Universidade Estadual de Londrina

VICTOR HUGO MARQUES RAMOS

DUALITIES IN QUANTUM FIELD THEORY

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Dissertação apresentada ao Programa de Mestrado em Física da Universidade Estadual de Londrina para obtenção do título de Mestre em Física.

Orientador: Prof. Dr. Carlos André Hernaski

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RESUMO

Nesta dissertação estudamos o modelo conforme do bóson compactificado $c = 1$, em particular investigamos uma maneira de descrever dinamicamente as configurações topológicas de enroamento que este modelo apresenta. Aplicamos, então, a mesma metodologia para um modelo exótico chamado $XY-$plaquette. Para isso, começamos revisando alguns aspectos de relevância em teoria conforme. Discutimos a simetria conforme para teorias de dimensão arbitrária e focamos em fazer uma apresentação detalhada do caso bidimensional, em que o bóson $c = 1$ se apresenta. Mostramos em seguida que este modelo apresenta configurações de enrolamento e construímos explicitamente o operador quântico que excita estas configurações a partir do vácuo. Reconhecendo que o modelo XY na rede captura os elementos fundamentos do bóson compactificado, fazemos uma descrição das características desse modelo na rede e mostramos que configurações de enrolamento também estão presentes, sendo vórtices na rede. Então, a partir deste modelo na rede, obtemos uma teoria no contínuo que descreve dinamicamente as excitações do tipo vórtice, chamada teoria de Sine-Gordon. Por fim, utilizamos o método do grupo de renormalização para estudar o running das constantes de acopla-mento e encontramos que este modelo sofre uma transição de fase topológica.

Palavras-chave: Teoria Conforme de Campos. Bóson Compactificado $c = 1$. Transições de Fase Topológicas

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ABSTRACT

In this dissertation we study the conformal model of the compactified boson $c = 1$, in particular we investigate a way to dynamically describe the topological configurations of winding that this model presents. We then apply the same methodology to an exotic model called $XY-$plaquette. For this purpose, we start by reviewing some aspects of relevance in conformal field theory. We discuss conformal symmetry for arbitrary-dimensional theories and focus on making a detailed presentation of the two-dimensional case, in which the boson $c = 1$ presents itself. We then show that this model presents winding configurations and we explicitly build the quantum operator that excites these configurations from the vacuum. Recognizing that the XY model in the lattice captures the fundamental elements of the compactified boson, we describe the characteristics of this model in the lattice and show that winding configurations are also present, being vortexes on the lattice. Then, from this model in the lattice, we obtain a theory in the continuum that dynamically describes the vortex-like excitations, called Sine-Gordon theory. Finally, we use the renormalization group method to study the running of the coupling constants and conclude that this model undergoes a topological phase transition.

Keywords: Conformal Field Theory. $c = 1$ Compactified Boson. Topological Phase Transitions.

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Chapter 1

Introduction

We start by studying the role of symmetry in general classical and quantum field theories, exploring how it can be implemented in both regimes and what are its consequences. The upshot of these discussions are Noether's theorem and Ward identities. We are particularly interested in the conformal symmetry in 2 dimensions. However, before studying the subtleties of this particular case, essential general concepts need to be introduced. Next, we define the conformal symmetry and determine its associated algebra in a classical field theory. Next, show how its structure fixes the two and three-point correlation functions up to a constant. Also, we derive the general Ward identities for the conformal symmetry. We then study the two-dimensional classical case. A variety of features appear with very interesting results. For example, in two dimensions, the definition of the conformal group leads to the Cauchy-Riemann condition for holomorphic functions, conducting us to the concept of primary fields, which are one of the building blocks of a conformal field theory in two dimensions. Symmetry implications for prima-ry fields are considered in the quantum theory, and in order to create a systematic procedure to decide whether a field is primary or not, the operator product expansion is established together with a couple of useful examples.

In the second chapter we apply the tools developed in the first chapter to a concrete physical system, and we shall consider at first the free boson on the cylinder. One reason to study this specific theory is that the $c = 1$ compactified boson we are interested to describe consists of a free boson whose base space is the cylinder and the target space is the circle. So the free boson on the cylinder, whose target space is the real line, is an interesting intermediate step towards the full compactified boson. By considering the former, we will notice that topological configurations only appear when both base and target space have compactified directions, which seems to be a general feature of systems alike. But before heading to this problem, a general discussion about the canonical formalism in a conformal field theory is presented. That is, we might analyse operators on the Hilbert space of a conformal field theory, considering the usual canon of quantum field theory: implement a quantization procedure and define asymptotic states, determine mode expansion of quantum fields and discuss the algebra and spectrum of pertinent operators.

The first section of the fourth chapter has the goal to find the mode expansion of the compactified boson using some facts about homotopy groups and our previous results about the free boson on the cylinder. We will also derive creation and annihilation operators for winding quantum states. The concept of duality will be shortly discussed, and we will see that the original compactified boson is dual to another boson.

Finally, in the last chapter we investigate how one can dynamically describe configurations that present singularities. We are going to consider a lattice model that captures the elementary properties of the compactified boson we are dealing. Then, a continuum limit will be discussed in order to obtain a quantum field theory. A renormalization group analysis is performed in the continuum theory and the topological Kosterlitz-Thouless transition is discussed. Furthermore, we discuss some properties of the exotic XY-plaquette model and show that it is possible to apply the same prescription in order to find a continuum prescription that describes vortices.

CHAPTER 2

Conformal Symmetry

2.1 Symmetries

Consider a field theory described by a set of fields $\{\Phi_a\}$ defined over a $D-$dimensional space-time manifold $\mathcal{M}$. The dynamics is dictated by the action

$$S[\Phi_a] = \int d^D x \; \mathcal{L}\left(\Phi_a(x), \partial_\mu \Phi_a(x)\right), \tag{2.1}$$

where the Lagrangian density is supposed to be a function of the fields and its first derivatives. One can consider higher order derivatives, but this choice leads to quantum theories with negative norm modes, called ghosts, which we will not discuss.

In field theories described by the means of (2.1), a symmetry transformation is defined as a map that leaves the action invariant. Then, to investigate symmetry, it is sufficient to know how the action changes for a general transformation of the fields and coordinates of space-time, such as

$$x \longmapsto x'(x) \tag{2.2}$$

$$\Phi_a(x) \longmapsto \Phi'_a(x') = \mathcal{F}_a\left(\Phi_b(x)\right) \tag{2.3}$$

Under an integral change of variables, we verify that

$$S\left[\Phi'_a\right] = \int d^D x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}_a\left(\Phi_b\left(x\right)\right), \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu \mathcal{F}_a\left(\Phi_b\left(x\right)\right)\right). \tag{2.4}$$

where $|\frac{\partial x'}{\partial x}|$ is the Jacobian of the transformation.

As an example, consider the scale transformation

$$x \longmapsto x'(x) = \lambda x, \tag{2.5}$$

$$\Phi_a(x) \longmapsto \Phi'_a(x') = \lambda^{-\Delta_a}\Phi_a(x), \tag{2.6}$$

with λ the dilatation factor and Δ_a the scale dimension of the field $\Phi_a(x)$. The coordinate transformation implies that

$$\left|\frac{\partial x'}{\partial x}\right| = \lambda^D, \quad \frac{\partial x^\nu}{\partial x'^\mu} = \lambda^{-1}\delta^\nu_\mu, \tag{2.7}$$

and therefore, the action is changed to

$$S\left[\Phi_a'\right] = \int d^D x\, \lambda^D \mathcal{L}\left(\lambda^{-\Delta}\Phi_a\left(x\right), \lambda^{-\Delta-1}\partial_\mu \Phi_a\left(x\right)\right). \tag{2.8}$$

For a massless scalar field theory, the Lagrangian density is

$$\mathcal{L}\left(\phi\left(x\right), \partial_\mu \phi\left(x\right)\right) = \partial_\mu \phi\left(x\right) \partial^\mu \phi\left(x\right). \tag{2.9}$$

If the action is invariant we conclude that the theory has scale symmetry. For the Lagrangian density above, we see that invariance requires $S[\phi] = \lambda^{D-\Delta/2-2} S[\phi]$ which is equivalent to $\Delta = D/2 - 1$.

2.1.1 Continuous Symmetries

Before discussing further details about symmetry, we are going to specialize our analysis. The conformal symmetry can be characterized as a continuous symmetry associated with a connected Lie group, as we are going to see ahead. Because of that, we will restrict our analysis to continuous transformations. Some facts about this kind of group are worth mentioning. Lie groups are defined over smooth manifolds and because of that, every element of the group is defined by a finite number of smooth parameters $(\omega_1, \ldots, \omega_n)$, with n being the dimension of the manifold. As a consequence of smoothness, it is possible to show that to every group element A there is a continuous sequence from the identity element to A. These facts are sufficient to ensure the completeness of a linear (infinitesimal) description of the group transformations [2].

In conclusion, we can consider continuous infinitesimal transformations, defined, in general, as the maps

$$x^\mu \longmapsto x'^\mu(x) = x^\mu + \omega_p \frac{\delta x^\mu}{\delta \omega_p}, \tag{2.10}$$

$$\Phi_a(x) \longmapsto \Phi_a'\left(x'\right) = \Phi_a\left(x\right) + \omega_p \frac{\delta \mathcal{F}_a}{\delta \omega_p}. \tag{2.11}$$

(Summation over repeated indices are implicit unless stated the contrary). The indices p, q, r are related to the number of independent parameters of the transformation, while the indices a, b, c account for the number of fields we are dealing and also for internal degrees of freedom each Φ_a may have. Because we are supposing these transformations are associated with a group, one can think of a linear representation of such group in the space of fields. Each parameter ω_p will be associated with a generator G_p that acts on this space. Given that the group elements are connected to the identity, the generators G_p will also be connected to the corresponding identity element in the representation space. Therefore, an infinitesimal transformation of the fields can be written, in terms of the generators, as

$$\Phi_a'\left(x\right) = \Phi_a\left(x\right) - i\omega_p G^p \Phi_a\left(x\right). \tag{2.12}$$

Using the infinitesimal transformations (2.11) with the explicit coordinate transformation (2.10), we obtain a relation between the fields at x'. Doing so, in the first order of the parameters, we can then relabel the coordinate $x' \to x$, and compare with (2.12), obtaining

$$iG^p\Phi_a = \frac{\delta x^\mu}{\delta\omega_p}\partial_\mu\Phi_a - \frac{\delta\mathcal{F}_a}{\delta\omega_p}. \tag{2.13}$$

This relation enables us to determine the algebraic generators using the structure of the infinitesimal transformations. As an example, see that applying the above equation to scale transformations, we find that its generator is $\mathcal{D} = -ix^\mu\partial_\mu$.

2.1.2 Noether's Theorem

A canonical implication to the symmetries discussed above is Noether's theorem. Given a system with a continuous infinitesimal symmetry, there is a conserved current. Up to this point, only global transformations were considered. However, one approach to derive the theorem is starting with a corresponding *local* symmetry, in which the parameters are functions of space-time coordinates: $\omega_a \to \omega_a(x)$. If the action is invariant under a global symmetry, it will not be invariant (in general) to the corresponding local symmetry. The transformed action changes by an additional term due to the local generalization, leading naturally to Noether's current.

Considering the infinitesimal transformations characterized by (2.10) and (2.11), see that we can determine the Jacobian by

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu\left(\omega_p\frac{\delta x^\mu}{\delta\omega_p}\right), \tag{2.14}$$

$$\left|\frac{\partial x'}{\partial x}\right| = 1 + \partial_\mu\left(\omega_p\frac{\delta x^\mu}{\delta\omega_p}\right). \tag{2.15}$$

For the last identity, we used the infinitesimal expansion for the determinant of a matrix $\det E \approx 1 + \mathrm{Tr}\, E$. From these expressions, we deduce the change in the action by a local transformation as

$$\delta S = \int d^Dx\left[\left(\delta^\mu_\nu\mathcal{L}\left(\Phi_a, \partial_\mu\Phi_a\right) - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_a}\partial_\nu\Phi_a\right)\frac{\delta x^\nu}{\delta\omega_p} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_a}\frac{\delta\mathcal{F}_a}{\delta\omega_p}\right]\partial_\mu\omega_p. \tag{2.16}$$

We stress that only local contributions proportional to the derivative of the parameters remained in the expression above. Factors proportional to ω_p do not contribute because of global invariance. If we define

$$j^\mu_p = \left(\delta^\mu_\nu\mathcal{L}\left(\Phi_a, \partial_\mu\Phi_a\right) - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_a}\partial_\nu\Phi_a\right)\frac{\delta x^\nu}{\delta\omega_p} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_a}\frac{\delta\mathcal{F}_a}{\delta\omega_p}, \tag{2.17}$$

the variation can be written simply as

$$\delta S = \int d^Dx\, \omega_p\, \partial_\mu j^\mu_p. \tag{2.18}$$

Considering field configurations on-shell, the action is stationary. Therefore, for any local parameter ω_p,

$$\partial_\mu j_p^\mu = 0. \tag{2.19}$$

That is, j_p^μ is the Noether current associated with the infinitesimal symmetry. See that there is, for each parameter, a conserved current associated. Furthermore, given the conservation law, we see that $\partial_0 j_p^0 = \partial_i j_p^i$. If the current vanishes at the infinity, then integrating the latter in both sides and using the diverge theorem on the right-hand side, we find another conserved quantity. Namely, Noether's charge

$$Q_p = \int d^{D-1}x \, j_p^0(x). \tag{2.20}$$

All the results obtained in this section are valid for classical field theories. The analysis for implications of symmetry in a quantum theory of fields is quite distinct. To begin with, we do not access dynamical quantities by the action, we do it rather by the S matrix. This is because we are only able to determinate the quantum interacting fields in rare cases[1]. Then, the information that we seek is the probability amplitude between asymptotic states of particles, obtained by the LSZ formula using perturbation theory on the correlation functions. The S matrix, defined as this amplitude, is the object that allows us to obtain dynamical information about the system. For example, in a scalar theory, the S matrix elements between asymptotic states $|\alpha\rangle = |p_1, \ldots, p_n\rangle$ and $|\beta\rangle = |p'_1, \ldots, p'_m\rangle$ is

$$S_{\alpha\beta} = \left(\frac{i}{\sqrt{z}}\right)^{n+m} \int \prod_{i=1}^{n} d^4x_i e^{-ip_ix_i} \left(\Box_{x_i} + m^2\right) \prod_{j=1}^{m} d^4y_j e^{ip'_jy_j} \left(\Box_{y_j} + m^2\right) G_{nm} \tag{2.21}$$

where z is a renormalization parameter and

$$G_{nm} = \langle 0| T\left\{\hat{\phi}(x_1), \ldots, \hat{\phi}(x_n), \hat{\phi}(y_1), \ldots, \hat{\phi}(y_m)\right\} |0\rangle \tag{2.22}$$

is the correlation function associated with the process. The operator T is the time-ordered operator, in which fields with the smallest time coordinate acts first. Through this formula, it is clear that the non-trivial factor of the S matrix elements are the correlation functions, that explicitly depend on the dynamics of the interacting fields. If symmetries are implemented dynamically, one can expect that it is going to leave a trail in the correlation functions. These investigations will be the subject of the next section.

[1] See that if we do not know the solution of the equations of motion, we cannot express the conserved current (2.17).

2.1.3 Ward Identities

Ward identities are constraints between correlation functions as a result of symmetry. In this discussion, we are going to derive them using the path integral approach, reviewing [3]. The correlation functions in the path integral formalism are

$$\langle \Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n)\rangle = \frac{1}{Z}\int \mathcal{D}\Phi\, \Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n)\, e^{iS[\Phi_a]} \tag{2.23}$$

where the left-hand side denotes the vacuum expectation value of the time ordered fields $\Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n)$ and Z is the generating functional for the correlation functions,

$$Z[J] = \int \mathcal{D}\Phi\, e^{iS[\Phi_a]+\int d^Dx\, J^a\Phi_a}. \tag{2.24}$$

For a general transformation of the fields and coordinates of space-time, as (2.2) and (2.3), it follows that by a change of variables

$$\left\langle \Phi_{a_1}\left(x_1'\right)\cdots\Phi_{a_n}\left(x_n'\right)\right\rangle = \left\langle \mathcal{F}_{a_1}\left(\Phi_a(x_1)\right)\cdots\mathcal{F}_{a_n}\left(\Phi_a(x_n)\right)\right\rangle, \tag{2.25}$$

where we have assumed a trivial Jacobian.

Considering the infinitesimal transformation (2.11), the generation functional transforms, in the first order, as

$$Z[J] = \int \mathcal{D}\Phi \exp\left\{iS\left[\Phi_a\right] + \int d^Dx\, J^a\Phi_a\right\}\left\{1 + i\int d^Dx\left[i\partial_\mu j^{p,\mu} + J^a\frac{\delta\mathcal{F}_a}{\delta\omega_p}\right]\omega_p\right\}. \tag{2.26}$$

One of the factors in the right-hand side is the generating functional and the remaining must vanish,

$$\int d^Dx \int \mathcal{D}\Phi \exp\left\{iS\left[\Phi_a\right] + \int d^Dx\, J^a\Phi_a\right\}\left[i\partial_\mu j^{p,\mu} + J^a\frac{\delta\mathcal{F}_a}{\delta\omega_p}\right]\omega_p = 0. \tag{2.27}$$

However, the expression above must vanish for any local parameter ω_p, and therefore

$$W[J] \equiv \int \mathcal{D}\Phi \exp\left\{iS\left[\Phi_a\right] + \int d^Dx\, J^a\Phi_a\right\}\left[i\partial_\mu j^{p,\mu} + J^a\frac{\delta\mathcal{F}_a}{\delta\omega_p}\right] = 0. \tag{2.28}$$

If we derive this functional and take $J^a = 0$ afterwards, we can establish constraints between correlation functions. For example, the first functional derivative with respect to $J^a(y)$ is

$$\int d\Phi e^{iS_J[\Phi_a]}\left[i\partial_\mu j^{p,\mu}(x) + J^a(x)\frac{\delta\mathcal{F}_a}{\delta\omega_p}\right] i\Phi_c(y) + \int d\Phi\, e^{iS_J[\Phi_a]}\frac{\delta\mathcal{F}_c}{\delta\omega_p}\delta^{(D)}(x-y) = 0, \tag{2.29}$$

which can be rewritten using the expansion (2.13) and the definition (2.23) as

$$\partial_\mu\left\langle j^{p,\mu}(x)\Phi_a(y)\right\rangle = -i\delta^{(D)}(x-y)\left\langle G^p\Phi_a(x)\right\rangle. \tag{2.30}$$

Taking the second derivative, we find that

$$\partial_\mu \langle j^{p,\mu}(x)\Phi_a(x_1)\,\Phi_b(x_2)\rangle = -i\delta^{(D)}(x-x_1)\langle G^p\Phi_a(x)\Phi_b(x_2)\rangle - \\ - i\delta^{(D)}(x-x_2)\langle \Phi_a(x_1)\,G^p\Phi_b(x)\rangle \quad (2.31)$$

In conclusion, for n derivatives, induction tells us that a n-point correlation function must satisfy the Ward identities

$$\partial_\mu \langle j^{p,\mu}(x)\Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n)\rangle = -i\sum_{i=1}^{n}\delta^{(D)}(x-x_i)\langle \Phi_{a_1}(x_1)\ldots G^p\Phi_{a_i}(x)\ldots\Phi_{a_n}(x_n)\rangle\,, \quad (2.32)$$

as an implication of symmetry transformations in a quantum theory. Apart from that, Ward identities encodes some important facts. Integrating over all space-time, the left-hand side is trivial if $x \neq x_i$, and then the infinitesimal transformation of the correlation function must vanish,

$$\delta_\omega \langle \Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n)\rangle \equiv -i\omega_p\sum_{i=1}^{n}\langle \Phi_{a_1}(x_1)\ldots G^p\Phi_{a_i}(x)\ldots\Phi_{a_n}(x_n)\rangle = 0. \quad (2.33)$$

If we now integrate over a submanifold of space-time Σ whose boundary are spatial surfaces specified by the time coordinate parameters $t_- \leq t \leq t_+$, further important details are obtained. To simplify the notation, we introduce a string of fields as

$$Y_{a_2\cdots a_n}(x_2,\ldots,x_n) \equiv \Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)\,. \quad (2.34)$$

We will often not write all the coordinate points, and $Y_{a_2\cdots a_n}$ must be seen as a function of $x_2,\ldots,x_n$. Consider $y^\mu = (t,\mathbf{x}) \in \Sigma$ with t different from all time coordinates appearing in the above string. Integrating the Ward identities in this region, and using the divergence theorem in the left-hand side, we obtain

$$\int_{\partial\Sigma} dS_\mu\,\langle j^{p,\mu}(y)\Phi_{a_1}(x_1)\,Y_{a_2\cdots a_n}\rangle = \\ = -i\sum_{i=1}^{n}\int_{t_-}^{t_+}dt\int d^{D-1}\delta^{(D)}(y-x_i)\langle \Phi_{a_1}(x_1)\ldots G^p\Phi_{a_i}(x)\ldots\Phi_{a_n}(x_n)\rangle\,, \quad (2.35)$$

with $dS_\mu = d^{D-1}y\,n_\mu$ the normal area element and $\partial\Sigma$ the boundary of Σ. The right-hand side can be determined realizing that because t is different from all the time coordinates in the string, the Dirac deltas $\delta^D(y-x_i)$ vanish for $i = 2,\ldots,n$. Therefore, the first term in the sum is the only contribution. For the left-hand side, see that the boundary can be written with a par of disjoint hyperplanes $R_\pm = \{y^\mu \in \partial\Sigma : y^\mu = (t_\pm,\mathbf{y})\}$. In other words, we may write $\partial\Sigma = R_+ \cup R_-$. Since the intersection is a null set, the integral over the complete boundary can be expressed as a sum of the integral over these hyperplanes. Furthermore, the normal vector n^μ is orthogonal to the spatial directions in a flat space-time, and when in contraction with $j^{p,\mu}$, only the zeroth component will remain. With these considerations, we can parametrize $n^{\mu}_{\pm} = (\pm 1, 0)$ as the normal vectors to $R_{\pm}$, obtaining

$$\int_{\partial\Sigma} dS_{\mu} \, \langle j^{p,\mu}(y)\Phi_{a_1}(x_1) Y_{a_2\cdots a_n}\rangle = \int_{R_+} d^{D-1}y \left\langle j^{p,0}(y)\,\Phi_{a_1}(x_1)\,Y_{a_2\cdots a_n}\right\rangle - \\ - \int_{R_-} d^{D-1}y \left\langle j^{p,0}(y)\,\Phi_{a_1}(x_1)\,Y_{a_2\cdots a_n}\right\rangle . \quad (2.36)$$

If we now use the definition of conserved charge,

$$\int_{\partial\Sigma} dS_{\mu} \, \langle j^{p,\mu}(y)\Phi_{a_1}(x_1) Y_{a_2\cdots a_n}\rangle = \langle Q^{p}(t_+)\,\Phi_{a_1}(x_1)\,Y_{a_2\cdots a_n}\rangle - \langle \Phi_{a_1}(x_1)\,Q^{p}(t_-)\,Y_{a_2\cdots a_n}\rangle . \quad (2.37)$$

Taking the limit of $t_- \to t_+ = t$, the commutator of the conserved charge with a local field appear in the above expression, valid for an arbitrary string. Then,

$$[Q^{p}, \Phi_{a}(x)] = -iG^{p}\Phi_{a}(x), \quad (2.38)$$

showing that Noether's charge is indeed generator of the infinitesimal transformation.

With these results, we will start to investigate the conformal symmetry and its implications on both classical and quantum regimes.

2.2 Classical Conformal Symmetry

The conformal group is defined as the set of diffeomorphic coordinate transformations x' such that the metric tensor g transforms with a local scale factor:

$$g'_{\mu\nu}(x') = \Lambda(x)\, g_{\mu\nu}(x), \quad (2.39)$$

together with the composition of transformations. The conformal group is a connected Lie group [4], and then, it is sufficient to study infinitesimal transformations in order to characterize the finite ones, and also, all the generators of the associated Lie algebra.

Considering an infinitesimal transformation $x^{\mu} \to x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x)$, the metric tensor transforms as

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - \partial_{\mu}\varepsilon_{\nu}(x) - \partial_{\nu}\varepsilon_{\mu}(x). \quad (2.40)$$

Defining $f(x) = 1 - \Lambda(x)$ and applying the above to the metric conformal transformation, we obtain

$$f(x) g_{\mu\nu}(x) = \partial_{\mu}\varepsilon_{\nu}(x) + \partial_{\nu}\varepsilon_{\mu}(x). \quad (2.41)$$

In order to characterize the transformations, some useful relations need to be derived. Simply taking the trace,

$$f\left(x\right)=\frac{2}{D}\partial^{\mu}\varepsilon_{\mu}\left(x\right), \tag{2.42}$$

Furthermore, considering an Euclidean metric tensor, and differentiating (2.41),

$$\partial_{\alpha}f\left(x\right)g_{\mu\nu}=\partial_{\alpha}\partial_{\mu}\varepsilon_{\nu}\left(x\right)+\partial_{\alpha}\partial_{\nu}\varepsilon_{\mu}\left(x\right). \tag{2.43}$$

With the last three equations, we will characterize all conformal transformations. The idea is find an equation exclusively for f. See that if we exchange the indices and linearly combine the equations,

$$2\partial_{\alpha}\partial_{\mu}\varepsilon_{\nu}\left(x\right)=\partial_{\alpha}f\left(x\right)g_{\mu\nu}+\partial_{\mu}f\left(x\right)g_{\alpha\nu}-\partial_{\nu}f\left(x\right)g_{\mu\alpha}. \tag{2.44}$$

Writing in this way, we can contract α and μ, obtaining

$$2\partial^{\mu}\partial_{\mu}\varepsilon_{\nu}\left(x\right)=\left(2-D\right)\partial_{\nu}f\left(x\right), \tag{2.45}$$

derivate it and use equation (2.42) to find that

$$\left(D-1\right)\partial^{2}f\left(x\right)=0. \tag{2.46}$$

In one dimension there is no constraint (any smooth function f is conformal). The two dimensional case will be studied in detail next section, so in what follows we will focus on $D>2$.

Because of the last constraint derived (2.46), in a series expansion of $f(x)$ all coefficients of order higher than linear must be zero, remaining

$$f\left(x\right)=A+B_{\mu}x^{\mu}. \tag{2.47}$$

Plugging this result in (2.44), and using the same argument (for Euclidean metric tensor),

$$\varepsilon_{\nu}\left(x\right)=a_{\nu}+b_{\nu\mu}x^{\mu}+c_{\nu\mu\alpha}x^{\mu}x^{\alpha}. \tag{2.48}$$

Now, using the relations derived above (2.41, 2.42, 2.44) we can determine each of these coefficients, and therefore, characterize infinitesimal conformal transformations order by order. See that all these relations contain derivatives in the infinitesimal parameter. Then, the zeroth order coefficient in (2.48) remains undetermined by these constraints. However, considering the first order, we obtain by (2.41) that

$$Ag_{\mu\nu}=\partial_{\mu}\left(b_{\nu\sigma}x^{\sigma}\right)+\partial_{\nu}\left(b_{\mu\sigma}x^{\sigma}\right)=b_{\nu\mu}+b_{\mu\nu}. \tag{2.49}$$

Considering now the second relation (2.42),

$$A = \frac{2}{D}\partial^{\mu}\left(b_{\mu\sigma}x^{\sigma}\right) = \frac{2}{D}b^{\sigma}{}_{\sigma}. \tag{2.50}$$

Comparing the last two expressions,

$$\frac{2}{D}b^{\sigma}{}_{\sigma}g_{\mu\nu} = b_{\nu\mu} + b_{\mu\nu}. \tag{2.51}$$

Considering this expansion coefficient as a tensor, the above relation tells us that the symmetric part depends on the trace. Then, in a tensorial decomposition,

$$b_{\mu\nu} = \frac{1}{D}b^{\sigma}{}_{\sigma}g_{\mu\nu} + m_{\mu\nu}, \tag{2.52}$$

where $m_{\mu\nu}$ is the antisymmetric part of $b_{\mu\nu}$.

The third relation (2.44) vanishes the first order coefficient (because it has only second derivatives in the infinitesimal parameter), but gives us information about the second order. In fact, considering (2.44) and contracting the first two free indices of $c_{\nu\mu\alpha}$, we obtain

$$B_{\alpha} = \frac{4}{D}c^{\mu}{}_{\mu\alpha} \equiv 4b_{\alpha}. \tag{2.53}$$

Therefore, the most general form of this coefficient is

$$c_{\nu\mu\alpha} = b_{\alpha}g_{\mu\nu} + b_{\mu}g_{\alpha\nu} - b_{\nu}g_{\mu\alpha}. \tag{2.54}$$

See that we can now derive all infinitesimal conformal transformations by considering independent coefficients. That is, if $b_{\mu\nu} = c_{\mu\nu\alpha} = 0$, the coordinate transformation will be

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) = x^{\mu} + a^{\mu}, \tag{2.55}$$

characterizing a coordinate translation. On the other hand, for $a_{\mu} = c_{\mu\nu\alpha} = b^{\sigma}b_{\sigma} = 0$,

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) = x^{\mu} + m^{\mu\nu}, \tag{2.56}$$

which is identified as a Lorentz transformation ($m_{\mu\nu}$ is antisymmetric)[2]. If only $b^{\sigma}_{\sigma} \neq 0$,

$$x'^{\mu} = x^{\mu} + \frac{1}{D}b^{\sigma}{}_{\sigma}x^{\mu} \tag{2.57}$$

we find scale transformations. Finally, for non-vanishing b_{μ}, special conformal transformations are

$$x'^{\mu} = x^{\mu} + 2b_{\alpha}x^{\alpha}x^{\mu} - b^{\mu}x^{2}. \tag{2.58}$$

[2] We will denominate these generally as rotations

Having determined infinitesimal transformations, the next step is consider their finite form. In order to do that, we first determine the generators associated with each infinitesimal transformation by simply plugging them into (2.13), obtaining

$$\mathcal{D} = -ix^{\mu}\partial_{\mu} \quad \text{(scale)}$$
$$P_{\mu} = -i\partial_{\mu} \quad \text{(translation)}$$
$$L_{\mu\nu} = i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) \quad \text{(rotations)}$$
$$K_{\mu} = -i\left(2x_{\mu}x^{\nu}\partial_{\nu} - x^{2}\partial_{\mu}\right) \quad \text{(SCT)}$$

These operators are the generators of the conformal algebra, realized by the following commutation relations:

$$\begin{aligned} \left[P_{\mu}, \mathcal{D}\right] &= -P_{\mu} \\ \left[\mathcal{D}, K_{\mu}\right] &= -iK_{\mu} \\ \left[K_{\mu}, K_{\nu}\right] &= 2ix^{2}x^{\alpha}L_{\mu\nu}\partial_{\alpha} \\ \left[L_{\mu\nu}, L_{\alpha\beta}\right] &= -i\left(g_{\mu\alpha}L_{\nu\beta} + g_{\mu\beta}L_{\nu\alpha} + g_{\nu\alpha}L_{\mu\beta} + g_{\nu\beta}L_{\mu\alpha}\right) \end{aligned} \qquad \begin{aligned} \left[P_{\alpha}, K_{\mu}\right] &= -2i\left(g_{\alpha\mu}\mathcal{D} - L_{\alpha\mu}\right) \\ \left[P_{\alpha}, L_{\mu\nu}\right] &= i\left(g_{\mu\alpha}P_{\nu} - g_{\nu\alpha}P_{\mu}\right) \\ \left[L_{\mu\nu}, K_{\alpha}\right] &= i\left(g_{\nu\alpha}K_{\mu} - g_{\mu\alpha}K_{\nu}\right) \end{aligned} \tag{2.59}$$

Now, exponentiating the above generators with associated parameters, we determine the finite form of the transformations,

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu}x^{\nu} \quad \text{(rotations)}$$
$$x'^{\mu} = x^{\mu} + a^{\mu} \quad \text{(translation)}$$
$$x'^{\mu} = x^{\mu} + \frac{1}{D}b^{\sigma}_{\ \sigma}x^{\mu} \quad \text{(scale)}$$

with $\Lambda = \exp(i\omega_{\mu\nu}L^{\mu\nu})$. For SCT, an alternative and derivation is considered by seeing that the ratio x^{μ}/x^{2} transforms linearly. To realize this, notice that

$$\delta\left(\frac{x^{\mu}}{x^{2}}\right) = -\delta b^{\mu}. \tag{2.60}$$

Then, the finite form of this ratio transformation is

$$\frac{x'^{\mu}}{x'^{2}} = \frac{x^{\mu}}{x^{2}} - b^{\mu}. \tag{2.61}$$

Taking the square and isolating the transformed coordinate leads to

$$x'^{\mu} = \frac{x^{\mu} - x^{2}b^{\mu}}{1 + 2\left(b\cdot x\right) + b^{2}x^{2}}. \quad \text{(SCT)}$$

Having determined all conformal transformations, we will start to study their role in quantum field theories. One final concept we will introduce is of *quasi-primary fields*. Considering a scalar (spinless) field, it transforms under the conformal group as[3]

$$\varphi'\left(x'\right) = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/D}\varphi\left(x\right), \tag{2.62}$$

[3] A detailed discussion on how to find the representation of the conformal algebra on the space of fields is in the appendix.

where the Jacobian above is related with the local conformal scale by

$$\left|\frac{\partial x'}{\partial x}\right| = \Lambda^{-d/2}. \tag{2.63}$$

All the fields transforming with a well defined conformal scale, as the above, are called quasi-primary fields. In what follows we will explore the conformal properties of correlation functions and the Ward identities for quasi-primary fields.

2.3 Quantum Conformal Symmetry

Consider a two point correlation function for quasi primary fields ϕ_1 and ϕ_2, with scale dimensions Δ_1 and Δ_2, respectively. Under a conformal transformation, it changes as

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/D} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/D} \left\langle \phi_1 \left(x_1'\right) \phi_2 \left(x_2'\right) \right\rangle, \tag{2.64}$$

where we have relabeled $x \leftrightarrow x'$ in (2.25). The strategy now is use the covariance of correlation functions and see, for each conformal transformation, what constraints it imposes. To simplify the manipulations, we take $f(x_1, x_2) \equiv \langle \phi_1 (x_1) \phi_2 (x_2) \rangle$. For scale transformations, we saw that the Jacobian is simply λ^D, which implies

$$f(x_1, x_2) = \lambda^{\Delta_1+\Delta_2} f (\lambda x_1, \lambda x_2). \tag{2.65}$$

Rotations keep distances and angles invariant, then knowing that translations preserves relative distances, we conclude that $f(x_1, x_2) = f (|x_1 - x_2|)$. This, together with (2.65), implies

$$f (|x_1 - x_2|) = \lambda^{\Delta_1+\Delta_2} f (\lambda |x_1 - x_2|). \tag{2.66}$$

Defining $x_{12} \equiv |x_1 - x_2|$ and deriving both sides with respect to λ,

$$(\Delta_1 + \Delta_2) \lambda^{\Delta_1+\Delta_2-1} f (\lambda x_{12}) + x_{12} \lambda^{\Delta_1+\Delta_2} f' (\lambda x_{12}) = 0. \tag{2.67}$$

Particularly for $\lambda = 1$, we see that f must be the solution of a first order differential equation,

$$\frac{f' (x)}{f (x)} = -\frac{\Delta_1 + \Delta_2}{x}, \tag{2.68}$$

which always have a solution (up to a constant). Integrating this expression we obtain

$$f(x_1, x_2) = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}. \tag{2.69}$$

Finally, if we consider SCT, with the Jacobian

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{\left(1+2b\cdot x+b^2x^2\right)^D} \equiv \frac{1}{\gamma_x^D}, \tag{2.70}$$

the following holds

$$f(x_1,x_2) = \frac{\gamma_1^{\frac{\Delta_1+\Delta_2}{2}}\gamma_2^{\frac{\Delta_1+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}. \tag{2.71}$$

Covariance implies that

$$f(x_1,x_2) = \frac{\gamma_1^{\frac{\Delta_1+\Delta_2}{2}}\gamma_2^{\frac{\Delta_1+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}f(x_1,x_2), \tag{2.72}$$

which is valid only in the case that the multiplicative factor depended of γ is the unit. That is, when $\Delta_1 = \Delta_2 \equiv \Delta$ covariance is established, otherwise $f(x_1,x_2) = 0$. We can conclude, then, that SCT and covariance forces the correlation function to vanish when we consider quasi-primary fields with different scale dimensions. With these considerations, we reach a final form for the propagator, up to a constant factor,

$$\langle\phi_1(x_1)\,\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1-x_2|^{2\Delta}}. \tag{2.73}$$

In an analogous fashion, the three point correlation function is also constrained by conformal symmetry to be

$$\langle\phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{-\Delta_1+\Delta_3+\Delta_2}x_{31}^{\Delta_1-\Delta_2+\Delta_3}}. \tag{2.74}$$

At first sight we could think that it is possible to fix an arbitrary $n-$point correlation function, but not quite. For the next case of 4 points, we already see that we can construct a conformal invariant argument, such that any function of it will remain unchanged. For instance, consider a function $\Gamma(x_1,\ldots,x_n)$ of the space-time coordinates x_i. If we want its argument to be invariant, we already know that translations and rotations forces $\Gamma(x_1,\ldots,x_n) = \Gamma\left(|x_i-x_j|\right)$. Scale invariance will require that dependence is on the ratio of modules,

$$\Gamma\left(|x_i-x_j|\right) = \Gamma\left(\frac{x_{ij}}{x_{kl}}\right). \tag{2.75}$$

At this point we see that we need at least three points to construct the invariant argument. SCT will cause to increase this number by one. To see this, note that the module transforms as

$$x'_{ij} = \frac{x_{ij}}{\left[1-2b\cdot x_i+b^2\left(x_i\right)^2\right]^{\frac{1}{2}}\left[1-2b\cdot x_j+b^2\left(x_j\right)^2\right]^{\frac{1}{2}}}. \tag{2.76}$$

Then, if we want the ratio to be invariant, the same points appearing in the numerator must also appear in the denominator in a non-trivial setting (that is, when the ratio is not simply the unit). For instance, this do not hold for 2 and 3 points, and the reason is simply because for two points the ratios are trivial, and for three points the number of possible ratios is 3 (precisely, x_{12}, x_{13}, x_{23}) and from these three we cannot choose any two such that the condition above remains. We, then, consider functions with four points arranged like

$$\frac{x_{12}x_{43}}{x_{24}x_{31}}, \qquad \frac{x_{12}x_{43}}{x_{14}x_{23}}. \tag{2.77}$$

Such conformal invariant argument are called anharmonic ratio, and they are the reason why we cannot keep making use of conformal symmetry to fix correlation functions, because whatever the form we find by the procedure discussed, we could always multiply by an arbitrary function dependent of an anharmonic ratio. To illustrate this, see that for a 4 point function,

$$\langle \phi_1 (x_1) \phi_2 (x_2) \phi_3 (x_3) \phi_4 (x_4) \rangle = \Gamma \left(\frac{x_{12}x_{43}}{x_{24}x_{31}}, \frac{x_{12}x_{43}}{x_{14}x_{23}} \right) \prod_{i<j} x_{ij}^{\Delta/3-\Delta_i-\Delta_j} \tag{2.78}$$

where Δ is the sum of all four scale dimensions [5].

To finish our discussion about conformal symmetry, we now turn to determine Ward identities associated with each conformal transformation. Considering currents and genera-tors (one for each symmetry), we can simply apply them to Ward Identities. However, it is convenient to write currents using the energy-momentum tensor,

$$j_{\alpha\mu\nu} = T_{\alpha\mu}x_\nu - T_{\alpha\nu}x_\alpha, \qquad \text{(rotations)}$$
$$j^\mu_D = T^\mu{}_\nu x^\nu. \qquad \text{(scale)}$$

The current associated with translation is the energy-momentum tensor itself, and these expressions have two assumptions. In order to obtain the first current above, it is necessary to consider a symmetric energy-momentum tensor (on shell), and the second requires a traceless one (also on shell)[4]. Furthermore, we do not consider SCT, and in fact we do not need to consider such transformation if we restrict ourselves to irreducible representations of the Lorentz group, where the associated generator must vanish, see appendix (B).

Considering translations,

$$\partial_\mu \langle T^{\mu\nu} (x) X_{c_1c_2\ldots c_n} \rangle = -\sum_{i=1}^{n} \delta^{(D)} (x - x_i) \partial_i^\nu \langle X_{c_1c_2\ldots c_n} \rangle , \tag{2.79}$$

[4] More details on the assumptions of whether or not we can consider a symmetric, traceless energy-momentum tensor is given in the appendix (**??**)

which tells us that for all coordinates different from the ones on the string, the energy-momentum tensor is conserved within correlation functions. Now, for rotations,

$$\partial^{\alpha}\left\langle\left(T_{\alpha\mu}x_{\nu}-T_{\alpha\nu}x_{\mu}\right)X_{c_1c_2\ldots c_n}\right\rangle=\sum_{i=1}^{n}\delta^{(D)}\left(x-x_i\right)\left\langle\Phi_{c_1}\ldots\left(-iS^{i}_{\mu\nu}\right)\Phi_{c_i}\ldots\Phi_{c_n}\right\rangle+$$
$$+\sum_{i=1}^{n}\delta^{(D)}\left(x-x_i\right)\left\langle\Phi_{c_1}\ldots\left(\left(x_i\right)_{\mu}\left(\partial_i\right)_{\nu}-\left(x_i\right)_{\mu}\left(\partial_i\right)_{\nu}\right)\Phi_{c_i}\ldots\Phi_{c_n}\right\rangle, \quad (2.80)$$

where Φ_{c_i} is a function of the coordinate x_i. It is possible to simplify this expression if we explicitly act with the coordinate derivatives in the left-hand side and use the former identity (2.79), obtaining

$$\left\langle\left(T_{\nu\mu}-T_{\mu\nu}\right)X_{c_1c_2\ldots c_n}\right\rangle=-i\sum_{i=1}^{n}\delta^{(D)}\left(x-x_i\right)S^{i}_{\mu\nu}\left\langle X_{c_1c_2\ldots c_n}\right\rangle. \quad (2.81)$$

That is, the energy-momentum tensor is symmetric within correlation functions in all coordinates different from the ones on the string X. Finally, the identity associated with scale symmetry is

$$\partial_{\mu}\left\langle\left(T^{\mu}_{\ \nu}x^{\nu}\right)X_{c_1c_2\ldots c_n}\right\rangle=-\sum_{i=1}^{n}\delta^{(D)}\left(x-x_i\right)\left\langle\Phi_{c_1}\left(x_1\right)\ldots\left(x_i^{\alpha}\partial^{i}_{\alpha}+\Delta_i\right)\Phi_{c_i}\left(x\right)\ldots\Phi_{c_n}\left(x_n\right)\right\rangle. \quad (2.82)$$

Acting with the derivate in the left-hand side we obtain

$$\left\langle T^{\mu}_{\ \mu}X_{c_1c_2\ldots c_n}\right\rangle=-\sum_{i=1}^{n}\delta^{(D)}\left(x-x_i\right)\Delta_i\left\langle X_{c_1c_2\ldots c_n}\right\rangle, \quad (2.83)$$

meaning that the energy-momentum tensor is traceless within correlation functions when evaluated at a point distinct to those on the string X. With these four identities we conclude the general discussion about conformal symmetry and turn to study the two-dimensional case.

2.4 1+1 Classical Conformal Symmetry

In two dimensions, an interesting property of the conformal transformations appear. To realize it, let us consider coordinates on the plane $z^{\mu}=\left(z^0,z^1\right)$ and a conformal transformation $z^{\mu}\mapsto w^{\mu}\left(z\right)$. We know, by the definition of the conformal group that

$$g'^{\mu\nu}=\Lambda\left(z\right)g^{\mu\nu}. \quad (2.84)$$

Or, using the transformation of the metric tensor by this change of coordinates,

$$\Lambda\left(z\right)g^{\mu\nu}=\frac{\partial w^{\mu}}{\partial z^{\alpha}}\frac{\partial w^{\nu}}{\partial z^{\beta}}g^{\alpha\beta}. \quad (2.85)$$

Choosing $\mu = \nu = 0$ and $\mu = \nu = 1$,

$$\Lambda(z) = \left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2, \tag{2.86}$$

$$\Lambda(z) = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2. \tag{2.87}$$

(Remember that we consider $g_{\mu\nu}$ as the Euclidean metric tensor). Comparing these two expressions,

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2. \tag{2.88}$$

On the other hand, we can choose $\mu = 0$ and $\nu = 1$ and obtain

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0. \tag{2.89}$$

By inspection, we see that both conditions above are satisfied if

$$\frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \qquad \text{and} \qquad \frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1}, \tag{2.90}$$

or also, if

$$\frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \qquad \text{and} \qquad \frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1}. \tag{2.91}$$

The two pair of equations above can be identified as the Cauchy-Riemann equations for holomorphic (2.91) and anti-holomorphic functions (2.90). This is one of the reasons why we describe a 2D conformal theory on the complex plane, as we are going to see next. Considering z and $\bar{z}$ as the complex plane coordinates, we can simplify the equations above using the map

$$\begin{aligned} z^0 &= \frac{1}{2}(z + \bar{z}), \qquad & z^1 &= \frac{1}{2i}(z - \bar{z}), \\ \partial_0 &= \partial + \bar{\partial}, \qquad & \partial_1 &= i\left(\partial - \bar{\partial}\right), \end{aligned} \tag{2.92}$$

where ∂ is a derivate with respect to z and $\bar{\partial}$ a derivative with respect to $\bar{z}$. We will call them conjugate coordinates, with z being the holomorphic component and $\bar{z}$ the anti-holomorphic one. Notice that, in principle, this does not seem to be an independent set of variables.

In this setting, the metric tensor and its inverse are transformed to

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \tag{2.93}$$

with Greek indices indicating the pair $z, \bar{z}$. The map between covariant and contravariant vectors induced by the metric now also creates a correspondence between holomorphic (covariant and contravariant) and anti-holomorphic (contravariant and covariant, respectively) components

$$a_z = g_{z\bar{z}} a^{\bar{z}} = \frac{1}{2} a^{\bar{z}}, \qquad a_{\bar{z}} = g_{\bar{z}z} a^z = \frac{1}{2} a^z. \tag{2.94}$$

Furthermore, recall that the completely anti-symmetric tensor is defined as

$$\varepsilon_{\mu\nu} = \sqrt{|g|} \epsilon_{\mu\nu}, \tag{2.95}$$

where $\epsilon_{\mu\nu} = \pm 1$ is the completely anti-symmetric symbol. Then,

$$\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}. \tag{2.96}$$

We can also rewrite the conditions over conformal transformations in this new coordinate system. Considering the Cauchy-Riemann equations, we simply apply the chain rule and sum them up, obtaining

$$\bar{\partial} w (z, \bar{z}) = 0. \tag{2.97}$$

Equivalently, we can rewrite the anti-holomorphic equations simply as

$$\partial \bar{w} (z, \bar{z}) = 0. \tag{2.98}$$

In this simplified form, we see that holomorphic transformations are those independent of $\bar{z}$ and anti-holomorphic ones are independent of z. We also emphasize that these conditions are *local* in the complex plane, in the sense that a transformation satisfying them does not need to be defined and invertible over all $\mathbb{C}$. So next, we would like to discuss further this possibility.

In our previous discussions about the conformal group, all transformations we found were well-defined invertible maps over the entire space-time. They were associated with the finite dimensional conformal algebra and lead us to the notion of quasi-primary fields and its interesting implications. As before, we will first analyse the algebraic structure. The upshot of this investigation is that the global sector is a subalgebra of an enhanced infinite dimensional local algebra.

Consider a infinitesimal transformation

$$z' = z + \epsilon (z), \qquad \bar{z}' = \bar{z} + \bar{\epsilon} (\bar{z}), \tag{2.99}$$

with $\varepsilon(z)$ and $\bar{\epsilon} (\bar{z})$ begin parameters admitting a Laurent expansion in the neighbourhood of the origin. Given a scalar field, it transforms as

$$\phi' (z, \bar{z}) = \phi (z, \bar{z}) - \epsilon (z) \partial_z \phi (z, \bar{z}) - \bar{\epsilon} (\bar{z}) \partial_{\bar{z}} \phi (z, \bar{z}), \tag{2.100}$$

implying in a variation

$$\delta\phi\left(z,\bar{z}\right)=-\epsilon\left(z\right)\partial_{z}\phi\left(z,\bar{z}\right)-\bar{\epsilon}\left(\bar{z}\right)\partial_{\bar{z}}\phi\left(z,\bar{z}\right). \tag{2.101}$$

Now, if the Laurent expansion of the infinitesimal parameters are

$$\epsilon\left(z\right)=\sum_{n\in\mathbb{Z}}c_{n}z^{n+1},\qquad\bar{\epsilon}\left(\bar{z}\right)=\sum_{n\in\mathbb{Z}}\bar{c}_{n}\bar{z}^{n+1}, \tag{2.102}$$

we can plug this into the previous variation and find that

$$\delta\phi\left(z,\bar{z}\right)=\sum_{n\in\mathbb{Z}}\left(-c_{n}z^{n+1}\partial_{z}-\bar{c}_{n}\bar{z}^{n+1}\partial_{\bar{z}}\right)\phi\left(z,\bar{z}\right). \tag{2.103}$$

This is a sum of parameters times generators acting on the field, just as we discussed earlier about continuous symmetries (2.12). Therefore, the generators of the local conformal algebra are

$$\ell_{n}=-z^{n+1}\partial_{z},\qquad\bar{\ell}_{n}=-\bar{z}^{n+1}\partial_{\bar{z}}, \tag{2.104}$$

and the expansion coefficients in (2.103) are the associated parameters. These generators satisfy the classical Witt algebra

$$\left[\ell_{n},\ell_{m}\right]=\left(n-m\right)\ell_{n+m},\qquad\left[\bar{\ell}_{n},\bar{\ell}_{m}\right]=\left(n-m\right)\bar{\ell}_{n+m},\qquad\left[\ell_{n},\bar{\ell}_{m}\right]=0. \tag{2.105}$$

The last commutator tells us that this algebra is, indeed, the direct sum of the holomorphic and anti-holomorphic sectors, each generated by ℓ_n and $\bar{\ell}_n$, respectively. Furthermore, this algebra has a unique central extension called Virasoro algebra, which is its quantum version [5]. Here, we will not derive the algebra for the quantum fields as this extension, we will consider a physical approach where we can extract the generators from conserved charges. We mentioned earlier that the global sector is a subalgebra of the local one. See that the generators

$$\ell_{-1}=-\partial_{z},\qquad\ell_{0}=-z\partial_{z},\qquad\ell_{1}=-z^{2}\partial_{z}, \tag{2.106}$$

are exactly those in the global conformal algebra: ℓ_{-1} generates rotations, ℓ_0 scale transfor-mations and ℓ_1 the SCT.

Following what we considered in the general $D-$dimensional case, we would like to define a notion of quasi-primary fields compatible with the complex structure introduced. If we recall that a quasi-primary field transforms with a power of the Jacobian, we can simply consider

$$\left|\frac{\partial x'}{\partial x}\right|^{-\Delta/2}\rightarrow\left(\frac{dw}{dz}\right)^{-\Delta/2}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\Delta/2}. \tag{2.107}$$

We can also consider a non-trivial spin label and define quasi-primary fields as those transforming as

$$\phi'\left(w,\bar{w}\right)=\left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi\left(z,\bar{z}\right) \tag{2.108}$$

where the conformal dimensions h and $\overline{h}$ are defined as

$$h \equiv \frac{1}{2}\left(\Delta + s\right), \qquad \overline{h} \equiv \frac{1}{2}\left(\Delta - s\right). \tag{2.109}$$

Notice that for $s = 0$, we obtain (2.107).

The definition of quasi-primary fields given above is exclusive for global transformations. If it happens that a field transforms in this fashion for any *local* transformation, then we call it *primary* field. In this case, the fields have a well-defined conformal dimension $(h, \overline{h})$, independent of conformal transformations. Then, we can say that these dimensions are an intrinsic property of the field itself. In addition, every primary field is also quasi-primary but the converse is not true. If a field is not primary we call it *secondary*.

Having introduced primary fields, we will start to look for quantum implications, just as we did for quasi-primary fields. That is, we are now going to study correlation functions and Ward identities for this generalized notion of primary fields.

2.5 1+1 Quantum Conformal Symmetry

In this section we will translate the results from our $D-$dimensional discussion about quantum conformal symmetry to the two-dimensional case, and also make it compatible with complex notation. Considering a pair of primary fields, we know by definition that

$$\left\langle \phi_1\left(z_1, \overline{z}_1\right) \phi_2\left(z_2, \overline{z}_2\right) \right\rangle = \prod_{i=1}^{2} \left(\frac{dz'}{dz}\right)^{h_i}_{z=z_i} \left(\frac{d\overline{z}'}{d\overline{z}}\right)^{\overline{h}_i}_{\overline{z}=\overline{z}_i} \left\langle \phi_1\left(z'_1, \overline{z}'_1\right) \phi_2\left(z'_2, \overline{z}'_2\right) \right\rangle. \tag{2.110}$$

Along the lines of what we did in the $D-$dimensional case, we will consider each conformal transformation and use the covariance of correlation functions. Considering translations and rotations,

$$\left\langle \phi_1\left(z_1, \overline{z}_1\right) \phi_2\left(z_2, \overline{z}_2\right) \right\rangle = f\left(\left(z_{ij}\overline{z}_{ij}\right)^{\frac{1}{2}}\right), \tag{2.111}$$

where $z_{ij} = z_i - z_j$. By scale transformations $(z', \overline{z}') = \lambda^2 \left(z, \overline{z}\right)$, we see that

$$\frac{dz'}{dz} = \frac{d\overline{z}'}{d\overline{z}} = \lambda^2. \tag{2.112}$$

Then, the correlation function must satisfy

$$\left\langle \phi_1\left(z_1, \overline{z}_1\right) \phi_2\left(z_2, \overline{z}_2\right) \right\rangle = \prod_{i=1}^{2} \lambda_i^{2\left(h_i + \overline{h}_i\right)} \left\langle \phi_2\left(\lambda\left(z_1, \overline{z}_1\right)\right) \phi_2\left(\lambda\left(z_2, \overline{z}_2\right)\right) \right\rangle. \tag{2.113}$$

Combining the last two constraints,

$$f\left(\left(z_{ij}\overline{z}_{ij}\right)^{\frac{1}{2}}\right) = \prod_{i=1}^{2} \lambda^{2\left(h_i + \overline{h}_i\right)} f\left(\lambda\left(z_{ij}\overline{z}_{ij}\right)^{\frac{1}{2}}\right) \tag{2.114}$$

Again, we can find a differential equation deriving the above with respect to λ, and the solution for this equation will be

$$\langle \phi_1 (z_1, \bar{z}_1) \phi_2 (z_2, \bar{z}_2) \rangle = \frac{C_{12}}{(z_1 - z_2)^{h_1 + h_2} (\bar{z}_1 - \bar{z}_2)^{\bar{h}_1 + \bar{h}_2}}. \tag{2.115}$$

Finally, SCT forces the fields within correlation functions to have the same conformal dimension $h_1 = h_2 = h$ and $\bar{h}_1 = \bar{h}_2 = \bar{h}$, and therefore

$$\langle \phi_1 (z_1, \bar{z}_1) \phi_2 (z_2, \bar{z}_2) \rangle = \frac{C_{12}}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}}. \tag{2.116}$$

is the full constrained correlation function, arbitrary only up to a constant factor.

For the three-point correlation functions of primary fields the derivation follows from the same procedure above. The result is the same obtained in the general $D-$dimensional case if we observe the decoupling of holomorphic and anti-holomorphic sectors,

$$\begin{aligned} \langle \phi_1 (z_1, \bar{z}_1) \phi_2 (z_2, \bar{z}_2) \phi_3 (z_3, \bar{z}_3) \rangle = & \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{-h_1+h_3+h_2} z_{31}^{h_1-h_2+h_3}} \times \\ & \times \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{-\bar{h}_1+\bar{h}_3+\bar{h}_2} \bar{z}_{31}^{\bar{h}_1-\bar{h}_2+\bar{h}_3}}. \end{aligned} \tag{2.117}$$

The four point function (and beyond) are not completely constrained as we saw before, and we will not discuss them further.

Let us now translate the Ward identities. Considering $D = 2$ we have

$$\partial_\mu \left\langle T^\mu_{\ \nu} (x) X_{c_1 c_2 \ldots c_n} \right\rangle = - \sum_{i=1}^{n} \delta^{(2)} (x - x_i) \partial^i_\nu \left\langle X_{c_1 c_2 \ldots c_n} \right\rangle, \tag{translation}$$

$$\varepsilon^{\mu\nu} \left\langle T_{\mu\nu}(x) X_{c_1 c_2 \ldots c_n} \right\rangle = -i \sum_{i=1}^{n} \delta^{(2)} (x - x_i) s_i \left\langle X_{c_1 c_2 \ldots c_n} \right\rangle, \tag{rotations}$$

$$\left\langle T^\mu_{\ \mu}(x) X_{c_1 c_2 \ldots c_n} \right\rangle = - \sum_{i=1}^{n} \delta^{(2)} (x - x_i) \Delta_i \left\langle X_{c_1 c_2 \ldots c_n} \right\rangle, \tag{scale}$$

where s_i is the spin of the field ϕ_{c_i}[5]. We may now rewrite them in conjugate coordinates, that is, we need to rewrite the delta functions and the components of the energy-momentum tensor in the complex setting we are working on. The delta function can be represented in conjugate coordinates as[6]

$$\delta^{(2)} (z, \bar{z}) = \frac{1}{\pi} \partial_z \frac{1}{\bar{z}} = \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z}. \tag{2.118}$$

5 In two dimensions we can write the rotation group generators as $S^i_{\mu\nu} = s_i \varepsilon_{\mu\nu}$, with s_i being the spin of the corresponding field.

6 A discussion about this representation is on appendix (A.1).

The energy-momentum terms can be rewritten as

$$\partial_\mu T^\mu_\nu = 2\left(\partial_z T_{\overline{z}\nu} + \partial_{\overline{z}} T_{z\nu}\right), \qquad T^\mu_\mu = 2\left(T_{\overline{z}z} + T_{z\overline{z}}\right). \tag{2.119}$$

Plugging this into the identities,

$$2\pi\partial_z \left\langle T_{\overline{z}z} X_{c_1c_2\ldots c_n}\right\rangle + 2\pi\partial_{\overline{z}} \left\langle T_{zz} X_{c_1c_2\ldots c_n}\right\rangle = -\sum_{i=1}^{n} \partial_{\overline{z}} \frac{1}{z-w_i}\partial_{w_i} \left\langle X_{c_1c_2\ldots c_n}\right\rangle, \tag{2.120}$$

$$2\pi\partial_{\overline{z}} \left\langle T_{z\overline{z}} X_{c_1c_2\ldots c_n}\right\rangle + 2\pi\partial_z \left\langle T_{\overline{z}\overline{z}} X_{c_1c_2\ldots c_n}\right\rangle = -\sum_{i=1}^{n} \partial_z \frac{1}{\overline{z}-\overline{w}_i}\partial_{\overline{w}_i} \left\langle X_{c_1c_2\ldots c_n}\right\rangle, \tag{2.121}$$

$$2\left\langle \left(T_{\overline{z}z} - T_{z\overline{z}}\right) X_{c_1c_2\ldots c_n}\right\rangle = -\sum_{i=1}^{n} \delta^{(2)}\left(z-w_i, \overline{z}-\overline{w}_i\right) s_i \left\langle X_{c_1c_2\ldots c_n}\right\rangle, \tag{2.122}$$

$$2\left\langle \left(T_{\overline{z}z} + T_{z\overline{z}}\right) X_{c_1c_2\ldots c_n}\right\rangle = -\sum_{i=1}^{n} \delta^{(2)}\left(z-w_i, \overline{z}-\overline{w}_i\right) \Delta_i \left\langle X_{c_1c_2\ldots c_n}\right\rangle. \tag{2.123}$$

Summing (2.122) and (2.123), and subtracting the result in (2.120), we obtain the first expression below. Also, subtracting (2.122) from (2.123), and subtracting the result from (2.121), we get the second relation:

$$\begin{aligned} &\partial_{\overline{z}} \left[2\pi \left\langle T_{zz} X_{c_1c_2\ldots c_n}\right\rangle - \sum_{i=1}^{n}\left(\partial_z \frac{h_i}{z-w_i} - \partial_{\overline{z}}\frac{1}{z-w_i}\partial_{w_i}\right)\left\langle X_{c_1c_2\ldots c_n}\right\rangle\right] = 0, \\ &\partial_z \left[2\pi \left\langle T_{\overline{z}\overline{z}} X_{c_1c_2\ldots c_n}\right\rangle - \sum_{i=1}^{n}\left(\partial_{\overline{z}} \frac{\overline{h}_i}{\overline{z}-\overline{w}_i} - \partial_z\frac{1}{\overline{z}-\overline{w}_i}\partial_{\overline{w}_i}\right)\left\langle X_{c_1c_2\ldots c_n}\right\rangle\right] = 0, \end{aligned} \tag{2.124}$$

where we have introduced the conformal dimensions $(h, \overline{h})$, which is only well-defined for primary fields. In order to get rid of 2π terms, we consider a renormalized energy-momentum tensor

$$T = -2\pi T_{zz}, \qquad \overline{T} = -2\pi T_{\overline{z}\overline{z}}. \tag{2.125}$$

Explicitly applying the derivatives in (2.124), we get

$$\partial_{\overline{z}} \left[\left\langle T X_{c_1c_2\ldots c_n}\right\rangle - \sum_{i=1}^{n}\left(\frac{h_i}{(z-w_i)^2} + \partial_{\overline{z}}\frac{1}{z-w_i}\partial_{w_i}\right)\left\langle X_{c_1c_2\ldots c_n}\right\rangle\right] = 0, \tag{2.126}$$

$$\partial_z \left[\left\langle \overline{T} X_{c_1c_2\ldots c_n}\right\rangle - \sum_{i=1}^{n}\left(\frac{\overline{h}_i}{(\overline{z}-\overline{w}_i)^2} + \partial_z\frac{1}{\overline{z}-\overline{w}_i}\partial_{\overline{w}_i}\right)\left\langle X_{c_1c_2\ldots c_n}\right\rangle\right] = 0. \tag{2.127}$$

Recalling our discussion about holomorphic and anti-holomorphic functions, we know that (2.126) must be a holomorphic function, and (2.127) a anti-holomorphic one. Therefore, up to a regular (anti-)holomorphic function $(\overline{F})F$, the following must hold

$$\begin{aligned} \left\langle T X_{c_1c_2\ldots c_n}\right\rangle &= \sum_{i=1}^{n}\left(\frac{h_i}{(z-w_i)^2} + \partial_{\overline{z}}\frac{1}{z-w_i}\partial_{w_i}\right)\left\langle X_{c_1c_2\ldots c_n}\right\rangle + F(z), \\ \left\langle \overline{T} X_{c_1c_2\ldots c_n}\right\rangle &= \sum_{i=1}^{n}\left(\frac{\overline{h}_i}{(\overline{z}-\overline{w}_i)^2} + \partial_z\frac{1}{\overline{z}-\overline{w}_i}\partial_{\overline{w}_i}\right)\left\langle X_{c_1c_2\ldots c_n}\right\rangle + \overline{F}(\overline{z}). \end{aligned} \tag{2.128}$$

One should notice that both expressions were derived upon the requirement that X is a string of primary fields. Suppose we are considering an arbitrary local field ϕ. If ϕ is primary, then its correlation function with the energy-momentum tensor must satisfy the above expressions, otherwise ϕ is secondary. Additionally, for primary fields, the structure of quadratic singularities in (2.128) provides us the conformal dimensions $(h, \bar{h})$. So we can use (2.128) to decide whether or not a arbitrary field is primary, and what are its conformal dimensions. Therefore, (2.128) can be simplified further. The regular functions can be left out in this analysis, and looking only on what is inside correlation functions, the following structure is noticed:

$$TX_{c_1c_2\ldots c_n} \sim \sum_{i=1}^{n} \left(\frac{h_i}{(z-w_i)^2} + \partial_{\bar{z}} \frac{1}{z-w_i} \partial_{w_i} \right) X_{c_1c_2\ldots c_n} \tag{2.129}$$

That is, the product of T and $X_{c_1c_2\ldots c_n}$ is being decomposed by an expansion consisting of singular complex coefficients $c_i(z-w_i)$ multiplied by regular operators. Such expression is called operator product expansion (OPE), and we always understand them within correlation functions. Considering two arbitrary local operators A and B, their OPE is given by

$$A(z,\bar{z})B(w,\bar{w}) = \sum_i c_i(z-w, \bar{z}-\bar{w}) \mathcal{O}_i(w,\bar{w}), \tag{2.130}$$

where $c_i(z-w, \bar{z}-\bar{w})$ are the singular complex coefficients when $z \to w$ and $\bar{z} \to \bar{w}$, and $\mathcal{O}_i(w,\bar{w})$ are regular operators. Let us see some examples on how to calculate OPEs.

Free Scalar Boson

Consider a massless real scalar field described by the action

$$S = \frac{g}{2} \int d^2x \, \partial_\mu \varphi \partial^\mu \varphi, \tag{2.131}$$

where g is a normalization constant. Usually we redefine the field to absorb this factor, so it is not a relevant constant. However, will be convenient to insist on keeping it because when φ is compact, the g factor cannot be absorbed (we will discuss this in detail next section). This example is actually one of the most simple in conformal field theory, where the canonical energy momentum tensor is already symmetric and traceless:

$$T^\mu_\nu = g(\partial^\mu \varphi)(\partial_\nu \varphi) - \frac{g}{2} \delta^\mu_\nu (\partial_\alpha \varphi)(\partial^\alpha \varphi). \tag{2.132}$$

The two-point correlation function (or the propagator) $K(x,y)$ will be such that

$$-g\nabla_x^2 K(x,y) = \delta^{(2)}(x-y), \tag{2.133}$$

where ∇_x^2 is the Laplacian acting on the coordinate x. The propagator is also translational and rotational invariant[7], $K(x,y) = K(r)$ with $r = |x-y|$. Integrating the above from the origin to r,

$$-2\pi g \int_0^r d\rho\, \partial_\rho \left(\rho \frac{dK}{d\rho}\right) = -2\pi g r \frac{dK}{dr} = 1. \tag{2.134}$$

Integrating once more, we obtain the two-point function

$$K\left(r\right) = -\frac{1}{2\pi g} \ln r + cte. \tag{2.135}$$

We can write it in conjugate coordinates,

$$\left\langle \varphi\left(z,\bar{z}\right) \varphi\left(w,\overline{w}\right) \right\rangle = -\frac{1}{4\pi g}\left[\ln\left(z-w\right) + \ln\left(\bar{z}-\overline{w}\right)\right] + cte. \tag{2.136}$$

At this point we can already conclude that φ is not a primary field. We have discussed earlier that the two point function of quasi-primary fields is fixed, up to a constant, to be like (2.116) – a power law in the relative coordinates, where the holomorphic and anti-holomorphic sector are decoupled. However, the correlation function we found is not a power law, so the scalar field φ cannot be quasi-primary, and therefore it is not primary. On the other hand, we see that taking holomorphic (or anti-holomorphic) derivatives,

$$\left\langle \partial_z \varphi\left(z,\bar{z}\right) \partial_w \varphi\left(w,\overline{w}\right) \right\rangle = -\frac{1}{4\pi g} \frac{1}{\left(z-w\right)^2}, \tag{2.137}$$

the structure of quasi-primary fields appear. If fact, we can show that $\partial\varphi$ is a primary field. Notice that the holomorphic component of the energy-momentum tensor is

$$T\left(z\right) = -2\pi g :\partial\varphi\left(z\right)\partial\varphi\left(z\right):, \tag{2.138}$$

where the normal ordering

$$:\partial\varphi\left(z\right)\partial\varphi\left(z\right): = \lim_{z\to w}\left(\partial\varphi\left(z\right)\partial\varphi\left(w\right) - \left\langle\partial\varphi\left(z\right)\partial\varphi\left(w\right)\right\rangle\right). \tag{2.139}$$

must be introduced to get rid of divergences due to the multiplication of quantum fields at the same point, and we have shortened the notation $\partial\varphi(z) \equiv \partial_z\varphi(z,\bar{z})$. The product we are looking for is

$$T(z)\partial_w\varphi(w) = -2\pi g :\partial\varphi\left(z\right)\partial\varphi\left(z\right): \partial\varphi(w). \tag{2.140}$$

At this point we can use Wick's theorem (assuming a calculation within correlation functions), and therefore

$$T(z)\partial\varphi(w) \sim -4\pi g \partial\varphi\left(z\right)\partial\varphi\left(w\right)\partial\varphi\left(z\right). \tag{2.141}$$

[7] One see this by applying these transformations on (2.25). For translations, $\varphi'(x+a) = \varphi(x)$ and therefore $\langle\varphi(x+a)\varphi(y+a)\rangle = \langle\varphi(x)\varphi(y)\rangle$.

Using that the contraction coincides with the propagator (2.137),

$$T(z)\,\partial\varphi(w) \sim \frac{\partial\varphi(z)}{(z-w)^2}. \tag{2.142}$$

This is not entirely at the form of an OPE, but if we simply expand $\partial\varphi(z)$ at w,

$$T(z)\,\partial\varphi(w) \sim \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w} \tag{2.143}$$

is on the expected form. Therefore, $\partial\varphi$ (and its anti-holomorphic counterpart) is a primary field with conformal dimension $h = 1$. We can also determine the OPE of the energy momentum tensor with itself,

$$\begin{aligned} T(z)T(w) &= (2\pi g)^2 : \partial\varphi(z)\partial\varphi(z) :: \partial\varphi(w)\partial\varphi(w) : \\ &\sim (2\pi g)^2 \left(4\,\overbrace{\partial\varphi(z)\partial\varphi(w)}\partial\varphi(z)\partial\varphi(w) + 2\,\overbrace{\partial\varphi(z)\partial\varphi(w)}\overbrace{\partial\varphi(z)\partial\varphi(w)} \right) \\ &\sim \frac{1}{2(z-w)^4} - \frac{4\pi g : \partial\varphi(z)\,\partial\varphi(w) :}{(z-w)^2}. \end{aligned} \tag{2.144}$$

Expanding at w and using the fact that $\partial T(w) = -4\pi g : \partial^2\varphi(w)\,\partial\varphi(w) :$, we obtain

$$T(z)\,T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{2.145}$$

Hence, the enery-momentum tensor OPE has an anomalous quartic singularity meaning that it is not a primary field. In the next example of the free fermion, we will see that the corresponding OPE of the energy-momentum tensor with itself has the same structure as the above.

Free Fermion

Let us consider the action for the free Majorana fermion

$$S = \frac{g}{2}\int d^2x\, \mathbf{\Psi}^\dagger \gamma^0 \gamma^\mu \partial_\mu \mathbf{\Psi}, \tag{2.146}$$

where the matrices γ^μ are generators of the Clifford algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{2.147}$$

and the Majorana fermion is represented as

$$\mathbf{\Psi} = \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix}, \qquad \mathbf{\Psi}^\dagger = \begin{pmatrix} \psi & \overline{\psi} \end{pmatrix}. \tag{2.148}$$

It should be noticed that the bar on the components of the spinor stands for the anti-holomorphic component, and should not be confused with the Dirac conjugate. For the two-dimensional Euclidean metric, we have in the Dirac representation that

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.149}$$

Then, we can explicitly write the action as

$$S = \frac{g}{2} \int dz d\bar{z}\, \boldsymbol{\Psi}^\dagger \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \boldsymbol{\Psi} = \frac{g}{2} \int dz d\bar{z} \left(\psi \bar{\partial} \psi + \overline{\psi} \partial \overline{\psi} \right). \tag{2.150}$$

The equations of motion are $\partial \overline{\psi} = 0$ and $\bar{\partial} \psi = 0$, simply meaning that the (anti)holomorphic component is a (anti)holomorphic function. The propagator K will be such that

$$g \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} K_{zz} & K_{z\bar{z}} \\ K_{\bar{z}z} & K_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \delta^{(2)}(z - w, \bar{z} - \overline{w}). \tag{2.151}$$

Using the delta Dirac representation in conjugate coordinates, the solutions can readily be derived,

$$K_{zz} = \langle \psi(z, \bar{z}) \psi(w, \overline{w}) \rangle = \frac{1}{2\pi g} \frac{1}{z - w}, \qquad K_{\bar{z}\bar{z}} = \langle \overline{\psi}(z, \bar{z}) \overline{\psi}(w, \overline{w}) \rangle = \frac{1}{2\pi g} \frac{1}{\bar{z} - \overline{w}}, \tag{2.152}$$

and the remaining functions are null, $K_{z\bar{z}} = K_{\bar{z}z} = 0$. Before we calculate the OPE with the energy-momentum tensor, it is useful to notice that

$$\langle \partial \psi(z, \bar{z}) \psi(w, \overline{w}) \rangle = -\frac{1}{2\pi g} \frac{1}{(z - w)^2}, \qquad \langle \partial \psi(z, \bar{z}) \partial \psi(w, \overline{w}) \rangle = -\frac{1}{\pi g} \frac{1}{(z - w)^3}. \tag{2.153}$$

The energy momentum tensor is given by

$$T_{\mu\nu} = \frac{\mathcal{L}}{\partial \partial_\mu \boldsymbol{\Psi}} \partial_\nu \boldsymbol{\Psi} - g_{\mu\nu} \mathcal{L}. \tag{2.154}$$

Considering holomorphic components $\mu = \nu = z$, we have that

$$T(z) = -\pi g : \psi(z) \partial \psi(z) : . \tag{2.155}$$

Therefore,

$$\begin{aligned} T(z)\psi(w) &= -\pi g : \psi(z) \partial \psi(z) : \psi(w) \\ &\sim -\pi g \partial \overbrace{\psi(z) \psi(w)} \psi(z) + \pi g \overbrace{\psi(z) \psi(w)} \partial \psi(z), \end{aligned} \tag{2.156}$$

where we have taken into account the fermionic statistics to flip signs when needed. Using the correlation functions we calculated earlier, and expanding at w, we obtain

$$T(z)\psi(w) \sim \frac{1}{2} \frac{\psi(w)}{(z - w)^2} + \frac{\partial \psi(w)}{z - w}, \tag{2.157}$$

concluding that $\psi(w)$ is a primary field with conformal dimension $h = \frac{1}{2}$. We can calculate the OPE of the energy-momentum tensor with itself, obtaining

$$T(z)T(w) \sim \frac{1/4}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w}, \tag{2.158}$$

and again we conclude that the energy-momentum tensor it is not a primary field because of an anomalous quartic singularity.

One should have noticed that the above expression is very similar to the one we obtained in the case of a free boson, the only difference being the constant coefficient in the anomalous contribution. In fact, we can argue that in a general CFT the structure of $T(z)T(w)$ will have a model-dependent constant c in the quartic anomaly. This constant is called the *central charge*, and it is 1 for the free boson, and $1/2$ for the free fermion. Notice that in two dimensions the energy-momentum tensor has scale dimension $\Delta[T_{\mu\nu}] = 2$, since $\Delta[T_{\mu\nu}] = [\mathcal{L}]$. Then, by consistency, all terms in the OPE of $T(z)$ with itself must be of the form

$$\frac{\mathcal{O}_i}{(z-w)^i}, \tag{2.159}$$

where the scale dimension $\Delta[\mathcal{O}_i] = 4 - i$. Particularly for $i = 4$, $\mathcal{O}_i$ must have zero scale dimension, and then, it must be a constant. Of course, we are considering only operators with positive scale dimension, because in a unitary CFT there are no operator with negative scale dimension. This will become clear when we introduce Virasoro algebra and study the consequences of unitarity in the next chapter.

CHAPTER 3

Canonical Formalism

3.1 Radial Quantization

The canonical formalism in the Hilbert space breaks Poincaré invariance, and then, one needs to choose a reference frame. In other words, we need to define space and time axes. However, does not seem to have a natural choice in an Euclidean signature, where time and space are somewhat indistinguishable [5]. Let us consider a cylinder of circumference L with time coordinate running along $-\infty < t < \infty$ and the space coordinate compactified $x \sim x + L$, as the illustration below. This coordinate

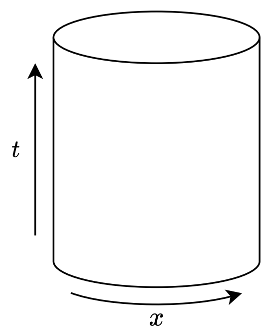

Figure 1 – Coordinate choice on the cylinder.

choice is natural from a string theory perspective, where the world-sheet of a closed string in Euclidean coordinates is a cylinder [6]. Then, any point in the cylinder can be identified by a complex number $\xi = t + ix$. In fact, we can map the cylinder to the complex plane by a conformal map defined simply as

$$\xi \longmapsto z = e^{\frac{2\pi}{L}t} \cdot e^{\frac{2\pi i}{L}x}. \tag{3.1}$$

In complex notation it is clear that $|z| = e^{\frac{2\pi}{L}t}$ and $\arg(z) = 2\pi x/L$. We can picture this conformal map as if we had opened the cylinder inside out. The origin corresponds to an infinite past time, and for each time t, we map the space coordinates to a concentric circle around the origin of radius $|z|$, see the figure below.

3.1.1 Quantization

The quantization procedure of a field theory whose base space is conformally mapped to radial coordinates is called *radial quantization*. This procedure shows us

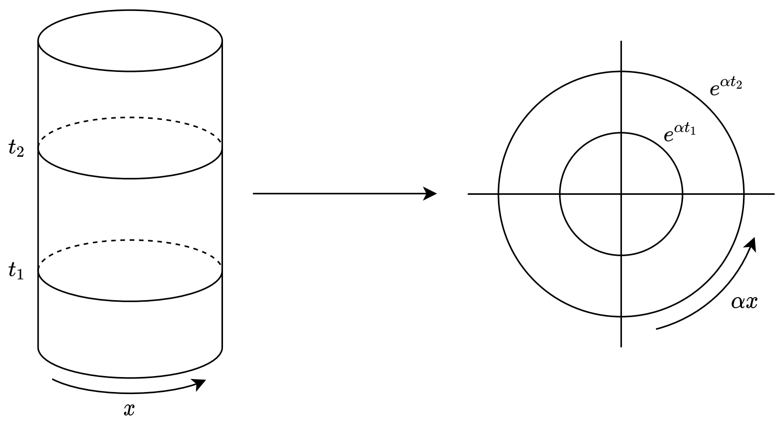

Figure 2 – Conformal mapping with $\alpha = 2\pi/L$.

quite clearly that conformal field theories exhibit a bijection between states and operators, fact that is somewhat obscure by the usual quantization method [5]. Also, by this radial map, we can use all the machinery of holomorphic functions, since we get back to the complex plane.

We can already appreciate some immediate implications of radial quantization. Notice that time translations are equivalent to scale transformations. Performing a time shift $t \mapsto t + a$, we see that

$$z = e^{\frac{2\pi}{L}t} \cdot e^{\frac{2\pi i}{L}x} \longmapsto e^{\frac{2\pi}{L}a} z. \tag{3.2}$$

Hence, we can expect that the Hamiltonian, which generates time translations, must be proportional to the dilatation generator. Similarly, the momentum operator, that generates space translations, will be proportional to the rotations generator.

Let us start by assuming the existence of a vacuum state in the free theory (whose Hamiltonian is a bilinear function of the quantum fields). We can strictly define creation and annihilation operators by their action in a chosen basis of the Hilbert space. However, we have not introduced a basis yet. So we proceed equivalently defining annihilation operators as those annihilating the vacuum state, and the creation operators as their adjoint [7]. We also consider the same Hilbert space in both free and interacting theory, and define asymptotic free primary fields Φ_{in} as satisfying

$$\langle \alpha | \Phi_{in} | \beta \rangle = \lim_{z,\bar{z} \to 0} \langle \alpha | \Phi(z, \bar{z}) | \beta \rangle, \tag{3.3}$$

for any states $|\alpha\rangle$ and $|\beta\rangle$[1]. We can also define asymptotic states as

$$|\Phi_{in}\rangle = \lim_{z,\bar{z} \to 0} \Phi(z, \bar{z}) |0\rangle, \tag{3.4}$$

where $|0\rangle$ is the vacuum state [5]. In order to introduce *out* asymptotic fields and states, we need to describe how the hermitian conjugate is implemented in this quantization approach.

[1] See [7] for a detailed discussion.

3.1.2 Hermitian Conjugate

In an Euclidean space, the hermitian conjugation flips the sign of the time coordinate $t = ix^0 \mapsto -t$. Considering radial coordinates, this is reflected as

$$z = e^{\alpha t} e^{i\alpha x} \longmapsto e^{-\alpha t} e^{i\alpha x} = \frac{1}{e^{\alpha t} e^{-i\alpha x}} = \frac{1}{\overline{z}}, \tag{3.5}$$

and also $\overline{z} \mapsto 1/z$. Notice that one immediate implication of this change in sign is

$$|\Psi_{in}\rangle^\dagger = \lim_{z,\overline{z}\to 0} \langle 0| \left[\Phi(z,\overline{z})\right]^\dagger . \tag{3.6}$$

As we just discussed, coordinates are also conjugated, hence

$$|\Psi_{in}\rangle^\dagger = \lim_{z,\overline{z}\to 0} \langle 0|\Phi^\dagger(1/\overline{z},1/z) = \lim_{\xi,\overline{\xi}\to\infty} \langle 0|\Phi^\dagger\left(\xi,\overline{\xi}\right), \tag{3.7}$$

where we change variables $\xi = 1/\overline{z}$ and $\overline{\xi} = 1/z$. To find a consistent prescription, we assume that hermitian conjugation must be of the form

$$\Phi^\dagger\left(\xi,\overline{\xi}\right) = c\left(\xi,\overline{\xi}\right) \Phi\left(\xi,\overline{\xi}\right), \tag{3.8}$$

that is, a complex function c times the original field calculated at the conjugated coordinates. One could consider a more general form, but this linear choice is already sufficient. In order to fix this complex function, see that $|\Psi_{in}\rangle^\dagger$ is actually an asymptotic *out* bra state,

$$\langle\Phi_{out}| = \lim_{\xi,\overline{\xi}\to\infty} \langle 0|\Phi^\dagger\left(\xi,\overline{\xi}\right). \tag{3.9}$$

Then, given that the transition amplitude between in and out free states must be finite,

$$\begin{aligned}\langle\Phi_{out}|\Phi_{in}\rangle &= \lim_{\substack{z,\overline{z}\to 0\\ \xi,\overline{\xi}\to\infty}} \langle 0|\Phi^\dagger\left(\xi,\overline{\xi}\right) \Phi(z,\overline{z})|0\rangle = \\ &= \lim_{\substack{z,\overline{z}\to 0\\ \xi,\overline{\xi}\to\infty}} c\left(\xi,\overline{\xi}\right) \langle 0|\Phi\left(\xi,\overline{\xi}\right) \Phi(z,\overline{z})|0\rangle,\end{aligned} \tag{3.10}$$

we need to choose a function c such that the above limit exists. Notice that inside the given limit, we can assume a correlation function

$$\langle\Phi_{out}|\Phi_{in}\rangle = \lim_{\substack{z,\overline{z}\to 0\\ \xi,\overline{\xi}\to\infty}} c\left(\xi,\overline{\xi}\right) \langle\Phi\left(\xi,\overline{\xi}\right) \Phi(z,\overline{z})\rangle, \tag{3.11}$$

since the fields are already time-ordered, one is associated with an infinite past time and the other with an infinite future. Using the fact that Φ is a primary field (2.116),

$$\langle\Phi_{out}|\Phi_{in}\rangle = C \lim_{\substack{z,\overline{z}\to 0\\ \xi,\overline{\xi}\to\infty}} \frac{c\left(\xi,\overline{\xi}\right)}{(\xi-z)^{2h}(\overline{\xi}-\overline{z})^{2\overline{h}}} = C \lim_{\xi,\overline{\xi}\to\infty} \frac{c\left(\xi,\overline{\xi}\right)}{\xi^{2h}\overline{\xi}^{2\overline{h}}}. \tag{3.12}$$

Therefore, choosing $c\left(\xi,\overline{\xi}\right)=\xi^{2h}\overline{\xi}^{2\overline{h}}$, the transition amplitude becomes a finite constant

$$\langle \Phi_{out}|\Phi_{in}\rangle = C, \tag{3.13}$$

and the prescription for the hermitian conjugate is

$$\Phi^{\dagger}\left(\xi,\overline{\xi}\right)=\xi^{2h}\overline{\xi}^{2\overline{h}}\Phi\left(\xi,\overline{\xi}\right). \tag{3.14}$$

3.1.3 Mode Expansion

Now that we have defined the quantization procedure, let us consider the mode expansion of the primary field Φ as the Laurent series

$$\Phi(z,\overline{z})=\sum_{m,n\in\mathbb{Z}} z^{-m-h}\overline{z}^{-m-\overline{h}}\Phi_{mn}, \tag{3.15}$$

where the expansion coefficients are given by

$$\Phi_{mn}=\frac{1}{2\pi i}\oint_C dz\, z^{m+h-1}\frac{1}{2\pi i}\oint_C d\overline{z}\,\overline{z}^{n+\overline{h}-1}. \tag{3.16}$$

Notice that we added conformal weights $(h,\overline{h})$ on the expansion. This is because primary fields transform from the plane to the cylinder by a factor of $z^h\overline{z}^{\overline{h}}$. As we discussed, $z=e^{\alpha\xi}$ maps the cylinder to the plane, so the inverse mapping is given by $\xi=(1/\alpha)\ln z$. If Φ is primary with conformal dimensions $(h,\overline{h})$,

$$\Phi_{\text{cylinder}}\left(\xi,\overline{\xi}\right)=\left(\frac{d\xi}{dz}\right)^{-h}\left(\frac{d\overline{\xi}}{d\overline{z}}\right)^{-\overline{h}}\Phi_{\text{plane}}\left(z,\overline{z}\right)=\frac{z^h\overline{z}^{\overline{h}}}{\alpha^2}\Phi_{\text{plane}}\left(z,\overline{z}\right). \tag{3.17}$$

These factors are also important in the context of hermitian conjugation. In the real surface ($z^*=\overline{z}$), the hermitian conjugate is

$$\Phi^{\dagger}(z,\overline{z})=\sum_{m,n\in\mathbb{Z}}\overline{z}^{-m-h}z^{-n-\overline{h}}\Phi^{\dagger}_{m,n}. \tag{3.18}$$

On the other hand, using our prescription,

$$\Phi^{\dagger}(z,\overline{z})=\overline{z}^{-2h}z^{-2\overline{h}}\Phi(1/\overline{z},1/z)=\sum_{m,n\in\mathbb{Z}}\overline{z}^{-m-h}z^{-n-\overline{h}}\Phi_{-m,-n}. \tag{3.19}$$

Therefore, both expressions are compatible when

$$\Phi^{\dagger}_{m,n}=\Phi_{-m,-n}, \tag{3.20}$$

showing that the choice of conformal weight in the expansions also ensures this known expression for hermitian conjugation.

Upon radial quantization, the expansion coefficients $\phi_{m,n}$ become operators in the Hilbert space such that in and out states must be well-defined. Notice that using the mode expansion,

$$\begin{aligned}
|\Phi_{in}\rangle &= \lim_{z,\bar{z}\to 0} \Phi(z,\bar{z})|0\rangle \\
&= \lim_{z,\bar{z}\to 0} \sum_{m=-\infty}^{-h} \sum_{n=-\infty}^{-\bar{h}} z^{-m-h}\bar{z}^{-m-\bar{h}}\Phi_{m,n}|0\rangle \\
&\quad + \lim_{z,\bar{z}\to 0} \sum_{m=-h+1}^{\infty} \sum_{n=-\bar{h}+1}^{\infty} z^{-m-h}\bar{z}^{-m-\bar{h}}\Phi_{m,n}|0\rangle.
\end{aligned} \tag{3.21}$$

The second sum above is clearly divergent, because it contains negative powers of z and $\bar{z}$. Therefore, we need to impose

$$\Phi_{m,n}|0\rangle = 0, \quad \text{if} \quad m > -h \text{ and } n > -\bar{h}. \tag{3.22}$$

In the following we would like to explore the conformal algebra in the Hilbert space and what are the properties of its generators when acting on quantum states. Two results are fundamental to carry this discussion: the conformal ward identities, and the relation between OPE's and commutation relations.

3.1.4 Radial Ordering

In the radial quantization scheme, time-ordering operators become radial ordering operators defined as

$$\mathcal{R}\left(\Phi_1(z)\Phi_2(w)\right) = \begin{cases} \Phi_1(z)\Phi_2(w), \text{ if} & |z| > |w| \\ \Phi_2(w)\Phi_1(z), \text{ if} & |z| < |w| \end{cases}. \tag{3.23}$$

That is, within radial-ordering the operator associated with the smallest module acts first. For fermions we have yet a sign due to permutations,

$$\mathcal{R}\left(\Psi_1(z)\Psi_2(w)\right) = \begin{cases} \Psi_1(z)\Psi_2(w), \text{ if} & |z| > |w| \\ -\Psi_2(w)\Psi_1(z), \text{ if} & |z| < |w| \end{cases}. \tag{3.24}$$

One important remark about this radial-ordering: when calculating OPE's previously, we always considered expressions within correlation functions, which are time-ordered. Therefore, if we want to make use of these expressions in an operational sense, the radial ordering $|z| > |w|$ must be understood [5].

The conformal Ward identity is an elegant and simple form to write all Ward identities into a single expression. We will not only introduce it for the sake of simplicity, but written that way, the identity will provide us a mean to define conformal charges, and find the Virasoro algebra. Notice that by an infinitesimal conformal variation ϵ_μ, the following holds

$$\partial_\mu(\epsilon_\nu T^{\mu\nu}) = \epsilon_\nu \partial_\mu T^{\mu\nu} + \partial_{(\mu}\epsilon_{\nu)} T^{\mu\nu} + \partial_{[\mu}\epsilon_{\nu]} T^{\mu\nu}, \tag{3.25}$$

where the round brackets indicate symmetrization, and the square brackets indicate antisymmetrization. Using the charactering equations (2.41,2.42), we know that

$$\partial_{(\mu}\epsilon_{\nu)} = \frac{1}{2}(\partial_\alpha \epsilon^\alpha) g_{\mu\nu}. \tag{3.26}$$

Furthermore, the antisymmetric part can be rewritten as

$$\partial_{[\mu}\epsilon_{\nu]} = \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha \epsilon_\beta \varepsilon_{\mu\nu}, \tag{3.27}$$

leading (3.25) to

$$\partial_\mu(\epsilon_\nu T^{\mu\nu}) = \epsilon_\nu \partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\alpha \epsilon^\alpha) T^\mu_\mu + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha \epsilon_\beta \varepsilon_{\mu\nu} T^{\mu\nu}. \tag{3.28}$$

Notice now that if we consider a constant variation ϵ_μ, the above simplifies to

$$\partial_\mu(\epsilon_\nu T^{\mu\nu}) = \epsilon_\nu \partial_\mu T^{\mu\nu}, \tag{3.29}$$

and integrating this expression within a correlation function,

$$\int_M d^2x\, \partial_\mu \langle \epsilon_\nu T^{\mu\nu} X\rangle = \int_M d^2x\, \epsilon_\nu \partial_\mu \langle T^{\mu\nu} X\rangle\,, \tag{3.30}$$

the right-hand side can be identified by the Ward identity (translation) to be

$$\int_M d^2x\, \epsilon_\nu \partial_\mu \langle T^{\mu\nu} X\rangle = -\int_M d^2x\, \epsilon_\nu \sum_{i=1}^{n} \delta^{(2)}\left(x - x_i\right) \partial_i^\nu \langle X_{c_1 c_2 \ldots c_n}\rangle\,. \tag{3.31}$$

Taking M as the region containing all the points x_i of the fields in the string X, we can calculate the integral above,

$$\int_M d^2x\, \epsilon_\nu \partial_\mu \langle T^{\mu\nu} X\rangle = -\sum_{i=1}^{n} \epsilon_\nu \partial_i^\nu \langle X_{c_1 c_2 \ldots c_n}\rangle = \delta_\epsilon \langle X\rangle\,, \tag{3.32}$$

where we have used the definition of variation for correlation functions (2.33) in the last step. The same argument holds for the divergence term and for the curl term in (3.28). So comparing (3.32) with (3.30), all Ward identities can be written as

$$\delta_\epsilon \langle X\rangle = \int_M d^2x\, \partial_\mu \langle \epsilon_\nu T^{\mu\nu} X\rangle\,. \tag{3.33}$$

Applying Gauss theorem (A.2),

$$\int_M d^2x\, \partial_\mu \langle \epsilon_\nu T^{\mu\nu} X\rangle = \frac{i}{2}\oint_{\partial M} \left[d\bar{z} \left\langle \epsilon_\nu T^{\bar{z}\nu} X\right\rangle - dz \langle \epsilon_\nu T^{z\nu} X\rangle \right]. \tag{3.34}$$

However, by the (scale) Ward identity, $\left\langle T^{\bar{z}z}X\right\rangle$ and $\left\langle T^{z\bar{z}}X\right\rangle$ contribute only at the points of the string X, which are within M and not over the contour ∂M. Hence, we may write

$$\int_M d^2x\, \partial_\mu \left\langle \epsilon_\nu T^{\mu\nu} X\right\rangle = \frac{i}{2}\oint_{\partial M}\left[d\bar{z}\left\langle \epsilon_{\bar{z}} T^{\bar{z}\bar{z}} X\right\rangle - dz\left\langle \epsilon_z T^{zz} X\right\rangle\right]. \tag{3.35}$$

Or, using the following definitions,

$$\epsilon \equiv \epsilon^z, \qquad \bar{\epsilon} \equiv \epsilon^{\bar{z}}, \qquad T = -2\pi T^{zz}, \qquad \overline{T} = -2\pi T^{\bar{z}\bar{z}}, \tag{3.36}$$

we obtain the *conformal Ward identity*,

$$\delta_{\epsilon,\bar{\epsilon}}\left\langle X\right\rangle = \frac{1}{2\pi i}\oint_{\partial M} d\bar{z}\,\bar{\epsilon}(\bar{z})\left\langle \overline{T}(\bar{z})X\right\rangle - \frac{1}{2\pi i}\oint_{\partial M} dz\,\epsilon(z)\left\langle T(z)X\right\rangle. \tag{3.37}$$

Let us investigate now how we can relate operator product expansions with commutation relations. Consider the integral

$$\oint_w dz\, a(z)b(w) \tag{3.38}$$

over a positive loop around w, with $a(z)$ and $b(w)$ being holomorphic local fields. We know how to understand product of operators within correlation functions in terms of OPE's, and in this context the integral will have singular contributions at $z = w$, and must be also radially ordered. However, parametrizing this integral as a circle centred at w is not convenient, because the path of integration does not capture the radial ordering explicitly. It would capture if we reparametrize it with circles around the origin, where the conditions of whether $|z| > |w|$ are obvious. Notice that possible singularities in the OPE are poles, and hence the integral is path independent if the region enclosed contains w and no other singularity (see the residues theorem [8]). Let $C_{|w|-\epsilon}$ and $C_{|w|+\epsilon}$ be two circles around the origin, C and C' be paths defined by the illustration below.

See that the integral over C' vanishes if there are no singularities in its enclosure. Therefore, the integral over C can be written as

$$\oint_C dz\, a(z)b(w) = \oint_{C+C'} dz\, a(z)b(w). \tag{3.39}$$

On the other hand, $C + C'$ is equivalent to $C_{|\omega|+\epsilon} + C_{|\omega|-\epsilon}$, because contributions of $C + C'$ between the circles are opposite and cancel out. Furthermore, we generally think of OPE's inside correlation functions with multiple local fields. It can happen that some other field have a singular OPE with $a(z)$ at a point w' , for example. So in order to mitigate this issue, we take the limit of $\epsilon \to 0$. Taking into account the radial ordering explicitly now,

$$\oint_C dz\, a(z)b(w) = \lim_{\epsilon\to 0}\left[\oint_{C_{|\omega|+\epsilon}} dz\, a(z)b(w) - \oint_{C_{|\omega|-\epsilon}} dz\, b(w)a(z)\right]. \tag{3.40}$$

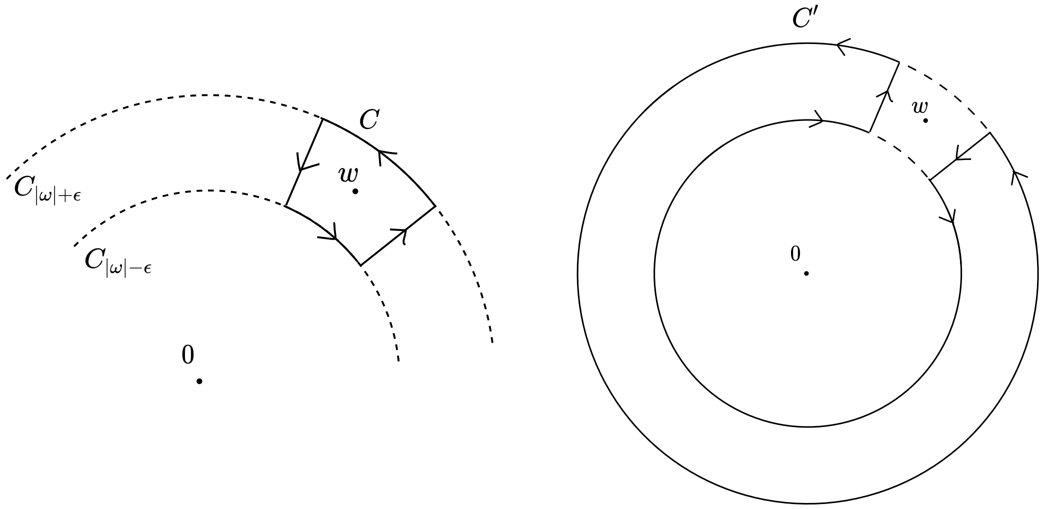

Figure 3 – Reparametrization of the loop integral around w.

The minus sign is due to the change of orientation (from clockwise, to anticlock-wise). The right hand-side can be defined as a commutator,

$$[A, b(w)] \equiv \lim_{\epsilon \to 0} \left[\oint_{C_{|\omega|+\epsilon}} dz\, a(z) b(w) - \oint_{C_{|\omega|-\epsilon}} dz\, b(w) a(z) \right], \tag{3.41}$$

where the operator

$$A \equiv \oint dz\, a(z). \tag{3.42}$$

Because we are taking the limit of $\epsilon \to 0$, the commutator is, by definition, an equitemporal one. As we have mentioned, the integral we started with can be understood (and in fact, calculated) if we consider the OPE of both local fields $a(z)b(w)$ and use the residues theorem to evaluate each singular contribution. Therefore, we conclude that commutators are closely related to OPE's, because the integral defining the commutator is precisely the integral of the product $a(z)b(w)$.

3.2 Virasoro Algebra

As we have mentioned, the conformal Ward identity will help us define the conserved charges which will lead us to the quantum conformal algebra, the Virasoro algebra.

Considering a field Φ, the conformal Ward identity implies

$$\delta_\epsilon \langle \Phi(w) \rangle = -\frac{1}{2\pi i} \oint_C dz\, \langle T(z) \epsilon(z) \Phi(w) \rangle = -\langle [Q_\epsilon, \Phi(w)] \rangle, \tag{3.43}$$

where we have defined the conformal charges

$$Q_\epsilon = \frac{1}{2\pi i} \oint_C dz\, T(z) \epsilon(z). \tag{3.44}$$

These are the generators of the symmetry transformation $\epsilon(z)$. Now, by expanding the components of the energy-momentum tensor,

$$
\begin{aligned}
T(z) &= \sum_{n\in\mathbb{Z}} z^{-n-2}L_n, \qquad L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z), \\
\overline{T}(\bar{z}) &= \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\overline{L}_n, \qquad \overline{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+1}\overline{T}(\bar{z}).
\end{aligned} \tag{3.45}
$$

and also expanding the symmetry transformation parameter ϵ,

$$
\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n, \qquad \epsilon_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}\epsilon(z), \tag{3.46}
$$

we see that the conformal charge can be written as

$$
Q_\epsilon = \frac{1}{2\pi i}\oint dz \sum_{m,n\in\mathbb{Z}} z^{n+1-m-2}\epsilon_n L_m = \sum_{n\in\mathbb{Z}} \epsilon_n L_n. \tag{3.47}
$$

For the last step, notice that

$$
\oint dz\, z^{n-m-1} = 2\pi i \delta_{m+n,0}. \tag{3.48}
$$

Therefore we conclude that the Fourier modes of the energy-momentum tensor also expand conformal charges, hence these modes are the generators of the local conformal group. The algebra of these modes are called Virasoro algebra, which we will derive in what follows. Using the relation between commutators and OPE's,

$$
\begin{aligned}
[L_n, L_m] &= \oint_0 \frac{dw}{2\pi i} w^{n+1} \oint_w \frac{dz}{2\pi i} z^{n+1}\, T(w)T(z) \\
&= \oint_0 \frac{dw}{2\pi i} w^{n+1} \oint_w \frac{dz}{2\pi i} z^{n+1} \left\{ \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \right\},
\end{aligned} \tag{3.49}
$$

where we have not considered regular contributions (their integral vanishes). Using the residue theorem [8], we can calculate the integral over z obtaining

$$
[L_n, L_m] = \oint_0 \frac{dw}{2\pi i} \left[\frac{c}{12} n(n-1)(n+1) w^{n+m-1} + 2(n+1) w^{n+m+1} T(w) + w^{n+m+2}\partial T(w) \right]. \tag{3.50}
$$

The first contribution can be integrated using (4.41), the second is proportional to L_{n+m}. For the last one, see that integrating by parts,

$$
\oint_0 \frac{dw}{2\pi i} \partial\left(w^{n+m+2} T(w) \right) - \oint_0 \frac{dw}{2\pi i} (n+m+2) w^{n+m+1} T(w) = -(n+m+2) L_{m+n}. \tag{3.51}
$$

Plugging all together, the commutator becomes

$$
[L_n, L_m] = \frac{c}{12} n(n^2-1)\delta_{n+m,0} + (n-m) L_{n+m}. \tag{3.52}
$$

Using the same procedure, we find that the anti-holomorphic sector satisfies

$$[\overline{L}_n, \overline{L}_m] = \frac{c}{12} n(n^2 - 1)\delta_{n+m,0} + (n - m)\overline{L}_{n+m}. \tag{3.53}$$

These relations are closed algebras in each sector, because the cross commutator vanishes

$$[L_n, \overline{L}_m] = 0. \tag{3.54}$$

We can realize this by noticing that the OPE $T(z)\overline{T}(\overline{w}) \sim 0$. In other words, $T(z)\overline{T}(\overline{w})$ is a regular function. Recall the general definition of OPE,

$$A(z, \overline{z})B(w, \overline{w}) = \sum_i c_i(z - w, \overline{z} - \overline{w})\mathcal{O}_i(w, \overline{w}), \tag{3.55}$$

where $\mathcal{O}_i(w, \overline{w})$ is a regular field. If $A(z, \overline{z}) = A(z)$ and $B(w, \overline{w}) = B(\overline{w})$ we can take $w \to z$ and $\overline{z} \to \overline{w}$ in both sides, obtaining

$$A(z)B(\overline{w}) = \sum_i c_i(0, 0)\mathcal{O}_i(z, \overline{w}), \tag{3.56}$$

which is a sum of constant coefficients and regular operators. Therefore, in general, $A(z)B(\overline{w}) \sim 0$. This result implies, by the algebra, a decoupling between holomorphic and anti-holomorphic sectors of a conformal field theory, allowing us to simplify the notations by only consider holomorphic dependencies, and easily restore the anti-holomorphic sector when needed.

3.2.1 Hilbert Space

Usually in a relativistic quantum field theory we choose the vacuum state to be invariant under global transformations [7]. In a conformal field theory, we will also make this assumption [5]. Notice that if we impose the action of the energy momentum tensor to be well defined in the vacuum state, particularly for $z \to 0$,

$$\lim_{z \to 0} \sum_{n \in \mathbb{Z}} z^{-n-2} L_n |0\rangle = |\psi\rangle, \tag{3.57}$$

one needs to choose

$$L_n|0\rangle = 0, \quad \text{if} \quad n \geq -1, \tag{3.58}$$

to avoid divergence. So we conclude that the vacuum state is invariant under generators indexed by $n \geq -1$. For the case of $n = 0, \pm 1$, the central extension terms vanish, leaving the remainder Witt algebra, where $\ell_0, \ell_{\pm 1}$ were associated with global transformations. By the means of this correspondence, $L_0, L_{\pm 1}$ are related to global transformations, and therefore, the above expression (4.42) tells us the vacuum is invariant under the global sector. Furthermore, taking the hermitian conjugate,

$$\lim_{z \to 0} \langle 0| \sum_{n \in \mathbb{Z}} z^{-n-2} L_n^\dagger = \langle \psi|. \tag{3.59}$$

The generators L_n are Hermitian operators, so we must also assume

$$\langle 0|L_n = 0, \quad \text{if} \quad n \leq -2. \tag{3.60}$$

Making these both assumptions, the expectation value of the energy-momentum tensor vanishes at the vacuum state[2],

$$\langle 0|T(z)|0\rangle = \left\langle 0 \left| \sum_{n=-\infty}^{-2} z^{-n-2}L_n + \sum_{n=-1}^{\infty} z^{-n-2}L_n \right| 0 \right\rangle = 0. \tag{3.61}$$

These are fundamental discussions in a Hilbert space of a conformal theory, given how frequent the energy-momentum tensor appear in the conformal structure.

Another fundamental, and general discussion, is the creation of asymptotic states by primary fields. We shall now consider this matter. If Φ is a primary field, notice that

$$\begin{aligned}
[L_n, \Phi(w,\overline{w})] &= \oint_w \frac{dz}{2\pi i} z^{n+1} T(z)\Phi(w,\overline{w}) = \\
&= \oint_w \frac{dz}{2\pi i} z^{n+1} \left[\frac{h}{(z-w)^2}\Phi(w,\overline{w}) + \frac{1}{z-w}\partial\Phi(w,\overline{w}) \right] = \\
&= h(n+1)w^n\Phi(w,\overline{w}) + w^{n+1}\partial\Phi(w,\overline{w}).
\end{aligned} \tag{3.62}$$

First we considered the OPE of the energy-momentum tensor with a primary field, and then used the residues theorem to calculate the integrals. Particularly if $n = 0$,

$$[L_0, \Phi(w,\overline{w})] = h\Phi(w,\overline{w}) + w\partial\Phi(w,\overline{w}). \tag{3.63}$$

Considering the origin $w = \overline{w} = 0$ and acting the above upon the vacuum state,

$$L_0\Phi(0,0)|0\rangle = h\Phi(0,0)|0\rangle, \qquad \overline{L}_0\Phi(0,0)|0\rangle = \overline{h}\Phi(0,0)|0\rangle, \tag{3.64}$$

where the later is obtained by the same way than the former. Therefore, the state

$$|h,\overline{h}\rangle \equiv \Phi(0,0)|0\rangle \tag{3.65}$$

is an eigenstate of the Hamiltonian (which is proportional to L_0 and $\overline{L}_0$), called the primary state. Furthermore, if $n > 0$, we verify that

$$[L_n, \Phi(0,0)]|0\rangle = 0 \implies L_n \left| h,\overline{h} \right\rangle = 0, \tag{3.66}$$

which indicate that these are ladder operators. In fact, considering the Virasoro algebra for $n, m = -n$, the following commutator holds

$$[L_n, L_{-n}] = nL_{-n}. \tag{3.67}$$

[2] These results are completely analogous in the anti-holomorphic sector.

This is precisely a ladder algebra, where $L_n, n < 0$ are lowering operators, and $L_n, n > 0$ are raising operators. Considering an iteration

$$L_{-k_1} L_{-k_2} \dots L_{-k_n} |h\rangle \propto |h + N\rangle, \tag{3.68}$$

where $1 \leq k_i \leq ki + 1$ for $i = 1, \dots, n$ and $N = k_1 + \dots k_n$. See that an specific ordering was fixed, but we can use the Virasoro algebra and consider another one, for example

$$\begin{aligned} L_{k_i+1} L_{k_i} &= [L_{k_i+1}, L_{k_i}] + L_{k_i} L_{k_i+1} \\ &= L_{2k_i+1} + L_{k_i} L_{k_i+1}. \end{aligned} \tag{3.69}$$

The states defined by the iteration are called *descendent*, and N is their descendent level. The set of all descendent states constitute a module in the Hilbert space, the Verma module, which constitute an irreducible representation of the Virasoro algebra [5]. Once we have determined the spectrum of primary states, the spectrum of the whole theory can be determined [9]. Here we will not explore further these matters, and we will now turn ourselves to the problem of the free boson on the cylinder.

3.3 Free Boson on the Cylinder

Let $\varphi(x, t)$ be a free boson taking values on a cylinder of circumference L, with periodic conditions $\varphi(x + L, t) = \varphi(x, t)$. Consider a Fourier expansion,

$$\varphi(x, t) = \sum_{n \in \mathbb{Z}} e^{2\pi i n x / L} \varphi_n(t), \tag{3.70}$$

where the modes are given by

$$\varphi_n(t) = \frac{1}{L} \int dx \, e^{-2\pi i n x / L} \varphi(x, t). \tag{3.71}$$

The action for this theory is

$$S = \frac{g}{2} \int d^2x \, \partial_\mu \varphi \partial^\mu \varphi = \frac{g}{2} \int d^2x \, \left[(\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right]. \tag{3.72}$$

In Fourier modes, the derivatives in the Lagrangian become

$$\begin{aligned} (\partial_t \varphi)^2 &= \sum_{m,n \in \mathbb{Z}} e^{2\pi i x (m+n)/L} \dot{\varphi}_m(t) \dot{\varphi}_n(t), \\ (\partial_x \varphi)^2 &= \sum_{m,n \in \mathbb{Z}} e^{2\pi i x (m+n)/L} \left(\frac{2\pi i}{L} \right)^2 m \cdot n \, \varphi_m(t) \varphi_n(t). \end{aligned} \tag{3.73}$$

Therefore, the Lagrangian can be written as

$$\mathcal{L} = \frac{gL}{2} \sum_{n \in \mathbb{Z}} \left[\dot{\varphi}_n(t) \dot{\varphi}_{-n}(t) - \left(\frac{2\pi n}{L} \right)^2 \varphi_n(t) \varphi_{-n}(t) \right], \tag{3.74}$$

where the following integral was applied

$$\int dx\, e^{2\pi i x(m+n)/L} = L\delta_{m+n,0}. \tag{3.75}$$

In order to consider the canonical formalism, we need to introduce the Hamiltonian operator and consider the canonical commutation rule. Notice that the conjugate momenta of φ_n is given by

$$\Pi_n = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_n} = \frac{gL}{2}\dot{\varphi}_{-n}. \tag{3.76}$$

These conjugate variables satisfy the equitemporal commutation rule

$$[\varphi_n(t), \Pi_n(t)] = i\delta_{m,n}, \tag{3.77}$$

and the Hamiltonian for this theory is

$$H = \sum_{n\in\mathbb{Z}} \Pi_n \dot{\varphi}_n - \mathcal{L} = \frac{1}{2gL}\sum_{n\in\mathbb{Z}}\left[\Pi_n\Pi_{-n} + (2\pi g n)^2 \varphi_n \varphi_{-n}\right]. \tag{3.78}$$

See that this is the Hamiltonian for a system of decoupled harmonic oscillators, whose frequency is determined, by comparison, to be

$$\frac{1}{2}g\omega_n^2 = \frac{1}{2gL}(2\pi g n)^2 \implies \omega_n = \frac{2\pi}{L}|n|. \tag{3.79}$$

Notice that this frequency vanishes for $n = 0$, and therefore the zero mode do not appear in the Hamiltonian. This is due to the absence of a mass kinetic term, and implies in a conserved charge Π_0, that commutes with the Hamiltonian. On the other hand, if one consider such mass contribution, conformal invariance will not hold, and hence this conserved charge and its implications are exclusive to the massless theory.

Let us now diagonalize the Hamiltonian by introducing creation and annihilation operators

$$a_n = \frac{1}{\sqrt{4\pi g|n|}}(2\pi g|n|\varphi_n + i\Pi_{-n}), \qquad a_n^\dagger = \frac{1}{\sqrt{4\pi g|n|}}(2\pi g|n|\varphi_{-n} - i\Pi_n), \tag{3.80}$$

satisfying the usual algebra

$$[a_m, a_n^\dagger] = \delta_{m,n}, \qquad [a_m, a_n] = [a_m^\dagger, a_n^\dagger] = 0. \tag{3.81}$$

We see that the Hamiltonian becomes

$$H = \frac{1}{2gL}\Pi_0^2 + \frac{\pi}{L}\sum_{n\neq 0}|n|(a_n a_n^\dagger + a_n^\dagger a_n). \tag{3.82}$$

Notice that we can absorb the $|n|$ factor in the summation above if we consider the operators

$$b_n = \begin{cases} -i\sqrt{n}a_n, & \text{if} \quad n > 0 \\ i\sqrt{-n}a_{-n}^\dagger, & \text{if} \quad n < 0 \end{cases}, \qquad \tilde{b}_n = \begin{cases} -i\sqrt{n}a_{-n}, & \text{if} \quad n > 0 \\ i\sqrt{-n}a_n^\dagger, & \text{if} \quad n < 0 \end{cases}. \tag{3.83}$$

Their non-vanishing commutation relations are

$$\begin{aligned} [b_n, b_m] &= n\delta_{n+m,0}, \qquad \text{if} \qquad n > 0 \text{ and } m < 0, \\ [\widetilde{b}_n, \widetilde{b}_m] &= n\delta_{n+m,0}, \qquad \text{if} \qquad n > 0 \text{ and } m < 0. \end{aligned} \tag{3.84}$$

Rewriting the Hamiltonian in terms of these operators,

$$H = \frac{1}{2gL}\Pi_0^2 + \frac{\pi}{L}\sum_{n\neq 0}\left(b_{-n}b_n + \widetilde{b}_{-n}\widetilde{b}_n\right). \tag{3.85}$$

It is easy to check that a ladder algebra is satisfied,

$$[H, b_{-n}] = \frac{2\pi}{L} n b_{-n}, \tag{3.86}$$

meaning that if we apply b_{-n} to an eigenstate of the Hamiltonian with energy E, the resulting state is still an eigenstate, but with energy $E + 2\pi n/L$. Therefore, b_{-n} are usually called raising operator if $n > 0$, and lowering operator if $n < 0$.

Now we can also write the field expansion in terms of these operators, and use the Heisenberg equation to derive the dynamics of $\varphi(x,t)$. Notice that

$$\varphi_n = \frac{i}{n\sqrt{4\pi g}}\left(b_n - \widetilde{b}_{-n}\right), \qquad \forall n \neq 0. \tag{3.87}$$

Then, at $t = 0$, the field expansion becomes

$$\varphi(x,0) = \varphi_0(0) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(b_n(0) - \widetilde{b}_{-n}(0)\right)e^{2\pi i x/L}. \tag{3.88}$$

The Heisenberg equations will be,

$$\frac{d\varphi_0}{dt} = \frac{1}{gL}\Pi_0, \qquad \frac{db_n}{dt} = -\frac{2\pi i}{L}nb_n, \qquad \frac{d\widetilde{b}_{-n}}{dt} = \frac{2\pi i}{L}n\widetilde{b}_{-n}. \tag{3.89}$$

Integrating these equations (recall that Π_0 is a conserved quantity),

$$\varphi_0(t) = \varphi_0(0) + \frac{t}{gL}\Pi_0, \qquad b_n(t) = b_n(0)e^{-2\pi i n t/L}, \qquad \widetilde{b}_{-n}(t) = \widetilde{b}_{-n}(0)e^{2\pi i n t/L}, \tag{3.90}$$

we solve the time evolution of the field:

$$\varphi(x,t) = \varphi_0(0) + \frac{t}{gL}\Pi_0 + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left[b_n(0)e^{2\pi i n(x-t)/L} - \widetilde{b}_{-n}(0)e^{2\pi i n(x+t)/L}\right]. \tag{3.91}$$

It will be convenient to have this expansion written in conjugate coordinates, which means inverting $x(z,\bar{z})$ and $t(z,\bar{z})$. See that considering the conformal mapping in an Euclidean time prescription ($t \to -i\tau$),

$$z = e^{2\pi(\tau - ix)/L}, \qquad \bar{z} = e^{2\pi(\tau + ix)/L}, \tag{3.92}$$

which inverts the exponentials in the expansion. Furthermore, if we multiply these last two, only a function of $\tau = it$ will remain. Inverting this function,

$$t = -\frac{iL}{4\pi} \ln(z\bar{z}). \tag{3.93}$$

Plugging all this together,

$$\varphi(z,\bar{z}) = \varphi_0 - \frac{i}{4\pi g} \ln(z\bar{z})\Pi_0 + \frac{i}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n} \left[b_n z^{-n} - \widetilde{b}_{-n}\bar{z}^{-n} \right]. \tag{3.94}$$

From our previous discussion, we already know that $\varphi(z,\bar{z})$ is not a primary field, but its derivative is. Looking only into the holomorphic sector, we see that

$$i\partial\varphi(z) = \frac{1}{\sqrt{4\pi g}} \sum_{n\in\mathbb{Z}} b_n z^{-n-1}, \tag{3.95}$$

where we have defined $b_0 \equiv \Pi_0/\sqrt{4\pi g}$.

We, then, finish our discussion on how to structure and solve the free boson on the cylinder as an example of a conformal quantum field theory.

3.3.1 Vertex Operator

There is yet another interesting feature we can explore at this level. The scale dimension of the free scalar field vanishes in two-dimensions, and hence, powers of the free scalar field do not introduce scale to the theory [5]. An operator of particular interest is the vertex operator, defined as

$$\mathcal{V}_\alpha(z,\bar{z}) =: e^{i\alpha\varphi(z,\bar{z})} : . \tag{3.96}$$

The normal ordering must be understood as

$$\begin{aligned} \mathcal{V}_\alpha(z,\bar{z}) = \exp\left(i\alpha\varphi_0 + \frac{\alpha}{\sqrt{4\pi g}} \sum_{n>0} \frac{1}{n} \left[b_{-n} z^n - \widetilde{b}_{-n}\bar{z}^n \right] \right) \times \\ \times \exp\left(\frac{\alpha}{4\pi g}\Pi_0 - \frac{\alpha}{\sqrt{4\pi g}} \sum_{n>0} \frac{1}{n} \left[b_n z^n - \widetilde{b}_n \bar{z}^n \right] \right). \end{aligned} \tag{3.97}$$

That is, we can understand this normal ordering as a decoupling of creation and annihilation operators. This operator is a primary operator, as we will argue next.

Recalling that $T(z) = -2\pi g : \partial\varphi(z)\partial\varphi(z) :$ is the holomorphic component of energy-momentum tensor, we need to find the OPE of the product $T(z)\mathcal{V}_\alpha(w,\bar{w})$. Notice that

$$\begin{aligned} \partial\varphi(z)\mathcal{V}_\alpha(w) &= \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \partial\varphi(z) : (\varphi(w,\bar{w}))^n : \\ &\sim \sum_{n=1}^{\infty} \frac{(i\alpha)^n}{n!} n \partial\overbrace{\varphi(z)\varphi}(w,\bar{w}) : (\varphi(w,\bar{w}))^{n-1} : \\ &\sim -\frac{1}{4\pi g} \frac{i\alpha}{z-w} \mathcal{V}_\alpha(w,\bar{w}), \end{aligned} \tag{3.98}$$

where the Wick contraction was obtained by deriving once (2.136). Therefore,

$$\begin{aligned} T(z)\mathcal{V}_\alpha(w,\overline{w}) &= -2\pi g : \partial\varphi(z)\partial\varphi(z) : \mathcal{V}_\alpha(w,\overline{w}) \\ &\sim -2\pi g \sum_{n=1}^{\infty} \frac{(i\alpha)^n}{n!} n \partial\varphi(z)\partial\varphi(z)\varphi(w,\overline{w}) : (\varphi(w,\overline{w}))^{n-1} : \\ &\quad - 2\pi g \sum_{n=2}^{\infty} \frac{(i\alpha)^n}{n!} n(n-1) \left(\partial\varphi(z)\varphi(w,\overline{w})\right)^2 : (\varphi(w,\overline{w}))^{n-2} : \\ &\sim \frac{\alpha^2}{8\pi g} \frac{V_\alpha(w,\overline{w})}{(z-w)^2} + \frac{\partial_w V_\alpha(w,\overline{w})}{z-w}, \end{aligned} \tag{3.99}$$

and we conclude that the vertex operator is a primary field with conformal dimension $h = \alpha^2/8\pi g$. This vertex operator seems to describe a fermion. A hint of this fact can be already appreciated if we calculate the OPE of vertex operators, which we will discuss in what follows. In the appendix we have derived an identity for vertex operators defined for an harmonic oscillator,

$$: e^{A_1} : \ldots : e^{A_n} : = : e^{A_1 + \ldots + A_n} : e^{\sum_{i<j} \langle A_i A_j \rangle}. \tag{3.100}$$

where each $A_j = \alpha_j a + \beta_j a^\dagger$ is a complex linear combination of creation and annihilation operators. The boson we discussed in this section decomposes itself into a set of decoupled harmonic oscillators, and therefore

$$: e^{\alpha\varphi_1} :: e^{\beta\varphi_2} : = : e^{\alpha\varphi_1} e^{\beta\varphi_2} : e^{\alpha\beta\langle\varphi_1\varphi_2\rangle}. \tag{3.101}$$

In terms of vertex operators,

$$\mathcal{V}_\alpha(z,\overline{z})\mathcal{V}_\beta(w,\overline{w}) \sim (z-w)^{2\alpha\beta/4\pi g} \mathcal{V}_{\alpha+\beta}(w,\overline{w}). \tag{3.102}$$

(Recall that the radial ordering $|z| > |w|$ ought to be understood in OPE's.) The above expressions hold for a general correlation function with an arbitrary string of local fields. However, we know that for the case of a two-point correlation function of primary fields,

$$\Phi(z)\Phi(w) = \frac{C}{(z-w)^{2h}}. \tag{3.103}$$

Because the vertex operator is primary, conformal invariance constrains $\beta = -\alpha$ in such a way that non-vanishing contributions of the two-point function of vertex operators are

$$\mathcal{V}_\alpha(z,\overline{z})\mathcal{V}_{-\alpha}(w,\overline{w}) \sim (z-w)^{-2\alpha^2/4\pi g}. \tag{3.104}$$

Notice, then, that just as the two-point correlation function of the free boson and the free fermion captures the statistics of each, the above two-point function seems to capture a fermionic statistics, when we exchange z and w, a negative sign appears.

The implementation of the vertex operator is fundamental in the conformal theory. We will now show that the index α used in the definition of the vertex operator is actually indexing a set of vacuum states, with each state obtained by the action of the vertex operator. We will also derive some interesting properties of the vacuum states and study how we can systematically generate the tower of states, from the vacuum, using the Virasoro algebra operator L_0.

Notice that the Hamiltonian of the free boson theory

$$H = \frac{1}{2gL} \sum_{n \in \mathbb{Z}} \left[\Pi_n \Pi_{-n} + (2\pi g n)^2 \varphi_n \varphi_{-n} \right] \tag{3.105}$$

does not depend on φ_0, implying in the conservation of Π_0. In addition, both H and Π_0 can be simultaneously diagonalized,

$$H|E, \alpha\rangle = E|E, \alpha\rangle, \qquad \Pi_0|E, \alpha\rangle = \alpha|E, \alpha\rangle. \tag{3.106}$$

Then, each α characterises a set of eigenstates of energy. For the eigenstate of energy $E = 0$, we identify the vacuum states $|0, \alpha\rangle \equiv |\alpha\rangle$ and $|0, 0\rangle \equiv |0\rangle$. Let us explore some properties of these vacuum states. Using the field expansion (3.95), we can rewrite the holomorphic energy-momentum tensor as

$$\begin{aligned} T(z) &= -2\pi g : \partial\varphi(z)\partial\varphi(z) : \\ &= \frac{1}{2} \sum_{m,n \in \mathbb{Z}} z^{-m-2} : b_{m-n} b_n : \end{aligned} \tag{3.107}$$

On the other hand, we have already seen that

$$T(z) = \sum_{m \in \mathbb{Z}} z^{-m-2} L_m, \tag{3.108}$$

so comparing both expressions,

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} : b_{m-n} b_n : . \tag{3.109}$$

Particularly for $m = 0$,

$$L_0 = \frac{1}{2} b_0^2 + \sum_{n>0} b_{-n} b_n, \qquad \overline{L}_0 = \frac{1}{2} b_0^2 + \sum_{n>0} \widetilde{b}_{-n} \widetilde{b}_n. \tag{3.110}$$

Here, it should be noticed that b_n are annihilation operators for $n > 0$, and creation operators for $n > 0$, so the normal ordering

$$\sum_{n \neq 0} : b_{-n} b_n := \sum_{n>0} : b_{-n} b_n : + \sum_{n<0} : b_{-n} b_n := 2 \sum_{n>0} b_{-n} b_n. \tag{3.111}$$

Now, see that the Hamiltonian

$$H = \frac{1}{2gL}\Pi_0^2 + \frac{\pi}{L}\sum_{n\neq 0}\left(b_{-n}b_n + \widetilde{b}_{-n}\widetilde{b}_n\right)$$
$$= \frac{2\pi}{L}\left(b_0^2 + \sum_{n>0}\left(b_{-n}b_n + \widetilde{b}_{-n}\widetilde{b}_n\right) + \sum_{n>0} 2n\right). \tag{3.112}$$

Dropping the constant factor,

$$H = \frac{2\pi}{L}(L_0 + \overline{L}_0). \tag{3.113}$$

Then, as we have already argued, the Hamiltonian is proportional to the dilatation generators on the radial quantization scheme.

Let us see how we can generate the tower of eigenstates of energy from the vacuum $|\alpha\rangle$. We have already concluded that asymptotic states $|h\rangle$ with a well defined conformal weight h are eigenstates of L_0, (3.64). However, acting with the vacuum state $|\alpha\rangle$ in (3.110),

$$L_0|\alpha\rangle = \frac{\alpha^2}{2}|\alpha\rangle, \tag{3.114}$$

because b_n is an annihilation operator for $n > 0$. Therefore, the vacuum states $|\alpha\rangle$ also have a well defined conformal weight $h = \alpha^2/2$. In fact, we can generate eigenstates of energy by iterating creation operators,

$$b_{-1}^{n_1} b_{-2}^{n_2} \dots \overline{b}_{-1}^{m_1} \overline{b}_{-2}^{m_2} \dots |\alpha\rangle, \qquad n_i, m_i \geq 0. \tag{3.115}$$

These eigenstates have conformal weights

$$h = \frac{\alpha^2}{2} + \sum_j j n_j, \qquad \overline{h} = \frac{\alpha^2}{2} + \sum_j j m_j. \tag{3.116}$$

To verify this we need to act L_0 (and $\overline{L}_0$) over (3.115). Using the expression (3.110) for L_0, we see that b_0 acts directly in (3.115), because it commutes with all the other creation operators, obtaining

$$\frac{1}{2} b_0^2 b_{-1}^{n_1} b_{-2}^{n_2} \dots \overline{b}_{-1}^{m_1} \overline{b}_{-2}^{m_2} \dots |\alpha\rangle. = \frac{\alpha^2}{2} b_{-1}^{n_1} b_{-2}^{n_2} \dots \overline{b}_{-1}^{m_1} \overline{b}_{-2}^{m_2} \dots |\alpha\rangle. \tag{3.117}$$

However, it remains to calculate the contribution of

$$\sum_{n>0} b_{-n} b_n b_{-1}^{n_1} b_{-2}^{n_2} \dots \overline{b}_{-1}^{m_1} \overline{b}_{-2}^{m_2} \dots |\alpha\rangle. \tag{3.118}$$

In order to do that, we use the algebra of these operators to pass $b_{-n}b_n$ all the way to the state $|\alpha\rangle$. Notice that for $n > 0$,

$$\begin{aligned}
[b_{-n}b_n, b_{-m}] &= n\delta_{n,m}, \\
[b_{-n}b_n, b_{-m}^2] &= 2n\delta_{n,m} b_{-n}, \\
&\vdots \\
[b_{-n}b_n, b_{-m}^k] &= kn\delta_{n,m} b_{-n}^{k-1},
\end{aligned} \tag{3.119}$$

where one should recall that $[b_n, b_m]$ is non-vanishing only when $n > 0$. Then, using this algebra relation,

$$\begin{aligned}\sum_{n>0} b_{-n}b_n b_{-1}^{n_1}b_{-2}^{n_2}\dots\bar{b}_{-1}^{m_1}\bar{b}_{-2}^{m_2}\dots|\alpha\rangle &= \sum_{n>0} b_{-n}\left(b_{-1}^{n_1}b_n + [b_n, b_{-1}^{n_1}]\right) b_{-2}^{n_2}\dots\bar{b}_{-1}^{m_1}\bar{b}_{-2}^{m_2}\dots|\alpha\rangle \\ &= \sum_{n>0} b_{-1}^{n_1}b_{-n}b_n b_{-2}^{n_2}\dots\bar{b}_{-1}^{m_1}\bar{b}_{-2}^{m_2}\dots|\alpha\rangle + n_1 b_{-1}^{n_1}b_{-2}^{n_2}\dots\bar{b}_{-1}^{m_1}\bar{b}_{-2}^{m_2}\dots|\alpha\rangle \\ &= \sum_{n>0} b_{-1}^{n_1}b_{-2}^{n_2}\dots\bar{b}_{-1}^{m_1}\bar{b}_{-2}^{m_2}\dots b_{-n}b_n|\alpha\rangle + \sum_j j n_j b_{-1}^{n_1}b_{-2}^{n_2}\dots\bar{b}_{-1}^{m_1}\bar{b}_{-2}^{m_2}\dots|\alpha\rangle\end{aligned} \tag{3.120}$$

The first sum just above vanishes, because $b_n|\alpha\rangle = 0$, remaining the expected.

As mentioned, the vacuum states $|\alpha\rangle$ can be obtained by the action of the vertex operator. Using the Hausdorff identity for constant commutator,

$$[B, e^A] = e^A[B, A], \tag{3.121}$$

when $A = i\alpha\varphi$ and $B = \Pi_0$,

$$[\Pi_0, V_\alpha(z, \bar{z})] = \alpha V_\alpha(z, \bar{z}). \tag{3.122}$$

Taking this relation at the origin,

$$\Pi_0 V_\alpha(0) = \alpha V_\alpha(0), \tag{3.123}$$

and acting on the vacuum state,

$$\Pi_0\left(V_\alpha(0)|0\rangle\right) = \alpha\left(V_\alpha(0)|0\rangle\right). \tag{3.124}$$

So in fact, $V_\alpha(0)|0\rangle$ is an eigenstate of Π_0.These states generated by the vertex operator are annihilated by b_n, if $n > 0$. To realize this, we can use again the Hausdorff formula for $B = b_n$ with $n > 0$,

$$[b_n, V_\alpha(z, \bar{z})] = \frac{\alpha z^n}{\sqrt{4\pi g}} V_\alpha(z, \bar{z}). \tag{3.125}$$

Therefore, evaluating at $z = 0$ and acting on the vacuum,

$$b_n V_\alpha(0)|0\rangle = 0, \tag{3.126}$$

from where we conclude that $V_\alpha(0)|0\rangle = |\alpha\rangle$.

CHAPTER 4

1 + 1 Compactified Boson

Let us consider the compactified boson[1] $\varphi : S^1 \times \mathbb{R} \to S^1$ with the action

$$S = \frac{g}{2} \int d^2x \, \partial_\mu \varphi \partial^\mu \varphi. \tag{4.1}$$

There are three interesting facts we can already appreciate.

(i) As we mentioned, the constant g factor cannot be absorbed into φ, because now the field configurations must lie within the circle,

$$\varphi(x + L, t) = \varphi(x, t) + 2\pi m R, \tag{4.2}$$

where L is the circumference of the cylinder and R is the radius of the circle. If one tries to incorporate g, the periodic condition above will capture it [10]. Then we can already suspect that different g's will somewhat describe different physics.

(ii) Another implication of the periodicity of φ is the restriction of additional fields we can add to the action. Contributions like φ^n are not periodic, hence we shall not consider them. However, terms proportional to $\cos \varphi$ or $\sin \varphi$ are periodic, and allowed [10].

(iii) If we only consider the spatial coordinate dependency of $\varphi(x, t)$, we have for each t a map of $S^1 \to S^1$. These maps are very well understood in the context of homotopy groups, and exploring this further we can find topological configurations of φ (the precise meaning of this concepts will be discussed).

4.1 Homotopy Group and Mode Expansion

In order to calculate the mode expansion for the compactified boson, we will make a digression about the first homotopy group of S^1.

Homotopy groups are one of the main tools to study the topological structure of spaces. They help us understand holes systematically, by considering equivalent classes of loops that are continuously deformable into each another. Let us set these

[1] It should be notice that any boson whose base and target space are compactified is called a compactified boson.

notions precisely, and to simplify the discussion, we will always consider the circle S^1. If $\alpha : S^1 \to S^1$ and $\gamma : S^1 \to S^1$ are loops at a point $x_0 \in S^1$, then they satisfy

$$\alpha(0) = \alpha(2\pi) = x_0, \qquad \gamma(0) = \gamma(2\pi) = x_0, \tag{4.3}$$

where we have parametrized the circle with the angular variable. In fact, any mapping of the circle is a loop. These loops are said to be homotopic if there is a continuous map $H : S^1 \times [0,1] \to S^1$ such that

$$H(\theta, 0) = \alpha(\theta), \qquad H(\theta, 2\pi) = \gamma(\theta), \qquad H(0,t) = H(2\pi, t) = x_0. \tag{4.4}$$

The map H is called homotopy between α and γ, and establishes a continuos transformation/deformation of the loop α to the loop γ. Hence, when there exists no homotopy, it is impossible to continuously deform a loop into another. It is interesting to define H because "α is homotopic to γ" (which is denoted by $\alpha \sim \gamma$) is an equivalence relation [2]. More precisely, because $\sim$ is an equivalence relation, it holds the following

(i) $\alpha \sim \alpha$;

(ii) if $\alpha \sim \gamma$, then $\gamma \sim \alpha$;

(iii) if $\alpha \sim \beta$ and $\beta \sim \gamma$, then $\alpha \sim \gamma$.

This equivalence relation defines classes of equivalence containing loops that are homotopic to one another. The set of all this classes of equivalence equipped with a product of loops is the first homotopy group. Here we will not construct this product explicitly, but we will consider an important result about the first homotopy group of the circle, which we denote by $\pi_1(S^1)$. This result, derived in detail on [2], states that

$$\pi_1(S^1) \cong \mathbb{Z}. \tag{4.5}$$

In other words, the first homotopy group is in one-to-one correspondence with the integers. The elements of $\pi_1(S^1)$ are classes of equivalence of homotopic curves. Hence this correspondence ensures that we can associate bijectively each homotopic class of loops with an integer. In fact, the homotopic classes are defined by all the maps $\gamma : S^1 \to S^1$ such that

$$\gamma(x + L) = \gamma(x) + 2\pi R m, \qquad m \in \mathbb{Z}, \tag{4.6}$$

and there is a simple geometric interpretation for each class. Considering $m = 0$, we have the class of periodic maps $\gamma(x + L) = \gamma(x)$, and when we circle around the origin $x \to x + L$, we arrive in the same target point. For $m = 1$, the class is defined by maps $\gamma(x + L) = \gamma(x) + 2\pi R$. That is, when we circle around the origin, the target point winds around the circle once. Then, a general class C_m can be defined as containing the maps winding m times the circle.

The isomorphism with the integers is deeper than a one-to-one correspondence. In fact, the classes C_m satisfy the same additive group operations that integers do. Notice that if $\varphi \in C_m$ and $\zeta \in C_n$,

$$\varphi(x+L) = \varphi(x) + 2\pi m R, \qquad \zeta(x+L) = \zeta(x) + 2\pi n R. \tag{4.7}$$

Summing up these two,

$$\varphi(x+L) + \zeta(x+L) = \varphi(x) + \zeta(x) + 2\pi(m+n)R. \tag{4.8}$$

But this is precisely what defines the maps on the class C_{m+n}, showing that $\varphi + \zeta \in C_{m+n}$. Following this idea, we can write a general representative[2] of some class C_n as

$$\varphi(x) = \phi(x) + \zeta(x), \tag{4.9}$$

where $\phi \in C_0$ and $\zeta \in C_n$. The most simple representative of C_n we can consider is

$$\zeta_n(x) = \frac{2\pi n R}{L} x, \tag{4.10}$$

which trivially satisfies (4.7). However, apart from being the most simple, this is the most general representative, and we notice this by considering two functions in C_n, say ζ and ζ'. We have showed that these functions may be written as

$$\zeta(x) = \zeta_0(x) + \zeta_n(x), \qquad \zeta'(x) = \zeta'_0(x) + \zeta'_n(x), \tag{4.11}$$

with $\zeta_0, \zeta'_0 \in C_0$ and $\zeta_n, \zeta'_n \in C_n$. See that their difference

$$\zeta(x) - \zeta'(x) = \zeta_0(x) - \zeta'_0(x) + \zeta_n(x) - \zeta'_n(x), \tag{4.12}$$

is a periodic function of C_0. Then, because all the functions in C_n differ by a periodic one, any function ζ shall be considered as

$$\zeta(x) = \zeta_0(x) + \frac{2\pi n R}{L} x, \tag{4.13}$$

with $\zeta_0(x+L) = \zeta_0(x)$.

Getting back to the physics, the upshot is that we can write the compactified boson (which is a function from $S^1 \to S^1$ in its spatial coordinate dependency) as

$$\varphi(x,t) = \varphi_0(x,t) + \frac{2\pi m R}{L} x, \qquad m \in \mathbb{Z}, \tag{4.14}$$

2 A representative of a class is simply a loop that belongs to the class.

where $\varphi_0(x+L,t) = \varphi_0(x,t)$ is a periodic field satisfying the same dynamics as the free boson on the cylinder, which we have already solved. Therefore, upon quantization,

$$\begin{aligned}\varphi(x,t) = \varphi_0 + \frac{\Pi_0}{gL}t + \frac{2\pi m R}{L}x \\ + \frac{i}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n}\left[b_n(0)e^{2\pi i n(x-t)/L} - \widetilde{b}_{-n}(0)e^{2\pi i n(x+t)/L}\right].\end{aligned} \tag{4.15}$$

Notice that we have expressed $\Pi_0 = n/R$. In order to have a consistent vertex operator $V_\alpha = e^{i\alpha\varphi}$, one needs to require invariance under winding symmetry, therefore $e^{i2\pi m R\alpha} = 1$. This, with the fact that Π_0 is a constant of motion, enable us to express it that way.

As for the usual free boson, we can also determine the spectrum of the Hamiltonian. Let us consider the above in conjugate coordinates, recalling that

$$t = -\frac{iL}{4\pi}\ln(z\bar{z}), \qquad x = -\frac{i}{4\pi}\ln\frac{z}{\bar{z}}, \tag{4.16}$$

we can write the field expansion as

$$\begin{aligned}\varphi(z,\bar{z}) = \varphi_0 - i\left(\frac{n}{4\pi g R} + \frac{mR}{2}\right)\ln z + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}b_n z^{-n} \\ - i\left(\frac{n}{4\pi g R} - \frac{mR}{2}\right)\ln \bar{z} - \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\widetilde{b}_{-n}\bar{z}^{-n}.\end{aligned} \tag{4.17}$$

Let us find the Hamiltonian operator. Proceeding analogously, notice that the holomorphic derivative

$$i\partial\varphi(z) = \left(\frac{n}{4\pi g R} + \frac{mR}{2}\right)\frac{1}{z} + \frac{1}{\sqrt{4\pi g}}\sum_{n\neq 0} b_n z^{-n-1}. \tag{4.18}$$

Then, defining

$$b_0 \equiv \sqrt{4\pi g}\left(\frac{n}{4\pi g R} + \frac{mR}{2}\right) \tag{4.19}$$

the free boson dilatation operator can be mapped to the compactified boson by considering this new zero mode operator b_0 defined above, heading to

$$L_0 = 2\pi g\left(\frac{n}{4\pi g R} + \frac{mR}{2}\right)^2 + \sum_{n>0} b_{-n}b_n, \tag{4.20}$$

$$\overline{L}_0 = 2\pi g\left(\frac{n}{4\pi g R} - \frac{mR}{2}\right)^2 + \sum_{n>0} \widetilde{b}_{-n}\widetilde{b}_n. \tag{4.21}$$

It is interesting to emphasise the topological character of the new zero mode: it contains information about the winding number m. This could be somewhat expected because the usual mode expansion could only capture local field configurations. Furthermore, we notice that when $g = 1/2\pi$, a symmetry of the Hamiltonian emerges by exchanging momentum and winding charge while swapping R to $1/R$. This is related to a duality between this boson and another compactified boson, which we will discuss in the next section. Now, for completeness, we may consider the vertex operator associated with compactified boson.

4.1.1 Vertex Operator

Now that we have determined the mode expansion for the compactified boson, we can define the vertex operator and study its implications. Let $\widetilde{\Pi}_0$ be the operator whose eigenvalues are the quantized winding numbers $2\pi Rm$. This operator commutes with all modes in the field expansion, and we can define its canonical conjugate pair as $\widetilde{\varphi}_0$ such that $[\widetilde{\varphi}_0, \widetilde{\Pi}_0] = i$. The quantum states are now going to carry three labels: E, α and m. Using these operators, we can write

$$\begin{aligned}\varphi(z,\bar{z}) = \frac{1}{2}\left(\varphi_0 + \widetilde{\varphi}_0\right) - \frac{i}{4\pi g}\left(\Pi_0 + g\widetilde{\Pi}_0\right)\ln z + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}b_n z^{-n} \\ + \frac{1}{2}\left(\varphi_0 - \widetilde{\varphi}_0\right) - \frac{i}{4\pi g}\left(\Pi_0 - g\widetilde{\Pi}_0\right)\ln \bar{z} - \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\widetilde{b}_{-n}\bar{z}^{-n}.\end{aligned} \tag{4.22}$$

Defining

$$Q_{\pm} \equiv \frac{1}{2}\left(\varphi_0 \pm \widetilde{\varphi}_0\right), \qquad P_{\pm} = \Pi_0 \pm g\widetilde{\Pi}_0, \tag{4.23}$$

it is possible separate φ into a right and a left component:

$$\varphi_+(z) = Q_+ - \frac{i\ln z}{4\pi g}P_+ + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}b_n z^{-n}, \tag{4.24}$$

$$\varphi_-(\bar{z}) = Q_- - \frac{i\ln \bar{z}}{4\pi g}P_- - \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\widetilde{b}_{-n}\bar{z}^{-n}. \tag{4.25}$$

One should notice that this is only a convenient way to express φ, because in fact we cannot strictly separate these components, the zero mode φ_0 appears in both of them. Consider the vertex operator

$$\mathcal{V}_{\alpha,\beta}(z,\bar{z}) =: e^{i(\alpha\varphi_+(z)+\beta\varphi_-(\bar{z}))} : . \tag{4.26}$$

The normal ordering is obtained using (3.101),

$$: e^{i(\alpha\varphi_+(z)+\beta\varphi_-(\bar{z}))} :=: e^{i\alpha\varphi_+(z)} :: e^{i\beta\varphi_-(\bar{z})} : e^{\alpha\beta\langle\varphi_+(z)\varphi_-(\bar{z})\rangle}, \tag{4.27}$$

and the normal ordering of each exponential operator in the right-hand side is

$$\begin{aligned} : e^{i\alpha\varphi_+(z)} :&= e^{i\alpha Q_+}e^{i\alpha P_+}e^{-\frac{\alpha}{\sqrt{4\pi g}}\sum_{n<0}\frac{1}{n}b_n z^{-n}}e^{-\frac{\alpha}{\sqrt{4\pi g}}\sum_{n>0}\frac{1}{n}b_n z^{-n}}, \\ : e^{i\alpha\varphi_-(\bar{z})} :&= e^{i\alpha Q_-}e^{i\alpha P_-}e^{-\frac{\alpha}{\sqrt{4\pi g}}\sum_{n<0}\frac{1}{n}\widetilde{b}_{-n}\bar{z}^{-n}}e^{-\frac{\alpha}{\sqrt{4\pi g}}\sum_{n>0}\frac{1}{n}\widetilde{b}_{-n}\bar{z}^{-n}}. \end{aligned} \tag{4.28}$$

Using the Hausdorff formula and the commutation relations between conjugate operators, we manage to find the following commutators:

$$\left[\Pi_0, V_{\alpha,\beta}(z,\bar{z})\right] = \frac{\alpha+\beta}{2}V_{\alpha,\beta}(z,\bar{z}), \qquad \left[\widetilde{\Pi}_0, V_{\alpha,\beta}(z,\bar{z})\right] = \frac{\alpha-\beta}{2}V_{\alpha,\beta}(z,\bar{z}). \tag{4.29}$$

Let us consider $\alpha = \beta = p$. In this case, the second commutator above vanishes, then both $\widetilde{\Pi}_0$ and $V_{p,p}(z,\bar{z})$ are simultaneously diagonal. The commutator with Π_0, however, implies that $V_{p,p}(0)$ creates momentum eigenstates. Notice that considering $|E,p,m\rangle = |0,0,0\rangle \equiv |0\rangle$, we have that

$$\Pi_0 V_{p,p}(0)|0\rangle = pV_{p,p}(0)|0\rangle, \tag{4.30}$$

because $\Pi_0|0\rangle = 0$. Therefore, $V_{p,p}|0\rangle$ is an eigenstate of Π_0 with eigenvalue p, showing that $V_{p,p}$ creates momentum states from the invariant vacuum $|0\rangle$. Furthermore, a consistent vertex operator requires p to be an integer:

$$: e^{i(\alpha\varphi_+ + \beta\varphi_-)} :=: e^{i(\alpha\varphi_+ + \beta\varphi_- + 2\pi Rm(\alpha+\beta))} : . \tag{4.31}$$

On the other hand, choosing $\alpha = -\beta = m$, the fist commutator in (4.29) vanishes, so Π_0 commutes with $V_{m,-m}(z,\bar{z})$. By the same argument above, $V_{m,-m}(0)|0\rangle$ is an eigenstate of $\widetilde{\Pi_0}$ with eigenvalue m, and therefore, we conclude that $V_{m,-m}$ creates winding from the invariant vacuum. The vertex operator $V_{m,-m}$ is strictly related to the dual boson, and we shall now investigate it.

4.2 T-Duality

We have considered the compactified boson theory described by

$$S = \frac{g}{2}\int d^2x\, \partial_\mu\varphi\partial^\mu\varphi, \tag{4.32}$$

with $\varphi(x+L,t) = \varphi(x,t) + 2\pi mR$ being defined onto S^1, with the Minkowski metric. In this section we are going to discuss the dualization procedure of this model, reviewing [11]. There are two symmetries associated with this boson. There is a symmetry by shifting the field $\varphi \to \varphi + const.$ with an associated current

$$j^\mu_{shift} = \frac{g}{2}\partial^\mu\varphi, \tag{4.33}$$

which is trivially conserved by the equations of motion.

As we have seen, the homotopic classes C_m are inequivalent to one another. Then, field configurations with different winding numbers cannot be homotopically deformed into each other. So we expect that the winding number m is a conserved charge, which is in turn associated with a conserved current. Indeed, for $x = 0$, the boundary condition over φ implies that

$$m = \frac{1}{2\pi R}\int_0^L dx\, \partial_x\varphi \equiv \int_0^L dx\, j^0_{wind} \tag{4.34}$$

If we define the current,

$$j^\mu_{wind} = \frac{1}{2\pi R}\varepsilon^{\mu\nu}\partial_\nu\varphi, \tag{4.35}$$

for simply connected field domain, we conclude by Schwartz theorem that $\partial_\mu j^\mu_{wind} = 0$. However, we need to be careful here. If we assume the base space is simply connected, we are broadly restricting the field configurations to those with no singularities. As we are going to see, the singularities are responsible for creating winding charge. Then, if we want to account for these topological physical implications, a more flexible imposition must be derived for the winding current. These matters will be treated in detail in the next chapter, but let us breafly see what are the immediate implications of singularities.

Consider field configurations with a finite number of isolated singularities, each located at the point x_i. A global quantity that encodes information about these singularities are circular integral of the winding current. If we circulate the winding current around each singularity, we have by the divergence theorem that

$$\oint_{\partial B_i} dx_\mu j^\mu_{wind} = \int_{B_i} d^2x\, \partial_\mu j^\mu_{wind} = \frac{1}{2\pi R}\int_{B_i} d^2x\, \varepsilon^{\mu\nu}\partial_\mu\partial_\nu\varphi. \tag{4.36}$$

Now, by Green's theorem,

$$\oint_{\partial B_i} dx_\mu j^\mu_{wind} = \frac{1}{2\pi R}\oint_{\partial B_i} dx_\mu \partial^\mu\varphi = n_i \in \mathbb{Z}, \tag{4.37}$$

where in the last equality we used the gradient theorem and the fact that φ has a non-trivial winding. If we integrate over all space-time, the procedure above yields

$$\int d^2x\, \partial_\mu j^\mu_{wind} = \sum_i n_i = n. \tag{4.38}$$

Or equivalently,

$$\partial_\mu j^\mu_{wind}(x) = \sum_i n_i \delta(x - x_i). \tag{4.39}$$

Therefore, the winding current is strictly conserved everywhere only when there are no singularities other than the one at the origin. This is clarified when we consider the radial scheme, where the integral defining m is performed in a loop around the origin. The origin corresponds to an infinite past time, so in this scenario, j_{wind} is strictly conserved everywhere.

Let us now go back to the boson without singularities. In an attempt to find its dual, we may consider the first order formalism approach. It consists of two major steps. First, we choose[3] $b_\mu = \partial_\mu\varphi$ as the dynamical field, instead of φ. Doing so, the original action becomes a functional of b_μ and the equations of motion are first order

[3] In the generating functional, this procedure corresponds to a change of variables, resulting in a Jacobian that does not depend on field configurations, and therefore can be absorbed into the normalization factor.

differential equations. The second step is define a generating functional $\overline{Z}$ integrated over b_μ and a new field $\widetilde{\varphi}$, with the following property: when integrated over $\widetilde{\varphi}$ this new functional must return to the original generating functional of b_μ and force $b_\mu = \partial_\mu \varphi$. This guarantees that we get back to the original boson. The interesting thing, however, happens when we integrate over b_μ. This will lead us to a new generating functional whose associated action is a functional of $\widetilde{\varphi}$, which we may call the *dual boson*. To capture these properties, we may consider the following action

$$\overline{S}\left[b_\mu, \tilde{\varphi}\right] = \int d^2x \left(\frac{g}{2} b_\mu b^\mu + \frac{1}{2\pi R} \varepsilon^{\mu\nu} b_\mu \partial_\nu \tilde{\varphi}\right), \tag{4.40}$$

constructed by coupling $\partial_\mu \widetilde{\varphi}$ with the winding current j^μ_{wind}, now in terms of b_μ. See that the equations of motion for b_μ and $\tilde{\varphi}$ are, respectively,

$$b^\mu = \frac{1}{2\pi R g} \varepsilon^{\mu\nu} \partial_\nu \tilde{\varphi}, \qquad \partial_\nu \left(\frac{1}{2\pi R} \varepsilon^{\mu\nu} b_\mu\right) = 0. \tag{4.41}$$

The second equation shows that the curl of b_μ vanishes, and hence b_μ is the gradient of a scalar function, recovering the original boson.

Let us integrate over b_μ now. Notice that the action is quadratic in b_μ, and therefore, we can integrate using the stationary configuration given by its equations of motion (4.41). Hence, replacing

$$b_\mu = -\frac{1}{2\pi R g} \varepsilon^{\mu\nu} \partial_\nu \tilde{\varphi} \tag{4.42}$$

in the generating functional (4.40), we obtain[4]

$$\overline{Z} = \int \mathcal{D}\widetilde{\varphi} \, \exp\left(i \int d^2x \, \frac{\widetilde{g}}{2} \, \partial^\mu \widetilde{\varphi} \, \partial_\mu \widetilde{\varphi}\right), \tag{4.43}$$

where the new constant is defined by

$$\widetilde{g} \equiv \frac{1}{4\pi^2 R^2 g}. \tag{4.44}$$

The action in (4.43) describes the dual compactified boson $\widetilde{\varphi}$, which is equivalent to the original boson φ by a change of the constant factor g.

The equation of motion for $\widetilde{\varphi}$ is derived under the assumption of smoothness, and therefore configurations with isolated singularities are not taken into account. However, we can show that if $\widetilde{\varphi}$ is compactified, the dual theory leads to the quantization of the winding current. Notice that considering the interpolating functional

$$\overline{Z} = \int \mathcal{D}b_\mu \mathcal{D}\widetilde{\varphi} \exp\left[i \int d^2x \, \left(\frac{g}{2} b_\mu b^\mu + \frac{1}{2\pi R} \varepsilon^{\mu\nu} b_\mu \partial_\nu \widetilde{\varphi}\right)\right]. \tag{4.45}$$

4 One must recall that $\varepsilon^{\mu\alpha} \varepsilon_{\mu\beta} = -\delta^\alpha_\beta$.

Notice that if we integrate by parts the second term above,

$$\overline{Z} = \int \mathcal{D}b_\mu \mathcal{D}\widetilde{\varphi} \exp\left[i\int d^2x \left(\frac{g}{2}b_\mu b^\mu - \frac{1}{2\pi R}\varepsilon^{\mu\nu}\partial_\nu b_\mu \widetilde{\varphi}\right)\right]. \tag{4.46}$$

Considering a change of variable $\widetilde{\varphi} \to \widetilde{\varphi} + \alpha$, for real constant α, the Jacobian is trivial, and then

$$\overline{Z} = \int \mathcal{D}b_\mu \mathcal{D}\widetilde{\varphi} \exp\left[i\int d^2x \left(\frac{g}{2}b_\mu b^\mu - \frac{1}{2\pi R}\varepsilon^{\mu\nu}\partial_\nu b_\mu \widetilde{\varphi} - \frac{1}{2\pi R}\varepsilon^{\mu\nu}\partial_\nu b_\mu \alpha\right)\right]. \tag{4.47}$$

Requiring the dual boson to be compactified, the transformation above for $\alpha = 2\pi m$ must be a symmetry of the action. Therefore, the last term in the exponential (4.47) must satisfy

$$2\pi m \int d^2x \, \frac{1}{2\pi R}\varepsilon^{\mu\nu}\partial_\nu b_\mu = 2\pi n', \tag{4.48}$$

for n' integer. This implies the quantization

$$\int d^2x \, \frac{1}{2\pi R}\varepsilon^{\mu\nu}\partial_\nu b_\mu = n \in \mathbb{Z}. \tag{4.49}$$

Therefore, compactness of the dual boson provides the condition (4.38), present in the original boson with isolated singularities.

Furthermore, when discussing the vertex operator, we have mentioned that the dual boson is related to the vertex operator that creates winding configurations on the original boson theory. Now that we have introduced the dual boson, we can appreciate this fact precisely, by introducing chiral bosons. Notice that we can decompose

$$\varphi(x,t) = \varphi_+(x_+) + \varphi_-(x_-), \qquad x_\pm = t \pm x. \tag{4.50}$$

This is not precise because the zero mode does note divide between the chiral components, as we have commented in the previous section. But it is instructive because then (4.42) is solved by considering

$$\tilde{\varphi} = 2\pi R g(\varphi_- - \varphi_+). \tag{4.51}$$

Therefore, the vertex operator $V_{m,-m}$ that creates winding configuration is actually the vertex operator associated with the dual boson. Hence we can see the dual boson creates winding configuration in the original theory.

CHAPTER 5

Lattice Models

5.1 XY Model

Consider a $2D$ lattice model defined by angular variables in each site. More precisely, we assign to every lattice site i, a unit vector $\hat{n}_i$ representing the angle with respect to some chosen direction. Considering first neighbours interaction, the Hamiltonian can be written as

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \hat{n}_i \cdot \hat{n}_j, \tag{5.1}$$

with $J > 0$ representing the intensity of the first neighbour's interaction. This model is called XY model. In two dimensions we can write every unit vector as

$$\hat{n}_i = \cos\phi_i \, \hat{x} + \sin\phi_i \, \hat{y}, \tag{5.2}$$

where $\hat{x} = (1,0)$ and $\hat{y} = (0,1)$. Hence, the Hamiltonian becomes

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \left(\cos\phi_i \cos\phi_j + \sin\phi_i \sin\phi_j \right) = -J \sum_{\langle i,j \rangle} \cos\left(\phi_i - \phi_j\right).$$

Denoting the unit vectors generically as $\hat{\mu} = \hat{x}, \hat{y}$, it is possible to write explicitly the sum over first neighbours, obtaining

$$\mathcal{H} = -J \sum_{i,\hat{\mu}} \cos\left(\phi_{i+\hat{\mu}} - \phi_i\right) \equiv -J \sum_{i,\hat{\mu}} \cos\left(\Delta_{\hat{\mu}} \phi_i\right), \tag{5.3}$$

where we have defined $\Delta_{\hat{\mu}}\phi_i \equiv \phi_{i+\hat{\mu}} - \phi_i$. The corresponding partition function for $J = 1$ is given by

$$Z = \int_{-\pi}^{\pi} \prod_i d\phi_i \, \exp\left[\beta \sum_{i,\hat{\mu}} \cos\left(\phi_{i+\hat{\mu}} - \phi_i\right) \right], \tag{5.4}$$

where $\beta = 1/k_B T$ is the inverse temperature[1]. In the high β limit, the most appreciable contributions to the partition function will be those with $\cos\left(\phi_{i+\hat{\mu}} - \phi_i\right) \approx 1$. Then, we are tempted to expand the cosine and consider the first non-vanishing contribution

$$\beta\mathcal{H} \simeq \frac{\beta}{2} \sum_{i,\hat{\mu}} \left(\phi_{i+\hat{\mu}} - \phi_i\right)^2 \simeq \frac{\beta}{2} \int d^2x \, \left(\nabla\phi\right)^2 \equiv S_E[\phi], \tag{5.5}$$

[1] The normalization factor will be omitted during the calculations. Also, every other constant factor that can be absorbed into the normalization factor will be omitted.

where S_E is the effective Euclidean action obtained in the low energy limit. However, as precisely argued in [12], this expansion misses important physical contributions. The effective Euclidean partition function is

$$Z_{eff} = \int \mathcal{D}\,\phi e^{-S_E[\phi]}. \tag{5.6}$$

Hence every finite contribution where $\phi_{i+\hat{\mu}}$ is not close to ϕ_i will be suppressed in the functional integration. But if we carefully analyse the condition over the cosine, $\cos\left(\phi_{i+\hat{\mu}} - \phi_i\right) \approx 1$, not only almost parallel spins satisfy it, but also the configurations

$$\begin{aligned} \varphi_i &= -\pi + \epsilon, \\ \varphi_{i+\hat{\mu}} &= \pi - \epsilon, \end{aligned} \tag{5.7}$$

with ϵ infinitesimal. In this case, $\varphi_{i+\hat{\mu}} - \varphi_i = 2\pi - 2\epsilon$ is a finite configuration near 2π that fulfil the cosine condition. So considering the low energy limit (5.5) we are missing these finite contributions. One way to treat this issue is allowing φ to be multivalued, having jumps of 2π [12], and to smoothly introduce this possibility we are going to consider a different representation of the XY model and then take its continuum limit.

5.1.1 Villain Representation

To formulate this representation precisely, let us consider the XY model defined above, with the potential

$$e^{-\beta V(\alpha)} = e^{\beta\cos(\alpha)}, \tag{5.8}$$

where $\alpha = \Delta\phi$. We would like to introduce a new potential representing the same physics in appropriate limits, and capturing the periodicity of the cosine. We can achieve this with the Villain potential, given by

$$e^{-\beta_V V_V(\alpha)} \equiv \sum_{p\in\mathbb{Z}} e^{-\frac{\beta_V}{2}(\alpha-2\pi p)^2}. \tag{5.9}$$

Notice that this potential contains an integer-valued variable p, which we expect to help us remedy our missing configuration problem. This in fact works because the integer field p restores the periodicity, presented in the original theory.

Let us show that by a suitable choice of β_V it is possible to relate this representation to the original one in both high and low temperatures. For large β (or low temperatures), the most relevant contribution to the partition function in the original theory (5.4) will be the ones that $\cos\alpha \approx 1$, which corresponds to $\alpha = 2\pi p + \delta\alpha$. Expanding the cosine, we obtain

$$e^{\beta\cos\alpha} \approx c e^{-\frac{\beta}{2}(\delta\alpha)^2}. \tag{5.10}$$

Then, we can relate in this limit $\beta \approx \beta_V$, reproducing the Villain representation.

For the small β limit (or high temperature), we use the Poisson resummation formula,

$$\sum_{n\in\mathbb{Z}} \exp\left(-\pi a n^2 + bn\right) = \frac{1}{\sqrt{a}} \sum_{k\in\mathbb{Z}} \exp\left[-\frac{\pi}{a}\left(k + \frac{b}{2\pi i}\right)^2\right], \tag{5.11}$$

derived in the appendix (A.3). Choosing appropriate constants,

$$\frac{\pi}{a} = \frac{\beta_V}{2}, \qquad \frac{b}{2\pi i} = \alpha, \tag{5.12}$$

and rescaling $k \to 2\pi k$ and $n \to (1/2\pi)J$, we conclude that

$$\sum_{p\in\mathbb{Z}} e^{-\frac{\beta_V}{2}(\alpha - 2\pi p)^2} = \sqrt{\frac{2\pi}{\beta_V}} \sum_{J\in\mathbb{Z}} \exp\left(-\frac{J^2}{2\beta_V} + i\alpha J\right). \tag{5.13}$$

Given this identity, we can consider small β_V if we expand in small

$$\epsilon \equiv \exp\left(-\frac{1}{2\beta_V}\right). \tag{5.14}$$

Writing the above summation in terms of this parameter,

$$\sqrt{\frac{2\pi}{\beta_V}} \sum_{n\in\mathbb{Z}} \exp\left(-\frac{J^2}{2\beta_V} + i\alpha J\right) = \sum_{J\in\mathbb{Z}} \epsilon^{J^2} \exp i\alpha J, \tag{5.15}$$

we can consider the linear order in ϵ, which is equivalent to consider the partial sums with $J = 0, \pm 1$. Therefore, it remains

$$\sqrt{\frac{2\pi}{\beta_V}} \sum_{n\in\mathbb{Z}} \exp\left(-\frac{J^2}{2\beta_V} + i\alpha J\right) \approx 1 + \epsilon e^{i\alpha} + \epsilon e^{-i\alpha} \approx \exp\left(2e^{-\frac{1}{2\beta_V}} \cos\alpha\right) \tag{5.16}$$

Taking $\beta \approx 2e^{-\frac{1}{2\beta_V}}$, we can relate both representations in this limit. Therefore, the Villain representation is an approximation of the original theory at these limits.

The partition function in the Villain representation is

$$Z = \sum_{p_{i,\hat{\mu}}\in\mathbb{Z}} \int_{-\pi}^{\pi} \prod_i d\phi_i \, \exp\left[-\frac{\beta_V}{2} \sum_{i,\hat{\mu}} \left(\phi_{i+\hat{\mu}} - \phi_i - 2\pi p_{i,\hat{\mu}}\right)^2\right] \tag{5.17}$$

Again, we have the angular variables ϕ_i for each site, but now we also have these integer variables $p_{i,\hat{\mu}}$ for each link of the lattice (Figure 4). That is, choosing a site i and

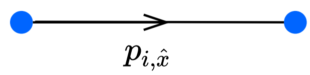

Figure 4 – Representation of the integer variables on the links.

a direction $\hat{\mu}$, we can think of $p_{i,\hat{\mu}}$ as lying on the line segment (or link) from the site i and to the site $i + \hat{\mu}$. We may also notice that, by uniqueness, $p_{i+\hat{\mu},-\hat{\mu}} = -p_{i,\hat{\mu}}$.

In order to acquire some physical intuition about the meaning of these variables, let us consider a symmetry of the Hamiltonian. See that if the angular variables transforms by local shifts of integer multiples of 2π,

$$\phi_i \to \phi_i + 2\pi N_i, \tag{5.18}$$

then the terms inside the summation in the partition function will change for

$$\phi_{i+\hat{\mu}} - \phi_i - 2\pi p'_{i,\hat{\mu}} + 2\pi \Delta_{\hat{\mu}} N_i \tag{5.19}$$

On the other hand, If $p_{i,\hat{\mu}}$ transforms as

$$p_{i,\hat{\mu}} \to p'_{i,\hat{\mu}} = p_{i,\hat{\mu}} + \Delta_{\hat{\mu}} N_i, \tag{5.20}$$

we can see that both transformations left the Hamiltonian unchanged. Investigating this symmetry will be possible to see that the Villain representation contains configurations that circle around a plaquette, called vortices.

Notice that an equivalence class of $p's$ can be constructed requiring that two sets $\{p_{i,\hat{\mu}}\}$ and $\{p'_{i,\hat{\mu}}\}$ be equivalent if exists another set $\{N_i\}$ such that the transformation above (5.20) is valid. Furthermore, the existence of such set $\{N_i\}$ can be shown to be equivalent to the invariance of the lattice 2-form

$$q_{i,\hat{\mu},\hat{\nu}} \equiv \Delta_{\hat{\mu}} p_{i,\hat{\nu}} - \Delta_{\hat{\nu}} p_{i,\hat{\nu}} \tag{5.21}$$

upon the transformation previously defined. To verify this fact, we start by noticing that p is a lattice $1-$form and q is its lattice exterior derivative:

$$p = p_\mu \, dx^\mu, \qquad q = q_{\mu\nu} \, dx^\mu \wedge dx^\nu = dp. \tag{5.22}$$

Thus, if we take arbitrary p and p', requiring invariance of q:

$$dp = dp' \implies d(p - p') = 0. \tag{5.23}$$

Which implies that

$$p - p' = dN, \tag{5.24}$$

just as in (5.20). As defined, the $q's$ are exact lattice forms, so they must always be closed lattice 2-forms, $dq = 0$. Furthermore, defining vortices as lattice angular variable configurations that add up to a multiple of 2π for some closed loop, make us realize that calculating q is equivalent to circle around a lattice site, making them objects that capture information about the vorticity. We may explore this possible role of the q's in the next section.

5.1.2 2D Vorticity

Let us consider an example in two dimensions of a lattice configuration that presents a 2π vortex, as in (Figure 5). In this case, see that we can express

$$q_{i,\hat{x},\hat{y}} = p_{i+\hat{x},\hat{y}} - p_{i,\hat{y}} - p_{i+\hat{y},\hat{x}} + p_{i,\hat{x}} \tag{5.25}$$

Hence, choosing a site i, notice that calculating $q_{i,\hat{x},\hat{y}}$ is equivalent to circulate the placket starting by the ith site (Figure 6).

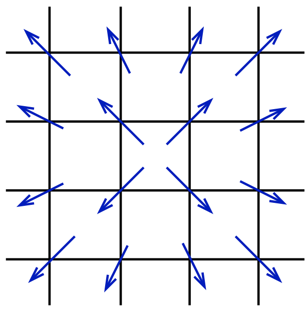

Figure 5 – A vortex configuration.

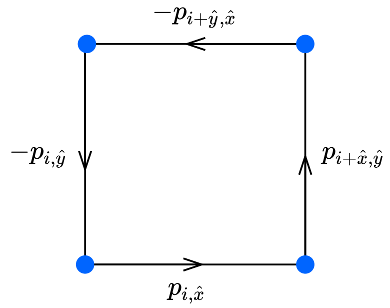

Figure 6 – 2D representation of q.

A vortex configuration is such that if we circulate it (for any closed loop) we accumulate a 2π contribution from the angular variables (actually, any multiple of 2π as well). Recall that we have set the reference for our angular variables in the $\hat{x}$ direction. We now would like to understand in the example given in (Figure 5) what is the role of the q's, by explicitly calculating it in different plaquettes. In order to do so, we need to determine the links variables $p_{i,\hat{\mu}}$, which can be calculated considering the most relevant angular configurations, i.e. that minimize the Hamiltonian. Or equivalently, the ones that

$$\phi_{i+\hat{\mu}} - \phi_i - 2\pi p_{i,\hat{\mu}} \sim \epsilon, \tag{5.26}$$

for $\epsilon \ll 1$. Hence, we notice two distinct settings (Figure 7): the ones near the reference (inside the red boxes), and the remaining. Outside the red boxes, the difference $\phi_{i+\hat{\mu}} - \phi_i$ is already of order ϵ, forcing $p_{i,\hat{\mu}} = 0$. On the other hand, picking angular variables connected by the green links, we see that the ones below are $\phi_i \sim -\pi + \epsilon$, and the ones above are $\phi_{i+\hat{y}} \sim \pi - \epsilon$. Then, these configurations satisfy $\phi_{i+\hat{y}} - \phi_i \sim 2\pi - 2\epsilon$, forcing $p_{i,\hat{\mu}} = 1$. In summary, every $p_{i,\hat{y}}$ between the red boxes must be equal to 1 and the remaining are zero. We call the set of all non-vanishing p's, represented by all the green segments on (Figure 7), of branch cut – this line will be the branch cut of the multivalued field in the continuum limit.

With that in mind, we can calculate q in any plaquette. Clearly, if q is calculated at a plaquette like 1 in (Figure 8), where all p's vanish, q will also vanish. If we consider the plaquettes along the branch cut like 2, we see that when circulating them in the

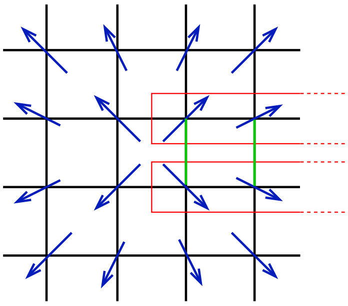

Figure 7 – Branch cut of the vortex.

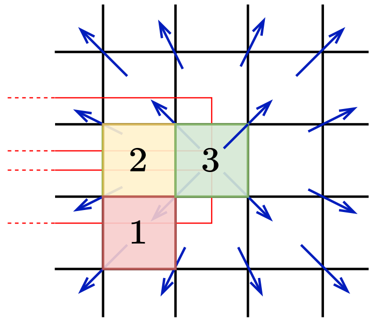

Figure 8 – Examples of plaquettes.

anticlockwise direction, the vertical contributions will cancel out, and the horizontal ones are null. Therefore, for all the plaquettes in the branch cut, q also vanish. However, if we choose the plaquette at the core of the vortex, represented by the plaquette 3, a non-vanishing contribution appear $p_{i+\hat{x}} = 1$, while all the others are zero. Therefore, we see that only when circulating the vortex at its origin, q gives a contribution equals the vorticity density. Indeed, we can generalise this argument for vortex configurations that accumulate $2\pi k$ as we go around a closed loop, for integer k. In general, the lattice two form $q_{i,\hat{x},\hat{y}}$ will gives us k, which is the vorticity density.

Now that we have considered this example, and described the role of the q's and p's, we may study how to implement a continuum limit prescription for the Villain representation of the XY model. Notice, however, that considering the partition function (5.17), there is no natural way to implement a continuum limit for the integer variables $p_{i,\hat{\mu}}$. However, if we consider the vortex ensemble, we will be able to write the partition function as an integral of continuum variables, which is our next task.

5.1.3 Vortex Ensemble

The equivalence relation established in the space of p's disjoint equivalent classes. Then, in the partition function, instead of summing over all $p's$, we can sum over equivalent classes C_p and inside each class, sum over all the $p's$,

$$\sum_{p_{i,\hat{\mu}}} \mapsto \sum_{C_p} \sum_{p_{i,\hat{\mu}} \in C_p} . \tag{5.27}$$

Furthermore, considering all the $p's$ in the same class,

$$p_{i,\hat{\mu}} = p^{(0)}_{i,\hat{\mu}} + \nabla_\mu N_i, \tag{5.28}$$

for some representative $p^{(0)}_{i,\hat{\mu}}$. Hence, summing over all the $p's$ in a class is equivalent to sum over all the N_i of that class,

$$\sum_{C_p} \sum_{p_{i,\hat{\mu}} \in C_p} \mapsto \sum_{C_p} \sum_{N_i \in \mathbb{Z}} . \tag{5.29}$$

With this considerations, the partition function can be rewritten as

$$Z = \sum_{C_p} \sum_{N_i \in \mathbb{Z}} \int_{-\pi}^{\pi} \prod_j d\phi_j \, \exp\left[-\frac{\beta_V}{2} \sum_{i,\hat{\mu}} \left((\phi_{i+\hat{\mu}} - 2\pi N_{i+\hat{\mu}}) - (\phi_i - 2\pi N_i) - 2\pi p^{(0)}_{i,\hat{\mu}} \right)^2 \right]. \tag{5.30}$$

By a change of variables $\phi_i \mapsto \tilde{\phi}_i = \phi_i - 2\pi N_i$,

$$Z = \sum_{C_p} \sum_{N_i \in \mathbb{Z}} \prod_j \int_{-\pi-2\pi N_j}^{\pi-2\pi N_j} d\tilde{\phi}_j \, \exp\left[-\frac{\beta_V}{2} \sum_{i,\hat{\mu}} \left(\tilde{\phi}_{i+\hat{\mu}} - \tilde{\phi}_i - 2\pi p^{(0)}_{i,\hat{\mu}} \right)^2 \right]. \tag{5.31}$$

we may notice that the integral over $\widetilde{\phi}_i$ runs over all the real line, because

$$\sum_{N_i \in \mathbb{Z}} \prod_i \int_{-\pi-2\pi N_i}^{\pi-2\pi N_i} d\tilde{\phi}_i = \int_{-\infty}^{\infty} \prod_i d\tilde{\phi}_i. \tag{5.32}$$

Plugging this result in (5.31), follows the partition function

$$Z = \sum_{C_p} \int_{-\infty}^{\infty} \prod_j d\tilde{\phi}_j \, \exp\left[-\frac{\beta_V}{2} \sum_{i,\hat{\mu}} \left(\nabla_{\hat{\mu}} \tilde{\phi}_i - 2\pi p^{(0)}_{i,\hat{\mu}} \right)^2 \right]. \tag{5.33}$$

written as an integral of the continuous variables $\widetilde{\phi}_i$. However, we would like to consider the vorticity density explicitly, given that our main goal is to naturally incorporate vortex configuration in the continuum theory. One simple way to introduce the vorticity density is by using a Gaussian integral to replace the exponential weight in the above partition function by an integral of a new variable $j_{i\,\hat{\mu}}$,

$$\exp\left[-\frac{\beta_V}{2} \left(\Delta_{\hat{\mu}} \tilde{\phi}_i - 2\pi p^{(0)}_{i,\hat{\mu}} \right)^2 \right] \sim \int_{-\infty}^{\infty} dj_{i,\hat{\mu}} \, \exp\left[-\frac{j^2_{i,\hat{\mu}}}{2\beta_V} + i j_{i,\hat{\mu}} \left(\Delta_{\hat{\mu}} \tilde{\phi}_i - 2\pi p^{(0)}_{i,\hat{\mu}} \right) \right]. \tag{5.34}$$

Now, the partition function reads

$$Z = \sum_{C_p} \int_{-\infty}^{\infty} \prod_j d\tilde{\phi}_j \int_{-\infty}^{\infty} \prod_{k,\hat{\mu}} dj_{k,\hat{\mu}} \, \exp\left[\sum_{i,\hat{\mu}} \left(-\frac{j^2_{i,\hat{\mu}}}{2\beta_V} + i j_{i,\hat{\mu}} \left(\Delta_{\hat{\mu}} \tilde{\phi}_i - 2\pi p^{(0)}_{i,\hat{\mu}} \right) \right) \right]. \tag{5.35}$$

In order to integrate over $\widetilde{\phi}_i$ we must rearrange the $j's$ by simply shifting the index i as in a product derivative rule:

$$\sum_{i,\hat{\mu}} j_{i,\hat{\mu}} \Delta_{\hat{\mu}} \tilde{\phi}_i = -\sum_i \left(\sum_{\hat{\mu}} \Delta_{\hat{\mu}} j_{i-\hat{\mu},\hat{\mu}} \right) \tilde{\phi}_i. \tag{5.36}$$

Considering this rule, the integral over $\widetilde{\phi}_i$ reads

$$I = \prod_i \int_{-\infty}^{\infty} d\tilde{\phi}_i \exp\left[-i \sum_{\hat{\mu}} \Delta_{\hat{\mu}} j_{i-\hat{\mu},\hat{\mu}} \tilde{\phi}_i \right] = \prod_i \delta \left(\sum_{\hat{\mu}} \Delta_{\hat{\mu}} j_{i,\hat{\mu}} \right), \tag{5.37}$$

where the product over i allows us switch back $i-\hat{\mu}\rightarrow i$. The summation inside the delta function can be identified by a simple notation as

$$\sum_{\hat{\mu}}\Delta_{\hat{\mu}}j_{i,\hat{\mu}}\equiv\Delta_{(i)}\cdot j, \tag{5.38}$$

and then we obtain the following partition function

$$Z=\sum_{C_p}\int_{-\infty}^{\infty}\prod_{k,\hat{\mu}}dj_{k,\hat{\mu}}\,\delta\left(\Delta_{(k)}\cdot j\right)\exp\left[\sum_{i,\hat{\mu}}\left(-\frac{j_{i,\hat{\mu}}^2}{2\beta_V}-2\pi i j_{i,\hat{\mu}}p_{i,\hat{\mu}}^{(0)}\right)\right] \tag{5.39}$$

Let us explicitly consider the components of $j_{i,\hat{\mu}}$ and solve the delta function constraint by considering

$$\begin{cases}j_{i,\hat{x}} &= \Delta_{\hat{y}}\xi_i\\ j_{i,\hat{y}} &= -\Delta_{\hat{x}}\xi_i\end{cases}, \tag{5.40}$$

for a set of real valued functions ξ_i. This solution works because the lattice derivates along different directions commute:

$$\Delta_{\hat{x}}\Delta_{\hat{y}}\xi_i=\Delta_{\hat{x}}\left(\xi_{i+\hat{y}}-\xi_i\right)=\xi_{i+\hat{y}+\hat{x}}-\xi_{i+\hat{y}}-\xi_{i+\hat{x}}+\xi_i,$$

$$\Delta_{\hat{y}}\Delta_{\hat{x}}\xi_i=\Delta_{\hat{y}}\left(\xi_{i+\hat{x}}-\xi_i\right)=\xi_{i+\hat{x}+\hat{y}}-\xi_{i+\hat{x}}-\xi_{i+\hat{y}}+\xi_i,$$

Furthermore, we may notice that by a product rule

$$\sum_{i,\hat{\mu}}j_{i,\hat{\mu}}p_{i,\hat{\mu}}^{(0)}=\sum_i\left(\Delta_{\hat{x}}p_{i,\hat{y}}^{(0)}-\Delta_{\hat{y}}p_{i,\hat{x}}^{(0)}\right)\xi_i\equiv\sum_i q_{i,\hat{x},\hat{y}}\xi_i, \tag{5.41}$$

and identifying

$$\sum_{i,\hat{\mu}}j_{i,\hat{\mu}}^2=\sum_{i,\hat{\mu}}\left(\Delta_{\hat{\mu}}\xi_i\right)^2\equiv\left(\nabla\xi\right)^2, \tag{5.42}$$

the partition function reads as simply

$$Z=\sum_{\left\{q_{i,\hat{x},\hat{y}}\right\}}\int_{-\infty}^{\infty}\mathcal{D}\xi\,\exp\left[-\frac{\left(\nabla\xi\right)^2}{2\beta_V}-2\pi i\sum_i q_{i,\hat{x},\hat{y}}\xi_i\right]. \tag{5.43}$$

Using the above partition function is not clear yet how to consider a continuum limit, given that the $q's$ are integer variables. However, we are going to show that it is possible to map these integer variables into continuum ones using a mathematical identity. Let $f\left(q\right)$ be a function assuming integer values q, then

$$\sum_{q\in\mathbb{Z}}f\left(q\right)=\sum_{q\in\mathbb{Z}}\int d\varphi f\left(\varphi/\pi\right)\delta\left(\varphi-\pi q\right)=\frac{1}{2\pi}\sum_{q\in\mathbb{Z}}\int d\varphi f\left(\varphi/\pi\right)\sum_{p\in\mathbb{Z}}e^{ip(\varphi-\pi q)}. \tag{5.44}$$

However, we can sum up over q considering even and odd contributions, resulting

$$\sum_{p,q\in\mathbb{Z}}e^{ip(\varphi-\pi q)}=2\sum_{p\in\mathbb{Z}}e^{2ip\varphi}. \tag{5.45}$$

Plugging this into our previous identity, and introducing a parameter $a = 1$,

$$\sum_{q\in\mathbb{Z}} f(q) = \frac{1}{\pi}\sum_{q\in\mathbb{Z}}\int_{-\infty}^{\infty} d\varphi\, f(\varphi/\pi)\, e^{2iq\varphi} = \frac{1}{\pi}\int_{-\infty}^{\infty} d\varphi\, f(\varphi/\pi)\sum_{p\in\mathbb{Z}} e^{(\ln a)p^2+2ip\varphi}. \quad (5.46)$$

This new parameter will be considered as an expansion parameter in the approximation that we are going to describe. But before taking any approximation, let us rewrite te above expression as

$$\sum_{q\in\mathbb{Z}} e^{(\ln a)p^2+2ip\varphi} = 1 + \sum_{p>0}\left(e^{(\ln a)p^2+2ip\varphi} + e^{(\ln a)p^2-2ip\varphi}\right) = 1 + 2\sum_{p>0} a^{p^2}\cos(2p\varphi), \quad (5.47)$$

making the cosine explicit in the calculations. Therefore, one can write

$$\sum_{q\in\mathbb{Z}} f(q) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\varphi\, f(\varphi/\pi)\left[1 + 2\sum_{p>0} a^{p^2}\cos(2p\varphi)\right]. \quad (5.48)$$

This is an exact identity for $a = 1$. However, we are going to consider an approximation where $a \to 0$. The identity does not hold in this limit, but we must notice that we are going to consider it within a partition function. Therefore, whether the cosine operators appearing in the above expansion are relevant or irrelevant it's very pertinent to the physical description. In fact we are going to show by an analysis of the renormalization group that the cosine operator $\cos(2\varphi)$ is relevant, and all the remaining cosine operators are not. Then, it is simplifying in this moment to consider only linear contributions in the parameter a:

$$\sum_{q\in\mathbb{Z}} f(q) \approx$$
$$\approx \frac{1}{\pi}\int_{-\infty}^{\infty} d\varphi\, f(\varphi/\pi)\left[1 + 2a\cos(2\varphi)\right] \approx \frac{1}{\pi}\int_{-\infty}^{\infty} d\varphi\, f(\varphi/\pi)\exp\left[2a\cos(2\varphi)\right]. \quad (5.49)$$

We can generalise this identity to a function of multiple integer variables $f(q_1,,\ldots,q_n)$. The steps being exactly the same: we introduce a delta function and use its Fourier representation. Introducing a perturbative parameter $a \sim 0$,

$$\sum_{q_1\cdots q_n}\prod_{i=1}^{n}\int_{-\infty}^{\infty} d\xi_i\, \exp(2iq_i\xi_i)\, f\left(\frac{\xi_1}{\pi},\ldots,\frac{\xi_n}{\pi}\right) \approx$$
$$\approx \prod_{i=1}^{n}\int_{-\infty}^{\infty} d\xi_i\, \exp(2a\cos(2\xi_i))\, f\left(\frac{\xi_1}{\pi},\ldots,\frac{\xi_n}{\pi}\right). \quad (5.50)$$

Considering this identity for

$$f\left(\frac{\xi}{\pi}\right) = \exp\left[-\frac{(\nabla\xi)^2}{2\beta_V}\right], \quad (5.51)$$

the left hand-side becomes the partition function (5.43) and it is mapped to the right hand-side, which is a partition function written only with continuum variables ξ_i with a new cosine operator. This procedure encodes the information about the vorticity inside the cosine operator, fact that we may analyse further. The partition function now reads

$$Z = \int_{-\infty}^{\infty} \mathcal{D}\xi \; \exp\left(-\frac{(\nabla\xi)^2}{2\beta_V} + 2a\sum_i \cos\left(2\xi_i\right)\right). \tag{5.52}$$

In the continuum limit,

$$Z = \int \mathcal{D}\xi \; \exp\left\{-\int d^2x \left[\frac{1}{2\beta}\left(\nabla\xi\right)^2 - 2a\cos\left(2\xi\right)\right]\right\}. \tag{5.53}$$

5.1.4 Renormalisation Group Analysis

As we mentioned, the vortexes represented by the q's in the vortex ensemble are now encoded inside the cosine operator. We may study now when this operator is relevant in the continuum theory, and in order to that we will consider the Wilson's approach to the renormalization group. Let Λ be a high energy cut off such that in momentum space

$$Z = \int \left[\mathcal{D}\xi\right]_\Lambda \exp\left\{-\int d^2x \left[\frac{1}{2\beta}\left(\nabla\xi\right)^2 - 2a\cos\left(2\xi\right)\right]\right\}, \tag{5.54}$$

with the integral element being

$$\left[\mathcal{D}\xi\right]_\Lambda = \prod_{|k|<\Lambda} d\xi\left(k\right). \tag{5.55}$$

Defining a fraction $0 < b < 1$, we can introduce a UV and a IR field, respectively, by

$$\xi_+\left(k\right) = \begin{cases} \xi\left(k\right), & b\Lambda \leq |k| < \Lambda \\ 0, & \text{otherwise} \end{cases},$$

$$\xi_-\left(k\right) = \begin{cases} \xi\left(k\right), & 0 \leq |k| < b\Lambda \\ 0, & \text{otherwise} \end{cases}.$$

Then, we see that $\xi\left(k\right) = \xi_+\left(k\right) + \xi_-\left(k\right)$, and plugging this into the partition function,

$$Z = \int \mathcal{D}\xi_-\mathcal{D}\xi_+ \; \exp\left\{-\int d^2x \left[\frac{1}{2\beta}\left(\nabla\xi_+\right)^2 + \frac{1}{2\beta}\left(\nabla\xi_-\right)^2 + \right.\right.$$
$$\left.\left. + \frac{1}{\beta}\nabla\xi_+\cdot\nabla\xi_- - 2a\cos\left(2\xi_+ + 2\xi_-\right)\right]\right\}. \tag{5.56}$$

The cross term $\nabla\xi_+\cdot\nabla\xi_-$ vanishes because if k_μ represents the momentum of ξ_+ and k'_μ represents the momentum of ξ_-, then $k^\mu k'_\mu = 0$ and in Fourier space

$$\int d^2x\,\nabla\xi_+\cdot\nabla\xi_- = \int d^2x \int \frac{d^2k}{\left(2\pi\right)^2}\int \frac{d^2k'}{\left(2\pi\right)^2} e^{ikx}e^{ik'x}k_\mu k'^\mu \xi_+\left(k\right)\xi_-\left(k'\right) = 0. \tag{5.57}$$

Using this result, the partition function becomes

$$Z = \int \mathcal{D}\xi_- \exp\left\{-\int d^2x \frac{1}{2\beta}\left(\nabla\xi_-\right)^2\right\} \times \\ \times \int \mathcal{D}\xi_+ \exp\left\{-\int d^2x \left[\frac{1}{2\beta}\left(\nabla\xi_+\right)^2 - 2a\cos\left(2\xi_+ + 2\xi_-\right)\right]\right\}. \quad (5.58)$$

Let us define the partition function over the UV as

$$Z_+ = \int \mathcal{D}\xi_+ \exp\left\{-\int d^2x \left[\frac{1}{2\beta}\left(\nabla\xi_+\right)^2 - 2a\cos\left(2\xi_+ + 2\xi_-\right)\right]\right\}. \quad (5.59)$$

Now, we would like to integrate out the high energy contributions in order to verify its effect in the low energy theory. By a perturbative expansion in the parameter a,

$$Z_+ = \int \mathcal{D}\xi_+ \exp\left\{-\int d^2x \frac{1}{2\beta}\left(\nabla\xi_-\right)^2\right\}\left(1 + \int d^2x\, 2a\cos\left(2\xi_+^x + 2\xi_-^x\right) + \\ + \frac{1}{2}\int d^2x \int d^2y\, 4a^2\cos\left(2\xi_+^x + 2\xi_-^x\right)\cos\left(2\xi_+^y + 2\xi_-^y\right) + \mathcal{O}\left(a^3\right)\right). \quad (5.60)$$

Here, we can identify each order as an expected value

$$\langle A\left(\xi_+\right)\rangle_+ = \frac{1}{Z_+|_{a=0}}\int \mathcal{D}\xi_+ \exp\left\{-\int d^2x \frac{1}{2\beta}\left(\nabla\xi_+\right)^2\right\} A\left(\xi_+\right), \quad (5.61)$$

for some function $A\left(\xi_+\right)$. Considering a normalised partition function $\left(Z_+|_{a=0} = 1\right)$, we can rewrite Z_+ as

$$Z_+ = 1 + 2a\int d^2x\, \langle\cos\left(2\xi_+^x + 2\xi_-^x\right)\rangle_+ + \\ 2a^2\int d^2x \int d^2y\, \langle\cos\left(2\xi_+^x + 2\xi_-^x\right)\cos\left(2\xi_+^y + 2\xi_-^y\right)\rangle_+ + \mathcal{O}\left(a^3\right). \quad (5.62)$$

The upper indices on the fields $\xi_\pm$ are denoting the functional dependence on the space-time coordinates: $\xi_\pm^x \equiv \xi_\pm(x)$. Now, let us calculate each order separately. For the first order, see that

$$2\int d^2x\, \langle\cos\left(2\xi_+^x + 2\xi_-^x\right)\rangle_+ = \int d^2x \sum_{\sigma=\pm 1} e^{2i\sigma\xi_-^x}\left\langle e^{2i\sigma\xi_+^x}\right\rangle_+. \quad (5.63)$$

The expected values of these exponentials are

$$\left\langle e^{2i\sigma\xi_+^x}\right\rangle_+ = \int \mathcal{D}\xi_+ \exp\left\{-\int d^2y \frac{1}{2\beta}\left(\nabla\xi_+^y\right)^2 + 2i\sigma\xi_+^x\right\} = \exp\left\{-2\sigma^2 K\left(x,x\right)\right\}, \quad (5.64)$$

where $K\left(x,y\right) = \left\langle\xi_+^x\xi_+^y\right\rangle_+$ is the inverse operator of $-\frac{1}{2\beta}\nabla^2\delta\left(x-y\right)$, which must satisfy

$$\nabla^2 K\left(x,y\right) = -2\beta\delta\left(x-y\right). \quad (5.65)$$

We can obtain K explicitly by considering the Fourier transform of the above expression,

$$\int_{\Lambda_>} \frac{d^2k}{(2\pi)^2} \left(-k^2\right) e^{ik(x-y)} K(k) = -2\beta\delta(x-y), \tag{5.66}$$

where the region of integration $\Lambda_> = \{k : b\Lambda \leq |k| < \Lambda\}$. Using the Dirac delta Fourier representation, the above expression implies that

$$K(x,y) = 2\beta \int_{\Lambda_>} \frac{d^2k}{(2\pi)^2} \frac{e^{ik(x-y)}}{k^2} = \langle \xi_+^x \xi_+^y \rangle_+ . \tag{5.67}$$

It then follows that $K(x,y) = K(x-y)$ and hence

$$K(x,x) = K(0) = \langle \xi_+(0)\, \xi_+(0) \rangle_+ = 8\pi\beta \int_{b\Lambda}^{\Lambda} \frac{dk}{(2\pi)^2} \frac{1}{k} = \frac{\beta}{2\pi} \ln \zeta, \tag{5.68}$$

where we have defined $\zeta \equiv 1/b$. Therefore, the expected value of the exponential terms like (5.64) are

$$\left\langle e^{2i\sigma\xi_+^x} \right\rangle_+ = \exp\left\{ -\frac{\beta}{\pi} \ln \zeta \right\} = \zeta^{-\frac{\beta}{\pi}}, \tag{5.69}$$

and the first order contribution is

$$2a \int d^2x \, \langle \cos(2\xi_+^x + 2\xi_-^x) \rangle_+ = 2a\zeta^{-\frac{\beta}{\pi}} \int d^2x \, \cos(2\xi_-^x). \tag{5.70}$$

Up to this order, the high energy partition function can approximated to

$$Z_+ \simeq 1 + 2a\zeta^{-\frac{\beta}{\pi}} \int d^2x \, \cos(2\xi_-^x) \simeq \exp\left(2a\zeta^{-\frac{\beta}{\pi}} \int d^2x \, \cos(2\xi_-^x) \right) \tag{5.71}$$

and the complete partition function is given in terms of an Euclidean action

$$Z = \int \mathcal{D}\xi_- \exp\left\{-S_{eff}[\xi_-]\right\}, \quad S_{eff}[\xi_-] \simeq \int d^2x \left[\frac{1}{2\beta} (\nabla\xi_-)^2 - 2a\zeta^{-\frac{\beta}{\pi}} \cos(2\xi_-^x) \right]. \tag{5.72}$$

We may notice that the upper energy scale of this theory is now different from the original one. Therefore, in order to characterize the effect of high energy modes, comparing the above with the original theory, we need to restore the scale to the one we started with. Let us consider $x \to x/\zeta$ and relabel $a \to a_0$. Then, the action becomes

$$S_{eff}[\xi_-] \simeq \int d^2x \left[\frac{1}{2\beta} (\nabla\xi_-)^2 - 2a_0\zeta^{2-\frac{\beta}{\pi}} \cos(2\xi_-^x) \right] \tag{5.73}$$

Now, comparing the above action with original one, we see that the coupling constant is redefined to be

$$a(\zeta) = a_0\zeta^{2-\frac{\beta}{\pi}}. \tag{5.74}$$

Furthermore, notice that for $\beta > 2\pi$ the coupling constant decreases indicating that $\cos\left(2\xi_-^x\right)$ is an irrelevant operator in this β region. Hence the high energy effect up to first order is to renormalize the coupling constant a.

Let us consider the second order. The contribution we need to calculate is

$$Z_+^{(2)} = 2a^2 \int d^2x \int d^2y \, \left\langle \cos\left(2\xi_+^x + 2\xi_-^x\right) \cos\left(2\xi_+^y + 2\xi_-^y\right) \right\rangle_+ = \\ = \frac{a^2}{2} \int d^2x \int d^2y \, \Big\{ \cos\left(2\left(\xi_-^x + \xi_-^y\right)\right) e^{-2\left\langle \left(\xi_+^x + \xi_+^y\right)^2 \right\rangle_+} + \\ + \cos\left(2\left(\xi_-^x - \xi_-^y\right)\right) e^{-2\left\langle \left(\xi_+^x - \xi_+^y\right)^2 \right\rangle_+} \Big\}. \quad (5.75)$$

Before computing all the contributions, we may notice that the low energy effective action actually generates connected diagrams. To realize this we recall that

$$Z = \int \mathcal{D}\xi_- \int \mathcal{D}\xi_+ \, \exp -S\left[\xi_-, \xi_+\right] = \int \mathcal{D}\xi_- \, \exp -S_{eff}\left[\xi_-\right].$$

Hence, we can identify

$$\exp -S_{eff}\left[\xi_-\right] = \int \mathcal{D}\xi_+ \, \exp -S\left[\xi_-, \xi_+\right],$$

meaning that the effective action is actually the generator of connected diagrams in this perturbative expansion. Therefore, we need calculate and subtract the disconnected contribution, which is

$$\left\langle \cos\left(2\xi_+^x + 2\xi_-^x\right) \right\rangle_+ \left\langle \cos\left(2\xi_+^y + 2\xi_-^y\right) \right\rangle_+ = \\ = 2\cos\left(2\left(\xi_-^x + \xi_-^y\right)\right) e^{-2\left\langle \xi_+^x \xi_+^x \right\rangle_+} e^{-2\left\langle \xi_+^y \xi_+^y \right\rangle_+} + \\ + 2\cos\left(2\left(\xi_-^x - \xi_-^y\right)\right) e^{-2\left\langle \xi_+^x \xi_+^x \right\rangle_+} e^{-2\left\langle \xi_+^y \xi_+^y \right\rangle_+}. \quad (5.76)$$

Then, subtracting this contribution,

$$Z_+^{(2)} = \frac{a_0^2}{2} \int d^2x \int d^2y \, \Big\{ \cos\left(2\left(\xi_-^x + \xi_-^y\right)\right) \left[e^{-2\left\langle \left(\xi_+^x + \xi_+^y\right)^2 \right\rangle_+} - e^{-2\left\langle \xi_+^x \xi_+^x \right\rangle_+} e^{-2\left\langle \xi_+^y \xi_+^y \right\rangle_+} \right] + \\ + \cos\left(2\left(\xi_-^x - \xi_-^y\right)\right) \left[e^{-2\left\langle \left(\xi_+^x - \xi_+^y\right)^2 \right\rangle_+} - e^{-2\left\langle \xi_+^x \xi_+^x \right\rangle_+} e^{-2\left\langle \xi_+^y \xi_+^y \right\rangle_+} \right] \Big\}. \quad (5.77)$$

Using our previous calculations of the propagator, (5.67) and (5.68), we can calculate the expected values appearing in the exponentials above,

$$-2\left\langle \left(\xi_+^x \pm \xi_+^y\right)^2 \right\rangle_+ = -\frac{2\beta}{\pi} \ln \zeta \mp 4\beta K\left(x, y\right), \quad -2\left\langle \xi_+^x \xi_+^x \right\rangle_+ - 2\left\langle \xi_+^y \xi_+^y \right\rangle_+ = -\frac{2\beta}{\pi} \ln \zeta. \\ (5.78)$$

Plugging these on the second order contribution of the effective action, one gets

$$S_{eff}^{(2)}\left[\xi_-\right] = \zeta^{-\frac{2\beta}{\pi}} \frac{a_0^2}{2} \int d^2x \int d^2y \, \Big\{ \cos\left(2\left(\xi_-^x + \xi_-^y\right)\right) \left[e^{-4\beta K(x,y)} - 1 \right] + \\ + \cos\left(2\left(\xi_-^x - \xi_-^y\right)\right) \left[e^{4\beta K(x,y)} - 1 \right] \Big\}. \quad (5.79)$$

This seems to be a non-local action, let us analyse carefully. See that we can write the propagator K as an integral of a Bessel function of the first kind:

$$K(r) = 2\beta \int_{\Lambda/\zeta}^{\Lambda} \frac{dk}{(2\pi)^2} \frac{1}{k} \int_0^{2\pi} dk_\theta \, e^{ikr\cos\theta} = 2\beta \int_{\Lambda/\zeta}^{\Lambda} \frac{dk}{2\pi} \frac{J_0(kr)}{k}.$$

where $r = |x - y|$. Considering an infinitesimal interval, such that

$$d\Lambda = \Lambda - \Lambda/\zeta \equiv \Lambda s, \tag{5.80}$$

we can approximate the propagator to

$$K(r) \simeq \frac{\beta}{2\pi} \frac{d\Lambda}{\Lambda} J_0(\Lambda r) = \frac{\beta s}{2\pi} J_0(\Lambda r).$$

Therefore, the sharp cut off regularisation we started with leads to an oscillating propagator that is not concentrated on a region of the configuration space, given that J_0 in the above equation is an oscillating function. However, a more convenient regularisation procedure leads to a propagator that is non vanishing only for small $r \sim \zeta/\Lambda$, from where we could expand and eliminate the apparent non-locality, see [9, 13] for more details. Then, from now on, we will consider this other scheme of regularisation, where the propagator is given by

$$\tilde{K}(r) \simeq \frac{\beta s}{2\pi} J(\Lambda r),$$

where $J(\Lambda r)$ is the new regularizer concentrated at $r \sim \zeta/\Lambda$. Expanding in small $v = x - y \sim \zeta/\Lambda$, and keeping the first non-vanishing contributions,

$$\cos\left(2\left(\xi_-^x + \xi_-^y\right)\right) \simeq \cos\left(4\xi_-^x\right),$$

$$\cos\left(2\left(\xi_-^x - \xi_-^y\right)\right) \simeq \cos\left(2\left(\xi_-^x - \xi_-^x - v\cdot\nabla\xi_-^x\right)\right) \simeq 1 - 2\left(v\cdot\nabla\xi_-^x\right)^2.$$

Then, the second order effective action becomes

$$S_{eff}^{(2)}[\xi_-] = \zeta^{-\frac{2\beta}{\pi}} \frac{a_0^2}{2} \int d^2x \int d^2v \left\{ \cos\left(4\xi_-^x\right)\left[e^{-4\beta\tilde{K}(v)} - 1\right] + \right.$$
$$\left. -2\left(v\cdot\nabla\xi_-^x\right)^2 \left[e^{4\beta\tilde{K}(v)} - 1\right] \right\}. \tag{5.81}$$

It is useful to define the $\zeta-$dependent constants

$$A_1(\zeta) = \zeta^{-\frac{2\beta}{\pi}} \int d^2v \left[e^{4\beta\tilde{K}(v)} - 1\right],$$

$$A_2(\zeta) = -2\zeta^{-\frac{2\beta}{\pi}} \int d^2v \, v^2 \left[e^{4\beta\tilde{K}(v)} - 1\right],$$

because using them, the second order action is simplified to

$$S_{eff}^{(2)}[\xi_-] = \frac{a_0^2}{2} \int d^2x \left\{ A_1(\zeta)\cos\left(4\xi_-^x\right) + A_2(\zeta)\left(\nabla\xi_-^x\right)^2 \right\}$$

Again, to compare with the original action, we apply the rescale $x \to x/\zeta$, obtaining

$$S^{(2)}_{eff}\left[\xi_{-}\right] = \frac{a_0^2}{2}\int d^2x \left\{\frac{A_1\left(\zeta\right)}{\zeta^2}\cos\left(4\xi^x_{-}\right) + A_2\left(\zeta\right)\left(\nabla\xi^x_{-}\right)^2\right\}.$$

Therefore, the effective action up to second order becomes

$$S_{eff}\left[\xi_{-}\right] = \int d^2x \left\{\left(\frac{1}{2\beta_0} + \frac{a_0^2}{2}A_2\left(\zeta\right)\right)\left(\nabla\xi^x_{-}\right)^2 - 2a\left(\zeta\right)\cos\left(2\xi^x_{-}\right) + \right.$$
$$\left. + \frac{a_0^2}{2\zeta^2}A_1\left(\zeta\right)\cos\left(4\xi^x_{-}\right)\right\}. \quad (5.82)$$

We may notice the high energy effects up to the second order. The kinetic term is now renormalized and scale dependent, and a new cosine operator emerged. Up to this order we can already determine the renormalization group flux. Let us define

$$\frac{1}{2\beta\left(\zeta\right)} = \frac{1}{2\beta_0} + \frac{a_0^2}{2}A_2\left(\zeta\right), \quad (5.83)$$

Hence, up to second order in a_0,

$$\beta\left(\zeta\right) = \frac{\beta_0}{1 + \beta_0 a_0^2 A_2\left(\zeta\right)} \simeq \beta_0 - \beta_0^2 a_0^2 A_2\left(\zeta\right). \quad (5.84)$$

Furthermore, considering the definition (5.80), we notice that $\zeta \simeq 1 + s \simeq e^s$, and then writing (5.74) as a function of s and differentiating with respect to s, follows that

$$\frac{da}{ds} = \left(2 - \frac{\beta_0}{\pi}\right)a \simeq \left(2 - \frac{\beta}{\pi}\right)a, \quad (5.85)$$

where in the last step we approximate β_0 by β, given that the latter has a second order contribution. In terms of s, the β equation (5.84) becomes

$$\beta\left(s\right) = \beta_0 - \beta_0^2 a_0^2 A_2\left(e^s\right). \quad (5.86)$$

Now, using the infinitesimal approximation $s \sim 0$,

$$A_2\left(\zeta\right) \simeq -\frac{4\beta^2}{\pi}s\int d^2v\, v^2 J\left(\Lambda v\right). \quad (5.87)$$

Plugging this into β and differentiating with respect to s gives us

$$\frac{d\beta}{ds} = \frac{2}{\pi}C(\Lambda)\beta_0^3 a^2, \quad (5.88)$$

where the constant factor is defined as

$$C(\Lambda) \equiv \int d^2v\, v^2 J\left(\Lambda v\right). \quad (5.89)$$

Integrating both (5.85) and (5.88) we can derive the behaviour of the coupling constants. In order to simplify this step, let us consider

$$x = \frac{\beta}{\pi} - 2, \qquad y = \delta a, \qquad \delta = \left(\frac{2}{\pi} C(\Lambda) \beta_0^3 \right)^{1/2}. \tag{5.90}$$

Differentiating with respect to s,

$$\frac{dx}{ds} = -\frac{1}{\pi} y^2, \qquad \frac{dy}{ds} = -xy. \tag{5.91}$$

In particular, we notice that the derivate of x^2 and y^2 satisfy

$$\frac{d}{ds} \left(\pi x^2 - y^2 \right), \tag{5.92}$$

meaning that x and y must lie over the curve

$$\pi x^2 - y^2 = J \in \mathbb{R}. \tag{5.93}$$

Plotting this equation for β and a for different J's, one obtains the figure below. When

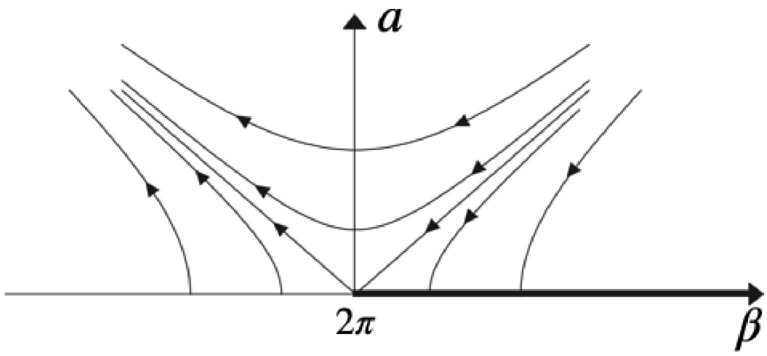

Figure 9 – Running of the coupling constants [1]

$J = 0$, we see that the solution of (5.93) are the two lines ending in $\beta = 2\pi$. Otherwise, for $J > 0$ or $J < 0$, equation (5.93) represents the hyperboles in (Figure 9). From a physics perspective, we can see that $\beta = 2\pi$ is a transition temperature: when $\beta > 2\pi$, the coupling constant a tend to decrease, meaning that the cosine operator is irrelevant in this region, and the vortices do not proliferate. On the other hand, when $\beta < 2\pi$, the coupling constant a tend to increase, making the cosine operator relevant. In this scenario, vortex configuration proliferate. So we have found two different phases: one in which the vortexes are confined ($\beta > 2\pi$), and other in which the vortexes are free to proliferate ($\beta < 2\pi$). This mechanism is called Kosterlitz-Thouless transition, discovered by the physicists John Michael Kosterlitz and David Thouless [14].

5.2 XY-Plaquette Model

In this section we discuss some key facts about the XY-plaquette model and apply the procedure discussed in the last section in order to obtain a continuum limit for this model. For a detailed discussion see [15, 16]. The XY-plaquette is a $(2+1)d$ lattice model that we can described by the Euclidean action

$$S = \beta_0 \sum_i \cos\left(\Delta_{\hat{\tau}}\phi_i\right) + \beta \sum_i \cos\left(\Delta_{\hat{x}}\Delta_{\hat{y}}\phi_i\right), \tag{5.94}$$

where $i = (\tau, x, y)$ are the lattice sites and $\hat{\tau}$ represents the time direction. We shall mention that this is periodic lattice, and therefore, we can identify by an equivalence relation $i \sim i + L$, where $L = (L_\tau, L_x, L_y)$ are the lattice size in each direction. Its Villain representation is obtained by introducing two integer fields $(l_{i,\hat{\tau}};\, p_{i,\hat{x},\hat{y}})$,

$$S = \frac{\beta_0}{2} \sum_i \left(\Delta_{\hat{\tau}}\phi_i - 2\pi l_{i,\hat{\tau}}\right)^2 + \frac{\beta}{2} \sum_i \left(\Delta_{\hat{x}}\Delta_{\hat{y}}\phi_i - 2\pi p_{i,\hat{x},\hat{y}}\right)^2. \tag{5.95}$$

The integer field $l_{i,\hat{\tau}}$ are defined over the links of lattice, and $p_{i,\hat{x},\hat{y}}$ are defined over the plaquettes. Similarly to the XY model, the XY-plaquette model presents a gauge symmetry,

$$\phi_i \to \phi_i + 2\pi N_i, \qquad l_{i,\hat{\tau}} \to l_{i,\hat{\tau}} + 2\pi \Delta_{\hat{\tau}} N_i, \qquad p_{i,\hat{x},\hat{y}} \to p_{i,\hat{x},\hat{y}} + 2\pi \Delta_{\hat{x}}\Delta_{\hat{x}} N_i, \tag{5.96}$$

and we can defined an invariant field strength

$$q_{i,\hat{\tau},\hat{x},\hat{y}} \equiv \Delta_{\hat{x}}\Delta_{\hat{y}} l_{i,\hat{\tau}} - \Delta_{\hat{\tau}} p_{i,\hat{x},\hat{y}}. \tag{5.97}$$

Again, the existence of the N's is equivalent to the invariance of the field strength above, which enable us to introduce equivalence classes of fields related by the gauge transformations (5.96), just as we did in the XY model discussion.

This is an exotic model that presents a subsystem symmetry, where the charges are conserved along a submanifold of space-time. This subsystem symmetry is implemented by the transformation

$$\phi_i \to \phi_i + c_{\hat{x}}(x) + c_{\hat{y}}(y). \tag{5.98}$$

Notice that the variation of the action under this transformation is

$$\delta S = \sum_i \left[-\beta_0 \Delta_{\hat{\tau}}\left(\Delta_{\hat{\tau}}\phi_i - 2\pi l_{i,\hat{\tau}}\right) + \beta \Delta_{\hat{x}}\Delta_{\hat{y}}\left(\Delta_{\hat{x}}\Delta_{\hat{y}}\phi_i - 2\pi p_{i,\hat{x},\hat{y}}\right)\right] \delta\phi_i. \tag{5.99}$$

Therefore, we can define the following currents

$$j_{i,\hat{\tau}} \equiv \beta_0 \left(\Delta_{\hat{\tau}}\phi_i - 2\pi l_{i,\hat{\tau}}\right), \qquad j_{i,\hat{x},\hat{y}} \equiv \beta \left(\Delta_{\hat{x}}\Delta_{\hat{y}}\phi_i - 2\pi p_{i,\hat{x},\hat{y}}\right), \tag{5.100}$$

which is, by definition, conserved on-shell:

$$\Delta_{\hat{\tau}} j_{i,\hat{\tau}} = \Delta_{\hat{x}}\Delta_{\hat{y}} j_{i,\hat{x},\hat{y}}. \tag{5.101}$$

We can also define conserved charges along submanifolds, and it is useful to consider a detour on how to construct charges in the continuum. For usual symmetries, a conserved current j in two dimensions satisfies

$$\partial_\mu j^\mu = 0. \tag{5.102}$$

Integrating over a space-time volume V and using the Stokes theorem,

$$\int_V d^2x\, \partial_\mu j^\mu = \int_{\partial V} dx^\alpha \epsilon_{\mu\alpha} j^\mu = 0. \tag{5.103}$$

Particularly, if the boundary of V is the union of disjoint spatial surfaces $C_\pm$, the above expression becomes

$$\int_{C_+} dx^\alpha \epsilon_{\mu\alpha} j^\mu - \int_{C_-} dx^\alpha \epsilon_{\mu\alpha} j^\mu = 0, \tag{5.104}$$

and we can define the conserved charge

$$Q\left(C\right) \equiv \int_C dx^\alpha \epsilon_{\mu\alpha} j^\mu. \tag{5.105}$$

If we now consider an exotic symmetry, where $\partial_\tau j^\tau - \partial_x \partial_y j^{xy} = 0$, we may notice that it is possible to define a conserved charge along an specific direction. Considering a fixed $x = x_0$, we see that

$$\partial_\tau j^\tau(\tau, x_0, y) - \partial_y \left(\partial_x j^{xy}\right)_{x=x_0} = 0. \tag{5.106}$$

Therefore, using expression for the conserved charge,

$$Q\left(x, C\right) = \int_C dy\, j^\tau(\tau, x, y) - \int_C d\tau\, \partial_x j^{xy}(\tau, x, y), \tag{5.107}$$

where we have identified $(j^0, j^y) = (j^\tau, \partial_x j^{xy})$ to apply (5.105).

On the lattice, the charge along x will be given by

$$Q_{\hat{x}}(x, C_{\hat{x}}) = \sum_{y \in C_x} j^\tau - \sum_{\tau \in C_x} \Delta_{\hat{x}} j^{xy}, \tag{5.108}$$

where $C_{\hat{x}}$ is a line on the links of the $y\tau$ plane at fixed x. The same construction is valid for the y coordinate, and we can also find $Q_{\hat{y}}(y, C_{\hat{y}})$. These charges are called momentum dipole charges, and are conserved along the lines $C_{\hat{x}}$ and $C_{\hat{y}}$. These lines are one dimensional submanifolds, and therefore, these charges characterize a subsystem symmetry. This model has a modified version which has yet another subsystem symmetry, the winding dipole symmetry.

5.2.1 Continuum Limit

Now that we have discussed some key facts about the Villain formulation of the XY-plaquette model, we are going to focus on the continuum limit, following the procedure of the last section. The partition function is given by

$$Z = \sum_{l_{i,\hat{\tau}}\in\mathbb{Z}} \sum_{p_{i,\hat{x},\hat{y}}\in\mathbb{Z}} \prod_i \int_{-\pi}^{\pi} d\phi_i \exp\left[-\frac{\beta_0}{2}\sum_i \left(\Delta_{\hat{\tau}}\phi_i - 2\pi l_{i,\hat{\tau}}\right)^2 - \frac{\beta}{2}\sum_i \left(\Delta_{\hat{x}}\Delta_{\hat{y}}\phi_i - 2\pi p_{i,\hat{x},\hat{y}}\right)^2\right]. \tag{5.109}$$

Noticing that the field strength will again define equivalent classes of gauge fields, summing over all gauge fields is equivalent to sum over the set o N's and then sum over all equivalent classes:

$$\sum_{l_{i,\hat{\tau}}\in\mathbb{Z}} \sum_{p_{i,\hat{x},\hat{y}}\in\mathbb{Z}} \longmapsto \sum_C \sum_{N_i}. \tag{5.110}$$

In this manner, we can choose representatives of each class being . Making this changes, Z becomes

$$Z = \sum_C \sum_{N_i} \prod_i \int_{-\pi}^{\pi} d\phi_i \exp\left[-\frac{\beta_0}{2}\sum_i \left((\phi_{i+\hat{\tau}} - 2\pi N_{i+\hat{\tau}}) - (\phi_i - 2\pi N_i) - 2\pi l_{i,\hat{\tau}}^{(0)}\right)^2 + \right.$$
$$\left. -\frac{\beta}{2}\sum_i \left(\Delta_{\hat{x}}\Delta_{\hat{y}}(\phi_i - 2\pi N_i) - 2\pi p_{i,\hat{x},\hat{y}}^{(0)}\right)^2\right]. \tag{5.111}$$

Changing the variables $\phi_i \to \phi_i - N_i$, the partition function reads

$$Z = \sum_C \prod_i \int_{-\infty}^{\infty} d\phi_i \exp\left[-\frac{\beta_0}{2}\sum_i \left(\Delta_{\hat{\tau}}\phi_i - 2\pi l_{i,\hat{\tau}}^{(0)}\right)^2 - \frac{\beta}{2}\sum_i \left(\Delta_{\hat{x}}\Delta_{\hat{y}}\phi_i - 2\pi p_{i,\hat{x},\hat{y}}^{(0)}\right)^2\right], \tag{5.112}$$

where now ϕ_i is integrated over the real line. Using a Gaussian integral, the exponentials above become

$$\int_{-\infty}^{\infty} dj_{i,\hat{\tau}} \exp\left(-\frac{1}{2\beta_0} j_{i,\hat{\tau}}^2 + i\left(\Delta_{\hat{\tau}}\phi_i - 2\pi l_{i,\hat{\tau}}^{(0)}\right) j_{i,\hat{\tau}}\right) \sim$$
$$\sim \exp\left(-\frac{\beta_0}{2}\left(\Delta_{\hat{\tau}}\phi_i - 2\pi l_{i,\hat{\tau}}^{(0)}\right)^2\right), \tag{5.113}$$

$$\int_{-\infty}^{\infty} dj_{i,\hat{x},\hat{y}} \exp\left(-\frac{1}{2\beta} j_{i,\hat{x},\hat{y}}^2 + i\left(\Delta_{\hat{x}}\Delta_{\hat{y}}\phi_i - 2\pi p_{i,\hat{x},\hat{y}}^{(0)}\right) j_{i,\hat{x},\hat{y}}\right) \sim$$
$$\sim \exp\left(-\frac{\beta}{2}\left(\Delta_{\hat{x}}\Delta_{\hat{y}}\phi_i - 2\pi p_{i,\hat{x},\hat{y}}^{(0)}\right)^2\right). \tag{5.114}$$

Plugging these results into the partition function we can integrate over ϕ_i, obtaining

$$Z = \sum_C \prod_i \int_{-\infty}^{\infty} dj_{i,\hat{\tau}} dj_{i,\hat{x},\hat{y}} \, \delta\left(\Delta_{\hat{\tau}} j_{i,\hat{\tau}} - \Delta_{\hat{x}} \Delta_{\hat{y}} j_{i,\hat{x},\hat{y}}\right) \times$$
$$\times \exp\left[\sum_i \left(-\frac{1}{2\beta_0} j_{i,\hat{\tau}}^2 - 2\pi i l_{i,\hat{\tau}}^{(0)} j_{i,\hat{\tau}} - \frac{1}{2\beta} j_{i,\hat{x},\hat{y}}^2 - 2\pi i p_{i,\hat{x},\hat{y}}^{(0)} j_{i,\hat{x},\hat{y}}\right)\right]. \quad (5.115)$$

The delta function constraint is solve by choosing

$$j_{i,\hat{\tau}} = \Delta_{\hat{x}} \Delta_{\hat{y}} \xi_i, \qquad j_{i,\hat{x},\hat{y}} = \Delta_{\hat{\tau}} \xi_i, \quad (5.116)$$

and considering this solution in the partition function, one obtains

$$Z = \sum_{\{q_{i,\hat{\tau},\hat{x},\hat{y}}\}} \int_{-\infty}^{\infty} \prod_i d\xi_i \exp\left[\sum_i \left(-\frac{1}{2\beta_0} \left(\Delta_{\hat{x}} \Delta_{\hat{y}} \xi_i\right)^2 - \frac{1}{2\beta} \left(\Delta_{\hat{\tau}} \xi_i\right)^2 - 2\pi i \xi_i q_{i,\hat{\tau},\hat{x},\hat{y}}\right)\right], \quad (5.117)$$

where q is the field strength (5.97). We can now use (5.50) to incorporate the vorticity into a continuum cosine operator:

$$Z = \int_{-\infty}^{\infty} \prod_i d\xi_i \exp\left[\sum_i \left(-\frac{1}{2\beta_0} \left(\Delta_{\hat{x}} \Delta_{\hat{y}} \xi_i\right)^2 - \frac{1}{2\beta} \left(\Delta_{\hat{\tau}} \xi_i\right)^2 - 2a \cos\left(2\xi_i\right)\right)\right], \quad (5.118)$$

and taking the continuum limit,

$$Z = \int D\xi \exp\left[-\int d^3x \left(\frac{1}{2\beta_0} \left(\partial_x \partial_y \xi\right)^2 + \frac{1}{2\beta} \left(\partial_t \xi\right)^2 + 2a \cos\left(2\xi\right)\right)\right] .z \quad (5.119)$$

This continuum version represents an exotic field theory whose "vortex" configurations are dynamically determined by the cosine operator.

Chapter 6

Final Remarks

Throughout the last sections we have investigated two lattice models. The initial goal of these analysis was to find a continuum field theory prescription that dynamically determined the vortex configurations, and this was achieved by encoding the vortexes within a cosine operator. In the case of the XY model we studied the renormalization group and concluded that a phase transition occurs for $\beta = 2\pi$. In one of the phases the cosine operator is relevant, in the order phase it is not. However one can show that there cannot be any spontaneous symmetry breaking at finite temperature [17], hence Landau theory does not explain this phase transition. In fact, one can show that a mechanism of confinement and proliferation of the vortexes can explain the transition in a topological sense [14]. As a future development we may investigate this mechanism further in order to see if similar models, as the XY-plaquette model, has the same sort of phase transition.

Recent development has been made in physical models that present exotic symmetries [18, 16, 19, 20]. The lattice version of these models can enlighten the discussion of possible generalizations of quantum field theories. In the last section we have given a short description of the XY-plaquette model properties and applied the method developed in order to find an operator that describes the vortices. In a future investigation we may proceed to the renormalization group analysis of this model in order to find out possible topological phase transitions it may present.

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Appendix

APPENDIX **A**

Mathematical Results

A.1 Delta Function in Conjugate Coordinates

In the complex plane, Gauss' theorem for a vector field F^μ defined over a region $\mathcal{R}$ tells us that

$$\int_{\mathcal{R}} d^2x\, \partial_\mu F^\mu = \int_{\partial\mathcal{R}} d\xi_\mu F^\mu. \tag{A.1}$$

where $d\xi_\mu$ is the line element normal to the curve given by the border of $\mathcal{R}$, denoted by $\partial\mathcal{R}$. We can rewrite the theorem in a convenient way considering a parallel line element ds^ρ such that $d\xi_\mu = \epsilon_{\mu\rho} ds^\rho$. Then,

$$\int_{\mathcal{R}} d^2x\, \partial_\mu F^\mu = \frac{i}{2} \int_{\partial\mathcal{R}} \left(d\bar{z}\, F^z - dz\, F^{\bar{z}} \right). \tag{A.2}$$

Now we can use this theorem to show that the delta function representation

$$\delta\left(z, \bar{z}\right) = \frac{1}{\pi} \partial_z \frac{1}{\bar{z}} = \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z} \tag{A.3}$$

does its job, that is, its convolution with a holomorphic function satisfies the expected definition. If f is a holomorphic function,

$$\int_{\mathcal{R}} d^2x\, \delta\left(z, \bar{z}\right) f\left(z\right) = \frac{1}{\pi} \int_{\mathcal{R}} d^2x\, \partial_{\bar{z}} \left(\frac{f\left(z\right)}{z} \right) \tag{A.4}$$

Now, choosing a complex field

$$\left(F^z, F^{\bar{z}} \right) = \left(0, \frac{f\left(z\right)}{z} \right), \tag{A.5}$$

Gauss' theorem implies

$$\int_{\mathcal{R}} d^2x\, \delta\left(z, \bar{z}\right) f\left(z\right) = \frac{1}{2\pi i} \int_{\partial\mathcal{R}} dz\, \frac{f\left(z\right)}{z}. \tag{A.6}$$

Therefore, by Cauchy's theorem [8],

$$\int_{\mathcal{R}} d^2x\, \delta\left(z, \bar{z}\right) f\left(z\right) = f\left(0\right). \tag{A.7}$$

Up to this point we showed that the second representation proposed in (A.3) is indeed consistent. To verify the other, one needs to consider an anti-holomorphic function $f(\bar{z})$ and a complex field

$$\left(F^z, F^{\bar{z}} \right) = \left(\frac{f\left(\bar{z}\right)}{\bar{z}}, 0 \right). \tag{A.8}$$

The procedure is completely analogous for both cases.

A.2 Coherent States and Vertex Operators

In this appendix we will demonstrate the following identity for the harmonic oscillator

$$\Big\langle : e^{A_1} :: e^{A_2} : \ldots : e^{A_n} : \Big\rangle = \exp \sum_{i<j} \langle A_i A_j \rangle, \tag{A.9}$$

where $A_i = \alpha_i a +$ are complex linear combinations of creation and annihilation operators. For that purpose, we will introduce coherent states

$$|z\rangle \equiv e^{za^\dagger}|0\rangle. \tag{A.10}$$

These are eigenstates of the annihilation operator a, defined as the operator that annihilates the vacuum state $a|0\rangle = 0$, and satisfies the creation algebra with its adjoint, $[a, a^\dagger] = 1$. Let us consider two operators A and B with a constant commutation relation. Using Hausdorff formula, we know that

$$e^{-A} B e^{A} = B + [B, A]. \tag{A.11}$$

Taking $A = za^\dagger$ and $B = a$, the following commutation relation must hold

$$[a, e^{za^\dagger}] = z e^{za^\dagger}. \tag{A.12}$$

Acting upon the vacuum state and recalling that a annihilates it,

$$a|z\rangle = z|z\rangle. \tag{A.13}$$

Furthermore, the Baker-Campbell-Hausdorff (BCH) formula, [21], gives us that

$$e^{B} e^{A} = e^{A} e^{B} e^{\frac{1}{2}[A,B]}. \tag{A.14}$$

Now if $A = za^\dagger$ and $B = wa$,

$$e^{wa} e^{za^\dagger} = e^{za^\dagger} e^{wa} e^{zw}. \tag{A.15}$$

We can define the vertex operator by the normal ordering

$$: e^{A_i} : = e^{\beta_i a^\dagger} e^{\alpha_i a}, \tag{A.16}$$

such that the annihilation operators are always on the right. Then, in the normal ordered product of vertex operators $: e^{A_1} : \ldots : e^{A_n} :$ we need consider that

$$e^{\alpha_i a} e^{\beta_{i+1} a^\dagger} \cdot \ldots \cdot e^{\beta_n a^\dagger} = e^{\beta_{i+1} a^\dagger} \cdot \ldots \cdot e^{\beta_n a^\dagger} e^{\alpha_i a} e^{\alpha_i (\beta_{i+1} + \ldots + \beta_n)}, \tag{A.17}$$

where we have simply applied (A.15) for suitable z and w. More generally, given that $[\alpha_i a, A_j] = \alpha_i \beta_j$ is also a constant, we can still apply (A.15) and obtain

$$e^{\alpha_i a} : e^{A_{i+1}} : \ldots : e^{A_n} : = : e^{A_{i+1}} : \ldots : e^{A_n} : e^{\alpha_i a} e^{\alpha_i (\beta_{i+1} + \ldots + \beta_n)}. \tag{A.18}$$

Therefore, we can use this procedure for each A_j,

$$\begin{aligned} :e^{A_1}:\ldots:e^{A_n}: &= e^{\beta_1 a^\dagger} e^{\alpha_1 a} :e^{A_2}:\ldots:e^{A_n}: \\ &= e^{\beta_1 a^\dagger} :e^{A_2}:\ldots:e^{A_n}: e^{\alpha_1 a} e^{\alpha_1(\beta_2+\ldots+\beta_n)} \\ &= e^{\beta_1 a^\dagger} e^{\beta_2 a^\dagger} e^{\alpha_2 a} :e^{A_3}:\ldots:e^{A_n}: e^{\alpha_1 a} e^{\alpha_1(\beta_2+\ldots+\beta_n)} \\ &= e^{(\beta_1+\beta_2)a^\dagger} :e^{A_3}:\ldots:e^{A_n}: e^{(\alpha_1 a+\alpha_2)a} e^{\alpha_1(\beta_2+\ldots+\beta_n)} e^{\alpha_2(\beta_3+\ldots+\beta_n)} \end{aligned} \tag{A.19}$$

and by finite induction,

$$:e^{A_1}:\ldots:e^{A_n}: = e^{(\beta_1+\ldots+\beta_n)a^\dagger} e^{(\alpha_1+\ldots\alpha_n)a} e^{\sum_{i<j}\alpha_i\beta_j}. \tag{A.20}$$

Notice that

$$:e^{A_1+\ldots+A_n}: = e^{(\beta_1+\ldots+\beta_n)a^\dagger} e^{(\alpha_1+\ldots\alpha_n)a} \tag{A.21}$$

and also, the vacuum expectation value

$$\langle A_i A_j \rangle = \langle 0|\alpha_i\beta_j a a^\dagger|0\rangle = \alpha_i\beta_j. \tag{A.22}$$

Then, we can rewrite the normal ordered product as

$$:e^{A_1}:\ldots:e^{A_n}: = :e^{A_1+\ldots+A_n}: e^{\sum_{i<j}\langle A_i A_j\rangle}. \tag{A.23}$$

Hence, the vacuum expectation value is

$$\Big\langle :e^{A_1}::e^{A_2}:\ldots:e^{A_n}: \Big\rangle = \exp\sum_{i<j}\langle A_i A_j\rangle. \tag{A.24}$$

A.3 Poisson Resummation Formula

The Poisson resummation formula states that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \tilde{f}(k) \tag{A.25}$$

for $\tilde{f}$ the Fourier transform of f. To verify this we define the function

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n), \tag{A.26}$$

and show that the Fourier coefficients of F are the $\tilde{f}(k)$. Being so, its Fourier expansion will be given by

$$F(x) = \sum_{k\in\mathbb{Z}} e^{2\pi ikx}\tilde{f}(k) \tag{A.27}$$

and for $x=0$,we obtain the desired result. In fact, F is a period 1 function and then its Fourier coefficients are

$$F_k = \int_0^1 dx\, e^{-2\pi ikx} \sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \int_0^1 dx\, e^{-2\pi ikx} f(x+n), \tag{A.28}$$

assuming uniform convergence. Changing the variables $x+n \to x$,

$$F_k = \sum_{n\in\mathbb{Z}} \int_n^{n+1} dx\, e^{-2\pi i k(x-n)} f(x) = \int_{-\infty}^{\infty} dx\, e^{-2\pi i k x} f(x) = \tilde{f}(k), \tag{A.29}$$

which completes the proof.

Now, given that the Fourier transform of the kernel $e^{2\pi i k x}$ is the Dirac delta function, we can write the resummation formula in a suitable way. Notice that

$$\sum_{n\in\mathbb{Z}} \delta(x+n) = \sum_{k\in\mathbb{Z}} e^{2\pi i k x}. \tag{A.30}$$

Multiplying both sides by $\exp\left(-\pi a x^2 + bx\right)$ and integrating,

$$\int_{-\infty}^{\infty} dx \sum_{n\in\mathbb{Z}} \exp\left(-\pi a x^2 + bx\right) \delta(x+n) = \int_{-\infty}^{\infty} dx \sum_{k\in\mathbb{Z}} \exp\left(-\pi a x^2 + bx + 2\pi i k x\right) \tag{A.31}$$

gives us

$$\sum_{n\in\mathbb{Z}} \exp\left(-\pi a n^2 + bn\right) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} dx \exp\left(-\pi a x^2 + bx + 2\pi i k x\right). \tag{A.32}$$

However, if we complete the square:

$$-\pi a x^2 + bx + 2\pi i k x = -\pi a \left(x + \frac{2\pi i k + b}{2\pi a}\right)^2 + \left(\frac{2\pi i k + b}{2\sqrt{\pi a}}\right)^2. \tag{A.33}$$

Because of that, the r.h.s integral is

$$\int_{-\infty}^{\infty} dx \exp\left(-\pi a x^2 + bx + 2\pi i k x\right) = \frac{1}{\sqrt{a}} \exp\left(\frac{2\pi i k + b}{2\sqrt{\pi a}}\right)^2. \tag{A.34}$$

Plugging it back to the relation and simplifying,

$$\sum_{n\in\mathbb{Z}} \exp\left(-\pi a n^2 + bn\right) = \frac{1}{\sqrt{a}} \sum_{k\in\mathbb{Z}} \exp\left[-\frac{\pi}{a}\left(k + \frac{b}{2\pi i}\right)^2\right]. \tag{A.35}$$

Furthermore, if one considers $a = 2\pi\beta$ and $b = 2\pi\beta\theta + i\tilde{\theta}$, the above formula reads

$$\sum_n \exp\left[-\frac{\beta}{2}(\theta - 2\pi n)^2 + in\widetilde{\theta}\right] = \\ = \frac{1}{\sqrt{2\pi\beta}} \sum_{\widetilde{n}} \exp\left[-\frac{1}{2(2\pi)^2\beta}\left(\widetilde{\theta} - 2\pi\widetilde{n}\right)^2 - \frac{i\theta}{2\pi}\left(2\pi\widetilde{n} - \widetilde{\theta}\right)\right]. \tag{A.36}$$

APPENDIX B

Classical Conformal Algebra

In this appendix we will discuss how to find the generators of the conformal algebra of classical fields. In order to do that, the method of induced representations will be used. It consists of finding a representation for a subgroup and enhance it to the complete group afterwards.

An infinitesimal transformation of a field Φ is given by

$$\Phi'(x') = \Phi(x) - i\omega_a G_a \Phi(x). \tag{B.1}$$

Recall that the conformal group is connected, which enables us to write an infinitesimal transformation this way. Furthermore, notice that the coordinate transformation $x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x)$ will induce a change in the field, such that infinitesimally,

$$\Phi'(x) = \Phi(x) - \left(i\omega_a G_a + \varepsilon^{\mu}(x)\partial_{\mu}\right)\Phi(x). \tag{B.2}$$

So it must be clear that G_a transforms the field keeping coordinates still, and $\varepsilon^{\mu}(x)\partial_{\mu}$ is the generator induced by the change in coordinates. The sum of both constitute a complete transformation, having both functional and coordinate induced generators.

Let us consider the Poincaré subgroup that makes invariant the origin $x = 0$. We know that this subgroup coincides with the Lorentz group[1],

$$L_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0), \tag{B.3}$$

where $S_{\mu\nu}$ are Lorentz generators. Using now the Hausdorff identity,

$$e^{-A}Be^{A} = B + [B,A] + \frac{1}{2!}\left[[B,A],A\right] + \frac{1}{3!}\left[\left[[B,A],A\right],A\right] + \ldots, \tag{B.4}$$

we can make a translation on the Lorentz generators,

$$L_{\mu\nu}(x+a) = e^{ia^{\mu}P_{\mu}}L_{\mu\nu}(x)\,e^{-ia^{\mu}P_{\mu}} = L_{\mu\nu}(x) - \left(a_{\mu}P_{\nu} - a_{\nu}P_{\mu}\right). \tag{B.5}$$

Taking $x = 0$, we find the generator for any coordinate a,

$$L_{\mu\nu}(a) = S_{\mu\nu} - \left(a_{\mu}P_{\nu} - a_{\nu}P_{\mu}\right). \tag{B.6}$$

[1] The Poincaré group is the Lorentz group together with translations. If we are fixing the origin, there is no more translational invariance, and therefore Poincaré reduces to Lorentz.

Now, we can generalize this procedure to the hole conformal group. Again, the subgroup keeping the origin $x = 0$ invariant is the conformal group without translations, so it is generated by rotations, dilatations and SCT. We denote the generators of this subgroup as $S_{\mu\nu}$ (rotations), $\widetilde{\mathcal{D}}$ (dilatations) and κ_μ (SCT). Using Hausdorff identity and commutation relations between the reduced algebra generators and P_μ in (2.59),

$$\mathcal{D} = \widetilde{\mathcal{D}} + x^\alpha P_\alpha, \qquad K_\mu = \kappa_\mu + 2x_\mu \widetilde{\mathcal{D}} - 2x^\alpha S_{\mu\alpha} - 2ix_\mu x^\alpha \partial_\alpha + ix^2 \partial_\mu. \tag{B.7}$$

It remains to determine $\widetilde{\mathcal{D}}$ and κ_μ. See that the reduced algebra is given by

$$\begin{aligned}
[\widetilde{\mathcal{D}}, S_{\mu\nu}] &= 0, \\
[\widetilde{\mathcal{D}}, \kappa_\mu] &= -i\kappa_\mu, \\
\left[\kappa_\mu, \kappa_\nu\right] &= 0, \\
\left[\kappa_\alpha, S_{\mu\nu}\right] &= i \left(\eta_{\alpha\mu}\kappa_\nu - \eta_{\alpha\nu}\kappa_\nu\right) \\
\left[S_{\mu\nu}, S_{\alpha\beta}\right] &= -i \left(\eta_{\mu\alpha} S_{\nu\beta} + \eta_{\mu\beta} S_{\nu\alpha} + \eta_{\nu\alpha} S_{\mu\beta} + \eta_{\nu\beta} S_{\mu\alpha}\right).
\end{aligned} \tag{B.8}$$

Restring ourselves to irreducible representations of the Lorentz group, Schur's lemma ensures that any operator commuting with all Lorentz generators $S_{\mu\nu}$ must be a multiple of the identity operator [5]. Then, by the first commutation rule in the above algebra, we see that $\widetilde{\mathcal{D}}$ satisfy this criteria, and hence must be a multiple of the identity, $\widetilde{\mathcal{D}} = -i\Delta$. We call Δ the scale dimension. Also, by the second commutation rule in the reduced algebra, we see that $\kappa_\mu = 0$ considering Lorentz irreducible representations.

As an example, consider a spinless scalar field φ. It belongs to an irreducible representa-tion of the Lorentz group, so $\widetilde{\mathcal{D}} = -i\Delta$ and $\kappa_\mu = 0$. The finite form of (B.1) is

$$\varphi'(x') = \exp\left(-i\omega_a G_a\right) \varphi(x), \tag{B.9}$$

and the only non-trivial generator is $\widetilde{\mathcal{D}}$. Then, the above reduces to

$$\varphi'(x') = \exp\left(-\alpha\Delta\right) \varphi(x). \tag{B.10}$$

Recalling that the scale coordinate transformation is given by $x'^\mu = e^\alpha x^\mu$, we can write the Jacobian as $\left|\frac{\partial x'}{\partial x}\right| = e^{\alpha D}$. Therefore, in terms of this Jacobian,

$$\varphi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/D} \varphi(x). \tag{B.11}$$