\mathcal{K}_{\mu\nu} - D_{(\mu}(D_{\nu)}V^\perp + V^\perp_{\nu)}). \tag{3.68}$$

We are at one step to find the gauge transformations of the modes decomposed earlier, we must use the equations (3.35), (3.40), and the transformations above. The strategy to get the desired transformations will be performing the divergences of (3.67) and (3.35), and using the inverse of the operators D^2 and $D^2(D^2+2K)$ on the spatial sections.

The first transformation is already determined (3.66), taking the divergence of (3.67), and using the decomposition (3.35) we obtain

$$D^2\widetilde{\mathcal{B}} = D_\mu B^\mu + D_\mu \partial_t (D^\mu V^\perp + V^{\perp\mu}) - D^2 V^{\parallel} \tag{3.69}$$

$$D^2\widetilde{\mathcal{B}} = D_\mu B^\mu - \frac{2}{3}\theta D^2 V^\perp + D^2\partial_t(V^\perp) + +D_\mu\partial_t V^{\perp\mu} - D^2 V^{\parallel} \tag{3.70}$$

$$D^2\widetilde{\mathcal{B}} = D_\mu B^\mu - \frac{2}{3}\theta D^2 V^\perp + D^2\dot{V}^\perp - D^2 V^{\parallel} \tag{3.71}$$

$$\widetilde{\mathcal{B}} = \mathcal{B} - \frac{2}{3}\theta V^\perp + \dot{V}^\perp - V^{\parallel}, \tag{3.72}$$

wherein we use the null divergence of the spatial vector gauges, and the relations (3.49), (3.54). Following the same procedure to $\widetilde{C}_{\mu\nu}$, performing the double divergence

$$D^\mu D^\nu \widetilde{C}_{\mu\nu} = D^\mu D^\nu C_{\mu\nu} - V^{\parallel} D^\mu D^\nu \mathcal{K}_{\mu\nu} - \mathcal{K}_{\mu\nu} D^\mu D^\nu V^{\parallel} - \tag{3.73}$$

$$-\frac{1}{2}(D^\mu D^\nu D_\mu D_\nu V^\perp + D^\mu D^2 D_\mu V^\perp) - \frac{1}{2}(D^\mu D^\nu D_\mu V^\perp_\nu + D^\mu D^2 V^\perp_\mu) \tag{3.74}$$

$$= D^\mu D^\nu C_{\mu\nu} - \mathcal{K}_{\mu\nu} D^\mu D^\nu V^{\parallel} - D^\mu D^2 D_\mu V^\perp \tag{3.75}$$

$$= D^\mu D^\nu C_{\mu\nu} - \frac{\theta}{3} h_{\mu\nu} D^\mu D^\nu V^{\parallel} - 2K D^2 V^\perp + D^4 V^\perp \tag{3.76}$$

$$= D^\mu D^\nu C_{\mu\nu} - \frac{\theta}{3} D^2 V^{\parallel} - D^2(D^2+2K)V^\perp, \tag{3.77}$$

using (3.59) one get

$$D^2\widetilde{\psi} - D^2(D^2+2K)\widetilde{\mathcal{E}} = D^2\psi - D^2(D^2+2K)\mathcal{E} - \frac{\theta}{3}D^2 V^{\parallel} - D^2(D^2+2K)V^\perp \tag{3.78}$$

$$= D^2\left(\psi - \frac{\theta}{3}V^{\parallel}\right) + D^2(D^2+2K)(\mathcal{E} + V^\perp). \tag{3.79}$$

Once the operator D^2, and $D^2(D^2+2K)$ are invertible and the scalar modes in which they are acting are linearly independent, we get

$$\widetilde{\mathcal{E}} = \mathcal{E} + V^\perp, \tag{3.80}$$

$$\widetilde{\psi} = \psi - \frac{\theta}{3}V^{\parallel}. \tag{3.81}$$

Returning to (3.35), using (3.67), and (3.72) one get the gauge transformation of the vector mode $\widetilde{b}_\mu$

$$\widetilde{B}_\mu = D_\mu\widetilde{\mathcal{B}} + \widetilde{b}_\mu \tag{3.82}$$

$$B_\mu + \partial_t(D_\mu V^\perp + V^\perp_\mu) - D_\mu V^{\parallel} = D_\mu\widetilde{\mathcal{B}} + \widetilde{b}_\mu \tag{3.83}$$

$$D_\mu\mathcal{B} + b_\mu - \frac{2}{3}\theta D_\mu V^\perp + D_\mu\dot{V}^\perp + \dot{V}^\perp_\mu - D_\mu V^{\parallel} = D_\mu\left(\mathcal{B} - \frac{2}{3}\theta V^\perp + \dot{V}^\perp - V^{\parallel}\right) + \tilde{b}_\mu \tag{3.84}$$

$$\widetilde{b}_\mu = b_\mu + \dot{V}^\perp_\mu. \tag{3.85}$$

Performing one divergence of (3.68) and using the inverse of the operators we get

$$\widetilde{F}_\mu = F_\mu - V^\perp_\mu, \tag{3.86}$$

using these transformations and substituting on (3.68) we see

$$\widetilde{W}_{\mu\nu} = W_{\mu\nu}. \tag{3.87}$$

Summarizing the gauge transformations of the modes:

$$\text{Scalars:} \begin{cases} \phi \longrightarrow \phi + \dot{V}^{\|} \\ \mathcal{B} \longrightarrow \mathcal{B} + \dot{V}^\perp - \frac{2}{3}\theta V^\perp - V^{\|} \\ \psi \longrightarrow \psi - \frac{\theta}{3} V^{\|} \\ \mathcal{E} \longrightarrow \mathcal{E} + V^\perp \end{cases} \tag{3.88}$$

$$\text{Vectors:} \begin{cases} b_\mu \longrightarrow b_\mu + \dot{V}^\perp_\mu \\ F_\mu \longrightarrow F_\mu - V^\perp_\mu \end{cases} \tag{3.89}$$

$$\text{Tensor:} \quad W_{\mu\nu} \longrightarrow W_{\mu\nu}. \tag{3.90}$$

Thereby, the tensor mode is gauge invariant since do not depend on the projections of the gauge V^μ. However, Bardeen, J. in 1980 [23] investigated these transformations in the context of Cosmological perturbations and found combinations of these geometrical modes in such a way to get gauge-invariant (GI) variables. We will express these combinations in this formalism.

The three GI scalar variables are:

$$\Phi \equiv \phi + \partial_t \left(\mathcal{B} - \dot{\mathcal{E}} + \frac{2}{3}\theta \mathcal{E} \right); \tag{3.91}$$

$$\Psi \equiv \psi - \frac{\theta}{3} \left(\mathcal{B} - \dot{\mathcal{E}} + \frac{2}{3}\theta \mathcal{E} \right); \tag{3.92}$$

$$\Xi \equiv 3\dot{\Psi} + \theta\Phi. \tag{3.93}$$

One can easily check that these three quantities are GI, using the gauge transformations (3.88). The variables Φ and Ψ are the Bardeen potentials. Since these quantities are linear combinations of perturbations is difficult to interpret them physically.

One possible way to deal with it is to choose a specific gauge to write the decompositions, as we will do further. Besides that, there is a relation between the choice of a gauge and the way to sectioning the physical manifold using a vector field v^μ, see [21] for details.

We may construct a GI vector variable as the following sum:

$$b_\mu + \dot{F}_\mu, \tag{3.94}$$

again, one can check that is GI using (3.89).

3.3 Energy-momentum tensor

In the last section, we describe the perturbations of the variables concerning the geometric sector, to get the Einstein equations linearized, we need to perform the same decompositions of the energy-momentum tensor since it describes the material sector of Einstein's equations, thus this section will focus to study the perturbed energy-momentum tensor.

The first thing we need to do is write the required tensors on the spatial sections defined by a time-like covector field $v_\mu \in T_p^*M$, which in this case, represents the tangent covector of the integral curves defining the observers following the fluid

$$\delta v_\mu = v_\mu - n_\mu \tag{3.95}$$

we assume that this perturbation is of the same order as $\delta g_{\mu\nu}$. Requiring the normality condition of v_μ

$$v_\mu v^\mu = -1, \tag{3.96}$$
$$\bar{g}_{\mu\nu}(n^\mu + \delta v^\mu)(n^\nu + \delta v^\nu) = -1, \tag{3.97}$$
$$\bar{g}_{\mu\nu}n^\mu n^\nu + 2\bar{g}_{\mu\nu}n^\mu \delta v^\nu = 0, \tag{3.98}$$
$$\phi = -n_\mu \delta v^\mu. \tag{3.99}$$

Thus, we may write

$$\delta v_\mu = -n_\mu \phi + u_\mu, \tag{3.100}$$

where $u_\mu = h[\delta v_\mu]$ has the spatial degrees of freedom on the choice of the observers which describe the fluid. The covector field describing the fluid is

$$v_\mu = (1 - \phi)n_\mu + u_\mu. \tag{3.101}$$

In appendix C the perturbations of the projector and the spatial metrics were made. We may define the projector in $\widehat{M}$ as

$$\hat{h}^\mu{}_\nu = v^\mu v_\nu + \delta^\mu{}_\nu. \tag{3.102}$$

The most general decomposition of $T_{\mu\nu}$ in M is given by (2.70). However, we can assume a covector field v_μ normal to hyper-surfaces, making a choice: $u_\mu = 0$, such that the heat term of the energy-momentum tensor is zero

$$\widehat{T}_{\mu\nu} = \hat{\rho} v_\mu v_\nu + \hat{p}\, \hat{h}_{\mu\nu} + \widehat{\Pi}_{\mu\nu}, \tag{3.103}$$

wherein $\hat{\rho} = \widehat{T}_{vv}$, $\hat{p} = \dfrac{\widehat{T}_\mu{}^\mu}{3}$, and $\widehat{\Pi}_\mu{}^\mu = 0$ have the same meaning of the variables on the background, but projected on spatial sections in $\widehat{M}$ defined by the covector field v_μ.

Thus we can always choose $v_\mu \in T_pM$ to be normal to the hypersurfaces being an eigenvector of $T^{\mu}_{\ \nu}$.[7]

$$\widehat{T}^{\mu}_{\ \nu}v^{\nu} = -\alpha v^{\mu}, \tag{3.104}$$

if we take the projector on both sides we get $\widehat{q}_\mu = \hat{h}[\widehat{T}_{\mu\nu}v^{\nu}] = -\alpha\hat{h}[v_\mu] = 0$ as expected. Besides that, multiplying both sides by v^μ one get $\alpha = \hat{\rho}$, then $T^{\mu}_{\ \nu}v^{\nu} = -\hat{\rho}v^{\mu}$ and $\hat{\rho}$ is the eigenvalue of $\widehat{T}^{\mu}_{\ \nu}$.

3.3.1 Gauge transformations of energy-matter modes

One may use the equations (3.100) and (3.103) to get the most general decomposition induced by the geometric modes

$$\begin{aligned}
\delta T_{\mu\nu} &= \widehat{T}_{\mu\nu} - T_{\mu\nu} && (3.105)\\
&= (\hat{\rho} + \hat{p})v_\mu v_\nu + \hat{p}v_\mu v_\nu + p\,\delta h_{\mu\nu} + \widehat{\Pi}_{\mu\nu} - \Big[(\rho + p)n_\mu n_\nu + pn_\mu n_\nu + \Pi_{\mu\nu}\Big] && (3.106)\\
&= (\delta\rho - 2\phi)n_\mu n_\nu + \delta p h_{\mu\nu} + 2(\rho + p)n_{(\mu}u_{\nu)} + \delta\Pi_{\mu\nu} + && (3.107)\\
&\quad + 2p(B_{(\mu}n_{\nu)} + C_{\mu\nu}). && (3.108)
\end{aligned}$$

the perturbation (C.7) has been used, one may get the mixed part of this tensor using (C.8)

$$\delta T_{\mu}^{\ \nu} = \delta\rho n_\mu n^\nu + (p + \rho)[u_\mu n^\nu + n_\mu(u^\nu + B^\nu)] + \delta p h_{\mu}^{\ \nu} + \delta\Pi_{\mu}^{\ \nu}, \tag{3.109}$$

thus the trace is

$$T = -\delta\rho + 3\delta p. \tag{3.110}$$

As we did in (3.35) and (3.39) one may perform the decompositions of the vector mode u_μ and the tensor mode $\delta\Pi_{\mu\nu}$:

$$\begin{aligned}
u_\mu &= D_\mu\mathcal{V} + \mathrm{V}_\mu && (3.111)\\
\delta\Pi_{\mu\nu} &= -\left(D_\mu D_\nu - \frac{h_{\mu\nu}D^2}{3}\right)\delta\Pi^{(s)} + D_{(\mu}\delta\Pi^{(v)}_{\nu)} + \delta\Pi^{(t)}_{\mu\nu}. && (3.112)
\end{aligned}$$

with $D^\mu\mathrm{V}_\mu = D^\mu\Pi^{(v)}_\mu = D^\mu\delta\Pi^{(t)}_{\mu\nu} = \delta\Pi^{(t)\ \mu}_{\mu} = 0$ Therefore, the modes are

$$\textbf{Scalars: } \delta\rho,\ \delta p,\ \delta\Pi^{s},\ \mathcal{V}. \tag{3.113}$$

$$\textbf{Vectors: } \delta\Pi^{(v)}_\mu, \mathrm{V}_\mu \tag{3.114}$$

$$\textbf{Tensor: } \Pi^{(t)}_{\mu\nu}. \tag{3.115}$$

[7] We are assuming there is no vorticity on $\widehat{M}$, i.e., $\hat{\omega}_{[\mu\nu]} = 0$.

We are in a position to determine the gauge transformation of these modes, note that some modes above define simple perturbations, i.e.,

$$\delta\rho = \hat{\rho} - \rho = \widehat{T}_{vv} - T_{nn} \tag{3.116}$$

$$\delta p = \hat{p} - p = \frac{\hat{h}[\widehat{T}]}{3} - \frac{h[T]}{3} \tag{3.117}$$

$$\delta\Pi_{\mu\nu} = \widehat{\Pi}_{\mu\nu} - \Pi_{\mu\nu} = \hat{h}[\widehat{T}_{\mu\nu}] - ph_{\mu\nu} \tag{3.118}$$

where $\widehat{T} = \hat{h}^{\mu\nu}\widehat{T}_{\mu\nu}$ is the trace of the physical energy-momentum tensor, and the same for the background version. Thus from (3.17) the gauge transformations are

$$\widetilde{\delta\rho} = \delta\rho - \mathcal{L}_V\rho = \delta\rho - \dot{\rho} \tag{3.119}$$

$$\widetilde{\delta p} = \delta p - \mathcal{L}_V p = \delta p - \dot{p} \tag{3.120}$$

$$\widetilde{\delta\Pi_{\mu\nu}} = \delta\Pi_{\mu\nu} \tag{3.121}$$

in which on the transformation (3.121) we know that in FLRW universe $\Pi_{\mu\nu} = 0$. Is directly to see that if $\delta\Pi_{\mu\nu}$ is GI thus $\delta\Pi^{(s)}$, $\delta\Pi^{(v)}_{\nu}$, and $\delta\Pi^{(t)}_{\mu\nu}$ will be GI. For the modes $\mathcal{V}$ and V_μ we get

$$\widetilde{\mathcal{V}} = \mathcal{V} + V^{\parallel}, \tag{3.122}$$

$$\widetilde{\mathrm{V}}_\mu = \mathrm{V}_\mu. \tag{3.123}$$

In appendix C some kinematic perturbations were calculated with arbitrary background manifold, expressing the shear perturbation (C.23) for a FLRW background manifold we obtain

$$\delta\sigma_{\mu\nu} = \left(D_{(\mu}D_{\nu)} - \frac{h_{\mu\nu}D^2}{3}\right)\mathcal{S} + D_{(\mu}b_{\nu)} + \left(\partial_t - \frac{2}{3}\theta\right)(D_{(\mu}F_{\nu)} + W_{\mu\nu}) \tag{3.124}$$

$$= \left(D_{(\mu}D_{\nu)} - \frac{h_{\mu\nu}D^2}{3}\right)\mathcal{S} + D_{(\mu}\mathcal{S}_{\nu)} + \left(\partial_t - \frac{2}{3}\theta\right)W_{\mu\nu} \tag{3.125}$$

In which $\mathcal{S} \equiv \left(\mathcal{B} - \dot{\mathcal{E}} + \frac{2}{3}\theta\mathcal{E}\right)$, and $\mathcal{S}_\nu \equiv b_\nu + \dot{F}_\nu - \frac{2}{3}\theta F_\nu$, are the scalar shear and vector shear potentials respectively, measured by the isotropic observers. The former tensor did appear in (3.91), (3.92), so it was important to get the Bardeen potentials, making it possible to construct GI variables. Using (3.88) the scalar shear transforms as

$$\widetilde{\mathcal{S}} = \mathcal{S} - V^{\parallel}, \tag{3.126}$$

analogously using (3.89)

$$\widetilde{S_\mu} = S_\mu - V^{\perp}_\mu. \tag{3.127}$$

With these results, one may construct the GI variables of energy-matter sector[8]:

$$\overline{\delta\rho} = \delta\rho - \mathcal{S}\,\dot{\rho}; \tag{3.128}$$
$$\overline{\delta p} = \delta p - \mathcal{S}\,\dot{p}; \tag{3.129}$$
$$\overline{\mathcal{V}} = \mathcal{V} + \mathcal{S}. \tag{3.130}$$

From a geometric point of view, the scalar potential shear acts as a correction of the perturbations on the spatial sections $\Sigma_t \subset \widehat{M}$ related by a gauge transformation between frames of reference in M. Such that, both frames of reference get the same GI variable.

Assuming the energy-momentum conservation on M, one may obtain some useful scalar equations using (3.103), then

$$\widehat{\nabla}_\mu \widehat{T}^{\mu\nu} = 0. \tag{3.131}$$

Separating the spatial and temporal parts:

$$\hat{h}[\widehat{\nabla}_\alpha \widehat{T}^\alpha_{\ \nu}] = 0 \tag{3.132}$$
$$h[v_\nu \widehat{\nabla}_{\hat{v}}\hat{\rho} + (\hat{\rho}+\hat{p})(v_\nu\hat{\theta} + v^\alpha \widehat{\nabla}_\alpha v_\nu) + \widehat{\nabla}_\nu \hat{p} + \widehat{\nabla}_\alpha \widehat{\Pi}^\alpha_{\ \nu}] = 0 \tag{3.133}$$
$$(\hat{\rho}+\hat{p})\hat{a}_\nu + \widehat{D}_\nu \hat{p} + \widehat{D}_\alpha \widehat{\Pi}^\alpha_{\ \nu} = 0 \tag{3.134}$$

$$-v_\nu \widehat{\nabla}_\mu \widehat{T}^{\mu\nu} = 0 \tag{3.135}$$
$$\widehat{\nabla}_{\hat{v}}\hat{\rho} + \hat{\rho}\hat{\theta} - \hat{\rho} v_\mu \hat{a}^\mu - v_\mu \hat{p}\widehat{\nabla}_\nu \hat{h}^{\mu\nu} - v_\mu \widehat{\nabla}_\nu \widehat{\Pi}^{\mu\nu} = 0 \tag{3.136}$$
$$\widehat{\nabla}_{\hat{v}}\hat{\rho} + (\hat{\rho}+\hat{p})\hat{\theta} + \mathcal{K}_{\mu\nu}\widehat{\Pi}^{\mu\nu} = 0 \tag{3.137}$$
$$\widehat{\nabla}_{\hat{v}}\hat{\rho} + (\hat{\rho}+\hat{p})\hat{\theta} + \widehat{\mathcal{K}}^{(t)}_{\mu\nu}\widehat{\Pi}^{\mu\nu} = 0 \tag{3.138}$$

in which on the third line of the spatial component we used the derivative product rule, and on the last line of the temporal component the traceless extrinsic curvature has been used, since the divergence of the anisotropic pressure is zero, i.e.,

$$\widehat{\mathcal{K}}_{\mu\nu}\widehat{\Pi}^{\mu\nu} = \left(\widehat{\mathcal{K}}^{(t)}_{\mu\nu} + \frac{\hat{h}_{\mu\nu}}{3}\hat{\theta}\right)\widehat{\Pi}^{\mu\nu} \tag{3.139}$$
$$\widehat{\mathcal{K}}_{\mu\nu}\widehat{\Pi}^{\mu\nu} = \widehat{\mathcal{K}}^{(t)}_{\mu\nu}\widehat{\Pi}^{\mu\nu}. \tag{3.140}$$

Writing the expression (3.134) in terms of the perturbations

$$h_\mu^{\ \sigma} D_\sigma(p+\delta p) + \delta h_\mu^{\ \sigma} D_\sigma(p+\delta p) + (\rho+p)(\dot{u}_\mu - D_\mu\phi) + D_\sigma(\Pi^\sigma_{\ \mu} + \delta\Pi^\sigma_{\ \mu}) = 0 \tag{3.141}$$
$$D_\mu(p+\delta p) + [n_\mu(B^\sigma + u^\sigma) + u_\mu n^\sigma]D_\sigma p + (\rho+p)(\dot{u}_\mu - D_\mu\phi) + D_\sigma(\delta\Pi^\sigma_{\ \mu}) = 0 \tag{3.142}$$
$$D_\mu \delta p + u_\mu \dot{p} + (\rho+p)(\dot{u}_\mu - D_\mu\phi) + D_\sigma(\delta\Pi^\sigma_{\ \mu}) = 0. \tag{3.143}$$

8 We will use a line above the symbols for GI variables.

In which (C.32) has been used, and on FLRW background manifold we have $D_\mu p = 0$ and $\Pi^\sigma{}_\mu = 0$. Doing the same for (3.138), we obtain

$$\begin{aligned} \dot{\delta\rho} + \phi\dot{\rho} + (B^\nu + u^\nu)D_\nu\rho + \theta(\delta\rho + \delta p) + \delta\theta(\rho + p) = 0 \\ \dot{\delta\rho} + \phi\dot{\rho} + \theta(\delta\rho + \delta p) + \delta\theta(\rho + p) = 0, \end{aligned} \tag{3.144}$$

where $\sigma_{\mu\nu} = 0$, and $D_\nu\rho = 0$ in FLRW background, and the scalar expansion perturbation is given in (C.20). The equations (3.143) and (3.144) are the first-order perturbation of Euler's equation and the continuity equation respectively.

3.4 Linear Einstein's equations

The equations of motion in M are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.145}$$

raising one of the indexes of the equation and using (A.17) we get[9]

$$R_\mu^{\ \nu} = \kappa\left(T_\mu^{\ \nu} - \frac{\delta_\mu^{\ \nu}T}{2}\right) \tag{3.146}$$

$$R_\mu^{\ \nu} = \kappa\mathcal{T}_\mu^{\ \nu}, \tag{3.147}$$

where $\mathcal{T}_\mu^{\ \nu} \equiv T_\mu^{\ \nu} - \frac{\delta_\mu^{\ \nu}T}{2}$. Once the metric of the background manifold is FLRW, we may define simple perturbations of the tensor above. Adding $-\overline{R}_\mu^{\ \nu}$ on both sides, we obtain

$$R_\mu^{\ \nu} - \overline{R}_\mu^{\ \nu} = \kappa\mathcal{T}_\mu^{\ \nu} - \bar{R}_\mu^{\ \nu} = \kappa(\mathcal{T}_\mu^{\ \nu} - \overline{\mathcal{T}}_\mu^{\ \nu}) \tag{3.148}$$

$$\delta R_\mu^{\ \nu} = \kappa\delta\mathcal{T}_\mu^{\ \nu}. \tag{3.149}$$

We shall study how the perturbations above change under a gauge transformation. Thus we could search for a GI version of Einstein's equations.

It is possible to generalize the way the perturbations on $\Sigma_t \subset M$ transform, first note that we can write any background mixed tensor of the form $\bar{f}_\mu^{\ \nu}$ on the spatial sections as

$$f_\mu^{\ \nu} = f^{(n)}n^\nu n_\mu + f^{(s)}\delta_\mu^{\ \nu}, \tag{3.150}$$

where $f^{(n)}$, and $f^{(s)}$ are scalar functions representing the time and spatial projections of $f_\mu^{\ \nu}$ respectively, which $D_\mu\bar{f}^{(n)} = D_\mu\bar{f}^{(s)} = 0$, i.e., these functions are constant on the hyper-surfaces due the homogeneity. We can define the pure perturbation $\delta f_\mu^{\ \nu} = f_\mu^{\ \nu} - \bar{f}_\mu^{\ \nu}$, doing a gauge transformation on it

$$\widetilde{\delta f_\mu^{\ \nu}} = \delta f_\mu^{\ \nu} - \mathcal{L}_V f_\mu^{\ \nu} \tag{3.151}$$

[9] The mixed tensors will simplify some calculations since the spatial metric becomes a Kronecker delta tensor.

calculating the Lie derivative

$$\mathcal{L}_V f_\mu{}^\nu = \mathcal{L}_V(f^{(n)} n^\nu n_\mu + f^{(s)} \delta_\mu{}^\nu) \tag{3.152}$$
$$= V^\sigma \nabla_\sigma f^{(n)} n_\mu n^\nu + \delta_\mu{}^\nu V^\sigma \nabla_\sigma f^{(s)} + f^{(n)} \mathcal{L}_V(n_\mu n^\nu) \tag{3.153}$$
$$= V^\| \dot{f^{(n)}} n_\mu n^\nu + V^\| \dot{f^{(s)}} \delta_\mu{}^\nu + f^{(n)} \mathcal{L}_V(n_\mu n^\nu) \tag{3.154}$$
$$= V^\| (\partial_t f_\mu{}^\nu) + f^{(n)} [V^\sigma \nabla_\sigma (n_\mu n^\nu) - n_\mu n^\sigma \nabla_\sigma V^\nu + n^\nu n_\sigma \nabla_\mu V^\sigma] \tag{3.155}$$
$$= V^\| \nabla_n f_\mu{}^\nu - f^{(n)} (n_\mu \partial_t h[V^\sigma] + n^\nu D_\mu V^\|). \tag{3.156}$$

On the third line the vector field has been decomposed, and we used the derivative by parts, moreover, in the last line we used the following results

$$(i) \quad V^\sigma \nabla_\sigma (n_\mu n^\nu) = (V^\| n^\sigma + h[V^\sigma]) \nabla_\sigma (n_\mu n^\nu) = h[V^\sigma D_\sigma](n_\mu n^\nu) = 0; \tag{3.157}$$
$$(ii) \quad n_\mu n^\sigma \nabla_\sigma (V^\| n^\nu + h[V^\nu]) = n^\sigma n_\mu \nabla_\sigma h[V^\nu] = n_\mu \partial_t h[V^\nu]; \tag{3.158}$$
$$(iii) \quad n^\nu n_\sigma \nabla_\mu (V^\| n^\sigma + h[V^\sigma]) = n^\nu n_\sigma \nabla_\mu (V^\| n^\sigma) + n^\nu n_\sigma \nabla_\mu h[V^\sigma]; \tag{3.159}$$
$$= n^\nu (n_\sigma \nabla_\mu h[V^\sigma] - \nabla_\mu V^\|) = -n^\nu D_\mu V^\|. \tag{3.160}$$

Furthermore, the gauge transformation of the perturbation is given by

$$\widetilde{\delta f_\mu{}^\nu} = \delta f_\mu{}^\nu - V^\| \nabla_n f_\mu{}^\nu + f^{(n)} \left[n_\mu \partial_t (D^\nu V^\perp + V^{\perp\nu}) + n^\nu D_\mu V^\| \right], \tag{3.161}$$

analyzing the equation (3.161), we need to define the tensors with which the transformations cancel out the gauge terms: $V^\|, V^\perp, V^{\perp\nu}$. Following this procedure, one may use the tensors $\mathcal{S}$ and B^ν, which transformations are given by (3.89), (3.126) respectively, then the GI tensor is

$$\overline{\delta f_\mu{}^\nu} = \delta f_\mu{}^\nu - \mathcal{S} \nabla_n f_\mu{}^\nu - f^{(n)} (n_\mu b^\nu - n_\mu D^\nu \mathcal{S} + n^\nu D_\mu \mathcal{S}) \tag{3.162}$$

We may calculate the projections of this GI perturbation: time-time

$$\overline{\delta f_n{}^n} = \delta f_n{}^n - \mathcal{S}\, \partial_t (f^{(n)} - f^{(s)}) - f^{(n)} \underbrace{(-b^n + D^n \mathcal{S} + D_n \mathcal{S})}_{0} \tag{3.163}$$

$$\overline{\delta f_n{}^n} = \delta f_n{}^n - \mathcal{S} (\dot{f}^{(n)} - \dot{f}^{(s)}), \tag{3.164}$$

time-spatial projection: $h[\overline{\delta f_n{}^\nu}]$

$$\overline{\delta f_n{}^\nu} = f_n{}^\nu - \mathcal{S} \partial_t [(f^{(s)} - f^{(n)}) n^\nu)] - f^{(n)} (-b^\nu + D^\nu \mathcal{S} - n^\nu \dot{\mathcal{S}}) \tag{3.165}$$
$$h[\overline{\delta f_n{}^\nu}] = h[f_n{}^\nu] + f^{(n)} (b^\nu - D^\nu \mathcal{S}), \tag{3.166}$$

where the time terms cancel out by the spatial projector, analogously the space-time projection $h[\widehat{\delta f_\mu{}^n}]$,

$$h[\overline{\delta f_\mu{}^n}] = h[f_\mu{}^n] + h[f^{(n)} (n_\mu \dot{\mathcal{S}} - D_\mu \mathcal{S})] \tag{3.167}$$
$$= h[f_\mu{}^n] - f^{(n)} D_\mu \mathcal{S} \tag{3.168}$$

and the spatial-spatial projection :

$$h[\overline{\delta f_\mu^{\ \nu}}] = h[\delta f_\mu^{\ \nu}] - \mathcal{S}\, h[\partial_t(f^{(n)}n^\nu n_\mu) + f^{(s)}\delta_\mu^{\ \nu}] \quad (3.169)$$
$$= h[\delta f_\mu^{\ \nu}] - \mathcal{S}\, h_\mu^{\ \nu}\dot{f}^{(s)}. \quad (3.170)$$

using the projections (3.164), (3.166), (3.168), and (3.170) one may find the GI tensors of Einstein's equations on first-order, let us start with the geometric part

3.4.1 Geometric sector

The Ricci tensor pertubation is

$$\delta R_\mu^{\ \nu} = \hat{R}_\mu^{\ \nu} - R_\mu^{\ \nu} \quad (3.171)$$
$$= \hat{R}_{\mu\alpha}\hat{g}^{\alpha\nu} - R_{\mu\alpha}g^{\alpha\nu} \quad (3.172)$$
$$= \hat{R}_{\mu\alpha}\hat{g}^{\alpha\nu} - R_{\mu\alpha}\hat{g}^{\alpha\nu} + R_{\mu\alpha}\delta g^{\alpha\nu} \quad (3.173)$$
$$= \delta R_{\mu\alpha}g^{\alpha\nu} + R_{\mu\alpha}\delta g^{\alpha\nu}, \quad (3.174)$$

in which on the second line we use (3.20), and on the last line we use the background metric since we are neglecting the second-order perturbation terms.

We may use the background Ricci tensor on the spatial sections, from (2.61), with $\theta = 3H$:

$$R_{\mu\alpha} = \left(2K + \frac{\dot{\theta} + \theta^2}{3}\right)h_{\mu\alpha} - \left(\dot{\theta} + \frac{\theta^2}{3}\right)n_\mu n_\alpha, \quad (3.175)$$

since the background Ricci tensor can be written as

$$R_{\mu\alpha} = R^{(n)}n_\mu n_\alpha + R^{(s)}g_{\mu\alpha} \quad (3.176)$$
$$= (R^{(n)} - R^{(s)})n_\mu n_\alpha + R^{(s)}h_{\mu\alpha} \quad (3.177)$$

comparing these expressions, we get

$$R^{(n)} = 2K - \frac{2}{3}\dot{\theta}; \quad (3.178)$$
$$R^{(s)} = 2K + \frac{\dot{\theta} + \theta^2}{3}; \quad (3.179)$$
$$R^{(n)} - R^{(s)} = -\left(\dot{\theta} + \frac{\theta^2}{3}\right). \quad (3.180)$$

Thus we can rewrite (3.174) as

$$\delta R_\mu^{\ \nu} = \delta R_{\mu\alpha}g^{\alpha\nu} + [(R^{(n)} - R^{(s)})n_\mu n_\alpha + R^{(s)}h_{\mu\alpha}](-2\phi n_\mu n_\nu - 2B_{(\mu}n_{\nu)} - 2C_{\mu\nu})$$
$$= \delta R_{\mu\alpha}g^{\alpha\nu} + (R^{(n)} - R^{(s)})(2\phi n_\mu n^\nu + n_\mu B^\nu) - R^{(s)}(B_\mu n^\nu + 2C_\mu^{\ \nu}). \quad (3.181)$$

We shall calculate the components of the Ricci perturbation using (3.181). The time-time projection may be calculated by applying the simplification of FLRW background in (C.37):

$$\delta R_n^{\ n} = \delta R_{n\alpha} g^{\alpha n} + (R^{(n)} - R^{(s)})2\phi \tag{3.182}$$

$$= \delta R_{nn} + 2\phi(R^{(n)} - R^{(s)}) \tag{3.183}$$

$$= -(\ddot{C} + \theta\dot{\phi} + D^\nu \dot{B}_\nu + D^2\phi + 2\mathcal{K}^{\mu\nu}\nabla_n C_{\mu\nu}) - 2\phi\left(\dot{\theta} + \frac{\theta^2}{3}\right) \tag{3.184}$$

$$= \dot{\Xi} + \frac{2}{3}\theta\Xi + (\dot{\theta} + D^2)\Phi + \partial_t\left(\dot{\theta} + \frac{\theta^2}{3}\right)\mathcal{S}. \tag{3.185}$$

From (3.164) one may compute the GI time-time component of the Ricci tensor

$$\overline{\delta R_n^{\ n}} = \delta R_n^{\ n} - \mathcal{S}\partial_t(R^{(n)} - R^{(s)}) \tag{3.186}$$

$$= -\left\{\dot{\Xi} + \frac{2}{3}\theta\Xi + (\dot{\theta} + D^2)\Phi + \partial_t\left(\dot{\theta} + \frac{\theta^2}{3}\right)\mathcal{S} + \mathcal{S}\partial_t(R^{(n)} - R^{(s)})\right\} \tag{3.187}$$

$$= -\left[\dot{\Xi} + \frac{2}{3}\theta\Xi + (\dot{\theta} + D^2)\Phi\right]. \tag{3.188}$$

calculating the spatial-time component taking the projector on (3.181), we obtain

$$h[\delta R_\mu^{\ n}] = h[\delta R_{\mu n}] + R^{(s)} b_\mu \tag{3.189}$$

$$= \frac{1}{2}(D^2\mathcal{S}_\mu + 2K\mathcal{S}_\mu) - \left(2K + \frac{\theta^2 + \dot{\theta}}{3}\right) b_\mu \tag{3.190}$$

where (C.38) has been used with FLRW metric, using (3.168) the GI component then will be

$$h[\overline{\delta R_\mu^{\ n}}] = h[R_\mu^{\ n}] - R^{(n)} D_\mu \mathcal{S} \tag{3.191}$$

$$= \frac{1}{2}(D^2\mathcal{S}_\mu + 2K\mathcal{S}_\mu) - \left(2K + \frac{\theta^2 + \dot{\theta}}{3}\right) b_\mu - \left(2K - \frac{2}{3}\dot{\theta}\right) D_\mu \mathcal{S} \tag{3.192}$$

$$= \frac{1}{2}(D^2\mathcal{S}_\mu + 2K\mathcal{S}_\mu). \tag{3.193}$$

The spatial-spatial projection may be calculated using the same simplification on (C.39)

$$h[R_\mu^{\ \nu}] = D_\mu D^\nu(\Phi - \Psi) + \left(\frac{\dot{\Xi} + 2\theta\Xi}{3} - D^2\Psi + \frac{\dot{\theta}\Phi}{3} - 4K\Psi + \frac{\ddot{\theta}\mathcal{S}}{3} + \frac{2}{3}\theta\dot{\theta}\mathcal{S}\right) h_\mu^{\ \nu} + \tag{3.194}$$

$$+ \left(D_{(\mu}\dot{\mathcal{S}}_{\alpha)} + \frac{1}{3}\theta D_{(\mu}\dot{\mathcal{S}}_{\alpha)}\right) + \ddot{W}_\mu^{\ \nu} + \theta\dot{W}_\mu^{\ \nu} + (2K - D^2)W_\mu^\nu. \tag{3.195}$$

Therefore, using (3.170) the GI spatial-spatial projection will be

$$h[\overline{\delta R_\mu{}^\nu}] = h[R_\mu{}^\nu] - \mathcal{S}\dot{R}^{(s)} h_\mu{}^\nu \tag{3.196}$$
$$= h[R_\mu{}^\nu] - \mathcal{S}\left(\frac{\ddot{\theta} + 2\theta\dot{\theta}}{3}\right) h_\mu{}^\nu \tag{3.197}$$
$$= D_\mu D^\nu(\Phi - \Psi) + \left(\frac{\dot{\Xi} + 2\theta\Xi}{3} - D^2\Psi + \frac{\dot{\theta}\Phi}{3} - 4K\Psi\right) h_\mu{}^\nu +$$
$$+ \left(D_{(\mu}\dot{\mathcal{S}}_{\alpha)} + \frac{1}{3}\theta D_{(\mu}\dot{\mathcal{S}}_{\alpha)}\right) h^{\alpha\nu} + \ddot{W}_\mu{}^\nu + \theta\dot{W}_\mu{}^\nu + (2K - D^2)W_\mu^\nu. \tag{3.198}$$

3.4.2 Energy-matter sector

We have in hand the GI components of the geometric sector, now we may see how to get the GI components of the source tensor $\mathcal{T}_\mu{}^\nu$. From (3.109), and (3.110) we get

$$\delta\mathcal{T}_\mu{}^\nu = \delta T_\mu{}^\nu - \frac{\delta_\mu{}^\nu}{2}\delta T \tag{3.199}$$
$$= \left(\frac{\delta\rho + 3\delta p}{2}\right) n_\mu n^\nu + \left(\frac{\delta\rho - \delta p}{2}\right) h_\mu{}^\nu + \tag{3.200}$$
$$+ (\rho + p)[u_\mu n^\nu + n_\mu(u^\nu + B^\nu)] + \delta\Pi_\mu{}^\nu. \tag{3.201}$$

The background source term is

$$\mathcal{T}_\mu{}^\nu = \mathcal{T}^{(n)} n_\mu n^\nu + \mathcal{T}^{(s)}\delta_\mu{}^\nu. \tag{3.202}$$

Comparing the definition of $\mathcal{T}_\mu{}^\nu$ and the background energy-momentum tensor (2.71).

$$\mathcal{T}^{(n)} = \rho + p, \tag{3.203}$$
$$\mathcal{T}^{(s)} = \frac{\rho - p}{2}. \tag{3.204}$$

Using the expressions (3.164), (3.168), and (3.170), one get the GI components of $\delta\mathcal{T}_\mu{}^\nu$. Calculating the perturbation of the time-time component

$$\delta\mathcal{T}_n{}^n = \frac{\delta\rho + 3\delta p}{2}, \tag{3.205}$$

the GI variable will be

$$\overline{\delta\mathcal{T}_n{}^n} = \delta\mathcal{T}_n{}^n - \mathcal{S}\left(\dot{\mathcal{T}}^{(n)} - \dot{\mathcal{T}}^{(s)}\right) \tag{3.206}$$
$$= \frac{\delta\rho + 3\delta p}{2} - \mathcal{S}\left(\frac{\dot{\rho} + 3\dot{p}}{2}\right) \tag{3.207}$$
$$= \frac{\overline{\delta\rho} + 3\overline{\delta p}}{2}. \tag{3.208}$$

The spatial-time component

$$h[\mathcal{T}_{\mu}{}^{n}] = h\left[-n_{\mu}\left(\frac{\delta\rho + 3\delta p}{2}\right) - (\rho + p)u_{\mu} + \delta\Pi_{\mu}^{n}\right] \tag{3.209}$$

$$= -(\rho + p)u_{\mu} \tag{3.210}$$

$$= -(\rho + p)(D_{\mu}\mathcal{V} + \mathrm{V}_{\mu}), \tag{3.211}$$

then the GI version is

$$h[\delta\overline{\mathcal{T}_{\mu}{}^{n}}] = -(\rho + p)(D_{\mu}\mathcal{V} + \mathrm{V}_{\mu}) - (p + \rho)D_{\mu}\mathcal{S} \tag{3.212}$$

$$= -(p + \rho)[D_{\mu}(\mathcal{V} + \mathcal{S}) + \mathrm{V}_{\mu}] \tag{3.213}$$

$$= -(p + \rho)(D_{\mu}\overline{\mathcal{V}} + \mathrm{V}_{\mu}). \tag{3.214}$$

In the same way the spatial-spatial component

$$h[\delta\mathcal{T}_{\mu}{}^{\nu}] = \left(\frac{\delta\rho - \delta p}{2}\right)h_{\mu}^{\nu} + \delta\Pi_{\mu}^{\nu}, \tag{3.215}$$

leading to

$$h[\delta\overline{\mathcal{T}_{\mu}{}^{\nu}}] = \left(\frac{\delta\rho - \delta p}{2}\right)h_{\mu}^{\nu} + \delta\Pi_{\mu}^{\nu} - \mathcal{S}\left(\frac{\dot{\rho} - \dot{p}}{2}\right)h_{\mu}^{\nu} \tag{3.216}$$

$$= \left(\frac{\overline{\delta\rho} - \overline{\delta p}}{2}\right)h_{\mu}^{\nu} + \delta\Pi_{\mu}^{\nu}. \tag{3.217}$$

3.4.3 Scalar equations

Finally, we can write the Einstein equations linearized for each mode, as said before we will focus on the equations of motions for the scalar modes since it will provide the mechanism we are interested in. The GI Einstein's equations are

$$\delta\overline{R_{\mu}^{\nu}} = \kappa\delta\overline{\mathcal{T}_{\mu}{}^{\nu}}. \tag{3.218}$$

The time-time equation is

$$\delta\overline{R_{n}^{n}} = \kappa\delta\overline{\mathcal{T}_{n}{}^{n}} \tag{3.219}$$

$$\dot{\Xi} + \frac{2}{3}\theta\Xi + (\dot{\theta} + D^2)\Phi = -\frac{\kappa}{2}(\overline{\delta\rho} + 3\overline{\delta p}) \tag{3.220}$$

taking the trace of (3.198), and (3.217) we obtain

$$h[\overline{\delta R_{\mu}^{\mu}}] = D^2(\Phi - \Psi) + \dot{\Xi} + 2\theta\Xi - 3D^2\Psi + \dot{\theta}\Phi - 12K\Psi; \tag{3.221}$$

$$h[\delta\overline{\mathcal{T}_{\mu}{}^{\mu}}] = \frac{3}{2}(\overline{\delta\rho} - \overline{\delta p}). \tag{3.222}$$

Thus, from (3.218) we have

$$h[\overline{\delta R_{\mu}^{\ \mu}}] = \kappa h[\delta \overline{\mathcal{T}_{\mu}^{\ \mu}}], \tag{3.223}$$

$$D^2(\Phi - \Psi) + \dot{\Xi} + 2\theta\Xi - 3D^2\Psi + \dot{\theta}\Phi - 12K\Psi = \frac{3}{2}\kappa(\overline{\delta\rho} - \overline{\delta p}). \tag{3.224}$$

The equations (3.220), and (3.224) are the first-order perturbative version of Friedmann's equations in the physical manifold M.

We may sum them to obtain an equation in which the matter sector depends only on $\delta\rho$, them substituting back on (3.224) to get an equation with source term solely of δp:

$$-\left(\dot{\Xi} + \frac{2\theta\Xi}{3} + (\dot{\theta} + D^2)\Phi\right) + \dot{\Xi} + 2\theta\Xi + D^2\Phi - 4D^2\Psi + \dot{\theta}\Phi - 12K\Psi = 2\kappa\overline{\delta\rho}$$

$$\frac{\theta\Xi}{3} - D^2\Psi - 3K\Psi = \frac{\kappa}{2}\overline{\delta\rho}, \tag{3.225}$$

replacing in (3.224)

$$\dot{\Xi} + 2\theta\Xi + D^2\Phi - 4D^2\Psi + \dot{\theta}\Phi - 12K\Psi = 3\kappa\left[\frac{1}{\kappa}\left(\frac{\theta}{3}\Xi - D^2\Psi - 3K\Psi\right) - \frac{\overline{\delta p}}{2}\right]$$

$$\dot{\Xi} + \theta\Xi + D^2(\Phi - \Psi) + \dot{\theta}\Phi - 3K\Psi = -\frac{3\kappa}{2}\overline{\delta p}. \tag{3.226}$$

Besides the two equations above, one may get two more linear independent scalar equations by performing divergences of the vectorial, and tensorial components. First, let us explicit the spatial part with a null trace of (3.218) by doing

$$h[\overline{\delta R_{\mu}^{\nu}}] - \frac{h_{\mu}^{\ \nu}\overline{\delta R_{\sigma}^{\sigma}}}{3} = \kappa\left\{h[\delta\overline{\mathcal{T}_{\mu}^{\ \nu}}] - \frac{h_{\mu}^{\ \nu}\delta\overline{\mathcal{T}_{\sigma}^{\ \sigma}}}{3}\right\},$$

the left side is

$$\begin{aligned} h[\overline{\delta R_{\mu}^{\nu}}] - \frac{h_{\mu}^{\ \nu}\overline{\delta R_{\sigma}^{\sigma}}}{3} &= D_\mu D^\nu(\Phi - \Psi) + \left(\frac{\dot{\Xi} + 2\theta\Xi}{3} - D^2\Psi + \frac{\dot{\theta}\Phi}{3} - 4K\Psi\right)h_{\mu}^{\ \nu} + \\ &+ f_{\mu(W_{\mu}^{\ \nu}, D^\nu S_\mu)}^{\ \nu} - \frac{h_{\mu}^{\ \nu}}{3}[D^2(\Phi - \Psi) + \dot{\Xi} + 2\theta\Xi - 3D^2\Psi + \dot{\theta}\Phi - 12K\Psi] \\ &= \left(D_\mu D^\nu - \frac{h_{\mu}^{\ \nu}D^2}{3}\right)(\Phi - \Psi) + f_{\mu(W_{\mu}^{\ \nu}, D^\nu S_\mu)}^{\ \nu}, \end{aligned} \tag{3.227}$$

while the right side:

$$\begin{aligned} h[\delta\overline{\mathcal{T}_{\mu}^{\ \nu}}] - \frac{h_{\mu}^{\ \nu}\delta\overline{\mathcal{T}_{\sigma}^{\ \sigma}}}{3} &= \left(\frac{\overline{\delta\rho} - \overline{\delta p}}{2}\right)h_{\mu}^{\ \nu} + \delta\Pi_{\mu}^{\ \nu} - \frac{h_{\mu}^{\ \nu}}{3}\left[\left(\frac{\overline{\delta\rho} - \overline{\delta p}}{2}\right)3 + \delta\Pi_{\sigma}^{\ \sigma}\right] \\ &= \delta\Pi_{\mu}^{\ \nu} \\ &= -\left(D_\mu D^\nu - \frac{h_{\mu\nu}D^2}{3}\right)\delta\Pi^{(s)} + l_{\mu\left(D^\nu\delta\Pi_{\mu}^{(v)}, \delta\Pi_{\mu}^{(t)\nu}\right)}^{\ \nu}. \end{aligned} \tag{3.228}$$

In which (3.112) has been used, and $f_\mu^{\ \nu}$ and $l_\mu^{\ \nu}$ are just a compact way to writing all the terms of these expressions since we will take the double divergence $D_\nu D^\mu$ on (3.227) and (3.228), with $D^\mu f_\mu^{\ \nu} = D^\mu l_\mu^{\ \nu} = 0$. Thus

$$\begin{aligned}\left(D^2 - \frac{D^4}{3}\right)(\Phi - \Psi) + 2KD^2(\Phi - \Psi) &= -\kappa\left(D^2 - \frac{D^4}{3}\right) - 2KD^2\delta\Pi^{(s)} \\ D^2\left(2K + 1 - \frac{D^2}{3}\right)(\Psi - \Phi) &= \kappa D^2\left(2K + 1 - \frac{D^2}{3}\right)\delta\Pi^{(s)} \\ \Psi - \Phi &= \kappa\delta\Pi^{(s)}. \end{aligned} \tag{3.229}$$

We know from (3.79) that the operator $\mathcal{D} \equiv D^2\left[3(2K+1) - D^2\right]$ has an inverse, i.e., $\mathcal{D}^{-1}\mathcal{D} = I$. Therefore, we have one of the useful scalar equations that relate the Bardeen potentials, with the matter sector. We may write one more relation taking the derivative D^μ on the spatial-time components, using (3.193), and (3.214), we have

$$\begin{aligned} D^\mu h[\overline{\delta R_\mu^{\ n}}] &= \kappa D^\mu h[\overline{\delta \mathcal{T}_\mu^{\ n}}] \\ -D^2\left(\frac{2}{3}\Xi\right) &= -D^2\left[\kappa(\rho + p)D^2\overline{\mathcal{V}}\right] \\ \Xi &= \frac{3}{2}\kappa(\rho + p)\overline{\mathcal{V}}. \end{aligned} \tag{3.230}$$

One may obtain one more GI scalar equation by performing a divergence on (3.143), thus

$$\begin{aligned} D^\mu u_\mu \dot{p} + D^2\delta p + (\rho + p)(D^\mu \dot{u}_\mu - D^2\phi) + D^\mu D_\alpha \delta\Pi^\alpha_{\ \mu} &= 0 \\ D^2\mathcal{V}\dot{p} + D^2\delta p + (\rho + p)(D^2\dot{\mathcal{V}} - D^2\phi) - D^2\left(\frac{2}{3}D^2 + 2K\right)\delta\Pi^{(s)} &= 0 \\ D^2\left[\mathcal{V}\dot{p} + \delta p + (\rho + p)(\dot{\mathcal{V}} - \phi) - \left(\frac{2}{3}D^2 + 2K\right)\delta\Pi^{(s)}\right] &= 0 \\ \overline{\delta p} + \dot{p}\overline{\mathcal{V}} + (\rho + p)(\dot{\overline{\mathcal{V}}} - \Phi) - \left(\frac{2}{3}D^2 + 2K\right)\delta\Pi^{(s)} &= 0, \end{aligned} \tag{3.231}$$

where the gauge transformations (3.91), (3.130), and (3.129) were used on the last line.

The set of the four scalar equations (3.225), (3.226), (3.229), and (3.230) specify the relations of the six GI scalar variables: $\Phi, \Psi, \overline{\mathcal{V}}, \overline{\delta p}, \overline{\delta\rho}$, and $\delta\Pi^{(s)}$. Therefore, we need two more equations to close the set of equations and specify each of these variables solely.[10]

3.4.4 Scalar perturbation dynamic

In the next chapter, we will see in detail how to specify the two more equations from the relativistic kinetic theory, in addition to the theorem that guarantees the equilibrium of the fluid. But for now, we will use this result without proof, i.e., the fluid

[10] The equations provided by $\widehat{\nabla}_\mu \widehat{T}^\mu_{\ \nu} = 0$ can be obtained from (3.220) and (3.224).

is in local thermodynamic equilibrium if $v^\mu \widehat{\nabla}_\mu \hat{S} = 0$, where $\hat{S}$ is the entropy of the fluid, therefore in this case, one may use thermodynamics relations.

Assuming $\hat{p} = \hat{p}(\hat{\rho}, \hat{S})$, we can write

$$\delta p = c_s^2 \delta\rho + \iota\, \delta S, \tag{3.232}$$

where

$$c_s^2 \equiv \left.\frac{\partial p}{\partial \rho}\right|_S, \quad \iota \equiv \left.\frac{\partial p}{\partial S}\right|_\rho. \tag{3.233}$$

In which c_s^2 is the sound's velocity of the fluid. From the equilibrium condition

$$\begin{aligned} v^\mu \widehat{\nabla}_\mu \hat{S} &= 0 && (3.234)\\ \nabla_n (S + \delta S) + (n^\mu + \delta v^\mu)\nabla_\mu (S + \delta S) &= 0 && (3.235)\\ (1 - \phi)\dot{S} + u^\mu D_\mu S + \dot{S} + \dot{\delta S} &= 0 && (3.236)\\ \dot{\delta S} &= 0, && (3.237) \end{aligned}$$

wherein the background entropy is constant, i.e., $\dot{S} = 0$. Also, one may write the time variation of p as

$$\begin{aligned} \dot{p} &= c_s^2 \dot{\rho} + \iota \dot{S} \\ &= c_s^2 \dot{\rho} \\ &= -c_s^2 \theta(\rho + p). \end{aligned} \tag{3.238}$$

From the Friedmann's equations (2.72), (2.75), we might subtract them to get

$$\begin{aligned} \dot{H} &= -\frac{\kappa}{2}(\rho + p) + K, \\ \dot{\theta} &= -\frac{3\kappa}{2}(\rho + p) + 3K. \end{aligned} \tag{3.239}$$

where $\theta = 3H$.

One may combine some scalar equations to obtain more compact equations. Replacing (3.230) in (3.225), we have

$$\begin{aligned} \frac{\theta}{2}\kappa(\rho + p)\overline{\mathcal{V}} - (D^2 + 3K)\Psi &= \frac{\kappa}{2}\overline{\delta\rho} \\ \frac{\kappa\theta\overline{\mathcal{V}}}{2} - \frac{D_K^2\Psi}{\rho + p} &= \frac{\kappa}{2}\overline{\delta\rho} \\ -D_K^2\Psi &= \frac{\kappa(\rho + p)}{2}(\overline{\delta_\rho} - \theta\overline{\mathcal{V}}), \end{aligned} \tag{3.240}$$

wherein $\overline{\delta_\rho} \equiv \dfrac{\overline{\delta\rho}}{\rho + p}$ is the density contrast, and $D_K^2 \equiv D^2 + 3K$. The equation (3.240) is the generalized Poisson's equation, note that if we take $\theta = 0$ and $K = 0$ we recover the Poisson's equation in the context of Newtonian perturbation.

Using the equation of state (3.232) in (3.231), we obtain

$$c_s^2\overline{\delta_\rho} + S - \frac{\overline{\mathcal{V}}\theta c_s^2(\rho+p)}{\rho+p} + \dot{\overline{\mathcal{V}}} - \Phi - \frac{2}{3(\rho+p)}D_K^2\delta\Pi^{(s)} = 0$$
$$\dot{\overline{\mathcal{V}}} - (\Phi - S) + c_s^2(\overline{\delta_\rho} - \overline{\mathcal{V}}\theta) - D_K^2 \mathrm{F} = 0. \quad (3.241)$$

Where (3.238) has been used, and $\mathrm{F} \equiv \dfrac{2\delta\Pi^{(s)}}{3(\rho+p)}$. In the same way, we may use (3.232) in (3.144) obtaining

$$\frac{\dot{\delta\rho}}{p+\rho} - \frac{\phi\theta(\rho+p)}{\rho+p} + \frac{\theta(\delta\rho + c_s^2\delta\rho + \iota\delta S)}{\rho+p} + D^2\overline{\mathcal{V}} + \delta\theta = 0$$
$$\dot{\delta_\rho} - \theta(1+c_s^2)\delta_\rho + \theta\delta_\rho(1+c_s^2) + \theta\,(\mathrm{S}-\phi) + \theta\delta_\rho(1+c_s^2) + \delta\theta + D^2\overline{\mathcal{V}} = 0$$
$$\dot{\delta_\rho} - \theta\,(\phi - \mathrm{S}) + \theta\delta_\rho(1+c_s^2) + \delta\theta + D^2\overline{\mathcal{V}} = 0, \quad (3.242)$$

Wherein $\mathrm{S} \equiv \iota\dfrac{\delta S}{\rho+p}$, and on the second line the derivative by parts was used on the first term.

Before expressing the dynamic of the scalar perturbation variable, we should choose a specific gauge to work with. As said earlier one needs to determine a gauge to interpret the GI variables, there are many possible gauges to use. Here and further, we will use the Newtonian gauge.

This gauge consists of taking $\mathcal{B} = \mathcal{E} = 0$ on the gauge transformations, physically, the isotropic background observers will measure a null shear potential on the physical manifold, i.e., $\mathcal{S} = 0$. From the definition of the shear potential, we can see how the gauge components behave

$$\tilde{\mathcal{S}} = 0, \quad (3.243)$$
$$\mathcal{B} + \dot{V}^{\perp} - \frac{2}{3}\theta V^{\perp} - V^{\parallel} - (\dot{\mathcal{E}} + \dot{V}^{\perp}) + \frac{2}{3}\theta(\mathcal{E} + V^{\perp}) = 0, \quad (3.244)$$
$$\mathcal{S} + V^{\parallel} = 0, \quad (3.245)$$
$$V^{\parallel} = 0. \quad (3.246)$$

Therefore the spatial vectors on the spatial sections of the background manifold will not have parallel components on the physical manifold, in other words, the Newtonian gauge is equivalent to choosing a sectioning on M, such that preserves the orthogonality of the spatial vectors on $\widehat{M}$.

From the potentials (3.91), (3.92), (3.93), we have $\Psi = \psi$, $\Phi = \phi$, and $\Xi = \delta\theta$, in this gauge the scalar Φ is related to the usual Newtonian potential on the weak field limit, i.e., $\nabla^2\Phi = 4\pi\rho$. The scalars ψ and Ξ, are related to the curvature perturbation, and the scalar expansion respectively [11].

[11] Even though we are taking $\mathcal{S}$ the Bardeen variables are still GI.

It will be useful to write the (3.230) in terms of the Bardeen potentials and the Hubble function:

$$\dot{\Psi} + H\Phi = \frac{\kappa(\rho + p)}{2}\overline{\mathcal{V}}. \tag{3.247}$$

Hereafter we will calculate the perturbations in the plane geometry, i.e., $K = 0$. This choice will be clear in chapter 5. Let us define the Mukhanov-Sasaki variable as

$$\zeta \equiv \Psi + \frac{\theta\overline{\mathcal{V}}}{3}. \tag{3.248}$$

The variable ζ has all the information about the scalar perturbations associated with the geometry and the matter, in the literature it is called the curvature perturbation. Calculating the time derivative of this variable, using the Newtonian gauge ($\mathcal{S} = 0$),

$$\begin{aligned}
\dot{\zeta} &= \dot{\Psi} + \frac{1}{3}\left(\dot{\theta}\overline{\mathcal{V}} + \theta\dot{\overline{\mathcal{V}}}\right) && (3.249)\\
&= \dot{\Psi} + \frac{1}{3}\left\{-\frac{3\kappa}{2}(\rho + p)\overline{\mathcal{V}} + \theta\left[(\Phi - \mathrm{S}) - c_s^2(\overline{\delta_\rho} - \theta\overline{\mathcal{V}}) + D^2\mathrm{F}\right]\right\} && (3.250)\\
&= \frac{\kappa(\rho + p)}{2}\overline{\mathcal{V}} - \frac{\theta}{3}\Phi + \left(\frac{\theta^2 c_s^2}{3} - \frac{\kappa(\rho + p)}{2}\right)\overline{\mathcal{V}} + \frac{\theta}{3}\left[(\Phi - \mathrm{S}) - c_s^2\overline{\delta_\rho} + D^2\mathrm{F}\right] && (3.251)\\
&= \frac{\theta}{3}\left(-\mathrm{S} - c_s^2\overline{\delta_\rho} + D^2\mathrm{F} + c_s^2\theta\overline{\mathcal{V}}\right) && (3.252)\\
&= \frac{\theta}{3}\left[-\mathrm{S} + D^2\mathrm{F} - c_s^2\overline{\delta_\rho} + c_s^2\left(\frac{2D^2\Psi}{\kappa(\rho + p)} + \overline{\delta_\rho}\right)\right] && (3.253)\\
&= \frac{\theta}{3}\left(-\mathrm{S} + D^2\mathrm{F} + \frac{2c_s^2 D^2\Psi}{\kappa(\rho + p)}\right), && (3.254)
\end{aligned}$$

on the second line, we used (3.239) and (3.241), and on the third and fifth lines (3.247), and (3.240) were used respectively.

Therefore we have

$$\dot{\zeta} = \frac{2Hc_s^2}{\kappa(\rho + p)}D^2\Psi + \frac{2HD^2}{3(\rho + p)}\delta\Pi^s - \frac{\iota\delta S}{\rho + p}. \tag{3.255}$$

To close the system of equations the last assumption is to consider the fluid on the physical manifold $\widehat{M}$ as described by a perfect fluid, such as, on the background we have $\widehat{\Pi}_{\mu\nu} = 0$, but this implies $\delta\Pi_{\mu\nu} = 0$, therefore $\delta\Pi^s = 0$. From (3.229) then $\Phi = \Psi$, considering adiabatic perturbations: $\delta S = 0$, we obtain

$$\zeta = \Psi + H\overline{\mathcal{V}}, \tag{3.256}$$

$$\dot{\zeta} = \frac{2Hc_s^2}{\kappa(\rho + p)}D^2\Psi, \tag{3.257}$$

$$\dot{\Psi} = \frac{\kappa(\rho + p)}{2}\overline{\mathcal{V}} - H\Psi, \tag{3.258}$$

using (3.256) in (3.258), we get

$$\dot{\Psi} = \frac{\kappa(\rho+p)}{2H}\xi - \left(\frac{\kappa(\rho+p)}{2H} + H\right)\Psi, \tag{3.259}$$

applying the laplacian operator on both sides and using (3.54), (3.257) and (3.259)

$$D^2\dot{\Psi} = \frac{\kappa(\rho+p)}{2H}D^2\zeta - \left(\frac{\kappa(\rho+p)}{2H} + H\right)D^2\Psi \tag{3.260}$$

$$\partial_t\left(\frac{\kappa(\rho+p)}{2Hc_s^2}\dot{\zeta}\right) = \frac{\kappa(\rho+p)}{2H}D^2\zeta - \left(\frac{\kappa(\rho+p)}{2H} + 3H\right)\frac{\kappa(\rho+p)}{2Hc_s^2}\dot{\zeta} \tag{3.261}$$

$$\ddot{\zeta} = -\underbrace{\left[3H + \frac{\kappa(\rho+p)}{2H} + \frac{H}{\rho+p}\partial_t\left(\frac{\rho+p}{H}\right)\right]}_{(i)}\dot{\zeta} + c_s^2 D^2\zeta, \tag{3.262}$$

let us simplify the factor (i) which multiply $\dot{\zeta}$, assuming a barotropic fluid $p = w\rho$, with $w = cte$,

$$\partial_t\left(\frac{\rho+p}{H}\right) = \frac{\dot{\rho}\left(\frac{\dot{p}}{\dot{\rho}} + 1\right)}{H} - \frac{\rho+p}{H^2}\dot{H} \tag{3.263}$$

$$= -3\rho(1+w)^2 + \frac{3\rho(1+w)^2}{2} \tag{3.264}$$

$$= -\frac{3\rho(1+w)^2}{2}, \tag{3.265}$$

where we used $w = c_s^2 = \frac{\dot{p}}{\dot{\rho}}$, and the equations (2.72), (2.73) and (2.76) with $\tilde{K} = 0$. Then

$$(i) = 3H + \frac{\kappa(\rho+p)}{2H} - \frac{H}{\rho+p}\frac{3\rho(1+w)^2}{2} \tag{3.266}$$

$$= 3H + \frac{3H^2(1+w)}{2H} - \frac{3(1+w)H}{2} \tag{3.267}$$

$$= 3H. \tag{3.268}$$

Furthermore, the dynamical equation for $\zeta(x,t)$ will be

$$\ddot{\zeta} + 3H\dot{\zeta} - c_s^2 D^2\zeta = 0 \tag{3.269}$$

Writing the above expression on the conformal time, and making a Fourier transform [12], we have $D^2\zeta(x,\eta) = -\frac{k^2}{a^2}\zeta(k,\eta)$, then

$$\zeta'' + 2\mathcal{H}\zeta' + \omega^2\zeta = 0 \tag{3.270}$$

[12] The Fourier transform to the momentum space is $\zeta(k,\eta) = \int_{-\infty}^{\infty} d^3x\, \zeta(x,\eta)\, e^{-ix\cdot k}$.

where we define $\omega^2 \equiv c_s^2 k^2$, in which k is the wave number associated with the field ζ, and $\mathcal{H}$ is the conformal Hubble function.

Assuming a scale factor on the form $a = t^{\alpha}$, with $0 < \alpha < 1$ then

$$\zeta'' + \left(\frac{2\alpha}{1-\alpha}\right)\frac{1}{\eta}\zeta' + \omega^2\zeta = 0. \tag{3.271}$$

The differential equation above has an analytical solution, given by Bessel functions

$$\zeta(k,\eta) = \eta^{\frac{1}{2}\left(1-\frac{2\alpha}{1-\alpha}\right)}\left[A_k J_{\frac{1}{2}\left(1-\frac{2\alpha}{1-\alpha}\right)}(\omega\eta) + B_k Y_{\frac{1}{2}\left(1-\frac{2\alpha}{1-\alpha}\right)}(\omega\eta)\right] \tag{3.272}$$

wherein $J_\nu(\omega\eta)$ and $Y_\nu(\omega\eta)$ are the Bessel functions of the first and second kind respectively and A and B constants. One may analyze this solution on the limits of sub-Hubble and super-Hubble discussed in chapter 2.

The sub-Hubble limit: $\lambda \ll R_{\mathcal{H}} \Rightarrow k \gg \mathcal{H}$, thus $\omega\eta \gg \dfrac{c_s\alpha}{1-\alpha}$, the solution will be[13]

$$\zeta(k,\eta) = \left(\frac{2}{\pi\omega\eta}\right)^{1/2}\left\{A_k\cos\left[\omega\eta - \frac{\pi}{2}\left(\frac{1}{2} - \frac{2\alpha}{1-\alpha}\right)\right] + B_k\sin\left[\omega\eta - \frac{\pi}{2}\left(\frac{1}{2} - \frac{2\alpha}{1-\alpha}\right)\right]\right\} \tag{3.273}$$

we obtain an oscillatory solution, with decreasing amplitude on time, as a damped oscillator with frequency ω.

On the other hand the super-Hubble limit: $\lambda \gg R_{\mathcal{H}} \Rightarrow \omega\eta \ll \dfrac{c_s\alpha}{1-\alpha}$, leads to

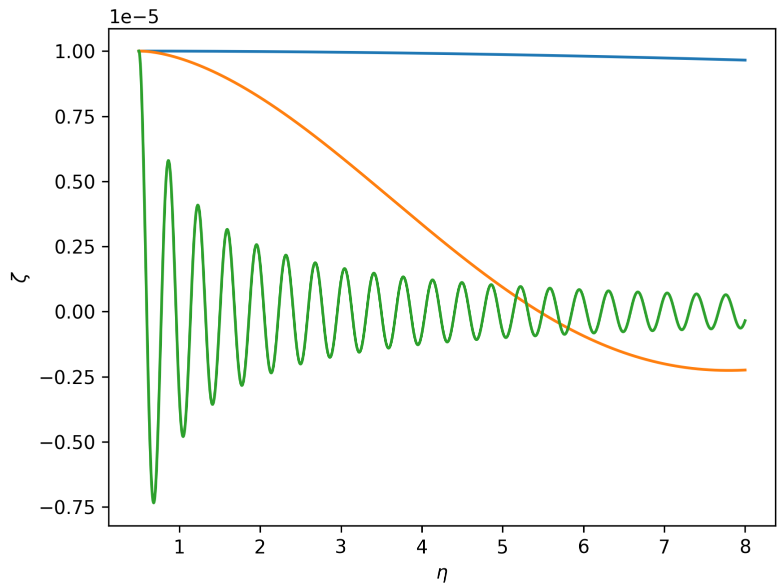
$$\zeta(k,\eta) = C_k\frac{\eta^{1-\frac{2\alpha}{1-\alpha}}}{1-\frac{2\alpha}{1-\alpha}} + D_k. \tag{3.274}$$

in which C and D are constants, which depend on the initial condition of the differential equation (3.271).

One may apply the solution (3.272) on the radiation domination era, setting $\alpha = 1/2$ and $c_s^2 = \frac{1}{3}$, the figure 4 shows the behavior of the field for different modes, comparing with the figure 5 one can see the wave-length λ associated with $k = 30$ (blue) is bigger than the Hubble radius (purple), leading to a super-Hubble solution, in which the mode it is "frozen"in time, on the other hand, the modes with the wave-length less than the Hubble radius have oscillatory solutions, as expected.

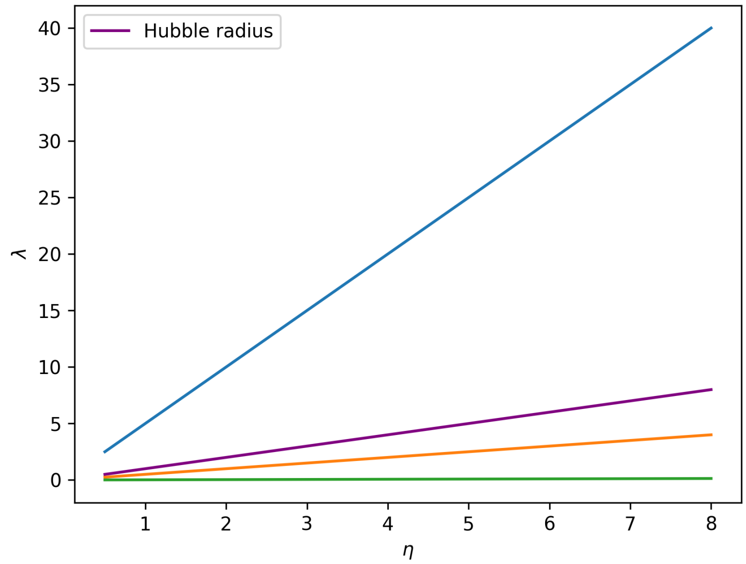
[13] Both the factor α and the sound's velocity c_s are less than 1 thus $\omega\eta >> 1$ is a sufficient condition for the limit.

Figure 4 – Curvature perturbation $\zeta(k, \eta)$ for the radiation domination era, i.e., $\alpha = 1/2$ and $c_s^2 = \frac{1}{3}$ for three frequencies: $k = 30(\text{green}), 1(\text{yellow}), 0.1(\text{blue})$, with the initial condition $\zeta(k, 0) = 10^{-5}$ and $\zeta'(k, 0) = 0$.



Source: Author.

Figure 5 – Wave-lenghts associated with the frequencies of the figure 4 (same colors), and the Hubble radius (purple), with $R_{\mathcal{H}} = \eta$ on the radiation domination era.



Source: Author.

With the evolution of the scalar perturbation in hands we may calculate the power spectrum defined as the two-point function of the field, i.e., $\mathcal{P}_k = \langle \zeta^* \zeta \rangle$, this quantity will be important for our analysis, we will return to this later in chapter 5. For now, we will focus on the kinetic theory in FLRW universe to understand some features of the thermal history of the universe which is relevant to compute the desired spectral distortions of CMB in the primordial universe.

4 Kinetic theory in curved spacetime

4.1 Kinetic theory in FLRW universe

The goal of this chapter is to describe the thermodynamic properties of the CMB and its interaction with other matter sources of the universe from first principles, in particular in the range $10^3 < z < 10^6$ of redshift. The relativistic kinetic theory fulfills this objective, once the CMB may be described as a gas of photons.

Furthermore, we will see that the relativistic hydrodynamics approach[1] is not enough to describe the behavior of the photons and baryons on the decoupling limit. Therefore being important to use the kinetic theory to have a full description of the primordial plasma and consequently the calculation of the CMB anisotropies. Besides that, we will prove the H-theorem which formalizes the condition of quasi-equilibrium used in chapter 2 concerning the assumption of the equation of state.

Since the universe can be described by a non-Euclidean geometry on large scales undergoing an accelerated expansion, we should study a relativistic kinetic theory in curved space-time, being necessary to evoke classical statistical hypotheses and reconcile them locally with the structure of Minkowski space, incorporating restricted covariance into the formalism.

One of the pioneers of this work was Synge in 1934 [25], which introduced the notion of the world lines of gas particles as the most fundamental ingredient for describing a relativistic gas. This idea led to the development of the modern generally covariant formulation of relativistic kinetic theory [26], allowing the generalization of the kinetic theory of gas to more general geometries, which will be relevant in this chapter. As the main references, we will use the books [27], [28], [10], and the article [26] which developed the kinetic theory in curved space-time emphasizing the Hamiltonian structure on it.

4.1.1 Distribution Function

The central hypothesis in kinetic theory is that the average properties of the gas are described by the one-particle distribution function $f(x,q)$ [26]. It is a scalar function defined on the fiber $P_x \subset TM$ defined as

$$f : P_x \to \mathbb{R}; \tag{4.1}$$

$$(x,q) \to f(x,q) \tag{4.2}$$

[1] Relativistic Euler's equation and the continuity equation.

where $x \in M$, and $q \in P_x$, such that, $g_x(q,q) \leq 0$. [2]

Hereafter the vector q will be time-like or null-like, in a local chart (U, M) $q = q^\mu \partial_\mu$. We will use the tensors without indices for denoting the geometric object, and with indices for denoting the components on a local basis, otherwise will be emphasized.

The distribution function $f(x,q)$ has the meaning of the density of the particles in the position x with momentum q. From special relativity, we have that all the particles with mass m should satisfy the dispersion relation given by

$$g_x(q,q) = -m^2, \tag{4.3}$$

where in a local frame

$$g_{\mu\nu} q^\mu q^\nu = -m^2 \tag{4.4}$$

$$g_{00}(q^0)^2 + 2g_{0i}q^0 q^i + g_{ii}(q^i)^2 = -m^2. \tag{4.5}$$

For a set of particles with the same mass m, one may define the mass shell $\Sigma \subset TM$ as

$$\Sigma = \{(x,q);\;\; x \in M,\;\; q \in P_{x,m}\} \tag{4.6}$$

wherein

$$P_{x,m} = T_x M \cap \left\{ g_x(q,q) = -m^2, q^0 > 0 \right\}. \tag{4.7}$$

It is important to note that we will use M^4 in this context as a time-oriented manifold, i.e., $q^0 > 0$ always. The future part of the mass hyperboloidal is shown in figure 6.

Once the distribution function has the meaning of the density of number in M^4, to get the desired statistical features of the microscopic system we might integrate it on the fiber bundle (phase space) TM^4, thus we need to define the volume form of this space

$$\omega = \omega_g \wedge \omega_q \tag{4.8}$$

wherein the natural 4-forms from a local frame on phase space are [8]

$$\omega_g = \sqrt{-\det g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \tag{4.9}$$

$$\omega_p = \sqrt{-\det g}\, dq^0 \wedge dq^1 \wedge dq^2 \wedge dq^3. \tag{4.10}$$

One may write an induced volume form for the moments using the constraint of the mass shell

$$\omega_{m,q} = \frac{\sqrt{-\det g}}{E}\, dq^1 \wedge dq^2 \wedge dq^3 \wedge dq^4, \tag{4.11}$$

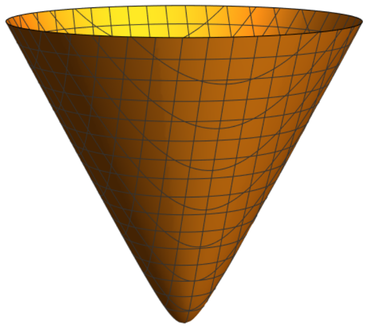
thus, the induced volume form on P_x will be

$$\omega_m = \omega_g \wedge \omega_{m,q}. \tag{4.12}$$

The proof of (4.11) is found in appendix D.

[2] We are using the metric signature $(-,+,+,+)$ and the label x is to emphasize that the inner product is defined on the point x.

Figure 6 – Representation of Σ in a local basis on TM for a particle with mass $m > 0$ consequently $q^0 > 0$. The coordinate q^0 is oriented parallel to the z-axis (up)



Source: Author.

4.1.2 Estatistical Moments

With the volume form of these spaces well defined, is possible to define the tensor fields in M^4 by integrations of geometric quantities involving the distribution function. The moment of order zero is [3]

$$n(x) \equiv \int_{P_x} f\, \omega_p, \tag{4.13}$$

it represents the particle density in space-time. Following the same idea, one may define the density number of trajectories crossing a submanifold $S \subset TM$ of the phase space:

$$N(S) = \int_S f\, \bar{\omega}, \tag{4.14}$$

in which X is a vector field in TM, we will explore it more ahead, and $\bar{\omega} \equiv i_X \omega =< \omega, X >$, this inner product is inducing the volume form $\bar{\omega}$ on S.

The current and the stress-energy tensor in $x \in M^4$ are defined as

$$J_x(\alpha) = \int_{P_x} f(x,q)\alpha(q)\, \omega_q, \tag{4.15}$$

$$T_x(\alpha,\beta) = \int_{P_x} f(x,p)\alpha(q)\beta(q)\, \omega_q, \tag{4.16}$$

in which $\alpha, \beta \in T^*M^4$ are one-forms, and the integration is on the fiber P_x. Given a local frame $\{x^\mu\}$ in M^4, we have $T_x(dx^\mu, dx^\nu)$ and $J_x(dx^\mu)$, then

$$J^\mu(x) = \int_{P_x} f(x,q)\, q^\mu\, \omega_q = \int_{\mathbb{R}^3} f(x,q)\, \frac{p^\mu}{|q_0|}\, d^3q, \tag{4.17}$$

$$T^{\mu\nu}(x) = \int_{P_x} f(x,q) q^\mu q^\nu\, \omega_p = \int_{\mathbb{R}^3} f(x,q)\, \frac{q^\mu q^\nu}{|q_0|}\, d^3q. \tag{4.18}$$

Wherein $|q_0| = |q_0(x, q^i)|$, and $d^3q = dq^1 \wedge dq^2 \wedge dq^3$.

[3] We will not put the normalization term $(2\pi)^3$ on most of the equations, when it is important to calculation we will write it

The vector field J and the symmetric contravariant tensor field T are defined as the first moment and second moment respectively. One may define a macroscopic unit flow velocity from the current:

$$u^{\mu} = \frac{1}{r} J^{\mu}, \tag{4.19}$$

where $r^2 \equiv -J^{\mu} J_{\mu}$, in which $r \geq 0$ is the proper mass density, if J^{μ} is time-like, then u^{μ} will be time-like.

If the distribution function $f = f(x, v_{\sigma} q^{\sigma})$, with v_{μ} a time-like covector on M^4, then the first and second moments of f read respectively as the particle number momentum vector and the stress-energy tensor of a perfect fluid. Besides that, the unit flow velocity u_{μ} is parallel with v_{μ}. Let us prove these results

Using a local orthonormal frame $g_{\mu\nu} = \eta_{\mu\nu}$, and the covector field $v_{\mu}(x) = -\lambda \delta_{\mu}^{\ 0}$, with $\lambda = \sqrt{-v^{\sigma} v_{\sigma}}$, then the first moment and second moment are

$$J^{\mu}(x) = \int_{\mathbb{R}^4} f(x, -\lambda q^0)\, q^{\mu}\, d^4q, \tag{4.20}$$

$$T^{\mu\nu}(x) = \int_{\mathbb{R}^4} f(x, -\lambda q^0) q^{\mu} q^{\nu}\, d^4q. \tag{4.21}$$

Calculating the temporal and space components

$$T^{00} = \int_{\mathbb{R}^4} f(x, -\lambda q^0)(q^0)^2\, d^4q \equiv \rho \quad \text{(energy density)}, \tag{4.22}$$

$$T^{ii} = \int_{\mathbb{R}^4} f(x, -\lambda q^0)(q^i)^2\, d^4q \equiv 3p \quad \text{(isotropic pressure)}, \tag{4.23}$$

$$T^{0i} = \int_{\mathbb{R}^4} f(x, -\lambda q^0) q^0 q^i\, d^4q \equiv \chi^i = 0 \quad \text{(heat flux)}, \tag{4.24}$$

$$T^{ij} = \int_{\mathbb{R}^4} f(x, -\lambda q^0) q^i q^j\, d^4q \equiv \Pi^{ij} = 0 \quad \text{(anisotropic pressure)}, \quad i \neq j, \tag{4.25}$$

$$J^i(x) = \int_{\mathbb{R}^4} f(x, -\lambda q^0)\, q^i\, d^4q = 0, \quad \text{(particle number current)} \tag{4.26}$$

$$J^0(x) = \int_{\mathbb{R}^4} f(x, -\lambda q^0)\, q^0\, d^4q \equiv n(x) \quad \text{(particle number density)}, \tag{4.27}$$

the components (4.24-4.26) are zero because the integral goes from $-\infty$ to ∞ for all the moment components, with $q^0 > 0$ and the integrand of these components being odd functions, the integrals are zero directly.

From the equations above $T^{\mu\nu}$ and J^{μ} describe a perfect fluid. From (4.19) the flow velocity is $u^{\mu} = \frac{1}{\lambda} v^{\mu}$, once they are collinear one may choose to use the same time-like vector to sectioning the space-time and defining the flow of matter as we did in the last chapters.

Note the variables in (4.22-4.27) are in that form for the inertial frame with four-velocity $u^\mu = -\lambda \delta_0^{\ \mu}$, nonetheless the equations (4.15) and (4.16) are covariant version valid for any frame.[4]

From equations (4.22) and (4.23), one may obtain the state equations of the fluids which were used in previous chapters. For a local frame on the mass shell, we have $d^4q = \frac{d^3q}{E(q)}$, wherein $q^0 = E(q)$. The pressure and the energy density are given by [5]

$$p = \frac{T^i_{\ i}}{3} = \frac{1}{3}\int \frac{f(x,q)q^2}{E(q)}\, d^3q, \tag{4.28}$$

$$\rho = T^{00} = \int f(x,q)\, E(q)\, d^3q. \tag{4.29}$$

For the relativistic particles: $E = q$, is straightforward

$$p_{rad} = \frac{1}{3}\,\rho, \tag{4.30}$$

which describes the equation of state for radiation, from (3.232).

For the non-relativistic particles we have $E = m + \frac{q^2}{2m}$, then

$$\rho = \int \left(m + \frac{q^2}{2m}\right) f(x,q)\, d^3q \tag{4.31}$$

$$= m\, n(x) + \int f(x,q)\frac{q^2}{2m}\, d^3q, \tag{4.32}$$

the pressure will be

$$p = \frac{1}{3}\int \frac{f(x,q)q^2}{m + q^2/2m}\, d^3q \tag{4.33}$$

$$= \frac{1}{3}\int f(x,q)q^2\left(\frac{1}{m} - \frac{q^2/2m}{m^2}\right) d^3q \tag{4.34}$$

$$= \frac{2}{3}\int f(x,q)\frac{q^2}{2m}\, d^3q \tag{4.35}$$

$$= \frac{2}{3}\left(\rho - m\, n(x)\right). \tag{4.36}$$

Where on the second line we expanded the denominator near $q = 0$, on the third line we ignored the term with q^4, and on the last line we used (4.32).

From this expression, the energy density of the comoving observers is $\rho = m\, n(x)$, thus $p_{cold} = 0$, for cold matter. Nevertheless, if we consider observers with a small velocity, we get an effective relation: $p_{warm} = \frac{2}{3}\rho_{eff}$, which describes a warm matter.

4 Although the results were found for the inertial frame, by the general covariance they will be true for any other frame of reference.

5 We are making the computation on the mass-shell, one needs the normalization factor $1/(2\pi)^3$ for a correct numerical result.

It is important to note that we obtained these relations without the form of the distribution function $f(x,q)$, consequently, these relations are valid independently from the Boltzmann equations.

4.1.3 Liouville-Vlasov equation

When the system is described by a distribution function in curved space without interaction with an external force, then the particles will follow a geodesic flow in TM given by the field $X = (q,\ Q)$, in a local coordinate system

$$q^\alpha = \frac{dx^\alpha}{d\lambda} \tag{4.37}$$

$$Q^\alpha = \frac{dq^\alpha}{d\lambda} = -\Gamma^\alpha{}_{\mu\nu} q^\mu q^\nu \tag{4.38}$$

where on the second line we use the geodesic equation (A.11), λ is a temporal parameter, being the proper time of the observers for time-like curves, and an affine parameter for null-like curves, and Γ are the Cristoffel symbols, i.e., we are assuming $\nabla_\alpha g_{\mu\nu} = 0$.

It was shown in appendix D that X is a vector field on TM, i.e., $X \in T(TM)$, in which generate integral curves in TM given by the one-parameter group $\phi_\lambda(x)$. In what follows we present the theorem which guarantees the dynamical equation we want to obtain

Theorem 4.1. *(Liouville theorem) The volume form ω is invariant under the geodesic flow, i.e.*

$$\mathcal{L}_X \omega = 0. \tag{4.39}$$

The proof can be found in appendix D. From a geometric perspective, this result shows the invariance of the volume element of the phase space under the flux generated by X, this invariance guarantees the integration of the distribution function using the same measuring element given by the volume form.

Corollary 4.1. *The form $\bar{\omega} = i_X \omega$ is closed, moreover for any $f \in C^\infty(\mathbb{R})$*

$$d(f\bar{\omega}) = (\mathcal{L}_X f)\omega. \tag{4.40}$$

Proof. Using the Cartan identity, and Liouville's theorem, one gets

$$(\mathcal{L}_X f)\omega = \mathcal{L}_X(f\omega) = i_X d(f\omega) + d i_X(f\omega) = d(f i_X \omega) = d(f\bar{\omega}) = 0, \tag{4.41}$$

for the case $f = 1$, we have $d\bar{\omega} = 0$, thus it is a closed form.

As from (4.14), letting the field X normal to each submanifold S flow, one may construct $V = \bigcup_t S$, with $0 < t < T$, as a tubular submanifold of TM, such that, the boundary ∂V will not contribute for the integration since $\partial V = \underset{t}{\cup}\, \partial S$, then $\int_{\partial V} f\, \bar{\omega} = 0$, therefore the density of trajectory crossing ∂V is given by

$$N(S_T) - N(S_0) = \int_{\partial V} f\, \bar{\omega}. \tag{4.42}$$

using the Stoke's theorem (A.10), and the corollary 4.1.3 we have

$$N(S_T) - N(S_0) = \int_V d(f\bar{\omega}) = \int_V \mathcal{L}_X f\, \omega. \tag{4.43}$$

Therefore the density of trajectories crossing $S(t = T)$ is the density of trajectories that crossed the hypersurface $S(t = 0)$ summed by the contribution of the right side of (4.43), this term is due to the interaction between the particles that constitute the gas.

If we assume there is no interaction between the particles, or in other words, the collisionless situation, thus $N(S_T) = N(S_0)$ which implies

$$\mathcal{L}_X f = 0 \tag{4.44}$$

this is Liouville's equation, which tells us the behavior of the distribution function on the phase space without interaction between the particles of the gas.

Liouville's equation may be written as

$$\frac{df(x,q)}{d\lambda} = 0, \tag{4.45}$$

$$q^{\alpha}\frac{\partial f}{\partial x^{\alpha}} + Q^{\alpha}\frac{\partial f}{\partial q^{\alpha}} = 0, \tag{4.46}$$

$$q^{\alpha}\frac{\partial f}{\partial x^{\alpha}} - \Gamma^{\alpha}_{\ \mu\nu} q^{\mu} q^{\nu}\frac{\partial f}{\partial q^{\alpha}} = 0, \tag{4.47}$$

let us apply this equation to the FLRW universe,

$$q^0\frac{\partial f}{\partial t} + q^i\frac{\partial f}{\partial x^i} - \Gamma^0_{\ ij} q^i q^j \frac{\partial f}{\partial q^0} - 2\Gamma^j_{\ 0i} q^0 q^j \frac{\partial f}{\partial q^j} = 0 \tag{4.48}$$

using the Cristoffel symbols in FLRW space (A.14) and (A.15), and $E = q^0$, it is convenient to separate the dependence of conformal momentum on its modulus $\bar{q} \equiv |\bar{q}| = \sqrt{\bar{q}^i \bar{q}_i}$ and the unit direction :

$$\bar{q}^i = C\, \hat{q}^i, \tag{4.49}$$

$$\bar{q}^i = \frac{q}{a}\hat{q}^i, \tag{4.50}$$

in which $q^2 = a^2 C^2$, since $\hat{q}^i \hat{q}_i = 1$. Using the mass shell condition, then

$$\frac{\partial f}{\partial t} - \bar{q}H\frac{\partial f}{\partial \bar{q}} - 2H\hat{q}^j\frac{\partial f}{\partial \hat{q}^j} + \frac{q}{aE}\hat{q}^j\frac{\partial f}{\partial x^j} = 0 \tag{4.51}$$

one can drop the last two terms, as we will see the third term represents a second-order perturbation term, and by homogeneity, the last term will be zero. [6]

$$\frac{\partial f}{\partial t} - \bar{q}H\frac{\partial f}{\partial \bar{q}} = 0, \tag{4.52}$$

a possible solution is

$$f(q,t) \propto \exp\left(-\frac{\bar{q}a}{T_0}\right) = \exp\left(-\frac{\bar{q}}{T}\right), \tag{4.53}$$

wherein $T = \dfrac{T_0}{a}$, if we consider relativistic particles $\bar{q} \gg m$, then $E = \bar{q}$ then the distribution function above describes a Bose-Einstein or Fermi-Dirac gas of massless particles, in which the temperature falls by a factor $1/a$. If the gas is non-relativistic, for example, cold matter consisting of particles with mass at low velocities we need other treatment to obtain the temperature as we will see later.

Integrating (4.52) on the moment's space, one may obtain a relation to the density of particle number.

$$\frac{d}{dt}\underbrace{\left(\int d^3q f\right)}_{n(t)} - 4\pi \int d^2q\, q^3 \frac{\partial f}{\partial q} = 0 \tag{4.54}$$

$$\frac{dn(t)}{dt} + 3Hn(t) - 4\pi H \int_0^\infty \frac{\partial (q^3 f)}{\partial q}\, dq = 0 \tag{4.55}$$

$$\frac{dn(t)}{dt} + 3Hn(t) = 0 \tag{4.56}$$

where on the second line we integrated by parts, where the last term is $(q^3 f)|_0^\infty = 0$, in which $\lim_{q\to\infty} f(q,t) = 0$, since we are assuming a well-behaved distribution function, then

$$\frac{dn(t)}{dt} + 3Hn(t) = 0, \tag{4.57}$$

$$\frac{dn(t)}{dt} + 3\frac{\dot{a}}{a}n(t) = 0, \tag{4.58}$$

$$a\frac{dn(t)}{dt} + 3\frac{da}{dt}n(t) = 0, \tag{4.59}$$

$$\frac{d}{dt}(na^3) = 0, \tag{4.60}$$

$$n(t) \propto a^{-3}. \tag{4.61}$$

As expected once the expansion of space increases the volume and the density of particles' number will decrease proportionally.

[6] On first-order perturbation of FLRW universe the last term will not be zero.

4.2 Boltzmann equation

When considering collisions between the particles, the right side of (4.44) will have a nonzero term

$$\mathcal{L}_X f = C[f], \tag{4.62}$$

wherein $C[f]$ is an integral operator given by

$$C[f](x,q) = \int [f(x,q')f(x,k') - f(x,q)f(x,k)]\, A(x,q',k',q,k)\; d^4k'd^4q'd^4k. \tag{4.63}$$

The collision term can be linked with the probability of two particles of momentum q' and k' interacting at x and given after the shock two particles, one of them with momentum q and the other with momentum k [27].

The conservation of the moments in the collision can be written as

$$q' + k' = q + k, \tag{4.64}$$

The function $A(x,q',k',q,k)$ is the cross-section, a phenomenological quantity without an exact form, it may be obtained from Quantum Field Theory (QFT) for different collisions. We will define the collision (4.64) as a reversible collision, in this kind of collision time reversal combined with parity holds, in this case, the function $A(x,q',k',q,k)$ has the property of invertibility:

$$A(x,q',k',q,k) = A(x,q,k,q',k'). \tag{4.65}$$

Also one may argue that this invertibility comes from the unitarity of the S-matrix of QFT, which is a weaker statement than the combination of time reversal and parity [28].

4.2.1 Thermal equilibrium

As discussed earlier one of the objectives of kinetic theory is to provide the thermodynamics relations from a relativistic microscopic theory. In this section we will expose the H-theorem which establishes the conditions of global equilibrium or the local one, moreover, we will see the difficulty of defining a global equilibrium in a non-stationary universe [27].

The entropy flux can be defined as

$$S^\mu \equiv -k_b \int_{P_x} (f \ln f)\; q^\mu \omega_q, \tag{4.66}$$

where k_b is the Boltzmann constant

Theorem 4.2. *(H-theorem) If collisions are elastic and the symmetry (4.65) holds, then the entropy flux S^μ satisfy*

$$\nabla_\mu S^\mu \geq 0. \tag{4.67}$$

The proof is sketched in appendix D. The thermal equilibrium condition for the gas is obtained when

$$\nabla_\mu S^\mu = 0. \tag{4.68}$$

From (D.11), a necessary and sufficient condition is $\mathcal{L}_X f = 0$, in other words, the distribution function should conserve under the integral curves generated by the field X

$$\mathcal{L}_X f = 0 \Rightarrow C[f] = 0. \tag{4.69}$$

From the collision term, we have

$$f(x,q')f(x,k') - f(x,q)f(x,k) = 0, \text{ e } q' + k' = q + k, \tag{4.70}$$

one may solve the system above by applying the logarithm on both sides of the first equation

$$\ln f(q') + \ln f(k') = \ln f(q) + \ln f(k), \tag{4.71}$$

given the collision equation, we have three independent momentum variables, $q^\mu = q^{\mu'} + k^{\mu'} - k^\mu$, calculating the derivative of the above expression with respect to $q^{\nu'}$

$$\frac{1}{f(q^{\mu'})}\frac{\partial f(q^{\mu'})}{\partial q^{\nu'}} = \frac{1}{\tilde{f}}\frac{\partial \tilde{f}}{\partial q^{\nu'}}, \tag{4.72}$$

where $\tilde{f} \equiv f(q^{\mu'} + k^{\mu'} - k^\mu)$, keeping $k^{\mu'}$ constant, the equality will be of the form $F(x) = F(y) = cte, x \neq y$, i.e., will hold if it is constant on the momentum variables.[7]

$$\frac{1}{f(q^{\mu'})}\frac{\partial f(q^{\mu'})}{\partial q^{\nu'}} = b_\nu(x^\mu), \tag{4.73}$$

$$\int \frac{1}{f(q^{\mu'})}\frac{\partial f(q^{\mu'})}{\partial q^{\nu'}} dq^{\nu'} = b_\nu \int dq^{\nu'}, \tag{4.74}$$

$$\ln f(x^\mu, q^{\nu'}) = b_\nu(q^{\nu'} + C(x^\mu)), \tag{4.75}$$

$$f(x^\mu, q^{\nu'}) = c(x^\mu)\ \exp\left(b_\nu q^{\nu'}\right). \tag{4.76}$$

On the covariant form,

$$f(x,q) = c(x)\exp\left(b_\mu(x) q^\mu\right). \tag{4.77}$$

This is the unique solution to the equations (4.70), $c(x) > 0$, and b_μ are an arbitrary real function and co-vector field on space-time respectively, moreover if b_μ is a future time-like vector, thus $b_\nu q^\nu < 0$ allowing to define a well-behave integration of $f(x,q)$, the

7 Will be constant on the momentum variables, but is coordinate dependent.

distribution function of this form with $b_\mu b^\mu < 0$ is called Maxwell-Jüttner distribution. For this distribution, we get

$$\mathcal{L}_X f = \exp\left(b_\sigma q^\sigma\right)\left[q^\alpha \frac{\partial c(x)}{\partial x^\alpha} + c(x)\left(q^\alpha q^\mu \frac{\partial b_\mu}{\partial x^\alpha} - Q^\mu b_\mu\right)\right] \tag{4.78}$$

$$= \exp\left(b_\sigma q^\sigma\right)\left[q^\alpha \frac{\partial c(x)}{\partial x^\alpha} + c(x) q^\alpha q^\mu \nabla_\alpha b_\mu\right] \tag{4.79}$$

$$= \exp\left(b_\sigma q^\sigma\right)\left[q^\alpha \frac{\partial c(x)}{\partial x^\alpha} + \frac{c(x)}{2} q^\alpha q^\mu \left(\nabla_\alpha b_\mu + \nabla_\mu b_\alpha\right)\right]. \tag{4.80}$$

where on the second line we used $\dfrac{\partial b_\mu}{\partial x^\alpha} = \nabla_\alpha b_\mu + \Gamma_{\alpha\mu}{}^\sigma b_\sigma$, in which we choose $\overline{\nabla}_\alpha = \partial_\alpha$, and on the last line the symmetrization has been used [8]. If we assuming $c(x) = cte$, then

$$\mathcal{L}_X f = 0 \Rightarrow q^\alpha q^\mu \nabla_{(\alpha} b_{\mu)} = 0. \tag{4.81}$$

Therefore to establish the thermal equilibrium in the system, a necessary condition is a time-like killing vector field b_μ defined on M[9]. For this reason in the FLRW universe is impossible to get a perfect equilibrium, once there is no time symmetry (no time-like killing vector field) in FLRW solution, i.e., only stationary spaces reach the perfect thermal equilibrium.

Nevertheless one can calculate in the relativistic and non-relativistic regimes an approximate thermal equilibrium in this space by solving directly from (4.52). First, let us write the distribution function using a time-like vector field b_μ in FLRW metric $f = c\exp\left(b_\mu q^\mu\right) = c\exp\left(-E\beta(t)\right)$, being $\beta(t)$ the time component of the vector field, thus from the Boltzmann equation without collision term mentioned before we have

$$\left(\frac{\dot{\beta}}{\beta} - \frac{Hq}{E}\frac{\partial E}{\partial q}\right) f(E,t) = 0, \tag{4.82}$$

for the relativistic case $q >> m$ the energy is $E = q$, then

$$\frac{\dot{\beta}}{\beta} = H, \tag{4.83}$$

$$\beta(t) = \frac{1}{T_0}\exp\left(\int_{t_0}^{t} H\, dt'\right) = \frac{1}{T_0} a(t), \tag{4.84}$$

the factor $\dfrac{1}{T_0}$ is a constant of integration of the variable β, the distribution function is given by $f(t,E) = c\exp\left(-\dfrac{a(t)}{T_0}E\right) = c\exp\left(-\dfrac{E}{T}\right)$, in which $T = \dfrac{T_0}{a(t)}$ this is the solution for radiation fluid (photon, neutrino) as we have obtained before.

[8] Since the term $q^\alpha q^\mu$ is symmetric then only the symmetric part of $\nabla_\alpha b_\mu$ will remain

[9] For the case $q^\alpha q_\alpha = 0$, one may have a conformal killing vector:$\nabla_{(\alpha} b_{\mu)} = \lambda g_{\alpha\mu}$

For the non-relativistic case, we have $E = \dfrac{q^2}{2m}$, from (4.82):

$$\frac{\dot{\beta}}{\beta} = 2H, \tag{4.85}$$

$$\beta(t) = \frac{1}{T_0}\left[\exp\left(\int_{t_0}^{t} H dt'\right)\right]^2 = \frac{a(t)^2}{T_0}. \tag{4.86}$$

The distribution function will be $f(t,E) = c\exp\left(-\dfrac{a(t)^2}{T_0}E\right) = c\exp\left(-\dfrac{E}{T_b}\right)$, wherein $T_b = \dfrac{T_{b,0}}{a^2}$, i.e., the temperature for the non-relativistic particles in an expanding universe falls with the square of the scale factor.

5 Spectral distortions on CMB

Spectral distortions in the CMB are tiny deviations from a perfect blackbody spectrum, created by processes that drive matter and radiation out of thermal equilibrium after thermalization becomes inefficient at redshifts $z < 2 \times 10^6$ [29].

To understand the formation of the current CMB spectrum, we assume photons initially had a pure blackbody spectrum. We study processes that lead to photon production or destruction and energy release or extraction, which introduce temporary distortions to the CMB spectrum [1].

The Silk damping of primordial small-scale perturbations [2] causes energy release in the early universe [30, 31], resulting in small spectral distortions in the CMB spectrum that depend on the shape and amplitude of the primordial power spectrum. Modes with wavenumbers $50\ Mpc^{-1} < k < 10^4\ Mpc^{-1}$ dissipate their energy during the μ-era ($5 \times 10^4 < z < 2 \times 10^6$), producing a residual chemical potential at high frequencies [30, 32, 31].

Precise measurements of the CMB spectrum by COBE/FIRAS limit possible deviations from a blackbody to $\mu \lesssim 9 \times 10^{-5}$ at 95% confidence [33]. The proposed CMB experiment PIXIE [5] might detect distortions 10^3 smaller than the COBE/FIRAS limits. At this sensitivity, PIXIE could detect distortions from the dissipation of acoustic modes for a power spectrum with $n_S = 0.96$ [31].

5.1 Thermalization problem

The thermalization problem can be solved by studying the Boltzmann equation for the photons coupled with the temperature evolution equation for the matter (baryons, electrons) in an isotropic universe. In general, this system of equations is nonlinear, but we may impose simplifications using the time scale of the interactions between the constituents of the universe as we will see later.

In this chapter, we want to develop the main ideas of the spectral distortions by energy injection in the early universe without the numerical precision on the estimations, for more details about the numerical results see [34].

The main interactions between the photons and baryonic matter are governed by Compton scattering (CS), double Compton scattering (DC), and normal electron-ion Bremsstrahlung (BR). Furthermore, due to the expansion of the universe, the photons redshift and the non-relativistic electrons and baryons cool adiabatically [35].

Since the Compton interaction is very efficient on the period we want to analyze $z \in [10^3, 10^6]$ and the Compton time scale is much smaller than the Hubble radius

$t_H = 1/H$ we can parametrize the Boltzmann equation for the photon in terms of the optical depth $d\tau = \dfrac{dt}{t_C}$ of the electron scattering, thus

$$\frac{\partial f}{\partial \tau} - Ht_C\nu\frac{\partial f}{\partial \nu} = \left.\frac{df}{d\tau}\right|_C + \left.\frac{df}{d\tau}\right|_{DC} + \left.\frac{df}{d\tau}\right|_{BR}, \tag{5.1}$$

where $t_C = \dfrac{1}{\sigma_T N_e c}$, ν is the photon frequency, and on the right side we have the collision terms. Let us analyze how these terms are associated with the change of the photon distribution function in phase space.

5.1.1 Compton scattering

The first term in the right-hand of (5.1) describes the Compton interaction between the photons and the electrons. It may be obtained in light of the Kompaneets equation, which derivation can be found in [36]

$$\left.\frac{df}{d\tau}\right|_C = \frac{\theta_e}{x^2}\frac{\partial}{\partial x}\left\{x^4\left[\frac{\partial f}{\partial x} + \phi f(f+1)\right]\right\}. \tag{5.2}$$

In which we write the electron temperature in units of electron rest mass $\theta_e = \dfrac{kT_e}{m_e c^2}$, and the dimensionless frequency $x \equiv \dfrac{h\nu}{kT_z}$, and $\phi \equiv \dfrac{T_z}{T_e}$, $T_z = T_0(1+z)$ is the photon temperature due the expansion of the universe.

This equation describes the redistribution of photons over frequency x, and the number of photons is conserved under this interaction. To see that, one may multiply both sides by x^2 and integrate over the frequency

$$\left.\frac{dN_\gamma}{d\tau}\right|_C \propto \int dx x^2 \left(\frac{df}{d\tau}\right)_C = 0, \tag{5.3}$$

applying the integration by parts on the first term in the right side of (5.2), and using the fact that the photon distribution function vanishes sufficiently fast for $x \to 0$ and $x \to \infty$.

This interaction will be significant in determining the temperature evolution of the baryons in the universe on the regime we are interested. In equation (5.2) we are neglecting relativistic corrections and we will do henceforward for all the interactions since these terms are expected to affect the results at the level of a few percent at redshift $z > 10^6$ [37].

5.1.2 Bremsstrahlung and Double Compton Scattering

In cosmology, the most effective photon number changing processes are Bremstrahlung (BR): $e^- + X \to e^- + X + \gamma$ (where X is an ion), and the double Compton Scattering (CS): $e^- + \gamma \to e^- + \gamma + \gamma$ [35].

The BR process occurs due to the deceleration of the electron when it interacts with an ion in the primordial plasma, then the electron emits a photon with a specific energy. The DC emission is a process in which a single high-energy photon interacts with an electron, resulting in the emission of two lower-energy photons, instead of one photon being scattered, two photons are produced.

The contribution of these interactions in the Boltzmann equation (5.1) is given by [38]

$$\left.\frac{df}{d\tau}\right|_{BR+DC} = \frac{1}{x^3}\left[1 - f(e^{\phi x} - 1)\right] K(x, \theta_z, \theta_e), \tag{5.4}$$

in which $K = K_{BR} + K_{DC}$ is the emission coefficient, we will see that if $x \to \infty$ then $K \to 0$, showing that the main emission and absorption of photons is occurring at low frequencies, moreover the equation (5.4) becomes very small in this limit.

At high redshifts, $z > 10^5$, DC emission dominates over BR in photon production, the DC emission coefficient can be given as [38, 37]

$$K_{DC}(x, \theta_e, \theta_z) = \frac{4\alpha\theta_z^2}{3\pi} g_{dc}(\theta_e, \theta_z, x) \tag{5.5}$$

where α is the fine structure constant and $g_{dc}(x, \theta_z, \theta_e)$ is the effective DC Gaunt factor.

At low redshifts $z < 10^5$ BR process becomes more effective in producing photons, and the BR emission coefficient can be given by [39]

$$K_{BR}(x, \theta_e) = \frac{\alpha\lambda_e^3\theta_e^{-7/2}e^{-\phi x}}{2\pi\sqrt{6\pi}\phi^3} \sum_i Z_i^2\, N_i\, g_{ff}(Z_i, x, \theta_e) \tag{5.6}$$

wherein $\lambda_e = \dfrac{h}{m_e c}$ is the Compton wavelength of the electron, Z_i, N_i and $g_{ff}(Z_i, x, \theta_e)$ are the charge, the number density and the BR Gaunt factor for a nucleus of the atomic species i, respectively. Some simplifications can be made in g_{dc}, and g_{ff} depending on the redshift under consideration, these expressions can be found in [37].

5.1.3 Matter temperature evolution

Since the timescale of the Coulomb interaction between the protons and electrons is very small compared to the Compton time scale, the baryons temperature (protons, Hydrogen, Helium) thermalizes very fast with the electron temperature. Thus, we may consider the same temperature between the electrons and the baryons, i.e., $T_b = T_e$ [4]. Knowing that the baryons are in a non-relativistic regime, we will use a Maxwell Boltzmann distribution function for the baryons/electrons to extract some statistical quantities.

To get the evolution of the electron temperature we will consider the first law of thermodynamics for each species $i = \gamma, e, H, H_e$ with an external source of energy

$$dQ_i = d(a^3\rho_i) + P_i d(a^3) \tag{5.7}$$

where ρ_i and P_i are the energy density and pressure for each species respectively, the total energy rate by volume a^{-3} will be

$$\frac{1}{a^3}\frac{dQ}{dt} = \sum_i \left(\frac{1}{a^3}\frac{d(a^3\rho_i)}{dt} + \frac{1}{a^3}\frac{da^3}{dt}P_i\right). \tag{5.8}$$

For photons, $P_\gamma = \frac{\rho_\gamma}{3}$, from the equation (4.56), one gets $\frac{1}{a^3}\frac{da^3}{dt} = 3H$, making an algebraic manipulation, the contribution of the photons ($i = \gamma$) is:

$$\frac{1}{a^3}\frac{dQ}{dt} \propto \frac{1}{a^4}\frac{d(a^4\rho_\gamma)}{dt}. \tag{5.9}$$

For the baryons, we may use the relations (D.57) and (D.58), such that

$$\frac{1}{a^3}\frac{dQ}{dt} = \frac{1}{a^4}\frac{d(a^4\rho_\gamma)}{dt} + \sum_{i\neq\gamma} m_i c^2 \left(F(\theta_i)\frac{1}{a^3}\frac{d(a^3N_i)}{dt} + N_i\frac{dF(\theta_i)}{dt} + 3HN_i\theta_i\right), \tag{5.10}$$

wherein $F(\theta_i)$ is defined in (D.59) and $\theta_i = \frac{kT_i}{m_e c^2}$, and we will assume no creation/destruction of baryons, then the first term vanishes. We can simplify the second term using the definition of the specific heat capacities, $c_{V,i}$ (see D.61).

$$m_i c^2 N_i \frac{dF(\theta_i)}{dt} = kN_i \frac{dT_i}{dt}\frac{dF(\theta_i)}{\theta_i} = N_i c_{V,i}\frac{dT_i}{dt}. \tag{5.11}$$

As said previously we are assuming the temperature of all species of baryons equal the electron temperature $T_i \equiv T_e$, moreover the baryons are in the non-relativistic regime [1], i.e, $c_{V,i} = \frac{3}{2}k_b$, thus

$$\frac{1}{a^3}\frac{d\tilde{Q}}{dt} = \frac{1}{a^4}\frac{d(a^4\tilde{\rho}_\gamma)}{dt} + \sum_{i\neq\gamma}\frac{3}{2}N_i\left(\frac{d\theta_e}{dt} + 2H\theta_e\right), \tag{5.12}$$

where the tilde denotes that the quantity is expressed in units of the electron rest mass. Using (D.62), (D.63), and (D.64) the number densities of the species can be written as

$$\frac{3}{2}N_i = \frac{3}{2}\left[\chi_e\left(1 - \frac{Y_p}{2}\right) + \left(1 - \frac{3Y_p}{4}\right)\right]N_b \equiv \alpha, \tag{5.13}$$

where χ_e is the ionization fraction of the hydrogen and helium atoms, and $Y_p = 0.2485$ is the primordial mass fraction of helium [40]. The characteristic time scale in equation (5.12) is the Compton time t_C, once the interaction of electrons and photons determines it, rewriting the equation in terms of the optical depth τ

$$\frac{d\theta_e}{d\tau} = \frac{1}{\alpha}\left(\frac{1}{a^3}\frac{d\tilde{Q}}{d\tau} - \frac{1}{a^4}\frac{d(a^4\tilde{\rho}_\gamma)}{d\tau}\right) - 2Ht_C\theta_e. \tag{5.14}$$

[1] Even though we would consider relativistic corrections in the kinetic collision terms, in this case, due to the masses of the nuclei of H, He and the other light elements are orders of magnitude larger than the electron rest mass, one can neglect relativistic corrections [37].

Due to the Compton scattering which is very effective in early epochs, the electron temperature is close to an equilibrium temperature θ_e^{eq} which is close to the effective photon temperature θ_γ (we will discuss later). Thus it is convenient to rewrite the equation above in terms of the relative difference between the photon and electron temperature, $\frac{\Delta\theta_e}{\theta_\gamma} \equiv (\theta_e - \theta_\gamma)/\theta_\gamma$, this leads to

$$\frac{d}{d\tau}\left(\frac{\Delta\theta_e}{\theta_\gamma}\right) = \frac{1}{\theta_\gamma \alpha}\left(\frac{1}{a^3}\frac{d\tilde{Q}}{d\tau} - \frac{1}{a^4}\frac{d(a^4\tilde{\rho}_\gamma)}{d\tau} - Ht_C\alpha\theta_e\right). \tag{5.15}$$

The first term on the right side of the equation (5.15) denotes the exchange of energy due to some physical effect, we will explore some mechanism of energy exchange later.

But at the moment let us use the Boltzmann equations for the photons to write explicitly the second term of the right side of the equation. By energy conservation, the interaction between the photons and the matter follows $\frac{d\tilde{\rho}_e}{d\tau} = -\frac{d\tilde{\rho}_\gamma}{d\tau}$, and we should take into account the Compton scattering, and low-frequency photon by DC and BR processes:

$$-\frac{d(a^4\tilde{\rho}_\gamma)}{d\tau} = \left.\frac{d\tilde{\rho}_e}{d\tau}\right|_C + \left.\frac{d\tilde{\rho}_e}{d\tau}\right|_{DC+BR}. \tag{5.16}$$

Since most of the photons are produced at low frequencies by the DC and BR processes usually the second term on the right side will be very small as compared to energy transfer due to Compton scattering, therefore for our purposes in this work we will ignore them[2].

Before writing a form for the Compton scattering, let us analyze the effective photon temperature θ_γ. We know from a Planckian distribution that

$$\tilde{\rho}_\gamma = \mathcal{G}_3^{PL}\theta_\gamma^4, \tag{5.17}$$

wherein $\mathcal{G}_3^{PL} \equiv \int_0^\infty x^3 f_{PL}\, dx = \frac{\pi^4}{15}$, with $f_{PL}(x) = \frac{1}{e^x - 1}$. If we consider a small distortion in the distribution $f(x) = f_{PL} + \Delta f$ in (4.29) we will have an effective energy density, given by

$$\tilde{\rho}_\gamma = \mathcal{G}_3\theta_\gamma^4, \tag{5.18}$$

where $\mathcal{G}_3 \equiv \mathcal{G}_3^{PL} + \int_0^\infty x^3\Delta f\, dx$, comparing the two energy densities we can define an effective (thermodynamic) photon temperature by

$$T_\gamma = \left(\frac{\rho_\gamma}{\mathcal{G}_3^{PL}}\right)^{1/4} = \left(\frac{\mathcal{G}_3}{\mathcal{G}_3^{PL}}\right)^{1/4} T_z, \tag{5.19}$$

2 For a numerical precision is important to take into account these effects, but at moment we want to understand qualitative aspects due to energy injection in the early universe. In future work a full treatment will be made.

thus in general $T_\gamma \neq T_z$, they will be the same if we neglect the small distortion $\Delta f = 0$, wherein $\mathcal{G}_3 = \mathcal{G}_3^{PL} \Rightarrow T_\gamma = T_z$.

Finally let us write a form for the Compton scattering, by multiplying the Kompaneets equation by $E = qc$ and integrating over d^3q making use of integration by parts, then

$$\left.\frac{d\tilde{\rho}_e}{d\tau}\right|_C = \tilde{\rho}_\gamma \left(\frac{\mathcal{I}_4}{\mathcal{G}_3}\theta_\gamma - 4\theta_e \right). \tag{5.20}$$

Where $\mathcal{I}_4 \equiv \int_0^\infty f(f+1)x^4\,dx$.

The Compton scattering equilibrium temperature θ_{eq} can be found by solving the equation $\left.\frac{d\tilde{\rho}_e}{d\tau}\right|_C = 0$. Since this temperature is reached when the Compton heating and cooling are equal, i.e., energy transfer by Compton scattering vanishes

$$\tilde{\rho}_\gamma \left(\frac{\mathcal{I}_4}{\mathcal{G}_3}\theta_\gamma - 4\theta_{eq} \right) = 0 \tag{5.21}$$

$$\theta_{eq} = \frac{\mathcal{I}_4}{4\mathcal{G}_3}\theta_\gamma \tag{5.22}$$

for photons that follow a perfect black body distribution f_{PL}, we get $\theta_{eq} = \theta_\gamma$, once $\mathcal{I}_4^{PL} = 4\mathcal{G}_3^{PL}$, this result will simplify our numerical results later.

Rewriting the equation (5.20) in terms of the equilibrium temperature

$$\left.\frac{d\tilde{\rho}_e}{d\tau}\right|_C = 4\tilde{\rho}_\gamma(\theta_{eq} - \theta_e). \tag{5.23}$$

Inserting this term in (5.15) one gets

$$\frac{d\rho_e}{d\tau} = \frac{t_C}{\alpha\theta_\gamma a^3}\frac{d\tilde{Q}}{dt} + \frac{4\tilde{\rho}_\gamma}{\alpha}(\rho_{eq} - \rho_e) - Ht_C\rho_e. \tag{5.24}$$

wherein we define $\rho_e \equiv \frac{T_e}{T_\gamma}$, and we write the energy rate in the cosmic time. We neglect the BR and DC scattering in this equation for the reasons mentioned before.

In section 5.4.2 we will see how the Silk damping effect and the adiabatic cooling of baryons in an expanding universe affect the evolution of the matter temperature in a stationary regime [3].

5.2 Spectral distortions in the primordial universe

From a thermodynamic point of view, we can understand the creation of spectral distortions on CMB and consequently, the energy injection that produces it by the mixing of black bodies with different temperatures.

[3] The stationary regime means near zero temperature difference. Moreover, other physical effects can generate spectral distortions on CMB but in this work, we will explore these two effects predicted by ΛCDM

As shown in the previous section in the radiation era in the early universe, primordial perturbations excite sound waves on entering the sound horizon, consequently, photons from different phases of the sound waves, having different temperatures, diffuse through the electron-baryon plasma and mix together [41].

To understand the mixing of blackbodies one could take a spatial average of a Taylor expansion of the photon distribution function in $f_{PL}(T+\Delta T)$, where T is the average temperature (CMB temperature), keep the expansion until the second order term in $\Theta \equiv \Delta T/T$.

$$\langle f_{PL}(T+\Delta T)\rangle = \left\langle \frac{1}{e^{\frac{E}{k(T+\Delta T)}}-1} \right\rangle = \left\langle f_{PL}(T) + \left.\frac{\partial f}{\partial \Theta}\right|_{\Theta=0}\Theta + \left.\frac{\partial^2 f}{\partial \Theta^2}\right|_{\Theta=0}\Theta^2 \right\rangle \tag{5.25}$$

$$= f_{PL}(T_{new}) + \frac{1}{2}Y(x)\left\langle \Theta^2 \right\rangle, \tag{5.26}$$

where $\langle\Theta\rangle = 0$ has been used, $T_{new} \equiv T\left[1+\left\langle\Theta^2\right\rangle\right]$, and the $Y(x)$ being the y-type distortion given by

$$Y(x) \equiv \frac{xe^x}{(e^x-1)^2}\left(\frac{xe^x(e^x+1)}{e^x-1} - 4\right). \tag{5.27}$$

Therefore if we mix the blackbody spectra with different temperatures, the resultant spectrum is not blackbody and at the lowest order, the distortion is given by a y-type spectrum.

Moreover, we get in the first term a black-body distribution with a new temperature $T_{new} > T$. Assuming a superposition of two blackbody spectra [4] with temperatures $T_1 = T + \Delta T$ and $T_2 = T - \Delta T$, with average temperature T, one may calculate the energy spectrum and number density of the average spectra by

$$\rho_{aver} = \frac{a_R}{2}(T_1^4 + T_2^4) \approx a_R T^4\left[1+6\Theta^2\right] > a_R T^4; \tag{5.28}$$

$$N_{aver} = \frac{b_R}{2}(T_1^3 + T_2^3) \approx b_R T^3\left[1+3\Theta^2\right] > b_R T^3. \tag{5.29}$$

Where $O\left(\Theta^{(n)}\right) \approx 0$ with $n > 2$, and the coefficients a_R and b_R are defined in D, as we can see the average energy density and number density are greater than the black-body energy and number density respectively, one can estimate an effective temperature considering the new number density N_{aver}

$$T_{eff} = \left(\frac{N_{aver}}{b_R}\right)^{1/3} \tag{5.30}$$

$$\approx T\left[1+\Theta^2\right], \tag{5.31}$$

which is the T_{new} of the black body part of (5.26), indicating that the photons after the mixing go into creating new blackbody spectra with a higher temperature.

[4] There is no loss of generality to considering only two black bodies, the difference results in the average process, in the general case we may take in an ensemble of black bodies

One may estimate the energy from a blackbody spectra with the temperature T_{new} given by

$$\rho_f = a_R T_{new}^4 \approx a_R T^4 \left[1 + 4\Theta^2\right] < \rho_{aver}. \tag{5.32}$$

This result shows that part of the energy is going to create spectral distortions. If we multiply E^3 in (5.26) and integrate over the energy we get

$$\rho_f - a_R T^4 = \frac{2}{3}(\rho_{aver} - a_R T^4), \tag{5.33}$$

using (5.28) and (5.32), is straightforward

$$\rho_f - \rho_{aver} = -2a_R T^4 \Theta^2, \tag{5.34}$$

substituting ρ_f above in (5.33)

$$\frac{1}{3}\left(\rho_{aver} - a_R T^4\right) = 2a_R T^4 \Theta^2. \tag{5.35}$$

Furthermore, it is well known [42] that the magnitude of the y-distortion is $y_Y = \frac{1}{2}\Theta^2$, and in [34] was shown by using second-order perturbation of Boltzmann equation that

$$2a_R T^4 \Theta^2 \propto y_Y \int x^3 Y(x)\, dx. \tag{5.36}$$

Therefore one can interpret 2/3 of the energy as going to increase the energy density of the black-body part of the spectrum, while the remaining 1/3 of the energy density appears as a y-type distortion[5].

5.2.1 μ-type distortion

At redshifts z $\gtrsim 10^5$ the y-type distortion will rapidly comptonize to μ distortion due to the effectivity of the Compton scattering, leading to a Bose-Einstein spectrum $f(x,t) = 1/[e^{x+\mu(x)} - 1]$ with a small chemical potential μ.

In this scenario, one may assume that the electron temperature is equal to the photon temperature [6] $T_e = T_\gamma = T_0(1+z)$. And that the photons evolve along a sequence of equilibrium spectra. Since $|\mu(x,t)| \ll 1$ one may expand the photon distribution $f(x,t) = 1/[e^{x+\mu(x)} - 1] \approx f_{PL} + \mu\, \partial_x f_{PL}$.

The photon energy density and the photon distribution for a small chemical potential are given by $\rho_\gamma(t) = \rho_{PL} f_\mu$ and $N_\gamma(t) = N_{PL}\phi_\mu$, see $D.54$ for the definitions of the functions, the index μ just indicate that the function has a chemical potential dependence.

5 The y-distortion is just a redistribution of the photons of the new blackbody spectrum, one can see it verifying that the photon number vanish $\int dx\, x^2 Y(x) dx = 0$ [34].

6 Notice we are considering small distortions then the effective photon temperature will be the photon-redshift temperature.

One may solve the time evolution of the energy density and the number density for a small chemical potential at high frequencies ($x \gg 1$), since these photons will be the main contribution of the energy and number density, in this regime one may approximate $\mu(x,t) \approx \mu_0(t)$ as a constant, then deriving the energy and number density in time

$$\frac{1}{\rho_\gamma}\frac{d\rho_\gamma}{dt} = 4\,\frac{dlnT}{dt} + \frac{dlnf_\mu}{d\mu_0}\frac{d\mu_0}{dt}; \tag{5.37}$$

$$\frac{1}{N_\gamma}\frac{dN_\gamma}{dt} = 3\,\frac{dlnT}{dt} + \frac{lnd\phi_\mu}{d\mu_0}\frac{d\mu_0}{dt} \tag{5.38}$$

Combining these equations it is possible to eliminate the time derivative of the temperature, writing in terms of the optical depth [30, 37]

$$\frac{d\mu_0(\tau)}{d\tau} = \frac{1}{B_\mu}\left[3\,\frac{1}{\rho_\gamma}\frac{d\rho_\gamma}{dt} - 4\,\frac{1}{N_\gamma}\frac{dN_\gamma}{dt}\right]. \tag{5.39}$$

Where $B_\mu = 3\,\dfrac{dlnf_\mu}{d\mu_0} - 4\,\dfrac{dln\phi_\mu}{d\mu_0} = \dfrac{8I_1}{I_2} - \dfrac{9I_2}{I_3} \approx 2.143$. The definition of I_i used to calculate this coefficient is in appendix D.

The change in the number density of the photons is only due to DC emission since Compton scattering conserves the number of photons. Therefore using equation (5.4), in the limit of $|\mu| \ll 1$ one may find

$$\frac{1}{N_\gamma}\frac{dN_\gamma}{dt} = \frac{4\hat{x}_c\theta_e}{I_2}\mathcal{I}(\theta_\gamma,\hat{x}_c), \tag{5.40}$$

wherein $\hat{\mu} = \mu(x,t)/\mu_0$, and $\mathcal{I}$ given by an integral

$$\mathcal{I}(\theta_\gamma,\hat{x}_c) = \mu_0\int_0^\infty dy\;\hat{\mu}(x,t)\frac{e^{y/2}}{y(e^{y/2}-1)}, \tag{5.41}$$

it was solved explicitly in [37], and the critical frequency $\hat{x}_c$ was found in the regime of thermal equilibrium of the Boltzmann equation (5.1) for small chemical potential[7] [37]

$$\hat{x}_c \approx 3.03\times 10^{-6}(1+z)^2. \tag{5.42}$$

Substituting the equations (5.40) and (5.41) in (5.39) we get

$$\frac{d\mu_0}{dt} = 1.4\frac{dQ}{dt} - \frac{\mu_0}{t_{\mu,DC}}, \tag{5.43}$$

in which

$$t_{\mu,DC} = \frac{B_\mu I_2}{16\hat{x}_c^2\,\theta_\gamma\,\mathcal{I}}. \tag{5.44}$$

[7] Considering only Compton and double Compton scattering, then $\left.\dfrac{df}{d\tau}\right|_C + \left.\dfrac{df}{d\tau}\right|_{DC} = 0$

Solving the equation (5.43) one get [35]

$$\mu = 1.4 \left. \frac{\Delta\rho_\gamma}{\rho_\gamma} \right|_\mu . \tag{5.45}$$

In which

$$\left. \frac{\Delta\rho_\gamma}{\rho_\gamma} \right|_\mu = \int_{z_{\mu,y}}^{\infty} \frac{\dot{Q}\,\mathcal{T}_{bb}}{H(z')\,\rho_\gamma(1+z')} dz'. \tag{5.46}$$

Where $\mathcal{T}_{bb} = \exp\left[-(z'/z_\mu)^{5/2}\right]$ is the visibility function for spectral distortions wherein $z_\mu \approx 1.98 \times 10^6$ is the thermalization redshift [37]. And $z_{\mu,y} = 4 \times 10^4$ denoting the end of the μ-era. For $z \gg z_\mu$ the thermalization process is very efficient, so all the released energy just increases the average temperature of the CMB, only below z_μ that spectral distortion will be generated by the energy dissipation process.

5.3 Photon-baryon dynamics

In this section, we will develop the solution that presents the Silk damping effect in the photon-baryon dynamics in the early universe. It will be necessary to associate this with the temperature fluctuation of the plasma and the creation of spectral distortions by the dissipation of energy in the sound waves.

Here we will follow [10]. From the hydrodynamics equations derived in appendix D for photons and baryons

$$\delta'_\gamma = \frac{4}{3} k v_\gamma + 4\Phi'; \tag{5.47}$$

$$\delta'_b = k v_b + 3\Phi'. \tag{5.48}$$

In which v_γ, and v_b are the photon and baryon velocity respectively, the Euler equation for photons and baryons (D.41), and (D.48) are [10]

$$v'_\gamma + \frac{1}{4} k\delta_\gamma - \frac{2}{3} k\Pi_\gamma + k\Psi = -\Gamma(v_\gamma - v_b), \tag{5.49}$$

$$v'_b + \mathcal{H} v_b + k\Psi = \frac{\Gamma}{R}(v_\gamma - v_b), \tag{5.50}$$

$$v_{\gamma-b} = \frac{R}{\Gamma}(v'_b + \mathcal{H} v_b + k\Psi). \tag{5.51}$$

With $R = 3\rho_b/4\rho_\gamma$, $\Gamma = aN_e\sigma_T$ is the scattering rate, and $v_{\gamma-b} \equiv v_\gamma - v_b$ being the slip velocity, which is the velocity of the photons in the baryon frame, in the tight coupling limit in which Γ is very large we have $v_\gamma \approx v_b$ since the scattering rate is big the photons and baryons will thermalize, thus

$$v'_\gamma + \frac{k\delta_\gamma}{4(1+R)} + k\Psi + \left(\frac{R}{1+R}\right)\mathcal{H} v_\gamma - \frac{2k\Pi_\gamma}{3(1+R)} = 0. \tag{5.52}$$

The last term can be suppressed because the shear will be very small in the tight coupling limit, deriving in the conformal time the equation (5.49) and substituting (5.52) in it we get

$$\delta_\gamma'' = -\frac{4}{3}k\left[\frac{k\delta_\gamma}{4(1+R)} + k\Psi + \frac{R}{1+R}\mathcal{H}v_\gamma\right] + 4\Phi'', \tag{5.53}$$

doing some algebraic manipulation

$$\delta_\gamma'' + \left(\frac{R'}{1+R}\right)\delta_\gamma' + c_s^2k^2\delta_\gamma = -\frac{4}{3}k^2\Psi + \left(\frac{R'}{1+R}\right)4\Phi' + 4\Phi''. \tag{5.54}$$

wherein $c_s^2 \equiv \dfrac{1}{3(1+R)}$, and $R \propto a \to \mathcal{H} = R'/R$ has been used.

The equation (5.54) describes the complete photon-baryon dynamics, but for this work, we will pay attention to the Silk damping mechanism

5.3.1 Silk damping

The source of damping which we will derive in this section is due to the photon diffusion between hot and cold regions erases temperature differences on small scales, i.e., deep inside the Hubble radius, then one may ignore the expansion terms $\mathcal{H}v_b \ll v_b'$ in (5.51), and since $\mathcal{H}^2\delta_\gamma \propto k^2\Phi^2 \propto \Psi^2$, then $k\delta_\gamma \gg k\Psi$, thus we can ignore this term from the Euler equation for photons, then the equations (5.49) and (5.51) become

$$v_\gamma' + \frac{k}{4}\delta_\gamma - \frac{2}{3}k\Pi_\gamma = -\Gamma v_{\gamma-b}, \tag{5.55}$$

$$v_{\gamma-b} = \frac{R}{\Gamma}v_b', \tag{5.56}$$

in order to capture the damping of the fluctuation we need to study these equations at leading order in an expansion in $k/\Gamma < 1$. For this reason, we leave the anisotropic photon stress, combining the equations above

$$v_{\gamma-b}' = -\left(\frac{1+R}{R}\right)\Gamma v_{\gamma-b} - \frac{k}{4}\delta_\gamma + \frac{2}{3}k\Pi_\gamma, \tag{5.57}$$

we can perform solutions for each term of the expansion k/Γ, the first term is the tight-coupling limit in which $\Pi_\gamma \to 0$ and $v_{\gamma-b}' \approx 0$, with $v_{\gamma-b}$ very small and given by

$$v_{\gamma-b} \approx \frac{k}{4\Gamma}\left(\frac{1+R}{R}\right)\delta_\gamma. \tag{5.58}$$

As said earlier the slip velocity is the photon velocity measured by the baryons frame, this can be interpreted as a photon energy flux seen in the baryons frame[8].

[8] Since the dimension of $[\delta_\gamma/\Gamma] = [\text{energy/area}]$.

Since $\rho_\gamma \propto T^4$ one may write (5.58) as $v_{\gamma-b} \propto \delta T/T$, which means that it describes the energy flux coming from a gradient in the photon temperature between hot and cold regions, the finite slip velocity is therefore associated with thermal conduction [10].

Let us collect the next order term, by adding (5.55) and (5.56)

$$v_b' + \frac{v_\gamma'}{R} + \frac{k}{4R}\delta_\gamma - \frac{2k}{3R}\Pi_\gamma = 0, \tag{5.59}$$

deriving (5.58) and using the slip velocity definition we get

$$v_b' = v_\gamma' + \frac{k}{4\Gamma}\left(\frac{R}{1+R}\right)\delta_\gamma', \tag{5.60}$$

substituting in (5.59)

$$(1+R)v_\gamma' = -\frac{k}{4}\delta_\gamma - \frac{R^2}{4(1+R)}\frac{k}{\Gamma}\delta_\gamma' + \frac{2}{3}\Pi_\gamma. \tag{5.61}$$

From (D.43) we see that the photon anisotropic stress is related to the photon viscosity. Using (5.47) in the sub-Hubble scales $\delta_\gamma' = \frac{4}{3}kv_\gamma$, then

$$(1+R)v_\gamma' = -\frac{k}{4}\delta_\gamma - \frac{1}{4}\left(\frac{8}{9} + \frac{R^2}{(1+R)}\frac{k}{\Gamma}\right)\delta_\gamma', \tag{5.62}$$

deriving the continuity equation $\delta_\gamma'' = \frac{4}{3}kv_\gamma'$, and substituting in the equation above

$$\delta_\gamma'' + \frac{k^2}{3(1+R)\Gamma}\left(\frac{8}{9} + \frac{R^2}{1+R}\right)\delta_\gamma' + c_s^2k^2\delta_\gamma = 0. \tag{5.63}$$

Comparing this equation with (5.54) we see that going beyond leading order in the tight-coupling approximation has introduced a friction term proportional to δ_γ'. Moreover, the amplitude of the friction depends on the wavenumber k.

As the way to deduce this equation showed, macroscopically, the dissipation of sound waves can be identified as due to the shear viscosity and thermal conduction in the relativistic fluid composed of baryons, electrons and photons.

The damping of sound waves on small scales due to thermal conduction was pointed out by Lifshitz in 1946 [43] and first calculated by (Silk 1968) and is known as Silk damping [2].

The differential equation above is identical to the one describing a damped harmonic oscillator, in this case, we can solve it analytically using a WKB approximation, assuming the ansatz:

$$\delta_\gamma \propto \exp\left[i\int_0^\eta \omega(\tilde{\eta})d\tilde{\eta}\right]. \tag{5.64}$$

Where ω is the sound frequency, we assume it changes slowly, i.e., $\omega' \ll \omega^2$. Then

$$\delta'_\gamma \propto i\omega\delta_\gamma, \tag{5.65}$$

$$\delta''_\gamma \propto (i\omega' - \omega^2)\delta_\gamma \approx \omega^2\delta_\gamma. \tag{5.66}$$

As we can see this approximation leads to a plane wave behavior, substituting in (5.63)

$$(k^2c_s^2 - \omega^2) + \frac{k^2}{3(1+R)\Gamma}\left(\frac{8}{9} + \frac{R^2}{1+R}\right)i\omega = 0 \tag{5.67}$$

making a perturbation in the frequency $\omega = kc_s + \delta\omega$, then doing some algebraic manipulations using $\delta\omega^2 \approx 0$, we obtain

$$\delta\omega = i\frac{k^2}{6(1+R)\Gamma}\left(\frac{8}{9} + \frac{R^2}{1+R}\right). \tag{5.68}$$

Therefore the photon density contrast will be

$$\delta_\gamma \propto \exp\left[i\int_0^\eta (kc_s + \delta\omega)d\tilde{\eta}\right] \tag{5.69}$$

$$= e^{-k^2/k_D^2}\exp\left[ik\int_0^\eta c_s d\tilde{\eta}\right] \tag{5.70}$$

$$= e^{-k^2/k_D^2}[A_k cos(kr_s) + B_k sin(kr_s)]. \tag{5.71}$$

where A_k and B_k are the WKB amplitudes determined by the initial condition, $r_s \equiv \int_0^\eta c_s d\tilde{\eta}$ is the sound horizon, and the damping scale k_D is

$$k_D^{-2} = \int_0^\eta \frac{d\tilde{\eta}}{6(1+R)\Gamma}\left(\frac{8}{9} + \frac{R^2}{1+R}\right) \tag{5.72}$$

for the tight-coupling limit $R \to 0$, in [44] was estimated a damping scale [9]

$$k_D \approx 4\times 10^{-6}(1+z)^{3/2}. \tag{5.73}$$

Although the photon mean free path due to Compton scattering on electrons is very small, they are still able to traverse a considerable distance since the big bang (or bounce), performing a random walk among the electrons. This diffusion and mixing of photons as a result of Thomson scattering erase the sound waves on scales corresponding to the diffusion scale k_D (and smaller) [41].

5.3.2 Energy injection by dissipation of sound waves

As discussed earlier from the microscopy point of view, diffusion of photons mixes photons from different phases of sound waves which have different temperatures.

[9] Including polarization corrections to the scattering would give the same result with $8/9 \to 16/15$ [44]

Denoting the perturbation in the Fourier space $\Theta(x^i, \hat{n}^i) \rightarrow \Theta(k^i, \hat{n}^i)$ wherein $\hat{n}^i$ is the photon unit direction, from (5.35) we have

$$\left.\frac{\Delta\rho_\gamma}{\rho_\gamma}\right|_{acou} = 6\left\langle \Theta^2 \right\rangle \tag{5.74}$$

$$= 6\int \frac{d^3k}{(2\pi^3)} e^{ik_i x^i} \int \frac{d^3k'}{(2\pi)^3} \left\langle \Theta(k'^i, \hat{n}^i)\Theta(k^i - k'^i, \hat{n}^i) \right\rangle \tag{5.75}$$

$$= 6\int \frac{dkk^2}{2\pi^2} P_i(k) \left[\sum_{l=0}^{\infty} (2l+1)\Theta_l^2 \right] \tag{5.76}$$

where on the second line we make a variable transformation of $k_i'' = k_i - k_i'$, and on the second line we make an expansion in terms of the Legendre polynomials, i.e., $\Theta(\hat{n}^i\hat{k}_i, k) = \sum_l (-i)^l (2l+1)\mathcal{P}_l(\hat{n}^i\hat{k}_i)\Theta_l(k)$, in which k and $\hat{k}_i$ are the comoving wavenumbers and the unit vector along the Fourier mode respectively, the latter being parallel to the electron peculiar velocity in linear perturbation.

$P_i(k)$ is the power spectrum of initial curvature perturbations ζ_i, the result above was achieved in (D.31) with $\hat{n} = \hat{n}'$. As we see in the previous section before cosmological recombination starts at $z \approx 6 \times 10^3$, electrons/baryons and radiation are tightly coupled such that most of the energy of the sound waves are in the monopole and dipole terms, then one may neglect $l > 2$ modes

$$\left.\frac{\Delta\rho_\gamma}{\rho_\gamma}\right|_{acou} = 6\int \frac{dkk^2}{2\pi^2} P_i(k) \left[\Theta_0^2 + 3\Theta_1^2 \right]. \tag{5.77}$$

In order to find the contribution of the energy dissipation over time and how it is related to the creation of distortions of CMB we should get the rate of change in the energy dissipation, but instead of using the expression above one may find a general relation by

$$\frac{d}{dt}\left.\frac{\Delta\rho_\gamma}{\rho_\gamma}\right|_{acou} \propto -2\left\langle \Theta \frac{d\Theta}{dt} \right\rangle. \tag{5.78}$$

Using the linear Boltzmann equation (D.30) with polarization corrections

$$\frac{d}{dt}\left.\frac{\Delta\rho_\gamma}{\rho_\gamma}\right|_{acou} \propto 2N_e\sigma_T \int \frac{dkk^2}{2\pi^2} P_i(k) \left[\Theta_1(3\Theta_1 - v) + \frac{9}{2}\Theta_2^2 - \frac{1}{2}\Theta_2\left(\Theta_2^P + \Theta_0^P\right) + \sum_{l>2}(2l+1)\Theta_l^2 \right]. \tag{5.79}$$

Where $v_e(k) \equiv -i\hat{k}v(k)$ is peculiar velocity of baryons/electrons and Θ_l^P denote polarization multipole moments. This expression was derived in [34] using the second-order Boltzmann equation in the same work the authors showed that potential metric perturbations can not create spectral distortions, then they were ignored in (5.79)

Based on previous sections about the description of Silk damping we can interpret each term of (5.79) associated with the macroscopic effects known. In this case, as discussed in [41], the first three terms of (5.79) give the dominant contribution to the dissipation of sound waves. The first term mixes the blackbodies in the dipole resulting in the transfer of heat along the temperature gradient, and can thus be identified as the effect of thermal conductivity. The second term, similarly, mixes the blackbodies in the quadrupole or the shear stress in the photon fluid and can thus be identified as the effect of shear viscosity. The third term is a correction of the shear viscosity due to the polarization effects in the Compton scattering.

Following the work [45], let us define the spatially average source function $\langle \mathcal{S}_{ac} \rangle \equiv \frac{d}{dt} \left. \frac{\Delta\rho_\gamma}{\rho_\gamma} \right|_{acou}$. As mentioned, we are interested in understanding the dissipation of energy before recombination $z > 10^4$ then the tight-coupling limit can be used such that, at that time, the universe is still radiation-dominated and the small baryon loading given by $R = \dfrac{3\rho_b}{4\rho_\gamma} \approx 0$. In this limit, $v = 3\Theta_1$, then the first term vanishes, also $\Theta_2 \approx \dfrac{8k}{15\dot{\tau}}\Theta_1$ and $\Theta_2^P + \Theta_0^P \approx \dfrac{3}{2}\Theta_2$, also we can neglect high order terms $(l > 2)$, then

$$\langle \mathcal{S}_{ac} \rangle \approx \frac{1}{\dot{\tau}^2} \int \frac{dkk^4}{2\pi^2} P_i(k) \frac{16}{15} \Theta_1^2. \tag{5.80}$$

From (D.33), (D.38) and (D.39) one may associate the monopole with the photon density contrast, in addition to $\Theta_1 = -\dfrac{1}{k}\partial_\eta \Theta_0$, doing a WKB approximation in δ_γ (inside the horizon) as we did before, thus

$$\Theta_0 \approx \frac{1}{(1+R)^{1/4}} \left[A_k cos(kr_s) + B_k sin(kr_s)\right] e^{-k^2/k_D^2}, \tag{5.81}$$

$$\Theta_1 \approx \frac{c_s}{(1+R)^{1/4}} \left[A_k sin(kr_s) - B_k cos(kr_s)\right] e^{-k^2/k_D^2}. \tag{5.82}$$

In which A_k and B_k are the WKB amplitudes, $c_s^2 = 1/3(1+R) \approx 1/3$ the photon-baryon sound speed, $r_s = \eta/\sqrt{3}$ the sound horizon and k_D is the damping scale, squaring the dipole (5.82)

$$\langle \mathcal{S}_{ac} \rangle \approx \frac{16}{45\dot{\tau}^2} \int \frac{dkk^4}{2\pi^2} P_i(k) \left[A_k^2 sin^2(kr_s) + B_k^2 cos^2(kr_s) - 2A_k B_k cos(kr_s) sin(kr_s)\right] e^{-2k^2/k_D^2}. \tag{5.83}$$

For the net effect of many modes on the CMB spectrum, we are only interested in time-averaged values, taking the average of the right side of the equation above one obtain $\left\langle sin^2(kr_s) \right\rangle \rightarrow 1/2$ and $\left\langle cos^2(kr_s) \right\rangle \rightarrow 1/2$ and the interference term vanishes, then

$$\langle \mathcal{S}_{ac} \rangle \approx \frac{8}{45\dot{\tau}^2} \int \frac{dkk^4}{2\pi^2} P_i(k) \left[A_k^2 + B_k^2\right] e^{-2k^2/k_D^2}. \tag{5.84}$$

In [45] the authors analyzed the energy dissipation to isocurvature perturbations to understand a general scenario of exchange of energy, but here we will use the adiabatic initial condition given by [46]

$$A_k \approx \left(1 + \frac{4}{15} R_\nu\right)^{-1}, \text{ and } \quad B_k \approx 0, \tag{5.85}$$

wherein $R_\nu = \rho_\nu/(\rho_\gamma + \rho_\nu) \approx 0.41$ denotes the fractional contribution of massless neutrinos to the energy density of relativistic species.

The energy rate of change of the injection will then

$$\frac{1}{\rho_\gamma}\frac{dQ_{ac}}{dt} = 4\dot{\tau}\,\langle \mathcal{S}_{ac}\rangle\,. \tag{5.86}$$

where the factor of 4 arises because of the change in the photon energy density by $\Delta\rho_\gamma \approx 4\rho_\gamma$ due to the y-distortion. The factor $\dot{\tau}$ arises because the source function $\langle \mathcal{S}_{ac}\rangle > 0$, is defined concerning the Thomson-scattering time scale [45].

From (5.84) with the adiabatic perturbation initial condition (5.85), and the expression (5.86) we obtain

$$\frac{dQ_{ac}}{dt} = 9.4\rho_\gamma\, H(z) \int_0^\infty \frac{dk\, k^4}{k_D^2\, 2\pi^2} P_i(k)\, e^{-2k^2/k_D^2}. \tag{5.87}$$

For the tight coupling regime, one may use Hubble of the radiation domination era,i.e., $H(z) = 2.1 \times 10^{-20}(1+z)^2\, s^{-1}$, and the damping scale is $k_D = 4 \times 10^{-6}(1+z)^{3/2}\mathrm{Mpc}^{-1}$. The equation above explicitly shows the sensitivity of the energy injection by the Silk damping mechanism with the power spectrum of the initial curvature perturbations in the primordial universe.

As we know the distortions are sensible to the energy injection and consequently with the power spectrum, this result opens a window to explore and constrain different inflation models, or general quantum cosmological models that predict a specific power spectrum from an initial curvature perturbation.

5.4 Small-scale power spectrum effects

The models we are interested in constrain in this work are the bounce models. They are cosmological models in which the universe undergoes a contraction phase followed by a minimum size and then re-expands, avoiding the cosmological singularity and solving the cosmological problems discussed in chapter 2, without the need for the inflation field. A review of this subject can be found in [15, 16].

Nonetheless in [16] the authors showed that in general to produce a classical bounce, one has to violate the Null energy Condition (NEC)[10], which must lead to instabilities of the solutions, moreover to violate the NEC often one needs an exotic

[10] Being k^μ a null vector, the condition is given by $T_{\mu\nu}k^\mu k^\nu \geq 0$, it is essential for maintaining the causal structure of spacetime and to get stable solutions.

matter in the model with negative pressure, i.e., $p < w\rho$, with $w < -1$, which is subject of criticism.

For this reason, we will study the spectral distortions generated in a quantum bounce model with a barotropic fluid. In [7] the properties of quantum bounce cosmology have been studied. The quantum bounce in these models will originate from the canonical quantization of GR assuming a mini-superspace model composed of the symmetries of a homogeneous and isotropic background. It is outside the scope of this work to explain this mechanism, for more details see the reference above.

In the same work, was found a power spectrum of the primordial perturbations following a power law given by $P_\zeta \propto k^{n_s-1}$, with

$$n_s = 1 + \frac{12w}{1+3w}. \tag{5.88}$$

In which $w > 0$, in the case of a single dust-like fluid $(w \approx 0)$ the spectral index is almost scale-invariant, i.e., $n_s \gtrsim 1$.

As we can see the power spectrum is blue, quasi-scale invariant, this is a specific problem of this model since the PLANCK data shows a red, quasi-scale-invariant, adiabatic power spectrum, which was predicted by the inflation, given a power spectrum with spectral index $n_s = 0.96$ [47].

5.4.1 Spectral distortion due Silk damping

Substituting the power spectrum $P_\zeta = \frac{A_s 2\pi^2}{k^3}\left(\frac{k}{k_0}\right)^{n_s-1}$ of the bounce model in (5.87)

$$\frac{dQ_{ac}}{dt} = \frac{9.4\rho_\gamma\, H(z) A_s}{k_D^2\, k_0^{n_s-1}} \int_0^\infty dk\, k^{n_s}\, e^{-2k^2/k_D^2}. \tag{5.89}$$

Where observations with WMAP from large scales set the amplitude of the power spectrum as $A_s = 2.4.10^{-9}$ for pivot scale $k_0 = 0.002\ Mpc^{-1}$ [48]. Making a change of variables $u = \sqrt{2}\, k/k_D$ we have

$$\frac{dQ_{ac}}{dt} = \frac{9.4\rho_\gamma\, H(z) A_s k_D^{n_s-1}}{2^{(n_s+1)/2}\, k_0^{n_s-1}} \int_0^\infty du\, u^{n_s}\, e^{-u^2}, \tag{5.90}$$

the integral will give a gamma function

$$\int_0^\infty dk\, u^{n_s}\, e^{-u^2} = \frac{1}{2}\Gamma\left[(n_s+1)/2\right], \tag{5.91}$$

defined for $n_s > -1$, then

$$\frac{dQ_{ac}}{dt} = \frac{9.4 A_s \Gamma\left[(n_s+1)/2\right] \rho_\gamma\, H(z) k_D^{n_s-1}}{2^{(n_s+3)/2}\, k_0^{n_s-1}} \tag{5.92}$$

$$= \mathcal{F}(n_s)\, H(z)\, \rho_\gamma (1+z)^{3/2} \tag{5.93}$$

wherein

$$\mathcal{F}(n_s) \equiv \frac{9.4 A_s \Gamma\left[(n_s+1)/2\right]}{2^{(n_s+3)/2}\, k_0^{n_s-1}\, (4\times 10^6)^{n_s-1}}. \tag{5.94}$$

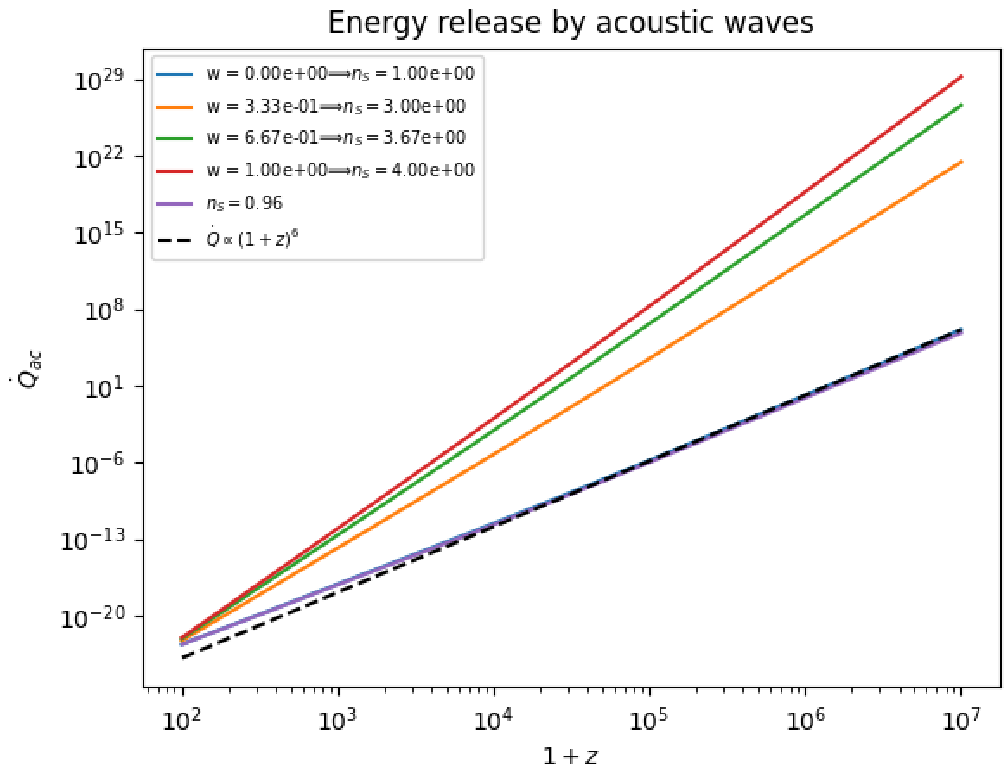
Where the damping scale $k_D = 4\times 10^{-6}(1+z)^{3/2}$ has been used, and the function $\mathcal{F}(n_s)$ can be seen as an amplitude of the energy rate.

Using the spectral index (5.88), in table 2 we compare the amplitude of energy dissipation for different fluids, in figure 7 we plot the energy release for these components

Table 2 – Amplitude of energy release

Component	w	n_S	$\mathcal{F}(n_s)$
Dust matter	0	1	5.64×10^{-9}
Radiation	1/3	3	4.51×10^{-20}
Warm matter	2/3	11/3	1.07×10^{-23}
Stiff matter	1	4	1.69×10^{-25}

Figure 7 – Energy release as function of time



Source: Author.

We used a logarithm plot both in the $\dot{Q}_{ac}[1/(s\, cm^3)]$ and the $1+z$ since the energy transfer varies greatly in the interval, moreover, we used the Hubble of the ΛCDM (2.85), and the energy density of photons (D.49).

The energy release grows with the spectral index, such that most of the energy is transferred in high redshift $z \gtrsim 10^4$. For quasi-scale invariant index $n_s \approx 1$ at high redshift one have $\dot{Q}_{ac} \propto (1+z)^6$ as discussed in Chluba paper [4]. Furthermore, there is a change in the amplitude of the energy release by 11 orders of magnitude between the dust matter and the radiation, indicating the possibility of constraints in the power spectrum of models that include radiation and dust matter.

With the energy release in hands, one may compute directly the μ distortion due to Silk damping using (5.45), thus

$$\mu = 1.4\mathcal{F}(n_s) \int_{z_{\mu,y}}^{z_\mu} e^{-(z'/z_{mu})^{5/2}} (1+z')^{(3n_s-5)/2} dz', \tag{5.95}$$

making a change of variable $v = \left(\frac{z}{z_\mu}\right)^{5/2}$, then $z = z_\mu v^{2/5}$, the integral will be

$$\mu = 0.56\, \mathcal{F}(n_s)\, z_\mu \int_{v_1}^{v_2} e^{-v} v^{3/5} (1 + z_\mu v^{2/5})^{(3n_s-5)/2} dv. \tag{5.96}$$

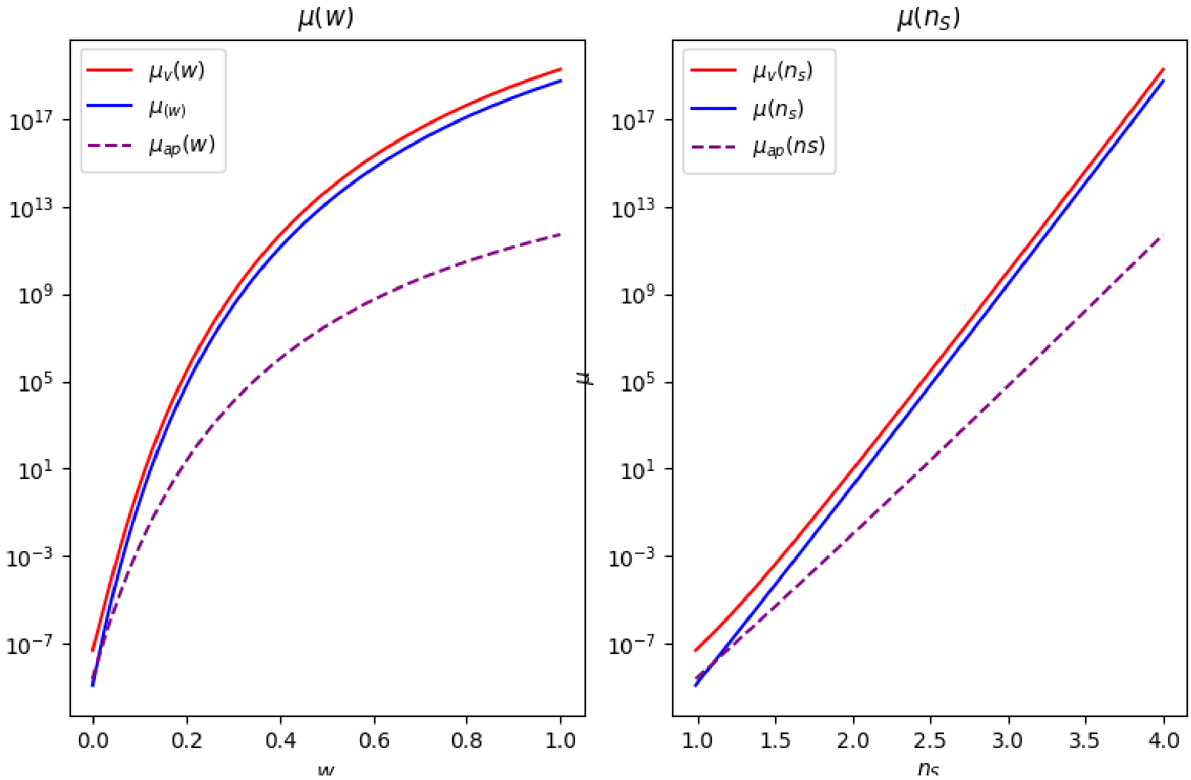
In which $v_1 = (z_{\mu,y}/z_\mu)^{5/2} \approx 0$, and $v_2 = (z_\mu/z_\mu)^{5/2} = 1$.

One may solve the integration above analytically by using the approximation $(1+x)^n \approx x^n$, with $x = z_\mu v^{2/5}$, this is a good approximation for small n_s we will call this solution μ_{ap}, but for a general and more accurate solution we should solve numerically, some works in literature estimate the distortion without the visibility function $\mathcal{T} = e^{-v}$, thus we solve for this case just to compare. The results for each n_s are in table 3 and figure 8 below

Table 3 – μ-type distortion

Component	n_S	μ	μ_v
Inflaton	0.96	3.3×10^{-8}	7.1×10^{-10}
Dust matter	1	6×10^{-8}	1.6×10^{-9}
Radiation	3	1×10^{10}	2.7×10^{9}
Warm matter	11/3	1.5×10^{16}	4.2×10^{15}
Stiff matter	4	1.9×10^{19}	5.5×10^{18}

Figure 8 – μ distortion



Source: Author.

On the left side we plot the distortion in terms of the equation of state, and on the right side the spectral index for the one fluid quantum bounce model. The distortion μ_v was calculated using the visibility function, on the other hand, μ was not, and the μ_{ap} is the approximate solution, which we plot to compare the error of this approximation.

The distortion increases for larger n_s, notice the large difference of the distortion between the dust matter and radiation (18 orders of magnitude), as said before this may be used as a probe of cosmological models.

5.4.2 Matter temperature due Silk damping

At this stage, from (5.24) we can compute the time evolution of the electron/baryon temperature due to the dissipation of acoustic waves at small scales.

$$\frac{d\rho_e}{d\tau} = \frac{t_C \dot{Q}}{\alpha \theta_\gamma} + \frac{4\tilde{\rho}_\gamma}{\alpha}(\rho_{eq} - \rho_e) - Ht_C\rho_e. \tag{5.97}$$

Unfortunately, we do not solve this equation fully until the moment, still one may get an approximate solution as Chluba did for general energy exchange in [4], assuming a quasi-stationary solution, i.e., $\frac{d\rho_e}{dz} \approx 0$, we get

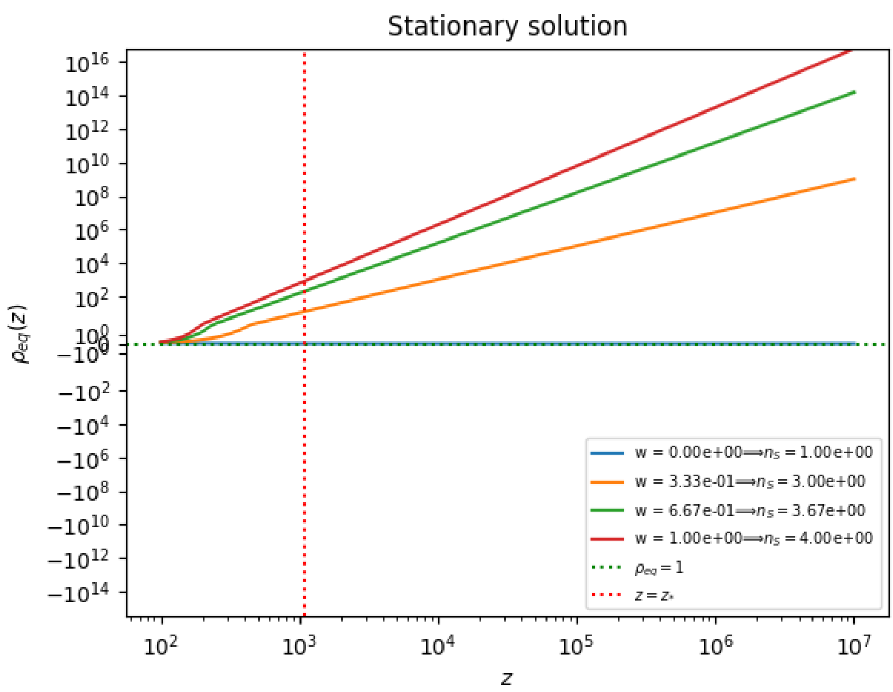
$$\rho_e(z) = \frac{\rho_{eq} + \frac{t_C \dot{Q}}{4\tilde{\rho}_\gamma \theta_\gamma}}{1+\Lambda}. \tag{5.98}$$

We just isolate $\rho_e(z)$, and

$$\Lambda = \frac{\alpha t_C H}{4\tilde{\rho}_\gamma} = \frac{3}{8}\left(\frac{f_{He}+\chi_e}{\chi_e\, \sigma_T\, c\, \tilde{\rho}_\gamma}\right) H(z). \tag{5.99}$$

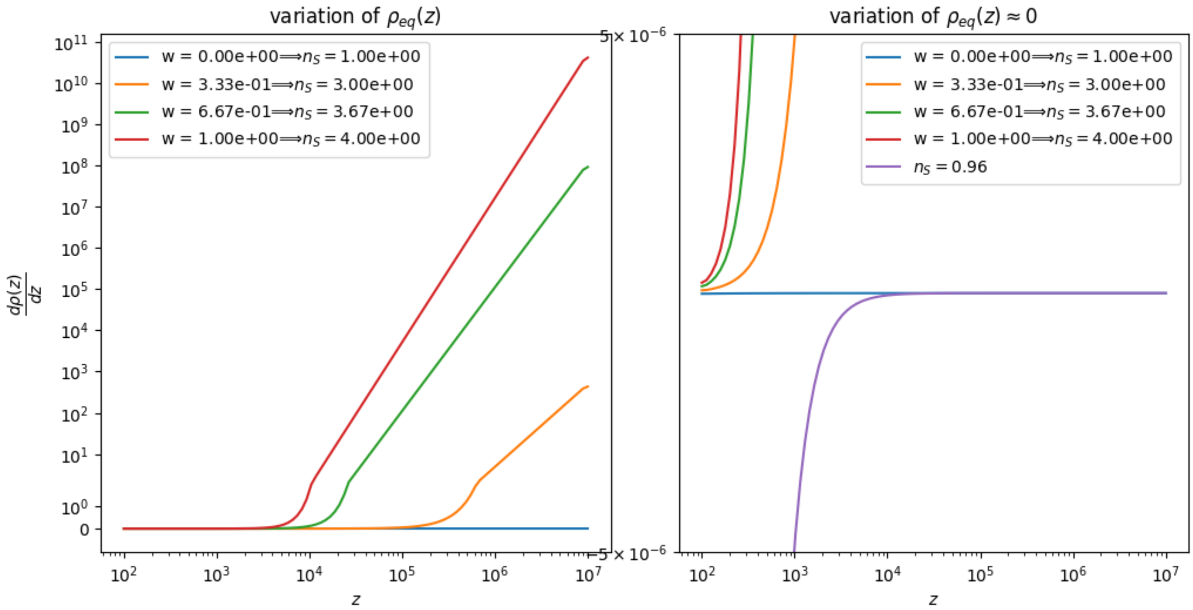
using the energy dissipation (5.93), $\chi_e = 1$, and $f_{He} = N_{He}/N_H$ (D.63, D.64). In figure 9 we plot the stationary solution, and the derivative of the solution to verify the valid domain

Figure 9 – Stationary solution for different equation of state



Source: Author.

Figure 10 – Time derivative of the matter temperature



Source: Author.

Apparently, only for dust matter ($n_s = 1$) do we observe well-behaved evolution during the considered interval. In the left figure 10, we see when the evolution is approximately stationary, indicating that for $n_s > 1$, this solution fails at high redshifts. Notice in the right figure that for inflation ($n_s = 0.96$), the derivative of ρ_e becomes negative near the last scattering surface ($z \approx 10^3$), representing an unphysical solution, while at this redshift we have stationary solutions for components with $n_s \geq 1$.

5.5 Conclusion

The main result of this work was the estimation of the μ distortion for a single fluid quantum bounce model, as shown in Table 3. Although this model has the issue of predicting a blueshifted power spectrum [7], it provides insight into the magnitude of the spectral distortion for the single-fluid case with different types of matter content, making it useful for evaluating multifluid models, such as dust and radiation.

Notably, there is a significant difference in the distortion between dust matter and radiation (18 orders of magnitude). As mentioned previously, this disparity could serve as a probe for cosmological models.

It is important to emphasize that we use the photon redshift temperature in all estimations, alongside the assumption of tight coupling between electrons and photons. For greater numerical precision, it is advisable to solve the equations assuming a small distortion in the photon distribution, such that $T_\gamma \neq T_z$.

Additionally, in the stationary temperature evolution shown in Figure 9, we do not account for the BR and DC processes. For a complete description, these terms should be included, and the evolution of the electron temperature should be solved in conjunction with the Boltzmann equation for photons.[11]

In future work, we aim to include these effects to obtain a more accurate numerical solution and provide a complete matter temperature evolution. Furthermore, we plan to extend the analysis of spectral distortions due to Silk damping for a two-fluid quantum bounce model, incorporating both radiation and dust matter, as this may offer signatures to test and constrain the model.

[11] For high-energy photons, Compton scattering is dominant since DC dominates over BR scattering. However, for lower redshifts, BR effects prevail.

A Differential geometry

A.1 Mathematical definitions

In this section are the mathematical definitions used in the work

Let M and N manifold of dimension m and n respectively, let $\phi : M \to N$ a $C^\infty(\mathbb{R})$ map. Thus the pullback of any dual vector w_μ at $\phi(p) \in N$ to the dual vector at $p \in M$ is the map $\phi^* : V^*_{\phi(p)} \to V^*_p$, such that, for all $v^\mu \in V_p$:

$$(\phi^* w)_\mu v^\mu = w_\mu (\phi_* v)^\mu \tag{A.1}$$

in which $\phi_* v(f) : V_p \to V_{\phi(p)} = v(f \circ \phi)$ is the pushforward of the vectors at p to vectors at $\phi(p)$, where $f \in C^\infty(N)$

The operation above is necessary to take the tensor from a manifold to another manifold and was important to define the frame of reference on the spatial sections $\Sigma_t \subset M^4$ since it allows us to carry the spatial vectors defining the basis for one hypersurface to any other. Next, we will define the Lied derivative which is useful to calculate the variations of tensor fields on a manifold.

Let ϕ_t a one parameter group of diffeomorphism generated y a vector field v^μ, then one may carry a tensor field $T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_k}$ under the flux using ϕ^t_*, thus the Lied derivative of this tensor field is

$$\mathcal{L}_v T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_k} = \lim_{t \to 0} \left(\frac{\phi^{-t}_* T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_k} - T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_k}}{t} \right) \tag{A.2}$$

From this definition the action on scalar, vector and dual vector fields are [49]

$$\mathcal{L}_v f = v(f) = \frac{df}{dt}, \tag{A.3}$$

$$\mathcal{L}_v u^\mu = [v, u]^\mu = \left(v^\nu \frac{\partial u^\mu}{\partial x^\nu} - u^\nu \frac{\partial v^\mu}{\partial x^\nu} \right), \tag{A.4}$$

$$\mathcal{L}_v w_\mu = v^\nu \nabla_\nu w_\mu + w_\nu \nabla_\mu v^\nu, \tag{A.5}$$

with the results above one may generalize the action on general tensor fields $T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_k}$. Let us calculate for the metric $g_{\mu\nu}$

$$\mathcal{L}_v g_{\mu\nu} = v^\lambda \nabla_\lambda g_{\mu\nu} + g_{\lambda\nu} \nabla_\mu v^\lambda + g_{\mu\lambda} \nabla_\nu v^\lambda \tag{A.6}$$

$$= \nabla_\mu v_\nu + \nabla_\nu v_\mu, \tag{A.7}$$

where we used $\nabla_\lambda g_{\mu\nu} = 0$.

Note if ϕ_t was a symmetry transformation of the tensor field $T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_k}$, then $\phi^{-t}_* T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_k} = T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_k}$, therefore $\mathcal{L}_v T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_k} = 0$, $\forall t \in \mathbb{R}$.

Let $\phi_t : M \to M$ the group of isometries, i.e., $\phi_t^* g_\mu \nu = g_{\mu\nu}$, where $g_\mu \nu$ is the metric of spacetime, then the vector field v which generate the flux ϕ_t is called a killing vector field, from (A.7)

$$\nabla_\mu v_\nu + \nabla_\nu v_\mu = 0. \tag{A.8}$$

A killing vector is a geometric object that does not depend on a frame of reference and is very useful for studying the geometric aspects of a space, including the analysis of conservations of variables in these spaces.

When dealing with differential forms a useful way to calculate the lie derivative of any form $\omega \in \bigwedge^d$ of an antisymmetric space defined on a pseudo-Riemannian manifold of dimension d is by Cartan's identity [50]

$$\mathcal{L}_X \omega = d i_X \omega + i_X d(\omega) \tag{A.9}$$

Finally, the Stokes theorem is a generalization of Gauss theorem in general spaces, being a useful result to make a bridge between the definitions of fields and variables from the boundary of a manifold ∂V with the manifold itself V

$$\int_M \omega = \int_{\partial M} d\omega \tag{A.10}$$

the proof of this theorem can be found in [50].

A.2 General Relativity

The theory nowadays that better describes the nature of gravity is General Relativity (RG), in this theory, gravity is not a fundamental force as in the Newtonian law of gravity, it is an effect of curvature on spacetime. In another way, on RG we can not define an inertial observer, for a detailed discussion about the definition of an inertial frame to define a force and the relationship with other fundamental forces see [8].

What is made in GR is to use the equivalence principle which asserts there is no way locally to distinguish an observer in free fall, under the effect of gravity from an accelerated observer with equal modulus in the opposite direction. Therefore studying the motion of these observers in free fall following a geodesic curve in space with four-velocity n^α, under effect only of curvature of space in which has the metric $g_{\mu\nu}$ as the fundamental structure of the theory.

In the frame of the free-falling observers, the geodesic equation is

$$\frac{dn^\alpha}{d\tau} = n^\mu \nabla_\mu n^\alpha = 0 \tag{A.11}$$

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\mu\lambda} \frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \tag{A.12}$$

In which $n^\mu = \frac{dx^\mu}{d\tau}$, with $n^\mu n_\mu = -1$. And $\Gamma^\alpha{}_{\mu\lambda}$ are the Christtofel symbols associated with the metric $g_{\mu\nu}$, with components

$$\Gamma^\alpha{}_{\mu\lambda} = -\frac{g^{\alpha\sigma}}{2}\left(\nabla_\mu g_{\alpha\nu} + \nabla_\nu g_{\mu\alpha} - \nabla_\sigma g_{\mu\nu}\right). \tag{A.13}$$

As an example the non-zero Christtofel symbols on FLRW space with $K = 0$, wherein the metric is given by (B.21). are

$$\Gamma^0{}_{ij} = \delta_{ij}\, a\dot{a} \tag{A.14}$$

$$\Gamma^i{}_{0j} = \delta^i{}_j\, H \tag{A.15}$$

A.2.1 Einstein field equations

As discussed at the beginning of this section $g_{\mu\nu}$ is the fundamental structure that tells us the influence of gravity in any test body and vice-versa, its dynamic is given by Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T^{(\Lambda)}_{\mu\nu} \tag{A.16}$$

$T^{(\Lambda)}_{\mu\nu} \equiv T_{\mu\nu} - \frac{\Lambda g_{\mu\nu}}{\kappa}$, and $\kappa = 8\pi G$ taking the trace of the above equation

$$R = -\kappa T \tag{A.17}$$

with $T = T^{\mu\,(\Lambda)}_{\ \mu}$.

The left side of equations has the geometric objects that carry the information of the curvature of space, $R_{\mu\nu}$ and R are the Ricci tensor and scalar Ricci tensor respectively, they may be obtained from the Riemann tensor $R_{\alpha\beta\mu}{}^\nu$, which has the properties:

$$R_{\alpha\beta\mu}{}^\nu = -R_{\beta\alpha\mu}{}^\nu \tag{A.18}$$

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} \tag{A.19}$$

$$R_{[\alpha\beta\mu]}{}^\nu = 0 \tag{A.20}$$

$$\nabla_{[\gamma} R_{\alpha\beta]\mu}{}^\nu = 0. \tag{A.21}$$

the last property is called Bianchi identity, it is crucial to to deduce the left side of Einstein's equations, for details of how to deduce Einstein's field equations look [8].

The right side of (A.16) is the matter sector where $T_{\mu\nu}$ is the stress-energy tensor, which modifies the curvature of spacetime, wherein Λ is the cosmological constant, responsible for the accelerated expansion of the universe, and believed be a kind of energy on nature.

B Geometric aspects in FLRW space-time

B.1 Metric on FLRW universe

One way to obtain the metric in the desired space is to induce the metric defined in the space M^4 into the space Σ_t. For this, we need to fix some variable in order to express one coordinate as a function of the others. Let the sphere $S^3 \subset \mathbb{R}^4$, with $R = \text{const}$, that is, we impose that the radius of the sphere is constant so that we can induce the metric into a lower-dimensional space.

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = R^2, \tag{B.1}$$

taking the differential of both sides, we get

$$x^1 dx^1 + x^2 dx^2 + x^3 dx^3 + x^4 dx^4 = 0, \tag{B.2}$$

$$dx^4 = -\frac{1}{x^4}(x^1 dx^1 + x^2 dx^2 + x^3 dx^3), \tag{B.3}$$

writing $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \Rightarrow rdr = x^1 dx^1 + x^2 dx^2 + x^3 dx^3$, then

$$dx^4 = -\frac{1}{x^4} rdr \Rightarrow (dx^4)^2 = \frac{r^2 dr^2}{(x^4)^2}, \tag{B.4}$$

from the constraint, $r^2 + (x^4)^2 = R^2$,

$$(dx^4)^2 = \frac{r^2 dr^2}{R^2 - r^2}, \tag{B.5}$$

the metric in R^4 in spherical coordinates is

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2, \tag{B.6}$$

fixing the radius, we can then induce the metric from R^4 to $\mathbb{S}^3$ by expressing one of the coordinates as a function of the others, as we did by choosing $x^4 = x^4(r, R)$,

$$ds^2|_{\mathbb{S}^3} = dr^2 + r^2(\sin^2\theta\, d\phi^2 + d\theta^2) + \frac{r^2}{R^2 - r^2} dr^2 \tag{B.7}$$

$$= \left(1 + \frac{r^2}{R^2 - r^2}\right) dr^2 + r^2(\sin^2\theta\, d\phi^2 + d\theta^2) \tag{B.8}$$

$$= \frac{1}{1 - \left(\frac{r}{R}\right)^2} dr^2 + r^2(\sin^2\theta\, d\phi^2 + d\theta^2) \tag{B.9}$$

$$= \left(\frac{1}{1 + Kr^2}\right) dr^2 + r^2(\sin^2\theta\, d\phi^2 + d\theta^2), \tag{B.10}$$

where we defined $K \equiv -\frac{1}{R^2}$. Here, K is the spatial curvature of S^3.

Let's analyze another possible geometry that satisfies all the previously mentioned spatial symmetries. To achieve this, we only need to modify the constraint equation by placing a negative sign on the right-hand side of the equation.

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = -R^2, \tag{B.11}$$

The above equation describes a hyperboloid, which we will denote as $\mathbb{H}^3$. We will follow the same procedure as we did for $\mathbb{S}^3$.

$$x^1dx^1 + x^2dx^2 + x^3dx^3 + x^4dx^4 = 0, \tag{B.12}$$

$$rdr - x^4dx^4 = 0 \tag{B.13}$$

$$dx^4 = \frac{r}{x^4}dr \Rightarrow (dx^4)^2 = \frac{r^2}{(x^4)^2}dr^2, \tag{B.14}$$

From the constraint $r^2 + (x^4)^2 = -R^2 \Leftrightarrow (x^4)^2 = -(r^2 + R^2)$, therefore

$$(dx^4)^2 = -\frac{r^2}{r^2 + R^2}dr^2 \tag{B.15}$$

The general metric is given by B.6, fixing the radius, and using the constraint equation, we obtain

$$ds^2|_{\mathbb{H}^3} = dr^2 + r^2(\sin^2\theta\, d\phi^2 + d\theta^2) - \frac{r^2}{r^2 + R^2}dr^2 \tag{B.16}$$

$$= \left(1 - \frac{r^2}{R^2 + r^2}\right)dr^2 + r^2(\sin^2\theta\, d\phi^2 + d\theta^2) \tag{B.17}$$

$$= \frac{1}{1 + \left(\frac{r}{R}\right)^2}dr^2 + r^2(\sin^2\theta\, d\phi^2 + d\theta^2) \tag{B.18}$$

$$= \left(\frac{1}{1 - Kr^2}\right)dr^2 + r^2(\sin^2\theta\, d\phi^2 + d\theta^2), \tag{B.19}$$

Note that the only modification in the metric is the sign accompanying the curvature term of the space, such that, when considering $K < 0$, we are dealing with hyperbolic space ($\mathbb{H}^3$), and for $K > 0$, with spherical space ($\mathbb{S}^3$). Observe that for $K = 0$, the space reduces to Euclidean space ($\mathbb{R}^3$). Thus, we can generalize the three metrics for maximally symmetric spaces:

$$ds^2|_\Sigma = \left(\frac{1}{1 - Kr^2}\right)dr^2 + r^2d\Omega^2 \begin{cases} K = 0\ (\mathbb{R}^3) \\ K > 0\ (\mathbb{S}^3) \\ K < 0\ (\mathbb{H}^3) \end{cases} \tag{B.20}$$

where $d\Omega^2 \doteq \sin^2\theta\, d\phi^2 + d\theta^2$ is the solid angle element.

Since the universe is expanding, and all geodesic observers must measure the same proper time interval between constant time hypersurfaces Σ_t and $\Sigma_{t+\delta t}$, the most general metric for spacetime M^4 must be

$$ds^2 = -dt^2 + a^2(t)\left[\left(\frac{1}{1-Kr^2}\right)dr^2 + r^2 d\Omega^2\right] \tag{B.21}$$

where $a(t)$ is an arbitrary function of proper time only, as we are imposing spatial homogeneity on the spatial hypersurfaces, and t is the proper time measured by isotropic observers.

We can use another coordinate system to rewrite the metric (B.21), to place the curvature dependence in the angular part of the metric. Suppose that $\tilde{r} = \tilde{r}(r)$, we want that;

$$d\tilde{r} = \frac{1}{\sqrt{1-Kr^2}}dr \tag{B.22}$$

then

$$d\tilde{r} = \frac{\partial \tilde{r}}{\partial r}dr \tag{B.23}$$

$$\frac{dr}{\sqrt{1-Kr^2}} = \frac{\partial \tilde{r}}{\partial r} \tag{B.24}$$

$$\int_0^r \frac{dr'}{\sqrt{1-Kr'^2}} = \tilde{r}(r) - \tilde{r}(0) \tag{B.25}$$

Choosing without loss of generality $\tilde{r}(0) = 0$, and solving the definite integral,

$$\tilde{r}(r) = \frac{1}{\sqrt{K}}\sin^{-1}(\sqrt{K}r) \tag{B.26}$$

Inverting the above relation to substitute into the metric B.21,

$$r(\tilde{r}) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}\tilde{r}) \tag{B.27}$$

$$ds^2 = -dt^2 + a^2(t)\left(d\tilde{r}^2 + \frac{\sin^2(\sqrt{K}\tilde{r})}{K}d\Omega^2\right), \tag{B.28}$$

We can analyze how the component will be for the three possible geometries $K = \{-1, 0, 1\}$,

$$\begin{cases} K = 1 \Rightarrow \sin^2 \tilde{r}; \\ K = 0 \Rightarrow \lim_{K\to 0} \dfrac{\sin^2(\sqrt{K}\tilde{r})}{K} = \dfrac{(\sqrt{K}\tilde{r})^2}{K} = \tilde{r}^2; \\ K = -1 \Rightarrow -\sin^2(i\tilde{r}) = -(i\sinh\tilde{r})^2 = \sinh^2(\tilde{r}). \end{cases} \tag{B.29}$$

Writing the metric explicitly,

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + f_K^2(r)d\Omega^2\right), \tag{B.30}$$

where $f_K(r) = \begin{cases} f_1(r) = \sin(r) \\ f_0(r) = r \\ f_{-1}(r) = \sinh r \end{cases}$

B.1.1 Spatial metric on homogeneous and isotropic spaces

In homogenous spaces, we have an isometry group on the spatial sections Σ_t, consequently one has the killing vectors on the spatial sections $\{\xi_a{}^{\mu}\}$, $a = 1, 2, 3$ which are generators of these symmetries. In these spaces, we have a natural geodesic foliation n^{μ}, since $\mathcal{L}_{\xi} n_{\mu} = 0$ and $a_{\mu} = \nabla_n n_{\mu} = 0$, moreover $\nabla_n(n_{\mu}\xi_a{}^{\mu}) = 0$, then n_{μ} is orthogonal to the killing fields and normal to all spatial sections.

Therefore is useful to define a spatial basis: $\{e^a{}_{\mu}\}$, which is linearly independent and tangent to the spatial sections for each $t \in \mathbb{R}_+$, in such a way that it is invariant by the isometry group, i.e., $\mathcal{L}_{\xi} e^a{}_{\mu} = 0$, being possible defining it in all the points $p \in \Sigma_t$ by an isometry transformation, the induced metric in this basis is

$$h_{\mu\nu} = h_{ab} e^a{}_{\mu} e^b{}_{\nu} \tag{B.31}$$

where h_{ab} are the components of the metric in this basis, it is straightforward to show that $h_{\mu\nu}$ is invariant by the isometry group $\mathcal{L}_{\xi} h_{\mu\nu} = 0$, thus, $\mathcal{L}_{\xi} h_{ab} = 0 \Rightarrow \xi_c{}^{\alpha}\partial_{\alpha} h_{ab} = 0$, then h_{ab} are constant on the spatial section.

The results to $e^a{}_{\mu}$ are true for a specific spatial section, nevertheless one may define this basis for any other spatial section imposing $\mathcal{L}_t\, e^a_{\mu} = \dot{e}^a{}_{\mu} = 0$, is easy to check that this basis will be normal to n_{μ} in all the sections, i.e., $\partial_t(n_{\mu} e^a{}_{\mu}) = 0$, and will be invariant by the isometry group for any section, since $[\partial_t, \mathcal{L}_{\xi}]e^a{}_{\mu} = 0 \Rightarrow \partial_t(\mathcal{L}_{\xi} e^a{}_{\mu}) - \mathcal{L}_{\xi}(\partial_t e^a{}_{\mu}) = 0 \Rightarrow \partial_t(\mathcal{L}_{\xi} e^a{}_{\mu}) = 0$.

Furthermore, if on any point $p \in \Sigma_t$ the isotropy group is isomorphic to the rotational group in three dimensions $(SO(3))$[1], it is possible to show that $h_{ab} = a^2(t)\delta_{ab}$, in which $a(t)$ is a constant function on Σ_t, thus the induced metric on an isotropic and homogeneous space in this basis is

$$h_{\mu\nu} = a^2(t)\delta_{ab} e^a{}_{\mu} e^b{}_{\nu} \tag{B.32}$$

[1] This symmetry group gives us three more killing fields associated with the rotations, intuitively this isotropy group tells us that in each point p of the three-dimensional space there is no a special direction

B.1.2 Spatial curvature and extrinsic curvature

We may associate the extrinsic curvature on M^4 and with the Riemann tensor in Σ_t, using the Bianchi identity (A.21) contracted with a time-like vector n^γ

$$\nabla_n R_{\mu\nu\alpha\beta} + n^\gamma \left(\nabla_\mu R_{\nu\gamma\alpha\beta} + \nabla_\nu R_{\gamma\mu\alpha\beta} \right) = 0 \tag{B.33}$$

making the spatial projection of the equation above with some algebra (see [18] for details), we get

$$\nabla_n \mathcal{R}_{\mu\nu\alpha\beta} + 2\left(D_\mu D_{[\alpha} \mathcal{K}_{\beta]\nu} - D_\nu D_{[\alpha} \mathcal{K}_{\beta]\mu} + \mathcal{K}^\gamma_{[\mu} \mathcal{R}_{\nu]\gamma\alpha\beta} \right) = 0 \tag{B.34}$$

contracting μ and ν indices:

$$\nabla_n \mathcal{R}_{\mu\alpha} + D_\mu D_\alpha \theta - 2 D_\nu D_{(\mu} \mathcal{K}^\nu{}_{\alpha)} + D^2 \mathcal{K}_{\alpha\mu} + 2 \mathcal{R}_{\gamma(\alpha} \mathcal{K}^\gamma{}_{\mu)} = 0 \tag{B.35}$$

C Perturbations of geometry

C.1 Kinematics quantities

Let us develop some useful results of the perturbations on kinematics quantities of GR, the first result is that any difference between covariant derivatives acting on an one- form can be written as [8]

$$(\widehat{\nabla}_\mu - \nabla_\mu)w_\nu = \Gamma_{\mu\nu}{}^{\alpha} w_\alpha. \tag{C.1}$$

where $\Gamma_{\mu\nu}{}^{\alpha}$ is a tensor with the property $\Gamma_{[\mu\nu]}{}^{\alpha} = 0$, i.e., we are assuming null torsion in $\widehat{M}$. Applying the difference of the covariant derivatives on the metric $g_{\mu\nu}$ is straightforward that

$$\Gamma_{\mu\nu}{}^{\alpha} = -\frac{\hat{g}^{\alpha\sigma}}{2}\left(\nabla_\mu \hat{g}_{\alpha\nu} + \nabla_\nu \hat{g}_{\mu\alpha} - \nabla_\sigma \hat{g}_{\mu\nu}\right) \tag{C.2}$$

$$= -\frac{g^{\alpha\sigma}}{2}\left(\nabla_\mu \delta g_{\alpha\nu} + \nabla_\nu \delta g_{\mu\alpha} - \nabla_\sigma \delta g_{\mu\nu}\right) \tag{C.3}$$

where we use $\nabla_\sigma g_{\mu\nu} = 0$.

The physical spatial metric is $\hat{h}_{\mu\nu} = v_\mu v_\nu + \hat{g}_{\mu\nu}$, then its perturbation is given by

$$\delta h_{\mu\nu} = \hat{h}_{\mu\nu} - h_{\mu\nu} \tag{C.4}$$

$$= v_\mu v_\nu + g_{\mu\nu} + \delta g_{\mu\nu} - n_\mu n_\nu - g_{\mu\nu} \tag{C.5}$$

$$= 2n_{(\mu}\delta v_{\nu)} + 2\phi n_\mu n_\nu + 2B_{(\mu}n_{\nu)} + 2C_{\mu\nu} \tag{C.6}$$

$$= 2n_{(\mu}u_{\nu)} + 2B_{(\mu}n_{\nu)} + 2C_{\mu\nu} \tag{C.7}$$

where we use (3.100), raising one indice

$$\delta h_\mu^{\ \nu} = n_\mu(B^\nu + u^\nu) + u_\mu n^\nu \tag{C.8}$$

the inverse of this perturbation will be

$$\delta h^{\mu\nu} = 2u^{(\mu}n^{\nu)} - 2C_{\mu\nu} \tag{C.9}$$

The perturbed extrinsic curvature $\widehat{\mathcal{K}}_{\mu\nu}$ has rich information about the kinematics on the physical manifold, since we may get from it the acceleration of the physical observers, the physical shear tensor, the vorticity, and the physical scalar expansion, then let us explicit this tensor

$$\widehat{\mathcal{K}}_{\mu\nu} = h[\widehat{\nabla}_\mu v_\nu] \tag{C.10}$$

$$= h\left[\mathcal{K}_{\mu\nu}(1-\phi) + \nabla_\mu u_\nu + \Gamma_{\mu\nu}{}^{\sigma} n_\sigma - n_\nu \nabla_\mu \phi\right] \tag{C.11}$$

$$= (1-\phi)\mathcal{K}_{\mu\nu} + 2h^\sigma{}_{(\mu}\delta h_{\nu)}{}^{\gamma}\mathcal{K}_{\sigma\gamma} + D_\mu u_\nu + h[\Gamma_{\mu\nu}{}^{\sigma} n_\sigma] \tag{C.12}$$

the second term is

$$2h^{\sigma}{}_{(\mu}\delta h_{\nu)}{}^{\gamma}\mathcal{K}_{\sigma\gamma} = 2n_{(\mu}\mathcal{K}_{\nu)\sigma}\left(B^{\sigma}+u^{\sigma}\right) \tag{C.13}$$

while the last term will be

$$h[\Gamma_{\mu\nu}{}^{\sigma}]n_{\sigma} - = h\left[\frac{-n^{\sigma}}{2}\left(\nabla_{\mu}\delta g_{\alpha\nu}+\nabla_{\nu}\delta g_{\mu\alpha}-\nabla_{\sigma}\delta g_{\mu\nu}\right)\right] \tag{C.14}$$

$$= -\frac{1}{2}h\left[\nabla_{\mu}(n^{\sigma}\delta g_{\alpha\nu})+\nabla_{\nu}(n^{\sigma}\delta g_{\mu\alpha})-\nabla_{n}\delta g_{\mu\nu}-\mathcal{K}_{\mu}{}^{\sigma}\delta g_{\sigma\nu}-\mathcal{K}_{\nu}{}^{\sigma}\delta g_{\sigma\mu}\right] \tag{C.15}$$

$$= 2\phi\mathcal{K}_{\mu\nu}+D_{(\mu}B_{\nu)}+\dot{C}_{\mu\nu} \tag{C.16}$$

where the derivative by part has been used on the second line, and the decomposition (3.25), substituting these results on the extrinsic curvature we obtain

$$\widehat{\mathcal{K}}_{\mu\nu} = \mathcal{K}_{\mu\nu}+2n_{(\mu}\mathcal{K}_{\nu)\sigma}\left(B^{\sigma}+u^{\sigma}\right)+D_{\mu}u_{\nu}+D_{(\mu}B_{\nu)}+\phi\mathcal{K}_{\mu\nu}+\dot{C}_{\mu\nu}. \tag{C.17}$$

One may contract the two indices to obtain the physical scalar expansion $\hat{\theta} = \widehat{\mathcal{K}}_{\mu}{}^{\mu}$, but first we need to upper one of the indices with the physical metric

$$\widehat{\mathcal{K}}_{\mu}^{\ \nu} = (1+\phi)\mathcal{K}_{\mu}^{\ \nu}+\mathcal{K}_{\alpha}^{\ \beta}\left[h_{\mu}^{\ \alpha}u_{\beta}n^{\nu}+h_{\beta}^{\ \nu}(u^{\alpha+B^{\alpha}n_{\mu}})\right]+D_{\mu}u^{\nu}+D_{(\mu}B_{\sigma)}h^{\sigma\nu}+$$
$$+\dot{C}_{\mu}^{\ \nu}-2\mathcal{K}_{\mu}^{\ \sigma}C_{\sigma}^{\ \nu}+2\mathcal{K}_{\sigma}^{\ \nu}C_{\mu}^{\ \sigma}, \tag{C.18}$$

taking the trace of this tensor

$$\hat{\theta} = \hat{h}^{\mu\nu}\widehat{\mathcal{K}}_{\mu\nu} = \theta+D_{\sigma}u^{\sigma}+D_{\sigma}B^{\sigma}+\phi\theta+\dot{C} \tag{C.19}$$

Thus, the scalar expansion perturbation is

$$\delta\theta = D_{\sigma}u^{\sigma}+D_{\sigma}B^{\sigma}+\phi\theta+\dot{C} \tag{C.20}$$

From (C.17) we have the vorticity as

$$\hat{\omega}_{\mu\nu} = \widehat{\mathcal{K}}_{[\mu\nu]} = D_{[\mu}u_{\nu]} \tag{C.21}$$

note that if $\hat{\omega}_{\mu\nu} \neq 0$ we can not use v_{μ} to sectioning the perturbed manifold $\widehat{M}$ since it will not be normal to the hypersurfaces.

As we did for the background tensors, if we subtract the trace above from the symmetric part of the extrinsic curvature we obtain the shear

$$\hat{\sigma}_{\mu\nu} = \widehat{\mathcal{K}}_{(\mu\nu)}-\frac{\hat{\theta}\hat{h}_{\mu\nu}}{3}$$
$$= (1+\phi)\sigma_{\mu\nu}+2n_{(\mu}\sigma_{\nu)\beta}\left(B^{\beta}+u^{\beta}\right)+D_{(\mu}u_{\nu)}+\left(D_{\mu}h_{\nu}^{\ \beta}-\frac{h_{\mu\nu}}{3}D^{\beta}\right)(u_{\beta}+B_{\beta})+$$
$$+\dot{C}^{t}_{\mu\nu}+\frac{2}{3}(C\sigma_{\mu\nu}-\theta C^{t}_{\mu\nu}). \tag{C.22}$$

wherein $C^t_{\mu\nu} = C_{\mu\nu} - \dfrac{Ch_{\mu\nu}}{3}$, thus the shear perturbation is given by

$$\delta\sigma_{\mu\nu} = \phi\sigma_{\mu\nu} + 2n_{(\mu}\sigma_{\nu)\beta}\left(B^\beta + u^\beta\right) + D_{(\mu}u_{\nu)} + (D_\mu h_\nu{}^\beta - \frac{h_{\mu\nu}}{3}D^\beta)(u_\beta + B_\beta) + \tag{C.23}$$

$$+ \dot{C}^t_{\mu\nu} + \frac{2}{3}(C\sigma_{\mu\nu} - \theta C^t_{\mu\nu}). \tag{C.24}$$

The last important kinematic quantity is the physical acceleration field $\hat{a}^\mu$, let us calculate it [1]

$$\hat{a}^\mu = v^\sigma\widehat{\nabla}_\sigma v^\mu = v^\sigma\widehat{\nabla}_\sigma v^\mu - \Gamma_{vv}{}^\mu \tag{C.25}$$

$$= \nabla_n(u^\mu + B^\mu) + n^\mu\dot{\phi} - \Gamma_{nn}{}^\mu + \mathcal{K}_\sigma{}^\mu(u^\sigma + B^\sigma) \tag{C.26}$$

$$\tag{C.27}$$

where $\Gamma_{vv}{}^\mu = \Gamma_{nn}{}^\mu$, since $\delta v^\sigma\Gamma_{\sigma\alpha}{}^\mu \approx 0$, explaining this term:

$$\Gamma_{nn}{}^\mu = -\frac{g^{\mu\sigma}}{2}\left[\nabla_n(\delta g_{\sigma n}) + \nabla_n(\delta g_{n\sigma}) + \delta g_{\alpha\beta}\nabla_n(n^\alpha n^\beta) - \nabla_\sigma(\delta g_{nn})\right] \tag{C.28}$$

$$= 2\dot{\phi}^\mu + \nabla_n B^\mu + \mathcal{K}^\mu_\sigma B^\sigma \tag{C.29}$$

$$= 2\dot{\phi}^\mu + g^{\mu\sigma}\dot{B}_\sigma \tag{C.30}$$

wherein $\dot{g}_{\mu\nu} = \mathcal{K}_{\mu\nu}$ has been used on the last line, using the result on the acceleration field

$$\hat{a}^\mu = h^{\mu\sigma}\dot{u}_\sigma - D^\mu\phi \tag{C.31}$$

Then the covector field will be

$$\hat{a}_\mu = \dot{u}_\mu - D_\mu\phi \tag{C.32}$$

as we may see, if we assume geodesic observers on the physical manifold,i.e., $\hat{a}_\mu = 0$, we will have the following choice of the spatial gauge $\dot{u}_\mu = D_\mu\phi$.

C.2 Ricci Tensor perturbations

One may define the curvature tensor in the perturbed manifold by

$$[\widehat{\nabla}_\mu, \widehat{\nabla}_\nu]v_\alpha = \widehat{R}_{\mu\nu\alpha}{}^\beta v_\beta \tag{C.33}$$

from (C.1) we have

$$\widehat{R}_{\mu\nu\alpha}{}^\beta = R_{\mu\nu\alpha}{}^\beta + 2\nabla_{[\mu}\Gamma_{\nu]\alpha}{}^\beta + 2\Gamma_{\alpha[\mu}{}^\lambda\Gamma_{\nu]\lambda}{}^\beta \tag{C.34}$$

[1] since v_μ is not necessarily geodesic $\hat{a}^\mu \neq 0$.

extracting the Ricci tensor on first order:

$$\widehat{R}_{\mu\alpha} = R_{\mu\alpha} + \nabla_\mu \Gamma_{\nu\alpha}{}^{\nu} - \nabla_\nu \Gamma_{\mu\alpha}{}^{\nu} \tag{C.35}$$

$$\delta R_{\mu\alpha} = \nabla_\mu \Gamma_{\nu\alpha}{}^{\nu} - \nabla_\nu \Gamma_{\mu\alpha}{}^{\nu}. \tag{C.36}$$

Furthermore, to get the components of (C.36), i.e., δR_{nn}, $h[\delta R_{\mu n}]$, and $h[\delta R_\mu^{\ \nu}]$ one need to calculate the projections of the connection $\Gamma_{ab}{}^{c}$, look the reference [18] for the details on the calculations, here we will put the results for a general geometry

The time-time component will be

$$\delta R_{nn} = -(\ddot{C} + \theta\dot{\phi} + D^\nu \dot{B}_\nu + D^2\phi + 2\mathcal{K}^{\mu\nu}\nabla_n C_{\mu\nu}) \tag{C.37}$$

For the spatial-time component, we have

$$\begin{aligned} h[\delta R_{\alpha n}] =& - D_\alpha \dot{C} + \mathcal{K}_\alpha^{\ \gamma} D_\gamma C + h_{\sigma\alpha} D^\nu \dot{C}_\nu^{\ \sigma} + \mathcal{K}^{\mu\gamma} D_\alpha C_{\gamma\mu} - D^\nu D_{[\alpha} B_{\nu]} - \\ & - D_\gamma(\theta + \nabla_n)\mathcal{K}_\alpha^{\ \gamma} + \mathcal{K}_\alpha^{\ \gamma} D_\gamma \phi - \theta D_\alpha \phi \end{aligned} \tag{C.38}$$

Finally, the spatial-spatial component:

$$\begin{aligned} h[\delta R_\mu^{\ \nu}] =& (D_\mu D^\nu - \mathcal{K}_\mu^{\ \nu}\partial_t)(\phi - C) + D_\gamma D_\mu C^{\nu\gamma} + D_\gamma D^\nu C_\mu^{\ \gamma} - D^2 C_\mu^{\ \nu} - \\ & - 2D_\gamma(\mathcal{K}_\mu^{\ [\gamma} B^{\nu]}) + \mathcal{K}_\gamma^{\ \nu} D_\mu B^\gamma + 2(\theta + \nabla_n)(\phi \mathcal{K}_\mu^{\ \nu}) + \\ & + (\theta + \partial_t)(D_{(\mu} B_{\alpha)} h^{\alpha\nu}) + (\theta + \partial_t)\dot{C}_\mu^{\ \nu} + \\ & + 2(\theta + \partial_t)(C_\mu^{\ \gamma}\mathcal{K}_\gamma^{\ \nu} - \mathcal{K}_\mu^{\ \gamma} C_\gamma^{\ \nu}) + B^\nu D_\mu \theta - 2\mathcal{R}_\mu^{\ \gamma} C_\gamma^{\ \nu}. \end{aligned} \tag{C.39}$$

D Results of Kinetic Theory

D.1 Proofs of theorems

D.1.1 Volume form on the mass shell

In any relativistic system, its moment's coordinate need to satisfy the mass shell (4.3), which allows us to induce a volume form on this constraint. Following [27], on a local frame one may write

$$\omega_{m,q} \wedge d\left(\frac{-q_\alpha q^\alpha + m^2}{2}\right) = \omega_q \tag{D.1}$$

writing the volume form on an orthonormal frame: $\omega_q = \epsilon_{\mu\nu\alpha\beta}\, dq^\mu \wedge dq^\nu \wedge dq^\alpha \wedge dq^\beta = \underbrace{\sqrt{-det(\eta_{\mu\nu})}}_{1}\, dE \wedge dq^1 \wedge dq^2 \wedge dq^3$. Using the constraint of the mass shell, one may write $q^0 = E(q^i)$, then

$$\omega_{m,q} \wedge d\left(\frac{-q_\alpha q^\alpha + m^2}{2}\right) = \omega_{m,q} \wedge \left(E\, dE - q_i dq^i\right) = E\, \omega_{m,q} \wedge dE \tag{D.2}$$

thus from (D.1)

$$E\, \omega_{m,q} \wedge dE = dE \wedge dq^1 \wedge dq^2 \wedge dq^3 \tag{D.3}$$

$$\omega_{m,q} = \frac{dq^1 \wedge dq^2 \wedge dq^3}{E} \tag{D.4}$$

D.1.2 Liouville's theorem

Lets calculate in a local frame (x^α, q^α), and vetor field $X^A = (q^\alpha, Q^\alpha)$ First note that $\omega_{\mu\nu\beta\sigma} = \omega_{0\ 1,\dots,7}$ do not depend on q, then

$$\mathcal{L}_X \omega_{0\ 1,\dots,7} = q^\alpha \frac{\partial}{\partial x^\alpha} \omega_{01,\dots,7} + \omega_{A\ 1,\dots,7} \frac{\partial X^A}{\partial x^0} + \dots + \omega_{0\ 1,\dots,A} \frac{\partial X^A}{\partial x^7} \tag{D.5}$$

Using the property $\omega^{abcd}\omega_{abcd} = n!$ [8], the first term will be

$$q^\alpha \frac{\partial}{\partial x^\alpha} \omega_{01,\dots,7} = q^\alpha g^{\lambda\gamma} \left(\frac{\partial g_{\lambda\gamma}}{\partial x^\alpha}\right) \omega_{0\ 1,\dots,7}. \tag{D.6}$$

from the action of a covariant derivative on a metric tensor, we have

$$g^{\lambda\gamma} \left(\frac{\partial g_{\lambda\gamma}}{\partial x^\alpha}\right) = 2\Gamma^\sigma{}_{\sigma\alpha} \tag{D.7}$$

since ω is antissymetric it can not repeat the index, thus the seven terms of (D.5) becomes

$$\omega_{0\ 1,\ldots,7}\frac{\partial X^A}{\partial x^A} = \omega_{0\ 1,\ldots,7}\frac{\partial Q^\alpha}{\partial q^\alpha} \tag{D.8}$$

whereas q^α is independently of x^α, from the geodesic equation

$$\frac{\partial Q^\alpha}{\partial q^\alpha} = \frac{\Gamma^\alpha{}_{\mu\nu}}{q^\mu q^\nu} = -2\Gamma^\alpha{}_{\alpha\mu}q^\mu \tag{D.9}$$

combining the terms the lie derivative with respect at X^A of the volume form ω is

$$\mathcal{L}_X\omega_{0\ 1,\ldots,7} = \left(2\Gamma^\sigma{}_{\sigma\mu} - 2\Gamma^\alpha{}_{\alpha\mu}\right)q^\mu\omega_{0\ 1,\ldots,7} = 0 \tag{D.10}$$

D.1.3 H-theorem

Calculating the divergence of the entropy flux

$$\nabla_\mu S^\mu = -k_b\int_{P_x}\nabla_\mu\left(q^\mu f\ln f\omega_q\right) \tag{D.11}$$

using a spatial orthonormal frame $\{e^a{}_\mu\}$, $\nabla_\alpha \to \partial_\alpha$ when acting in general tensor, then

$$\partial_\mu S^\mu = -k_b\left(\int_{P_x} q^\mu\partial_\mu(f\ln f)\omega_q + q^\mu f\ln f(\nabla_\mu\omega_q)\right) \tag{D.12}$$

$$= -k_b\int_{P_x} q^\mu\partial_\mu f(\ln f + 1)\omega_q \tag{D.13}$$

$$= -k_b\int_{P_x}\mathcal{L}_X f(\ln f + 1)\omega_q \tag{D.14}$$

$$= -k_b\int_{P_x} C[f](1 + \ln f)\omega_q \tag{D.15}$$

$$= -k_b\int_{P_x} C[f]\ln f\omega_q \tag{D.16}$$

where on the first line line, $\partial_\mu\epsilon_{abcd} = 0$, on the third line: $\mathcal{L}_X f = q^\mu\partial_\mu f$, since $Q^\mu = 0$ in this frame, and on the fourth line:

$$\int C[f]\omega_q = \tag{D.17}$$

$$= \int\underbrace{[f(x,q')f(x,k') - f(x,q)f(x,k)]}_{F(x,q,q',k,k')} A(x,q',k',q,k)\ d^4k'd^4q'd^4k\ d^4q = 0 \tag{D.18}$$

since $F(x,q,k,q',k') \to -F(x,q',k',q,k)$, and $A(x,q,k,q',k') \to A(x,q',k',q,k)$, thus we have an odd integrand being integrated into all the fiber. Therefore

$$\partial_\mu S^\mu = -k_b\int[f(x,q')f(x,k') - f(x,q)f(x,k)]\ln f(x,q)\ A(x,q',k',q,k)\ d^4k'd^4q'd^4k\ d^4q \tag{D.19}$$

$$= -\frac{k_b}{4}\int[f(q')f(k') - f(q)f(k)]\ln\left(\frac{f(q)f(k)}{f(q')f(k')}\right)A(x,q',k',q,k)\ d^4k'd^4q'd^4k\ d^4q \geq 0 \tag{D.20}$$

wherein on the second line we used the reversibility property of $A(x,q,k,q',k')$, the inequality above is always valid if $A > 0$, since $[f(x)f(y) - f(z)f(r)] \ln\left(\frac{f(z)f(r)}{f(x)f(y)}\right) \leq 0, \ \forall f \in \mathbb{R}$.

D.2 Boltzmann equation for photons and baryons

This section will present the way to get the linear Boltzmann equation following [10], let us expand the distribution function of the photon and substituting in (4.62), then

$$f(\eta, x, \epsilon, \hat{q}) = \exp\left[\frac{\epsilon}{aT_0(1+\Theta(\eta, x, \hat{q}))} - 1\right]^{-1} \tag{D.21}$$

where $\epsilon = E/a$, is the comoving energy, and $\Theta = \delta T/T$ is the temperature fluctuation, expanding this distribution in linear order

$$f(\eta, x, \epsilon, \hat{q}) \approx \bar{f}(\epsilon) - \Theta(\eta, x, \hat{q})\frac{d\bar{f}}{dln\epsilon} \tag{D.22}$$

substituting this expression on the left side of the Boltzmann equation, we get

$$\frac{d\Theta}{d\eta} = \frac{dln\epsilon}{d\eta} \tag{D.23}$$

$$= \frac{d\Psi}{d\eta} + \Phi' + \Psi' \tag{D.24}$$

in which in the second line we used the geodesic equation with a perturbative metric, and $'$ represents the partial derivative with the relation of the conformal time

As for the right side, a way to deduce the collision term is by assuming a Compton scattering, with a cross-section calculated by QFT considerations.

To define the quantities we need a frame of reference, let us use the background frame. We need to compare the collision in the electron frame in order to get the correct expression which is given by [10]

$$C[f(\epsilon, \hat{q})] = \frac{d\bar{f}}{dln\epsilon}\Gamma[\Theta(\hat{q}) - \Theta_0 - q \cdot v_e], \tag{D.25}$$

wherein the $\Gamma = aN_e\sigma_T$ is the scattering rate, and the monopole of the fluctuation is defined as

$$\Theta_0 = \int d\hat{q}_{in}\, \frac{1}{4\pi}\Theta(\hat{q}_{in}) \tag{D.26}$$

The monopole is the average of the fluctuation around the k space, we will see that it is proportional to the photon energy fluctuation.

Therefore the complete linear Boltzmann equation is

$$\frac{d\Theta}{d\eta} = \frac{dln\epsilon}{d\eta} - \Gamma[\Theta(\hat{q}) - \Theta_0 - \mathbf{q}\cdot\mathbf{v}_e] \tag{D.27}$$

We will see next when the scattering rate is large: $\Gamma \gg \mathcal{H}$, $\Theta \rightarrow \Theta_0 + \hat{q}\cdot v_b$ in the background frame, then the parenthesis on the right-hand side vanishes, leading to the thermalization of the photon gas, such that the equilibrium is reached, and the fluctuations will change only because the gravitational potentials.

writing the equation (D.27) in Fourier space we get

$$\Theta' + ik\mu\Theta = \Phi' - ik\mu\Psi - \Gamma[\Theta - \Theta_0 - i\mu v_b] \tag{D.28}$$

in which $\mathbf{v}_b = ikv_b$, and $\mu \equiv \hat{\mathbf{k}}\cdot\hat{\mathbf{q}}$

Expanding the fluctuation in Legendre polynomials in Fourier space

$$\Theta(\eta,\mathbf{k},\mu) = \sum_{l=0}^{\infty}(-i)^l\,(2l+1)\Theta_l(\eta,\mathbf{k})\mathcal{P}_l(\mu). \tag{D.29}$$

we may rewrite the Boltzmann equation in terms of this expansion inside the Hubble horizon [41]

$$\frac{d\Theta}{dt} = \Theta_0 - \frac{1}{2}\left(\Theta_2 - \Theta_0^P + \Theta_2^P\right)P_2(\hat{n}^i\hat{k}_i) - \Theta(\hat{n}^i\hat{k}_i) - ivP_1(\hat{n}^i\hat{k}_i) \tag{D.30}$$

where we neglect gravitational potentials since they are small compared with the fluctuations, and we add anisotropic Compton scattering terms, for more detail on how this term appears see [10] With this expansion, we can obtain the two point-function in the Fourier space of this fluctuation given by [10]

$$\langle\Theta(\hat{n})\Theta(\hat{n}')\rangle = \sum_l (2l+1)\left(\int d\ln k\, P_i^2(k)\Theta_l^2(k)\right)P_l(\hat{n}\cdot\hat{n}') \tag{D.31}$$

To relate the matter quantities with the temperature fluctuations, we use the definition of the energy-momentum tensor with the multipole expansion

$$T^{\mu}_{\ \nu} = \int \frac{d^3q}{E(q)} f q^{\mu} q_{\nu} \tag{D.32}$$

expand the distribution in linear order, and doing some algebraic manipulation one obtains

$$\delta_\gamma = 4\Theta_0, \tag{D.33}$$

similarly

$$v_\gamma = -3\Theta_1 \tag{D.34}$$

then the velocity of the photons is associated with the dipole term of the fluctuation. Finally

$$\Pi_\gamma = 3\Theta_2. \tag{D.35}$$

Showing that the quadrupole of the temperature fluctuations is associated with the anisotropic stress of the photons

Another special feature of the expansion made before is the Boltzmann hierarchy, substituting (D.29)in (D.28) we obtain

$$\Theta_l' + \frac{k}{2l+1}\left[(l+1)\Theta_{l+1} - l\Theta_{l-1}\right] = \delta_{l0}\Phi' + \delta_{l1}\frac{k\Psi}{3}\Gamma\left[(1-\delta_{l0})\Theta_l + \delta_{l1}v_b/3 - \delta_{l2}\Theta_2/10\right] \tag{D.36}$$

when writing in this form is directly to apply the tight coupling approximation, observing the equation for $l > 2$ for $\Gamma \gg k$ we have

$$\Theta_l \approx \frac{k}{\Gamma}\Theta_{l-1} \ll \Theta_{l-1} \tag{D.37}$$

which shows that all higher moments are indeed small in the tight-coupling regime. The dominant multipole moments are therefore the monopole and the dipole, writing the equations for $l = 0$ and $l = 1$

$$\Theta_0' = -k\Theta_1 + \Phi' \tag{D.38}$$

$$\Theta_1' = -\frac{k}{3}(2\Theta_2 - \Theta_0) + \frac{k}{3}\Psi - \Gamma\left(\Theta_1 + \frac{v_b}{3}\right) \tag{D.39}$$

using the previous relations

$$\delta_\gamma' = \frac{4}{3}kv_\gamma + 4\Phi' \tag{D.40}$$

$$v_\gamma' + \frac{1}{4}k\delta_\gamma - \frac{2}{3}k\Pi_\gamma + k\Psi = -\Gamma(v_\gamma - v_b) \tag{D.41}$$

In next-to-leading order ($l = 2$), it must be taken into account of the quadrupole moment which will lead to a damping of the fluctuations, the Boltzmann equation for it is

$$\Theta_2 \approx -\frac{4k}{9\Gamma}\Theta_1 \tag{D.42}$$

in which if we use the relation with the anisotropic stress we get

$$\Pi_\gamma \approx -\frac{4}{9}\frac{k}{\Gamma}v_\gamma \tag{D.43}$$

indicating when $\Gamma \gg k$, then $\Pi_\gamma \to 0$.

Lastly one may get the baryon evolution equation for v_b, applying momentum conservation to the coupled photon–baryon system:

$$q_i' + 4\mathcal{H}q_i = -D_i\delta P - D_j\Pi^j{}_i - (\bar\rho + \bar P)D_i\Psi \tag{D.44}$$

in Fourier space we have

$$q_i = ik_i\left(\frac{4}{3}\bar{\rho}_\gamma v_\gamma + \bar{\rho}_b v_b\right) \tag{D.45}$$

$$\delta P \approx \frac{1}{3}\bar{\rho}_\gamma \delta_\gamma \tag{D.46}$$

$$\Pi_{ij} = -\frac{4}{3}\bar{\rho}_\gamma \hat{k}_i \hat{k}_j \Pi_\gamma \tag{D.47}$$

where $P_b \approx 0$, using (D.41) we get

$$v_b' + \mathcal{H} v_b + k\Psi = \frac{\Gamma}{R} v_{\gamma - b} \tag{D.48}$$

with $R \equiv \dfrac{3\bar{\rho}_b}{4\bar{\rho}_\gamma}$.

D.3 Statistical moments of photons and baryons

Using the Planckian distribution function $f_{PL} = 1/e^x - 1$, we have

$$\rho_\gamma^{PL} = \frac{8\pi k_b^4}{15c^3h^3} T^4 \int dx\, \frac{x^3}{e^x - 1} \tag{D.49}$$

$$= \frac{8\pi^5 k_b^4}{15c^3h^3} T^4 \tag{D.50}$$

$$= a_R \theta_\gamma^4 \tag{D.51}$$

with $a_R = 0.26$ eVcm^{-3}, the number density is

$$N_\gamma^{PL} = \frac{8\pi k_b^3 T^3}{c^3h^3} \int \frac{x^2}{e^x - 1}\, dx \tag{D.52}$$

$$= b_R T^3 \tag{D.53}$$

with $b_R = 410$ cm^{-3}

For the case of small chemical potential in a distribution $f = 1/(e^{x+\mu(x)} - 1)$, we have $f(x,t) \approx f_{PL} + \mu\, \partial_x f_{PL} = f - [f(f+1)]\, \mu$ The photon energy density and the photon distribution for a small chemical potential are given by $\rho_\gamma = \rho_{PL} f_\mu$ and $N_\gamma = N_{PL}\phi_\mu$, where

$$f_\mu \equiv 1 - \frac{3I_2}{I_3}\left(\frac{I_2^\mu}{I_2} - \frac{I_3^{\mu'}}{3I_2}\right) \tag{D.54}$$

$$\phi_\mu \equiv 1 - \frac{2I_1}{I_2}\left(\frac{I_1^\mu}{I_1} - \frac{I_1^{\mu'}}{2I_2}\right) \tag{D.55}$$

in which $I_i^\mu = \int_0^\infty \mu\, x^i f_{PL}\, dx$, here $\mu' = \partial_x \mu$. If $\mu = \mu_0 = cte$, we have $I_i^\mu = I_i\, \mu$ and $I_i^{\mu'} = 0$.

To describe the non-relativistic matter we used a Maxwell-Boltzmann distribution, given by

$$f_i(q) = \frac{N_i \exp\left[-\bar{q}^2/(2\theta_i)\right]}{\left(2\pi m_i^6 \theta_i\right)^{3/2}} \tag{D.56}$$

where $\bar{q} = q/(m_i\, c)$ using this distribution and the definitions (4.28), and (4.29), the energy density and the pressure for the baryons are

$$\rho_i = m_i c^2 N_i F(\theta_i) \tag{D.57}$$

$$p = N_i k_b T_i \tag{D.58}$$

in which

$$F(\theta_i) \equiv \left(1 + \frac{3k}{2}\theta_i\right) \tag{D.59}$$

The specific heat capacity at constant volume may be defined as

$$c_{i,V} = C_{i,V}/N_i = \frac{1}{N_i}\left(\frac{d\rho_i}{dT_i}\right)_V \tag{D.60}$$

$$c_{i,V} = k_b \frac{dF(\theta_i)}{d\theta_i} = \frac{3k_b}{2} \tag{D.61}$$

where (D.57) has been used in the second line. This is the heat capacity of an ideal gas as expected.

The electron, hydrogen, and helium number densities can be related to the baryon number densities N_b by [37]

$$N_e = \left(1 - Y_p/2\right)\chi_e N_b \tag{D.62}$$

$$N_H = \left(1 - Y_p\right) N_b \tag{D.63}$$

$$N_{He} = \left(Y_p/4\right) N_b \tag{D.64}$$

in which χ_e is the ionization fraction of the hydrogen and helium atoms. At high redshifts ($z \gtrsim 10^4$) the ionization fraction is $\chi_e = 1$. The factor $Y_p = 0.2485$ is the primordial mass fraction of helium

References

[1] CHLUBA, J. Future steps in cosmology using spectral distortions of the cosmic microwave background. *Proc. Int. Sch. Phys. Fermi*, v. 200, n. arXiv: 1806.02915, p. 265–309, 2020.

[2] SILK, J. Cosmic black-body radiation and galaxy formation. *Astrophysical Journal, vol. 151, p. 459*, v. 151, p. 459, 1968.

[3] BARROW, J. D.; COLES, P. Primordial density fluctuations and the microwave background spectrum. *Monthly Notices of the Royal Astronomical Society*, The Royal Astronomical Society, v. 248, n. 1, p. 52–57, 1991.

[4] CHLUBA, J.; SUNYAEV, R. The evolution of cmb spectral distortions in the early universe. *Monthly Notices of the Royal Astronomical Society*, Blackwell Publishing Ltd Oxford, UK, v. 419, n. 2, p. 1294–1314, 2012.

[5] KOGUT, A. et al. The primordial inflation explorer (pixie): a nulling polarimeter for cosmic microwave background observations. *Journal of Cosmology and Astroparticle Physics*, IOP Publishing, v. 2011, n. 07, p. 025, 2011.

[6] ANDRÉ, P. et al. Prism (polarized radiation imaging and spectroscopy mission): an extended white paper. *Journal of Cosmology and Astroparticle Physics*, IOP Publishing, v. 2014, n. 02, p. 006, 2014.

[7] PINTO-NETO, N. Bouncing quantum cosmology. *Universe*, MDPI, v. 7, n. 4, p. 110, 2021.

[8] WALD, R. M. *General relativity*. [S.l.]: University of Chicago press, 2010.

[9] BAMBA, K. et al. Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests. *Astrophysics and Space Science*, Springer, v. 342, p. 155–228, 2012.

[10] BAUMANN, D. *Cosmology*. [S.l.]: Cambridge University Press, 2022.

[11] PERLMUTTER, S. et al. Discovery of a supernova explosion at half the age of the universe. *Nature*, Nature Publishing Group UK London, v. 391, n. 6662, p. 51–54, 1998.

[12] AGHANIM, N. et al. Planck 2018 results-vi. cosmological parameters. *Astronomy & Astrophysics*, EDP sciences, v. 641, p. A6, 2020.

[13] ZELDOVICH, Y. B. et al. Recombination of hydrogen in the hot model of the universe. *Sov. Phys. JETP*, v. 28, p. 146, 1969.

[14] DODELSON, S.; SCHMIDT, F. *Modern cosmology*. [S.l.]: Academic press, 2020.

[15] NOVELLO, M.; BERGLIAFFA, S. P. Bouncing cosmologies. *Physics reports*, Elsevier, v. 463, n. 4, p. 127–213, 2008.

[16] BATTEFELD, D.; PETER, P. A critical review of classical bouncing cosmologies. *Physics Reports*, Elsevier, v. 571, p. 1–66, 2015.

[17] MUKHANOV, V. F. *Physical foundations of cosmology*. [S.l.]: Cambridge university press, 2005.

[18] VITENTI, S. D. P. *Estudo das perturbações em universos com ricochete*. Tese (Doutorado) — Ph. D. thesis, Centro Brasileiro de Pesquisas Fısicas, Rio de Janeiro, Brazil, 2011.

[19] VITENTI, S. D. P.; PINTO-NETO, N. Large adiabatic scalar perturbations in a regular bouncing universe. *Physical Review D*, APS, v. 85, n. 2, p. 023524, 2012.

[20] STEWART, J. M.; WALKER, M. Perturbations of space-times in general relativity. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, The Royal Society London, v. 341, n. 1624, p. 49–74, 1974.

[21] VITENTI, S.; FALCIANO, F.; PINTO-NETO, N. Covariant bardeen perturbation formalism. *Physical Review D*, APS, v. 89, n. 10, p. 103538, 2014.

[22] GIESEL, K. et al. Manifestly gauge-invariant general relativistic perturbation theory: I. foundations. *Classical and quantum gravity*, IOP Publishing, v. 27, n. 5, p. 055005, 2010.

[23] BARDEEN, J. M. Gauge-invariant cosmological perturbations. *Physical Review D*, APS, v. 22, n. 8, p. 1882, 1980.

[24] CHAVEL, I. *Eigenvalues in Riemannian geometry*. [S.l.]: Academic press, 1984.

[25] SYNGE, J. L. The energy tensor of a continuous medium. In: ROYAL SOCIETY OF CANADA. [S.l.], 1934.

[26] SARBACH, O.; ZANNIAS, T. Relativistic kinetic theory: An introduction. In: AMERICAN INSTITUTE OF PHYSICS. *AIP Conference Proceedings*. [S.l.], 2013. v. 1548, n. 1, p. 134–155.

[27] CHOQUET-BRUHAT, Y. *General relativity and the Einstein equations*. [S.l.]: OUP Oxford, 2008.

[28] BERNSTEIN, J. *Kinetic Theory in the Expanding Universe*. [S.l.]: Cambridge University Press, 1988. (Cambridge Monographs on Mathematical Physics).

[29] CHLUBA, J. et al. Spectral distortions of the cmb as a probe of inflation, recombination, structure formation and particle physics. *arXiv preprint arXiv:1903.04218*, 2019.

[30] SUNYAEV, R.; ZELDOVICH, Y. B. The interaction of matter and radiation in the hot model of the universe, ii. *Astrophysics and Space Science*, Springer, v. 7, p. 20–30, 1970.

[31] CHLUBA, J.; ERICKCEK, A. L.; BEN-DAYAN, I. Probing the inflaton: Small-scale power spectrum constraints from measurements of the cosmic microwave background energy spectrum. *The Astrophysical Journal*, IOP Publishing, v. 758, n. 2, p. 76, 2012.

[32] HU, W.; SILK, J. Thermalization and spectral distortions of the cosmic background radiation. *Physical Review D*, APS, v. 48, n. 2, p. 485, 1993.

[33] MATHER, J. C. et al. Measurement of the cosmic microwave background spectrum by the cobe firas instrument. *The Astrophysical Journal, Part 1 (ISSN 0004-637X), vol. 420, no. 2, p. 439-444*, v. 420, p. 439–444, 1994.

[34] CHLUBA, J.; KHATRI, R.; SUNYAEV, R. A. Cmb at 2× 2 order: the dissipation of primordial acoustic waves and the observable part of the associated energy release. *Monthly Notices of the Royal Astronomical Society*, Blackwell Science Ltd Oxford, UK, v. 425, n. 2, p. 1129–1169, 2012.

[35] HU, W. T. *Wandering in the background: A Cosmic microwave background explorer*. Tese (Doutorado) — University of California, Berkeley, 1995.

[36] KOMPANEETS, A. The establishment of thermal equilibrium between quanta and electrons. *Zh. Eksp. Teor. Fiz*, v. 31, n. 5, p. 11, 1956.

[37] CHLUBA, J. *Spectral distortions of the cosmic microwave background*. Tese (Doutorado) — lmu, 2005.

[38] RYBICKI, G. B.; LIGHTMAN, A. P. *Radiative processes in astrophysics*. [S.l.]: John Wiley & Sons, 1991.

[39] BURIGANA, C.; DANESE, L.; ZOTTI, G. D. Formation and evolution of early distortions of the microwave background spectrum-a numerical study. *Astronomy and Astrophysics (ISSN 0004-6361), vol. 246, no. 1, June 1991, p. 49-58. Research supported by MPI, CNR, and ASI.*, v. 246, p. 49–58, 1991.

[40] CYBURT, R. H. Primordial nucleosynthesis for the new cosmology: Determining uncertainties and examining concordance. *Physical Review D*, APS, v. 70, n. 2, p. 023505, 2004.

[41] KHATRI, R.; SUNYAEV, R. A.; CHLUBA, J. Mixing of blackbodies: entropy production and dissipation of sound waves in the early universe. *Astronomy & Astrophysics*, EDP Sciences, v. 543, p. A136, 2012.

[42] ZELDOVICH, Y. B.; SUNYAEV, R. The interaction of matter and radiation in a hot-model universe. *Astrophysics and Space Science*, Springer, v. 4, p. 301–316, 1969.

[43] LIFSHITZ, E. M. On the gravitational stability of the expanding universe. *J. Phys.*, v. 10, p. 116–129, 1946.

[44] ZALDARRIAGA, M.; HARARI, D. D. Analytic approach to the polarization of the cosmic microwave background in flat and open universes. *Physical Review D*, APS, v. 52, n. 6, p. 3276, 1995.

[45] CHLUBA, J.; GRIN, D. Cmb spectral distortions from small-scale isocurvature fluctuations. *Monthly Notices of the Royal Astronomical Society*, The Royal Astronomical Society, v. 434, n. 2, p. 1619–1635, 2013.

[46] HU, W.; SUGIYAMA, N. Small-scale cosmological perturbations: an analytic approach. *The Astrophysical Journal*, IOP Publishing, v. 471, n. 2, p. 542, 1996.

[47] MUKHANOV, V. F. Quantum theory of gauge-invariant cosmological perturbations. *Zh. Eksp. Teor. Fiz*, v. 94, n. 1, 1988.

[48] KOMATSU, E. et al. Five-year wilkinson microwave anisotropy probe* observations: cosmological interpretation. *The Astrophysical Journal Supplement Series*, IOP Publishing, v. 180, n. 2, p. 330, 2009.

[49] ISHAM, C. *Modern Differential Geometry for Physicists*. World Scientific, 1999. Disponível em: <https://books.google.com.br/books?id=W3lsrjUZen8C>.

[50] NAKAHARA, M. *Geometry, Topology and Physics, Second Edition*. Taylor & Francis, 2003. Disponível em: <https://books.google.com.br/books?id=cH-XQB0Ex5wC>.