Universidade Estadual de Londrina

LUIZ FELIPE DEMÉTRIO

QUANTUM PERTURBATIONS IN BIANCHI MODELS

Londrina
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Dissertação apresentada ao Departamento de Física da Universidade Estadual de Londrina, como requisito parcial à obtenção do título de Mestre em Física.

Orientador: Prof. Sandro Dias Pinto Vitenti

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Começarei pelos agradecimentos atemporais: aqueles que estarão sempre presentes ao longo de minha vida. Aqui não posso deixar de expressar minha gratidão a minha família, em particular a meus avós Waldomiro Demétrio e Dora Demétrio, os quais ajudaram a me criar com muito carinho, além de plantarem em mim as primeiras sementes da curiosidade científica através da série Cosmos, a qual eu amava assistir ao lado de meu avô. Agradeço também a meus pais Luiz Roberto Demétrio e Neide Ferreira (Demétrio?) por me criarem tão bem mesmo em meio a dificuldades, desafios estes que superamos juntos, como uma família. Finalmente, agradeço à professora Elenice Vaz por ter sido a primeira a realmente me incentivar a estudar Matemática e consequentemente obter uma boa colocação na OBMEP. Antes, estudava apenas por obrigação mas, depois de ver o quão bonita é a Matemática, comecei a estudar por prazer, o que me faz separar minha trajetória em "antes da Elenice" e "depois da Elenice".

Agora vamos ao setor temporal, iniciando pelo lado profissional. Começo agradecendo à Capes pelos fundos financeiros concedidos ao meu árduo trabalho. Agradeço também imensamente a meu orientador Sandro Dias Pinto Vitenti, por ter me acolhido como um de seus (muitos) orientandos, apesar de eu ainda não estar habituado à cosmologia. Obrigado pela oportunidade! Agradeço também aos professores Thiago dos Santos Pereira e Patrick Peter pela participação na banca e pelas discussões. Agradeço em particular ao prof. Thiago pela excelente disciplina de Inflação da qual participei, que foi de crucial importância para o desenvolvimento deste trabalho. Valeu a pena fazer tantas listas! Por fim, gostaria de agradecer ao professor George Emanuel Avraam Matsas que, apesar de não ter sido meu orientador, contribuiu para o desenvolvimento das minhas (ainda em formação) visões científicas durante nossas conversas, bem como com seu curso de Relatividade Geral, o qual foi de suma importância para o desenvolvimento deste trabalho.

Seguindo, gostaria agora de agradecer aos meus amigos, muitos dos quais conheci no meio do caminho. Porém, aqui cabe uma divagação. Quando fazemos Física, as *leis da Física* nos concedem a magnífica habilidade de *fazer previsões*. Nesse sentido, conseguimos ver o futuro pois, dada uma *condição inicial* no sistema, as leis da Física ditam o que ocorrerá com ele depois: sua *evolução*. Por exemplo: se soltarmos uma maçã, ela cairá. Aqui, a condição inicial é a maçã ser solta, e sua evolução é governada pelas leis de Newton. Dito isto, como sou apenas um humilde sistema físico, também irei separar minha trajetória em condições iniciais e evolução.

A respeito das condições iniciais, comecei meu mestrado em Maringá durante a pandemia da Covid-19, de forma remota. Nesse contexto, gostaria de agradecer aos meus amigos, colegas e companheiros de Maringá por terem me proporcionado as melhores condições iniciais possíveis para que eu começasse o mestrado e seguisse minha evolução. São eles: Nathan Henryque Ferreira Barros, Valdinete Kahenler, Júlio Baleeiro Ribeiro, e João Victor Balieiro[1] da Silva, o João ~~engenheiro~~ físico. Mesmo que nosso contato tenha diminuído com a distância, seu apoio foi indispensável.

Antes de me mudar para Londrina, sentia muito medo e insegurança. Principalmente medo de não fazer amigos e ficar sozinho. E medo de não ser bom o bastante. E também medo de tentar e falhar. Medo de dar tudo errado. Medo de tentar. Medo.

[1]Gostaria de enfatizar que aqui realizei o ambicioso *cross-over* de incluir os dois Baleeiros/Balieiros em uma mesma frase!

Mas isso agora é "página virada"! Agora discutirei minha evolução: seguindo a mudança, conheci novos amigos e o medo foi se dissipando. Durante estes 2 anos, tive o prazer de evoluir não só como físico, mas como pessoa! Por este motivo, gostaria de agradecer aos amigos que fiz no meio do caminho, pelo apoio incondicional que me foi dado.

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Vou continuar na pós-graduação, mas agora iremos para o setor de matéria[2]. Agora vamos mudar de "campo", para que os agradecimentos não fiquem tão "condensados": vamos para teoria quântica de campos! Aqui, agradeço primeiramente a Lucas Queiroz Silveira pelas muitas discussões de Física que tivemos. Agradeço também a Giovanna Pimenta Barbarino pelas conversas de Física e outras mais, mas também por me acompanhar na tradição das piadas ruins. Só não vou falar que ela "apimentou" as coisas porque essa piada já está manjada. Para fechar esse "campo", olha "que massa ó", vou fechar com o Gustavo Massao, do qual agradeço pelas muitíssimas conversas sobre Física, Matemática e pelas partidas de ping-pong. Ainda na pós, mas num setor mais, digamos, "particular" (partículas) agradeço ao Bruno Araújo, o Brunão, pela amizade, por todas as muitíssimas conversas sobre teoria (clássica e quântica) de campos, programação, cervejas, mas principalmente pelo apoio durante as aulas que começavam as 8h da manhã!

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É claro, não pude citar todos os nomes que participaram desta jornada nesta breve seção, focando apenas nos mais próximos. Mas ressalto meus agradecimentos ao grupo de cosmologia da UEL e a todos os envolvidos nestes 2 anos que passei durante o mestrado.

Obrigado a todos.

[2] Isso está até parecendo análise das equações de Einstein!

"VACUUM is COMPLEX
NOTHING is DIFFICULT
NOTHING is EASY
NOTHING is not NOTHING."
George Emanuel Avraam Matsas

ABSTRACT

In this work we analyze the theory of quantum cosmological perturbations around Bianchi I background models. We start by reviewing the basic aspects of the Standard Cosmological Model (Λ-CDM) and pointing out its initial conditions problems. We proceed by considering extensions of the Standard Model and tools proposed to solve them, which are: the Bianchi classification of homogeneous cosmologies, the inflationary and bouncing paradigms, and cosmological perturbation theory around a homogeneous and isotropic background. We follow by discussing quantum field theory in curved spacetime in order to properly quantize such perturbations and define an appropriate initial physical state. Finally, we combine the previous concepts by investigating cosmological perturbation theory for a homogeneous (but anisotropic) and flat Bianchi I background and proceed to their quantization. We conclude by analyzing a simple inflationary model and showing that the usual vacuum prescriptions cannot be properly applied for quantum cosmological perturbations in a Bianchi I background.

Keywords: Primordial Cosmology, Quantum Field Theory in Curved Spacetime, Cosmological Perturbation Theory, Homogeneous Cosmologies, Cosmic Microwave Background (CMB).

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RESUMO

Neste trabalho nós analisamos a teoria de perturbações cosmológicas quânticas um modelos de fundo do tipo Bianchi I. Começamos revendo os principais aspectos do Modelo Cosmológico Padrão (Λ-CDM) e apontando seus problemas associados a condições iniciais. Seguindo, nós discutimos extensões do Modelo Padrão, que são: a classificação de Bianchi de cosmologias homogêneas, os paradigmas de inflação e de ricochete, e a teoria de perturbações cosmológicas com um modelo de fundo homogêneo e isotrópico. Em seguida, discutimos teoria quântica de campos em espaços curvos para quantizar tais perturbações e definir um estado físico inicial apropriado. Por fim, combinamos os conceitos anteriores e investigamos a teoria de perturbações cosmológicas para um modelo de fundo plano e homogêneo (mas anisotrópico) do tipo Bianchi I, efetuando em seguida sua quantização. Concluimos com a análise de um simples modelo inflacionário, e mostramos que as prescrições de vácuo usuais não podem ser aplicadas para perturbações cosmológicas quânticas em um modelo de fundo do tipo Bianchi I.

Palavras: Cosmologia Primordial, Teoria Quântica de Campos em Espaços Curvos, Teoria de Perturbação Cosmológica, Cosmologias Homogêneas, Radiação Cósmica de Fundo (CMB).

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INTRODUCTION

Cosmology is a unique branch of Physics, since it describes the largest scales known to mankind: the whole universe. Since we only have access to one and only universe, one does not have the freedom to *impose initial conditions* on the universe: they are fixed once and for all. This is a peculiarity particular to cosmology since, in other areas of Physics, one usually can perform experiments by varying the initial conditions of a system [1].

For instance, one may test Newtonian physics with a simple experiment: let an apple fall, and vary its initial conditions (position and velocity). In cosmology, since the initial conditions are already fixed, one does not (in general) perform experiments, but *observations*.[3] Hence, the development of cosmology is guided by such observations and the technologies that enable those to be performed.

Since in the beginning of the 20th century very few observations could be made, cosmology started as a very modest discipline based on working assumptions only [2], mainly the Cosmological Principle: in the largest scales, the universe looks roughly the same at every point (homogeneity) and every direction (isotropy) [3]. However, throughout the 20th century, more and more observations helped cosmology to grow and develop, for example:

- observations of galaxy redshifts by Hubble (1929) showed that the universe is dynamical and is expanding [2];
- analysis of galaxy rotation curves indicated the existence of an yet unknown component of the universe: dark matter [4];
- the discovery of the Cosmic Microwave Background Radiation (CMB) by Penzias and Wilson (1965) opened a window with rich information about the primordial universe [2];
- more precise redshift measurements (1999) indicated the existence of dark energy, another unknown component of the universe [5];

which, in combination with new theoretical and technological advances enabled cosmology to achieve the status of a robust scientific discipline, thereby able to make solid predictions: it entered the period known as *precision cosmology* [6–8].

The aforementioned developments culminated on the construction of the Standard Cosmological model, also known as Λ-CDM model, which pictures a 13 billion years old, infinite and spatially flat universe, presently composed mainly of baryonic matter (5%), dark matter (25%) and dark energy (70%). Such model has been highly successful in predicting statistical properties of the universe such as galaxy redshift and the abundance of light elements [2, 3, 9].

However, some problems remain, with the most famous of those being the yet unknown nature of dark matter and dark energy. While such problem concerns Physics in general, due to the special

[3]This is somewhat alleviated since one usually assumes the universe to be statistically homogeneous, thus allowing different portions of the universe to be treated as different realizations of a same "experiment".

nature of cosmology, it also presents problems of a very different nature, which are linked to choices of initial conditions. They are [1, 9]:

1. Flatness Problem;
2. Horizon Problem;
3. Origin of Perturbations Problem;
4. Singularity Problem.

They are associated to the fact that, to explain the highly homogeneous and isotropic cosmos that we observe nowadays with our currently accepted theories, one must impose rather special initial conditions on the past universe. Therefore, there are two options to deal with this problem: one may search for a plausible explanation for the chosen initial conditions: *a theory of initial conditions*, if you may, or modify our theories by proposing a mechanism that guarantees that generic initial conditions evolve to the special state that we now observe [1].

The aforementioned problems can be partially solved by considering two different paradigms: the inflationary and the bouncing one. In the first, one considers an inflationary period of quasi-exponential expansion on the early universe [2, 10–12], usually driven by a scalar field. In the later, one considers that the universe underwent a period of contraction followed by a bounce [6, 13, 14], usually driven by quantum gravitational effects or exotic matter.[4] Such mechanisms force somewhat general initial conditions to evolve into a highly homogeneous and isotropic state, and hence alleviate the initial conditions problems [1].

The proposed paradigms also presented an interesting explanation for the origin of the universe's large scale structure, such as galaxies and galaxy clusters. In this picture, quantum vacuum fluctuations of the matter fields generated (quantum) cosmological perturbations on spacetime, which were then amplified by inflation/bounce to generate large scale structure. Therefore, such vacuum fluctuations are essentially the "seeds" of our universe's large scale structure. Such picture is consistent with observations of the Cosmic Microwave Background Radiation (CMB), which is compatible with a slightly red, almost scale invariant Gaussian spectrum of curvature perturbations [6, 9]. One then notes the interesting fact that quantum field theory (in curved spacetime) can also make predictions for the primordial universe through the quantum fluctuations picture [11, 12], being not limited only to the Physics of the very small, e.g. particle physics.

This work comes as a response to both the initial conditions problems in cosmology as well the quantum fluctuations paradigm. Since in both inflationary and bouncing models one usually considers homogeneous and isotropic backgrounds where perturbations evolve, in this work we consider more general initial conditions by analyzing the case of a flat homogeneous (but not necessarily isotropic) universe, known as a Bianchi I universe.

[4] To be more specific, one may consider Canonical Quantum Gravity [1, 15], Loop Quantum Gravity [16–18], or classical bounces [6].

For such model to explain the universe that we observe nowadays, one must first show that, by considering such initial condition, the universe evolves to the highly isotropic and homogeneous state that we now observe. Then, one must also show that the observed CMB spectrum is consistent with the one predicted by quantum cosmological perturbations in a Bianchi I background. It should be noted that, since one assumes that the cosmological perturbations were described by a vacuum state in the far past, one needs to properly define such vacuum state in order to make predictions. However, the task of defining a vacuum state is not trivial on general curved backgrounds, since one in general has less symmetry than in flat Minkowski spacetime.[5]

The first task was accomplished by Wald [19], who showed that a period of semi-exponential expansion isotropizes an initial Bianchi background to a highly isotropic state [20]. Perturbations were also studied in [21–23], where it was shown that cosmological perturbations in Bianchi I backgrounds do not admit the usual prescription known as an adiabatic vacuum state[6] in the far past, which happens due to the violation of the WKB approximation. This work was then elaborated to be a literature review of the aforementioned results, aiming to describe the quantization of cosmological perturbations in a Bianchi I background.

To achieve our goals, this work is structured as follows. In chapter 1, we make a very brief review of the main hypotheses of the Standard Cosmological Model, and we conclude by properly discussing its initial conditions problems. We follow in chapter 2 by discussing extensions to the Standard Model developed to deal with such problems: homogeneous cosmologies, the inflationary and bouncing paradigms, and conclude with cosmological perturbation theory. In chapter 3, we make a self-contained introduction to Quantum Field Theory in Curved Spacetime in order to properly quantize cosmological perturbations. Finally, in chapter 4, we combine the previous mentioned tools by exploring cosmological perturbation theory in a Bianchi I background and their application on an inflationary model, and discuss the determination of a vacuum state for cosmological perturbations.

[5]In particular, a well defined prescription to define vacuum states is available for spacetimes with a timelike Killing vector field.

[6]This concept is discussed in detail in section 3.3.

1 STANDARD MODEL OF COSMOLOGY

In this chapter, we present a brief review of the Standard Model of Cosmology. We begin by introducing the cosmological principle and the FLRW metric (1.21) in section 1.1. We follow by studying the dynamics of such universes with the Friedmann equations in section 1.2. In section 1.3, we make a brief historical review of the evolution of cosmology, emphasizing the key observations and assumptions of the cosmological models and culminating in the Λ-CDM model. We discuss the mathematical formalism before history and observations in order to make the discussion more fluid. We conclude by analyzing some problems of the Λ-CDM model in section 1.4.

1.1 Cosmological Principle

Cosmology is the branch of Physics that studies the behaviour of our universe in the largest scales known to mankind [24][25]. Of course, to deal with such a complicated system as the whole universe in all of its details would be virtually impossible. Hence, some simplifying assumptions must be made in order for cosmology to make scientific predictions.

Since the time of Copernicus, the place of Earth and mankind has been removed from its special importance as the center of the universe. We now assume that Earth is just an ordinary planet orbiting an ordinary star in an ordinary galaxy.

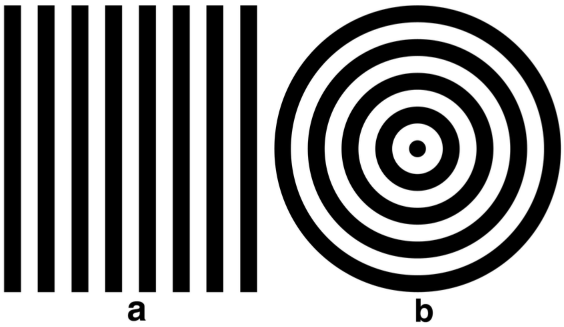

Figure 1.1: The figure on the left is homogenous since it looks roughly the same on every point, but it is not isotropic: there is a preferred direction. The one on the right is isotropic around the center, since all directions look the same, but is not homogeneous: the central point is different from the others [26].

The idea that the place we occupy in the cosmos is not special is implemented by assuming a set of hypothesis which are known as *Cosmological Principle*. It assumes that the universe is homogeneous and isotropic in large scales, which, respectively, means that it "looks roughly the same" in every point and every direction that we look at.[1] The two separate cases can be visualized in Figure 1.1, and the combination in Figure 1.2.

[1]This concept can be formalized in terms of statistical means. For instance, Figure 1.2 is homogeneous and isotropic in terms of average color on scales that are larger than the dots. Also, note that this hypothesis implicitly assumes that the universe has trivial topology.

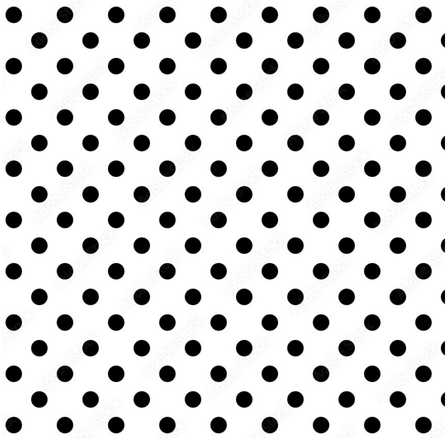

Figure 1.2: This figure is homogeneous and isotropic. Note that this applies just to large scales: in smaller ones, there are tiny dots. The same happens for our universe, with the galaxies taking the role of the dots. Source: Adobe Stock.

The Cosmological Principle is consistent with observations: from a naive point of view, when we look at the night sky with a naked eye, we see roughly the same in every direction: stars and more stars. The same is true for more rigorous observations, like the Hubble Ultra Deep Field (Figure 1.3), which shows homogeneity and isotropy on scales that are larger than galaxies. Such observations reveal that the Cosmological Principle is indeed satisfied for scales that are higher than 100Mpc[2] [3][27]. Cosmology is interested in describing such scales, which are denoted thereafter as *cosmological scales* [25].

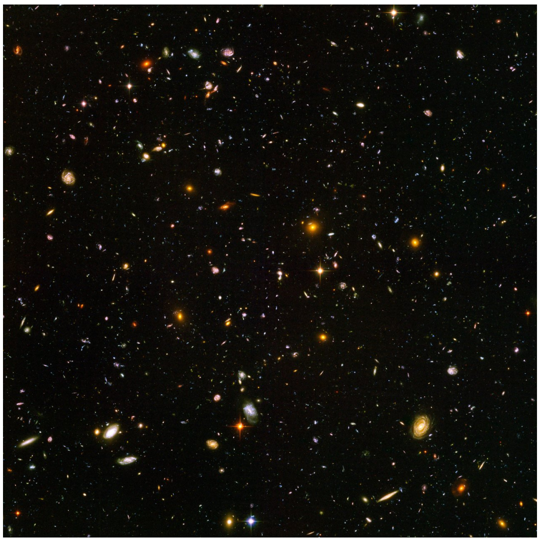

Figure 1.3: Hubble Ultra Deep Field. Each dot is a *galaxy*, and not a star. Source: ESA/ HUBBLE.

To describe the evolution of the universe in cosmological scales, one needs to consider what physical effects should be relevant in such regime. From the four fundamental interactions known by men, only gravity is relevant to describe the largest scales[3]. Since gravity is described by Einstein's

[2] $1\text{Mpc} = 3,086 \times 10^{16}\text{km} = 3,26 \times 10^{3}$ light-years is an astronomical unity.

[3] The two nuclear interactions are short ranged, and the electromagnetic interaction is irrelevant due to the assumption of matter large scale matter being electrically neutral.

General Relativity, we shall use such theory to make predictions in cosmology. Hence, we should formulate the Cosmological Principle in the language of General Relativity, which is Differential Geometry. The main definitions and notations are introduced in Appendix A.

We now proceed to construct homogeneous and isotropic cosmologies in a formal way. However, if the reader is satisfied only with the intuitive description of such symmetries, he/she can skip directly to the final form of the metric, Equation 1.21.

In General Relativity, one assumes that spacetime geometry is coupled to matter through the Einstein field equations (A.31). Since, as we have discussed, the large scale matter distribution of our universe follows the Cosmological Principle, in a reference frame adapted to observers that follow such large scale matter distribution, one can write its stress-energy tensor in a perfect fluid form [28], which in the abstract index notation becomes

$$T_{ab} = (\rho + p)\, u_a u_b + p g_{ab}\,, \tag{1.1}$$

where u_a is the fluid's four velocity field, ρ is the energy density and p is the pressure. Due to homogeneity, both ρ and p are assumed to be functions of time only, and they are also assumed to be the same in every direction due to isotropy.

As for geometry, since the matter distribution is assumed to be homogeneous and isotropic, it follows that spacetime is also homogeneous and isotropic. In General Relativity spacetime is described by a pair $(\mathcal{M}, g_{ab})$, where $\mathcal{M}$ is a manifold and g_{ab} is a Lorentzian metric defined on it. We assume a foliation (Σ_t, n^a) on $\mathcal{M}$, which induces a natural Riemannian metric h_{ab} on the spatial sections Σ_t, defined by

$$h_{ab} \equiv g_{ab} + n_a n_b\,. \tag{1.2}$$

The Cosmological Principle is implemented on spacetime geometry by assuming that the hypersurfaces Σ_t satisfy:

1. Homogeneity: for all $p, q \in \Sigma_t$, there exists an isometry curve of the induced metric h_{ab} that connects p and q;

2. Isotropy: for all points $p \in \Sigma_t$ and any pair of vectors $u, v \in T_p\Sigma_t$, there exists an isometry[4] of h_{ab} that leaves p invariant and also maps u to v.

Condition 1. expresses in mathematical terms the notion of "points being equivalent", and Condition 2. does the same for directions around a point, which are described by vectors in the tangent space. In cosmology, the normal vector field n^a is taken to be the four-velocity field u^a of the large distribution of galaxies, which defines a set of observers that perceive the universe as homogeneous and isotropic.

The Cosmological Principle fixes the form of the metric tensor g_{ab} as follows [29]. Consider the Riemann tensor ${}^{(3)}R_{abc}{}^{d}$ in Σ_t associated with h_{ab} and covariant derivative D_a, associated with its

[4]More precisely, the isometry ϕ induces a push-forward $\phi_* : T_p\Sigma_t \to T_p\Sigma_t$ that maps the two vectors.

induced Levi-Civita connection. Note that, by raising an index

$$
\begin{aligned}
{}^{(3)}R_{ab}{}^{cd} &\equiv h^{ce}\,{}^{(3)}R_{abe}{}^{d}\,, \\
\Longrightarrow\ {}^{(3)}R_{[ab]}{}^{cd} &= {}^{(3)}R_{ab}{}^{cd}\,.
\end{aligned}
$$

Now, consider a 2-form ω_{ab} in Σ_t. Due to this property, we have that

$$
f_{ab} \equiv R_{ab}{}^{cd}\omega_{cd}\,, \tag{1.3}
$$

is also a 2-form, hence, ${}^{(3)}R_{ab}{}^{cd}$ is a linear operator in the 2-forms space of Σ_t. Isotropy means that no direction in Σ_t is privileged, which implies that, if we diagonalize such operator, all of its eigenvalues are equal. If they were not, one could define a privileged direction: that of higher eigenvalue. This means that, in diagonalized form, it should be proportional to the identity operator $\delta^c{}_{[a}\delta^d{}_{b]}$:

$$
\begin{aligned}
{}^{(3)}R_{ab}{}^{cd} &\propto \delta^c{}_{[a}\delta^d{}_{b]}\,, \\
\Longrightarrow\ {}^{(3)}R_{ab}{}^{cd} &= \mathcal{K}\delta^c{}_{[a}\delta^d{}_{b]}\,,
\end{aligned}
$$

or, by lowering the indexes:

$$
{}^{(3)}R_{abcd} = \mathcal{K}h_{c[a}h_{b]d}\,. \tag{1.4}
$$

The only eigenvalue $\mathcal{K}$ is known as the curvature of Σ_t. The condition of homogeneity implies that $\mathcal{K}$ is a constant: if it was not, one could define privileged points of space where it is a maximum [29].

It should also be pointed out that isotropy at all points (Condition 2.) actually implies that the curvature $\mathcal{K}$ is a constant. To see this, substitute (1.4) in the Bianchi identities:

$$
D_{[e}R_{ab]cd} = 0\,, \tag{1.5}
$$

$$
\Longrightarrow\ \left(D_{[e}K\right) h_{|c|a}h_{b]d} = 0\,. \tag{1.6}
$$

In a manifold of dimension 3 or greater, (1.5) will only be satisfied if $D_e K = 0$, which means that the manifold is also homogeneous [29]. We then see that isotropy at all points is a stronger requirement than homogeneity. We shall relax the isotropy assumption when we study the Bianchi Classification of homogeneous cosmologies in chapter 2.

We showed that, in order for Σ_t to be homogeneous and isotropic, it must be a space of constant curvature $\mathcal{K}$. Up to topology[5], the only possibilities for such spaces in 3 spatial dimensions are

$$\mathcal{K} \begin{cases} > 0 & \text{for a spherical space } \mathbb{S}^3 \,, \\ = 0 & \text{for a plane space } \mathbb{R}^3 \,, \\ < 0 & \text{for a hyperbolic space } \mathbb{H}^3 \,. \end{cases} \tag{1.7}$$

It can be shown that any two spaces of constant curvature and same metric signature are locally isometric [29]. We also have that only the sign of $\mathcal{K}$ matters, because spaces with $\mathcal{K}$ of same sign but different values differ only by a scaling diffeomorphism[6].

Now, since we are be interested in describing the dynamics of homogeneous and isotropic universes using the Einstein Equations, it will be useful to find an explicit coordinate expression for the metric. To obtain the form of g_{ab} at a fixed time t, we embed Σ_t in a fiducial[7] $\mathbb{R}^4$. The three sphere $\mathbb{S}^3$, when embedded, is described in terms of the global cartesian coordinates of $\mathbb{R}^4$ as [3]

$$x^2 + y^2 + z^2 + w^2 = S^2 \,, \tag{1.8}$$

where S is its radius. In global cartesian coordinates, the line element in $\mathbb{R}^4$ is given by

$$\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2 \,. \tag{1.9}$$

However, since $\mathbb{S}^3$ is characterized by the constraint (1.8), the four coordinates (x, y, z, w) are not independent, and one of them can be eliminated in terms of the others, such as $w = w(x, y, z)$. In differential form, the constraint (1.8), becomes[8]

$$-w\mathrm{d}w = x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z \,. \tag{1.10}$$

We can also introduce the usual spherical coordinates to express

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,,$$

in particular, the differential form of $r^2 = x^2 + y^2 + z^2$ is given by

$$r\mathrm{d}r = x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z \tag{1.11}$$

[5]This is due to the fact that the metric does not completely fixes the topology, since the metric is a local property, while topology is a global one.

[6]This is implemented by rescaling the radial coordinate as $r \to r' = r/\sqrt{(|\mathcal{K}|)}$ in (1.15) and absorbing $|K|$ on the definition of $a(t)$ in (1.21) .

[7]We emphasize that such $\mathbb{R}^4$ is fiducial, since the manifolds $\mathbb{R}$,

$$\mathbb{H}^3, \mathbb{S}^3$$

can be analyzed intrinsically. However, this simplifies our discussion.

[8]More formally, this expression can be obtained by evaluating the exterior derivative of the constraint (1.8)

and, from (1.8), we see that

$$w^2 = S^2 - r^2\,. \tag{1.12}$$

Substituting (1.11) and (1.12) in (1.10), we get

$$\mathrm{d}w^2 = \left(\frac{r}{S}\right)^2 \frac{\mathrm{d}r^2}{1-\frac{r^2}{S^2}} \tag{1.13}$$

from which follows that (1.9) restricted to $\mathbb{S}^3$ by taking $S = \text{const.}$ is given by

$$\mathrm{d}l^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + \left(\frac{r}{S}\right)^2 \frac{\mathrm{d}r^2}{1-\frac{r^2}{S^2}}\,, \tag{1.14}$$

and after some algebra, we get

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{1-\mathcal{K}\mathrm{d}r^2} + r^2\mathrm{d}\Omega^2\,, \tag{1.15}$$

where $\mathcal{K} \equiv 1/S^2$, which can be shown to be the only eigenvalue of ${}^{(3)}R_{abc}{}^{d}$ by direct calculation. The same construction could be done for $\mathbb{R}^3$ and $\mathbb{H}^3$ by considering $S^2 \to \infty$ and negative values[9] of S^2, respectively. Hence, (1.15) describes the most general metric of a constant curvature 3-surface in spherical coordinates.

Now that we have obtained the metric on the spatial hypersurfaces, we need to obtain the metric on spacetime. To do so, we first define the symmetric extrinsic curvature tensor for a geodesic flow, K_{ab}, by

$$K_{ab} \equiv \frac{1}{2}\mathscr{L}_u h_{ab}\,, \tag{1.16}$$

which, using the identity

$$h_{ab} = u_a u_b + g_{ab}\,,$$

and the fact that the fundamental observers follow geodesics, $u^a\nabla_a u^b = 0$, can also be expressed as

$$K_{ab} = \nabla_a u_b\,, \tag{1.17}$$

and this shows that such tensor is purely spatial, since $K_{ab}u^a = 0$.

Now, consider the extrinsic curvature in mixed form, $K_a^{\ b} = g^{bc}K_{ac}$. Since $K_a^{\ b}$ is a geometric spatial tensor, it follows that, due to isotropy on Σ_t, all of its eigenvalues must be equal, just like the Riemann tensor. Hence, once diagonalized, $K_a^{\ b} = H(t)\,(\delta_a^{\ b} + u_a u^b) = H(t)h_a^{\ b}$, which is just the identity element on the tangent bundle of Σ_t, and $H(t) \equiv K/3$ is a function that depends only on time due to homogeneity.

Now, consider (1.16) in a coordinate system adapted to the fundamental observers with coordinate vector fields $\{u^a, (X_i)^a\}$. Here the $\{(X_i)^a\}$ are spatial vectors tangent to Σ_t and orthogonal to

[9] Formaly, this would make $w^2 < 0$ in (1.12). Taking $w^2 \to -q^2$ turns (1.8) into a hyperboloid.

u^a by construction, $u_a(X_i)^a = 0$. In this coordinate system, we have

$$g_{00} = g_{ab}u^a u^b = -1\,, \tag{1.18a}$$
$$g_{i0} = g_{ab}(X_i)^a u^b = 0\,, \tag{1.18b}$$
$$g_{ij} = g_{ab}(X_i)^a(X_j)^b = h_{ab}(X_i)^a(X_j)^b = h_{ij}\,, \tag{1.18c}$$

where we defined $h_{ij} \equiv h_{ab}(X_i)^a(X_j)^b$. Expressing (1.16) in our coordinate system, we obtain[10]:

$$H(t)h_{ij} = \frac{1}{2}\frac{\partial}{\partial t}h_{ij}\,, \tag{1.19}$$
$$\implies \frac{\partial}{\partial t}h_{ij} = 2H(t)h_{ij}\,, \tag{1.20}$$

here we used that $K_{ab} = H(t)h_{ab}$, $\partial h_{ij}/\partial t = \mathscr{L}_u h_{ij}$, and $\mathscr{L}_u\left(h_{ab}(X_i)^a(X_j)^b\right) = \left(\mathscr{L}_u h_{ab}\right)(X_i)^a(X_j)^b$, since $\mathscr{L}_u(X_i)^a = 0$ due to the fact that $\{u^a, (X_i)^a\}$ are coordinate vector fields.

Relation (1.19) is a differential equation for h_{ij} with general solution

$$h_{ij} = \exp\left\{2\int H(t)dt\right\}h_{ij}(t_0)\,.$$

Here, the $h_{ij}(t_0)$ are arbitrary integration functions that depend only on the spatial coordinates, which can be fixed by demanding that the metric reduces to (1.15) at $t = t_0$. In our coordinates, the line element then becomes:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}l^2\,, \tag{1.21}$$

where $a(t) \equiv a_0 e^{\int H(t)dt}$ is known as the scale factor and carries all information regarding the dynamics of the universe. The function $H(t) = \dot{a}/a$ is known as the Hubble function, and dl^2 is the metric of Σ_t given by (1.15). Here, due to our choice of coordinates (1.18), t is the proper time measured by clocks that follow the fundamental observers in the Hubble flow. The metric (1.21) describes a homogeneous and isotropic universe and is known as the FLRW (Friedmann-Lemaître-Robertson-Walker) metric.

To conclude, for some applications, it is also useful to express the FLRW metric in the conformal time coordinate η:

$$\mathrm{d}s^2 = a(t)^2\left(-\mathrm{d}\eta^2 + \mathrm{d}l^2\right)\,, \tag{1.22}$$

where $\mathrm{d}\eta = \mathrm{d}t/a(t)$. Note that in this coordinates the metric is explicitly conformal to the Minkowski metric, $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$, and hence the name.

1.2 Dynamics of a homogeneous and isotropic universe

Now we are in position to study the dynamics of a FLRW geometry by imposing the Einstein equations (A.31). By direct substitution of the metric (1.21) and the stress-energy tensor of a perfect

[10]The 00 and $0i$ equations are trivial because $h_{00} = h_{i0} = 0$.

fluid (1.1) in the definitions of the Christoffel Symbols and the curvature tensors, the 00 field equation and trace $R = -\kappa T$ imply

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\mathcal{K}}{a^2} = \frac{\kappa}{3}\rho + \frac{\Lambda}{3}\,, \tag{1.23a}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}\left(\rho + 3p\right) + \frac{\Lambda}{3}\,, \tag{1.23b}$$

which are known as Friedmann equations. They can also be expressed in terms of the Hubble function $H(t) \equiv \dot{a}/a$ as:

$$H^2 + \frac{\mathcal{K}}{a^2} = \frac{\kappa}{3}\rho + \frac{\Lambda}{3}\,, \tag{1.24a}$$

$$\dot{H} + H^2 = -\frac{\kappa}{6}\left(\rho + 3p\right) + \frac{\Lambda}{3}\,. \tag{1.24b}$$

It should be noted that the cosmological constant Λ can also be described by a perfect fluid stress-energy tensor. To see this, take the cosmological term to the right side of the Einstein equations (A.31) and define

$$^{\Lambda}T_{ab} \equiv -\frac{1}{\kappa}\Lambda g_{ab}\,, \tag{1.25}$$

which is a perfect fluid stress energy tensor with $p = -\Lambda/\kappa$ and $p = -\rho$. From here on, we group the cosmological term within the remaining energy densities and pressures describing the other fluids. It should also be noted that (1.23) also describes systems of n non-interacting perfect fluids with $\rho = \sum_{i=1}^{n} \rho_i$ and $p = \sum_{i=1}^{n} p_i$.

The conservation of energy-momentum $\nabla_a T^{ab} = 0$ leads to the continuity equation, which in a FLRW geometry is given by

$$\frac{d\rho}{dt} + 3H\left(\rho + p\right) = 0\,. \tag{1.26}$$

It can be shown that such equation is not independent of (1.23): only two out of the three can be used to study dynamics.

The first Friedmann equation (1.24a) can also be rewritten by introducing the critical density ρ_c

$$\rho_c \equiv \frac{\kappa}{3H^2}\,, \tag{1.27}$$

and the relative densities

$$\Omega_i \equiv \frac{\rho_i}{\rho_c}\,, \tag{1.28a}$$

$$\Omega \equiv \sum_i \Omega_i\,, \tag{1.28b}$$

$$\Omega_{\mathcal{K}} \equiv -\frac{\mathcal{K}}{a^2 H^2}\,, \tag{1.28c}$$

which, after substitution in (1.23a), leads to

$$\Omega + \Omega_{\mathcal{K}} = 1\,, \tag{1.29}$$

which means that, for a flat universe with $\mathcal{K} = 0$, we have $\Omega = 1$.

To solve the Friedmann equations explicitly, one needs to obtain $a(t)$, $\rho(t)$ and $p(t)$. However, since we only have two independent equations of motion, one needs to suplement then with an equation of state that relates ρ and p, such as $p = p(\rho, a, t)$. From here on, we assume that the fluids are barotropic: they satisfy an equation of state of the form

$$p(\rho) = w\rho\,, \tag{1.30}$$

where w is an dimensionless constant. The most relevant cases are

$$w = \begin{cases} 0 & \text{for matter/dust,} \\ \frac{1}{3} & \text{for radiation,} \\ -1 & \text{for a cosmological constant.} \end{cases} \tag{1.31}$$

For a fluid satisfying the equation of state (1.30), the continuity equation (1.26) can be integrated explicitly for $\rho(a)$:

$$\rho(a) = \rho_0 a^{-3(1+w)}\,, \tag{1.32}$$

where ρ_0 is an integration constant. In each case:

$$w = \begin{cases} 0 \implies \rho(a) = \rho_0 a^{-3} & \text{for matter/dust,} \\ \frac{1}{3} \implies \rho(a) = \rho_0 a^{-4} & \text{for radiation,} \\ -1 \implies \rho(a) = \rho_0 & \text{for a cosmological constant.} \end{cases} \tag{1.33}$$

Assuming a flat universe such that $\mathcal{K} = 0$ and substituting (1.32) in the first Friedmann equation (1.23a), one can find the scale factor $a(t)$ explicitly:

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}\frac{1}{(1+w)}}\,, \tag{1.34}$$

where we have chosen coordinates such that $a(t_0) = 1$, and

$$t_0 \equiv \sqrt{\frac{3}{\kappa\rho_0}} \tag{1.35}$$

is the age of the universe since $a(0) = 0$ for this class of models.

Solution (1.34) gives the evolution of the scale factor for a flat universe with a barotropic perfect fluid that satisfies the Cosmological Principle in General Relativity. Note that the presence of matter (the fluid) explicitly makes the geometry dynamic. In each case[11], we have:

$$w = \begin{cases} 0 \implies a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3} & \text{for matter/dust,} \\ \frac{1}{3} \implies a(t) = a_0 \left(\frac{t}{t_0}\right)^{1/2} & \text{for radiation,} \\ -1 \implies a(t) = a_0 e^{H_\Lambda (t-t_0)} & \text{for a cosmological constant,} \end{cases} \tag{1.36}$$

To conclude this section, we note that the same construction could be made using the FLRW metric in conformal time η, (1.22). In this case, the Friedmann equations become

$$\mathcal{H} + \mathcal{K} = \frac{\kappa}{3}\rho a^2 \, , \tag{1.37a}$$

$$\mathcal{H}' = -\frac{\kappa}{6}(\rho + 3p) + \frac{\Lambda}{3} \, , \tag{1.37b}$$

where $f' \equiv df/d\eta$ and $\mathcal{H} \equiv a'/a$ is the conformal Hubble parameter. The continuity equation is formally the same:

$$\frac{d\rho}{d\eta} + 3\mathcal{H}(\rho + p) = 0 \, , \tag{1.38}$$

and (1.37a) can be solved to find $a(\eta)$. We get:

$$w = \begin{cases} 0 \implies a(\eta) \propto \eta^2 & \text{for matter/dust,} \\ \frac{1}{3} \implies a(\eta) \propto \eta & \text{for radiation,} \\ -1 \implies a(\eta) \propto -\frac{1}{H_\Lambda \eta} & \text{for a cosmological constant.} \end{cases} \tag{1.39}$$

1.3 Hypothesis and Predictions of the Standard Model

To construct a scientific cosmological model, one first needs assumptions from which predictions can be made. Then, the model can be implemented according to the observed data. In Standard Cosmology, the main assumptions about the universe are [25]:

1. Cosmological Principle: the universe is roughly homogeneous and isotropic in cosmological scales [3];

2. The laws of Physics are the same everywhere in the universe and in every epoch. Without this assumption, it wouldn't be possible to make a scientific study of the universe [25];

3. The local laws of Physics are the same laws that determine the dynamics of the universe in cosmological scales. That is, there is not a "cosmological Physics" that applies only to large scales. In particular, it implies that gravity is described by classical General Relativity and matter is described by the Standard Model of Particle Physics.

[11] The presented $a(t)$ for a cosmological constant cannot be achieved by taking $w \to -1$ in (1.34), but equation (1.23a) can be solved anyway. $H_\Lambda = \Lambda/3$ is a constant in this case.

Such assumptions are of very general nature and can be used to make predictions where they can be tested. Then, one can fix the model parameters and make more precise predictions. We now proceed to do a brief historical review of the observations that motivated the next assumptions [5]. A complete succinct timeline can be found in [30], while a more detailed account can be found in [2].

Physical Cosmology as we know it started with Einstein's seminal work *Cosmological Considerations in the General Theory of Relativity* [31], where Einstein, after proposing the three classical tests[12] of General Relativity, applied his theory to the universe as a whole using the Cosmological Principle. At the time, even if the field equations enabled a dust filled dynamical universe, Einstein proposed a static model where $a(t)$ was constant by introducing a negative cosmological constant Λ. Einstein's model resulted in a static closed universe, which was later shown to be unstable. It should be emphasized that, at the time of Einstein, it was natural to assume that the universe was static, since no observation indicated that it was dynamic, which would be a true change of paradigm. For an interesting review, see [32].

Figure 1.4: Alexander Friedmann (1888-1925), the first physicist to consider dynamical universes.

Einstein's work was improved by Alexander Friedmann (1922) [33] who considered dynamic universe models with $\mathcal{K} \neq 0$, and wrote the complete Friedmann equations (1.23) for the first time. At this point, it was clear that General Relativity predicted that universes filled with ordinary matter would be dynamical, but Friedmann did not provide any methods to test such prediction.

Later, Georges Lemaître (1927) [34] showed that, by considering a dynamic universe, galaxies that are very distant from us should recede with an apparent[13] velocity that depends linearly on the galaxy's distance. Such law can be obtained from the FLRW metric alone. Considering a fixed instant of time t in (1.21), the spatial distance between two galaxies is given by

$$ds = a(t) dl \,, \tag{1.40}$$

assuming a flat universe[14], we can set $dl = dx$ and, after integrating, we get:

$$l = a(t)\Delta x \,. \tag{1.41}$$

From (1.41), we see that objects that are separated by a fixed coordinate amount[15] Δx in Σ_t have a varying physical distance l due to $a(t)$. Differentiating with respect to time:

$$\frac{dl}{dt} = \frac{d}{dt}\left(a(t)\Delta x\right) \,,$$

[12]They are: the anomalous Mercury perihelion precession, the deflection of light by the sun, and gravitational redshift.

[13]The velocity is apparent because the objects seem to be receding due to the expansion of the universe. Such apparent velocity can even be higher than c, as one can check in (1.43).

[14]The same could be done for the general case using comoving coordinates, which was not done for simplicity.

[15]Due to this, Δx is called *comoving distance*, while l is called *physical distance*. Objects with constant comoving distance are said to be part of the *Hubble Flow*.

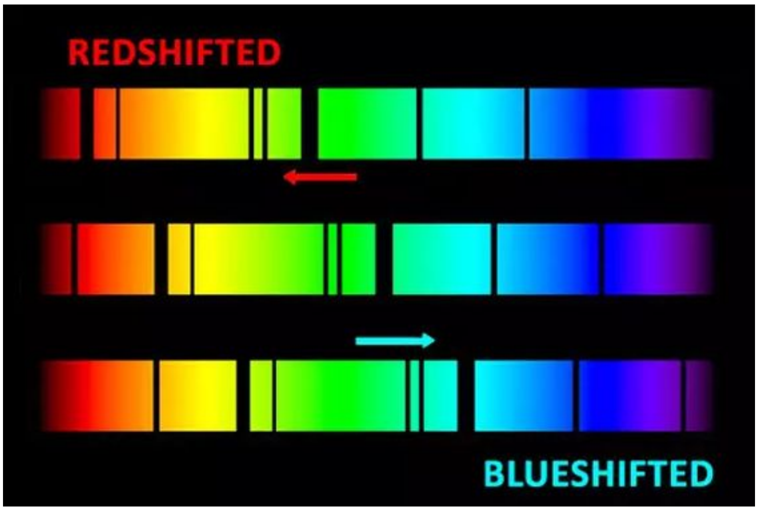

Figure 1.5: Redshift and blueshift of an atomic spectral line, respectively. Source: Science ABC.

$$= \dot{a}\Delta x \,,$$

where it was used that t and x are independent coordinates. Using the velocity $v = dl/dt$:[16]

$$v = \frac{\dot{a}}{a}\left(a\Delta x\right) \,. \tag{1.42}$$

and, for the present time, we have $H_0 = \dot{a}_0/a_0$ and $a_0 = 1$. Hence:

$$v = H_0 l \,. \tag{1.43}$$

The velocity law (1.43) implies that, due to an apparent Doppler effect, light from distant galaxies would suffer a redshift if the universe were expanding ($H_0 > 0$, receding galaxies) or a blueshift if it were contracting ($H_0 < 0$, approaching galaxies). See Figure 1.5. Redshifted patterns were observed by Vesto Sliher (1912) in the spectral lines of atomic elements of distant galaxies, and such observations were improved by Edwin Hubble (1929), who found that the velocities of galaxies obey a linear relation to distance and estimated H_0. It was then established that the universe is expanding[17], which was the first prediction of modern cosmology. Equation (1.43) is nowadays called Hubble-Lemaître law, and H_0 the Hubble-Leamître constant.

Two decades later, in 1948, George Gamow, Ralph Alpher, and Robert Herman (GRR) [35] used the discovered expansion of the universe to explain the formation of chemical elements. At the time, it was established that heavier elements (from carbon to iron) are produced in stars by a process known as stellar nucleosynthesis. However, to create such elements, the stars use light elements, such as hydrogen and helium, as fuel. Those elements are very abundant in the universe, comprising $\approx 75\%$ and $\approx 25\%$ of all elements, respectively. Hence, most of the chemical elements of the universe had an

[16]Here, we insist, such velocity is only *apparent*: distant objects seem to be receding due to the expansion of the universe, but are actually stationary with respect to the Hubble flow.

[17]It should be emphasized that, while recessional velocities were expected for distant galaxies, those were expected to be random, and an isotropic linear relation with the distance was quite unexpected and hard to account in a static universe.

unknown origin.

GRR's idea was that due to (1.32), the energy densities of radiation and density scale as $\rho_{rad} = \rho_{r0}a^{-4}$ and $\rho_m = \rho_{m0}a^{-3}$, which means that radiation should have been the dominating form of energy in the past. This implies that, in earlier times, the universe was much hotter than in the present. To see this, consider that radiation obeys a Planckian blackbody distribution[18]. Equating the radiation energy density with the Stefan-Boltzmann law, we get:

$$\rho_{r0}a^{-4} = \sigma T^4 , \tag{1.44a}$$

$$\implies T(t) = \frac{T_0}{a(t)} , \tag{1.44b}$$

where $T_0 \equiv (\rho_{r0}/\sigma)^{1/4}$. This relation shows explicitly that, in the past, when $a(t)$ was smaller, the temperature T was much higher, which means that the universe was composed by very hot radiation.

Gamow and his coworkers proposal was that, since the energy density of radiation was higher in the past, it should have had energy to split atoms into their constituent particles, only allowing their existence after the universe cooled down to a certain value. As for the photons of such radiation, since the universe is expanding, their wavelength varies with time by:

$$\lambda(t) = a(t)\lambda_0 , \tag{1.45}$$

which means that, compared to the past, their wavelength must have suffered a redshift z, which is defined by

$$z(t) \equiv \frac{\lambda_0 - \lambda(t)}{\lambda(t)} , \tag{1.46a}$$

$$\implies z(a) = \frac{1}{a(t)} - 1 , \tag{1.46b}$$

$$\implies a(z) = \frac{1}{z+1} , . \tag{1.46c}$$

Since the universe is expanding, $a(t)$ is a monotonic function of t, which means that z contains the same information as the scale factor, and can be used as a measure of time.

We now proceed to estimate the redshift of such photons at the time that the universe was cold enough to permit the formation of hydrogen. Since by hypothesis radiation follows a blackbody distribution, the wavelength with most energy contribution is given by Wien's law:

$$\lambda_{max} = \frac{b}{T} , \tag{1.47}$$

[18] This is justified by the fact that relativistic massless bosons in a FLRW spacetime tend to an equilibrium state described by such distribution.

which has an associated energy according to the Planck law:

$$E_{max} = \frac{hc}{\lambda_{max}} , \tag{1.48a}$$

$$= \left(\frac{hc}{b}\right) T . \tag{1.48b}$$

Combining (1.46c), (1.44), and (1.48) we see that

$$(1 + z) = \left(\frac{hc}{b}\right) \frac{E_{max}}{T_0} \tag{1.49}$$

hence, by equating E_{max} with the bonding energy of hydrogen, it is possible to find the redshift z_{dc} for such period, provided that we got the temperature of such radiation at present time, T_0. Since the value of T_0 was unknown, and GRR could only estimate it using other arguments, obtaining $T_0 \approx 5$K. For a more detailed account, see [2].

In 1965, a faint light signal was detected by Arno Penzias and Robert Wilson, two workers of Bell Telephone Laboratories. Such signal was in the microwave region of the electromagnetic spectrum, and appeared to come from all directions of the sky. It was then discovered that it was constituted of primordial radiation, the same predicted by Gamow and his coworkers [2].

Such radiation was named Cosmic Microwave Background (CMB) radiation. It is extremely isotropic, coming from all sides of the sky with a mean temperature of $T_0 = 2,76$K, which is very close to GRR's estimate. It also presents a black body espectrum, being the most perfect black body to be observed in nature.

Using $T_0 \sim 1K$, we can estimate the redshift z_{dc} of the time of formation of light elements. Since bonding to the ground state of the hydrogen atom is inefficient [36][2], we use the energy of the first excited state, $E_{\text{max}} \sim 1$e.V. . Using $b \sim 10^{-3}$mK, $c \sim 10^{8}$m/s, $h \sim 10^{-15}$e.V.s, we get

$$z_{dc} \sim 10^3 , \tag{1.50}$$

which, if combined with (1.46c), (1.44) and assuming a radiation dominated scale factor $a(t) = (t/t_0)^{1/2}$ gives

$$t_{eq} \sim 10^5 \text{ years} \tag{1.51}$$

which is approximately[19] 100 thousand years after $a = 0$.

Such event is called recombination, where light nuclei of hydrogen, lithium and helium were formed in abundance. The redshift is identified as z_{dc} because, at that time, radiation and energy had the same energy density (that of the hydrogen atom) and, after recombination, due to the expansion of the universe, radiation lost energy and matter became the dominant form of energy: matter and radiation *decoupled* at that time, which resulted in a cooling of the universe. Using (1.44) and (1.46c),

[19]More precise estimations give $z_{dc} \sim 1100$ and $t_{dc} \sim 300$ thousand years.

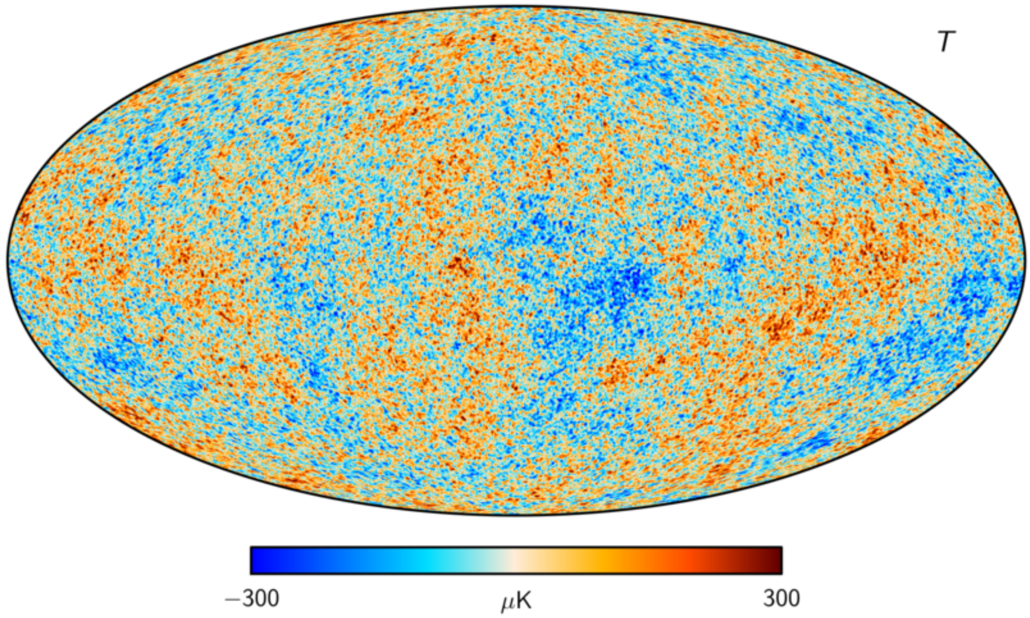

Figure 1.7: A Planck data CMB temperature map made using the COMMANDER software. Source: Wiki Cosmos.

we see that, at such time

$$T_{eq} \sim 10^3\,\mathrm{K}\,, \tag{1.52}$$

which is a very high temperature. Due to this, the paradigm that the universe evolved from a very hot and dense state from which atoms formed is called *hot Big Bang model*[20].

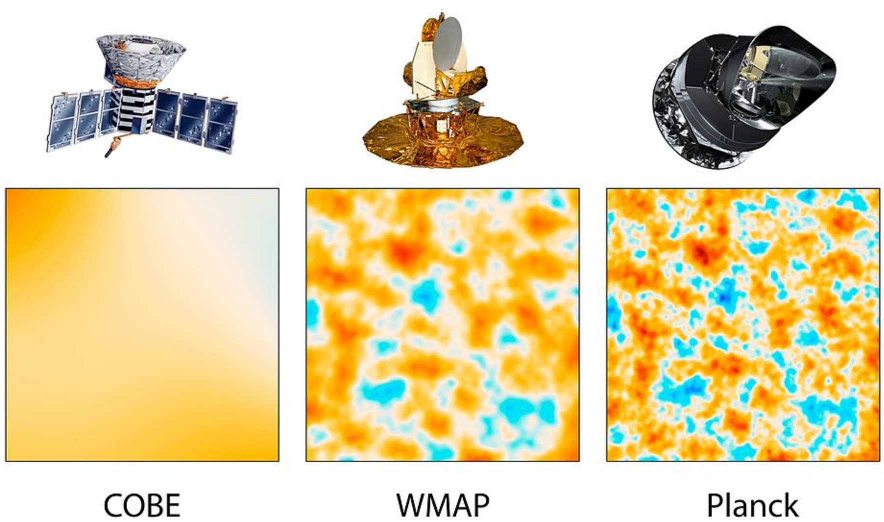

Figure 1.6: Evolution of the precision of the CMB observations. The hotter colors means a higher temperature. Source: NASA Photojournal.

The CMB existence and observation is one of the greatest achievements of the hot Big Bang Model, because it was predicted before observation and needed for a model with an expanding homogeneous and isotropic universe to be consistent while also enabling it to explain the abundance of

[20]The name *Big Bang* was proposed in a mocking way by Fred Hoyle, a proponent of a rival model, that of the Steady State Model. Such model was discard mainly because with was incompatible with the existence of the CMB.

light elements[21]. Since it's discovery, more surveys were done to examine the CMB, like COBE[22] (1989-1990), WMAP[23] (2001-2010), and Planck (2009-2013) [30]. See Figure 1.6. In particular, the recent Planck mission has imposed several constraints on the cosmological parameters [37] and on the background geometry [38]. Planck's observational data of the CMB is consistent with a flat $\mathcal{K} = 0$ universe, with some constraints on its topology[24].

Apart from such predictions, there are other features of our universe that are not explained by the hot Big Bang model, but need to be assumed to explain some phenomena. They are the existence of dark matter and dark energy.

Dark matter was first postulated by Fritz Zwicky (1933) and then by Vera Rubin (1985) to explain galaxy rotation curves. The velocities of the spiraling motion of galaxies could not be explained by the mass of visible matter alone, so the existence of an unseen "dark" component was assumed. This kind of matter is "dark" in the sense that it seems to interact very weakly with electromagnetic radiation, and only its gravitational effects seem to be relevant.

Apart from galaxy rotation curves, Dark Matter is also needed to explain the formation of structure in our universe. In particular, it is needed that Dark Matter moves in slow velocities (non-relativistic) to account for the formation of the observed structures (hot dark matter would wipe out all but the largest initial fluctuations by free streaming) [25]. This kind of dark matter is known as Cold Dark Matter (CDM), in contrast to Hot Dark Matter (relativistic). Hence, this component is included in the standard model. Other solid evidences for dark matter are observations of gravitational lensing, cluster dynamics and collisions, cluster gas in x-rays and constraints from the CMB[25] [4].

Finally, with improving technology, more detalied redshift surveys could be performed, which showed that the Cosmological Principle is indeed a valid approximation for the universe in scales higher than 100Mpc. This confirmed a crucial assumption of Standard Cosmology, which remained a wild guess until the second half of the XX century [3][7]. The redshift measurements also showed an unexpected result. Consider the Taylor expansion [40] of $a(t)$ around $t = t_0$:

$$a(t) \approx a_0 + \left.\frac{da}{dt}\right|_{t=t0} \Delta t + \frac{1}{2}\left.\frac{d^2a}{dt^2}\right|_{t=t_0} \Delta t^2 + \dots , \tag{1.53}$$

where $\Delta t \equiv t - t_0$. Using $a_0 = 1$, and introducing the Hubble constant $H_0 \equiv \dot{a}_0/a_0$ and the deceleration parameter $q \equiv -\ddot{a}a/H^2$:

$$a(t) \approx 1 + H_0\Delta t - \frac{1}{2}H_0^2 q_0 \Delta t^2 + \dots , \tag{1.54}$$

where $q_0 \equiv q(t_0)$.

By analyzing higher order contributions in redshift surveys, one can obtain observational constraints on both H_0 and q_0. In 1992, a survey was realized on supernovae of type 1A [41], whose data

[21] Precise calculations can be done to obtain the abundance of each element [3], but this is beyond the scope of this work.

[22] *Cosmic Microwave Background Explorer.*

[23] *Wilkinson Microwave Anisotropy Probe.*

[24] For a review on the topic, see [39].

[25] This constraint in particular indicates that such dark component must be non-barionic.

determined that $q_0 < 0 \iff \ddot{a}_0 > 0$, which implies an accelerating universe expansion. This is in conflict with single barotropic fluid models (1.34) for $w > -1/3$.

Consider the second Friedmann equation (1.23b). In the present epoch, the universe has matter abundance, but that by itself cannot describe the observed accelerated expansion $\ddot{a}_0 > 0$. Assuming that, at the present time, the universe is dominated by matter and an unknown barotropic fluid known as Dark Energy with $\rho_{\text{DE}} = \rho_{\text{DE}0} a^{-3(1+w_{\text{DE}})}$ and rewriting (1.23b) in terms of the relative densities:

$$\frac{\ddot{a}_0}{H_0^2} = -\frac{1}{2}\left[\Omega_{m0} + (1 + 3w_{\text{DE}})\,\Omega_{\text{DE}0}\right] \tag{1.55}$$

which, in terms of the deceleration parameter $q_0 = -\ddot{a}_0/H_0^2$ becomes:

$$q_0 = \frac{1}{2}\left[\Omega_{m0} + (1 + 3w_{\text{DE}})\,\Omega_{\text{DE}}0\right] . \tag{1.56}$$

The fact that $q_0 < 0$ can be described by assuming that Dark Energy is a cosmological constant with $w_{\text{DE}} = -1$, which implies that

$$q_0 = \frac{1}{2}\Omega_m - \Omega_\Lambda \, ,$$
$$q_0 < 0 \implies \Omega_\Lambda > 2\Omega_m \, ,$$

which can be combined with observations of Ω_m and evidence for a flat universe in the CMB to fix Ω_Λ. We then see that the assumption of $\Lambda > 0$ can describe the accelerated expansion of the universe.

This ends our historical review. The cited observations impose constraints on the cosmological parameters, and also needed some additional assumptions to be explained. Hence, in addition to the previous cosmological hypothesis, the Standard Cosmological model also assumes that [25]

- Presently, $H_0 > 0$, which means that the universe is expanding;
- The existence of dark matter, a non-baryonic matter component that is not described by the Standard Model of Particle Physics, whose only relevant interaction with baryonic matter is gravity;
- The large scale structure of the universe forms in a hierarchical, bottom-up way by gravitational collapse modulated by cold dark matter [5];
- The spatial sections of the universe are plane, admitting spatial curvature $\mathcal{K} = 0 \pm 10^{-5}$ [38];
- The topology of the spatial sections of the universe is trivial, that is, it is simply connected[26];
- Dark energy is described by General Relativity with a $\Lambda > 0$ cosmological constant, which implies an acceleration of the expansion of the universe.

[26] It means that it has no "holes", or, in more formal terms, that every closed loop can be continuosly deformed to a point.

Table 1.1: Chronology of the universe

Epoch	Dominant form of energy	Scale Factor $a(t)$	Events	Time (years)
Beginning	Unknown	0	Singularity	$t = 0$
Hot plasma	Radiation	$\left(\frac{t}{t_0}\right)^{1/2}$	Nucleosynthesis and Recombination	300 thousand
Visible Universe	(dark) Matter	$\left(\frac{t}{t_0}\right)^{2/3}$	Formation of structure	1 Billion
Present and future Universe	Cosmological Constant	$e^{H_\Lambda(t-t_0)}$	Accelerated Expansion	$13,8$ billion

Table 1.2: Composition of the universe

Component	Evidence	Symbol	Value (today)
Radiation	CMB Temperature	Ω_r	2.47×10^{-5}
Baryons	Nucleosynthesis predictions and the CMB	Ω_b	0.05
Cold Dark Matter	CMB Anisotropies	Ω_m	0.27
Cosmological Constant	Type 1A Supernovae	Ω_Λ	0.68
Curvature	CMB Anisotropies	$\Omega_\mathcal{K}$	0 ± 10^{-5}

This set of hypothesis constitutes the Standard Model of Cosmology, also know as Λ-CDM model, due to the presence of both the cosmological constant Λ and Cold Dark Matter (CDM). It is also known as the *Concordance Model* due to it is great conformity with observations, and we shall use such nomenclatures interchangeably. The main epochs of the universe according to such model are organized in Table 1.1.

The Standard Model parameters are fixed by observations, in particular of the CMB [37]. The relevant density parameters are organized in Table 1.2. Other important parameters are $H_0 = 67.36$km/s/Mpc and $T_0 = 2.72$K. It should be noted that, although, in this section we used estimates for the sake of simplicity, the large amount of data[27] in cosmology makes it possible to fix the parameters with great precision. Due to this, it is common to denote contemporary cosmology as *Precision Cosmology* [7].

1.4 Problems of the Standard Model

As pointed in the previous section, the Λ-CDM Model is highly successful in describing various aspects of our universe. However, as any scientific theory, there are some phenomena that the Standard Model cannot satisfactorily. For instance, when we look at the assumed composition of our universe (Figure 1.8), we see that most of it (95%) is a dark sector composed of dark matter and dark energy, both of we know very little about and are postulated to exist.

[27] It should be emphasized that observations are analyzed by assuming certain statistical and physical hypothesis, while also depending on cross relations of observations. In short, analysing cosmological data is highly non-trivial.

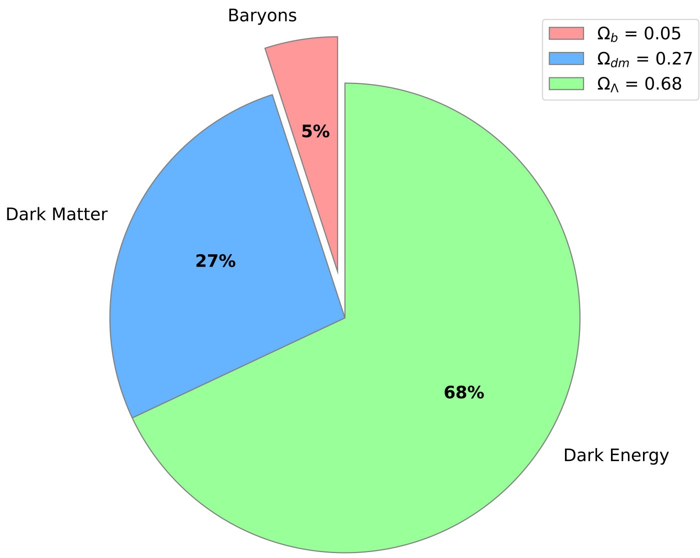

Figure 1.8: Composition of the universe according to the Λ-CDM model. Source: vickycowcroft.github.io.

Apart from the unexplained dark sector, the Standard Model has other problems of an entirely different nature. They are a direct consequence of the fact that cosmology, by definition, is an *observational* discipline, and not an *experimental* one.

In most branches of Physics, one is allowed to create and perform experiments, which are then compared to the predictions of the theory. The crucial difference is that, in experiments, one has the freedom to choose the experimental setup and hence the initial conditions of the system. However, *since in cosmology there is only one universe, we do not have the freedom to vary its initial conditions*: they are fixed once and for all.

Due to this fact, cosmology is based entirely on observations, which means that careful data analyses must be done in order to prevent possible biases in the conclusions. Eventually, this can lead to systematic errors. In particular, contemporary cosmology suffers from two "tensions": the H_0 tension and the σ_8 tension. The term ”tension” is used to describe results that appear to be discrepant between different observations. For a complete review, see [42].

In this work, we will focus on yet another kind of problem. Since we cannot "create universes", the initial conditions are fixed, and cosmology is not completed by just specifying the dynamics of the universe: it must also give a plausible explanation for the chosen initial conditions. Here "plausible" means that we should not choose special initial conditions as such to describe observations, but should derive those from first principles, which introduces a rather subjective step in our models.

We now focus our attention to 4 important problems of the Standard Model associated to initial conditions. We will see that present day observations can be explained, but only by choosing very specific and non-natural initial conditions.

1.4.1 Flatness Problem

As we have seen, present observations are in agreement with $\mathcal{K} = 0$ and a trivial topology [38]. This implies that, at the present time

$$\Omega \approx 1\,, \tag{1.57}$$

with a precision of $\sim 10^{-2}$ [38]. However, the initial conditions necessary to explain this observation need some fine tuning. Remembering $\Omega_{\mathcal{K}} = -\mathcal{K}/a^2H^2$, we can rewrite (1.29) as:

$$(\Omega - 1)\, a^2 H^2 = |\mathcal{K}|\,. \tag{1.58}$$

For fluid dominated universes ((1.15)), we have that $\ddot{a} < 0$, which means that matter slows the expansion. In particular, it means that $\dot{a} = Ha$ is always decreasing, hence, introducing $\Delta \equiv |\Omega - 1|$ we have that

$$\Delta(t) = \frac{|\mathcal{K}|}{\dot{a}^2(t)}\,, \tag{1.59}$$

which means that, if the energy density Ω of the universe is close to 1 in the present time, it must have been even closer at earlier times. We now proceed to make an estimation of the order of Δ at the time of equivalence between matter and radiation.

Since $\mathcal{K}$ is a constant, we can use (1.58) to compare different instants of time t_1 and $_{t2}$:

$$\Delta_1 H_1^2 a_1^2 = \Delta_2 H_2^2 a^2\,. \tag{1.60}$$

Assuming $t_1 = t_{eq}$ and $t_2 = t_0$

$$\Delta_{eq} H_{eq}^2 a_{eq}^2 = \Delta_0 H_0^2 a_0^2 \tag{1.61}$$

In the present epoch, the universe is roughly dominated by the cosmological constant. Hence, due to the Friedmann equations (1.23a):

$$\Omega \approx \Omega_\Lambda = \frac{\kappa}{3H_0^2}\rho_\Lambda\,,$$
$$\implies H_0^2 \approx \frac{\kappa}{3}\rho_\Lambda$$

and in the time of the formation of the CMB, it was dominated by radiation

$$\Omega \approx \Omega_r = \frac{\kappa}{3H_0^2}\rho_\Lambda\,,$$
$$\implies H_{eq}^2 \approx \frac{\kappa}{3}\rho_{r0}\,,$$

now, using $\rho_\Lambda = \rho_{\Lambda 0}$, $\rho_r = \rho_{r0} a^{-4}$, we get

$$\left(\frac{H_0}{H_{eq}}\right) = \frac{\Omega_{\Lambda 0}}{\Omega_{r0}} a_{eq}^2\,, \tag{1.62}$$

finally, using $(1+z) = 1/a$, $\Omega_{\Lambda 0}/\Omega_{r0} \approx 10^4$ we get

$$\Delta_{eq} = \frac{10^4}{(1+z_{eq})^2}\Delta_0 \,. \tag{1.63}$$

Since $\Delta_0 \sim 10^{-2}$ and $z_{eq} \sim 10^3$, we see that $\Delta_{eq} \sim 10^{-4}$, which is a fine-tuning of 4 decimal places.

The obtained value is just an upper bound for the initial value of the relative density of the universe. The situation is even worse if we assume this to be valid until the Planck scale: then $|\Delta| < 10^{-60}$, which would constitute a hard fine tuning of 60 decimal places. Hence, an explanation for why the universe looks so flat nowadays is needed.

1.4.2 Horizon Problem

Consider the comoving distance χ_{CMB} traveled by a CMB photon since the time of recombination until the presenting time. Since photons follow null geodesics, their trajectories are obtained by considering $ds^2 = 0$ in the FLRW metric (1.21), hence:

$$\begin{aligned} \chi_{\text{CMB}} &= \int_{t_{\text{CMB}}}^{t_0} \frac{dt}{a(t)} \,, \\ &= \int_{a_{\text{CMB}}}^{a_0} \frac{da}{a^2 H(a)} \,. \end{aligned}$$

Using the approximation that the universe has been roughly dominated by matter $w = 0$ since the formation of the CMB, we have

$$\begin{aligned} a(t) &= \left(\frac{t}{t_0}\right)^{2/3} \,, \\ \implies H(t) &= \frac{2}{3}\frac{1}{t} \,, \\ \implies H(a) &= H_0 a^{-3/2} \,, \end{aligned}$$

and χ_{CMB} can be obtained explicitly:

$$\begin{aligned} \chi_{\text{CMB}} &= \int_{a_{\text{CMB}}}^{a_0} \frac{1}{a^2}\frac{da}{(H_0 a^{-3/2})} \,, \\ &= \frac{1}{H_0}\int_{a_{\text{CMB}}}^{a_0} a^{-1/2} da \,, \\ &= \frac{2}{H_0}\left(\sqrt{a_0} - \sqrt{a_{\text{CMB}}}\right) \,, \end{aligned}$$

using $a_0 = 1$ and $a_{\text{CMB}} = (1+z_{\text{CMB}})^{-1} \approx 10^{-3}$, we obtain $\chi_{\text{CMB}} \approx 2/H_0$, which means that the photons of the CMB encompassed a comoving volume of $V_{\text{CMB}} \sim R_H^3$, where $R_H \equiv 1/H$ is the Hubble radius.

By a similar calculation, we can estimate the maximum comoving distance traveled by a photon since the "beginning of time", $t = 0 \implies a(0) = 0$ until the formation of the CMB. In such period, the universe was radiation dominated, hence:

$$
\begin{aligned}
a(t) &= \left(\frac{t}{t_0}\right)^{1/2}\,, \\
\implies H(t) &= \frac{1}{2}\frac{1}{t}\,, \\
\implies H(a) &= H_0 a^{-2}\,,
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
\chi_{\text{causal}} &= \int_0^{a_{\text{CMB}}} \frac{1}{a^2}\frac{da}{(H_0 a^{-2})}\,, \\
&= \frac{1}{H_0} a_{\text{CMB}}\,, \\
&\approx 5\times 10^{-4}\chi_{\text{CMB}}\,,
\end{aligned}
$$

hence, there were no time for all the photons of the last scattering surface to get into causal contact and then thermal equilibrium. In particular, if we evaluate the comoving volume V_{CMB} traveled by the CMB to the comoving volume of a casually connected region V_{causal}, identified with a sphere of radius χ_{causal}, we get

$$
\frac{V_{\text{CMB}}}{V_{\text{causal}}} \approx \left(\frac{\chi_{\text{CMB}}}{\chi}\right)^3 \approx 10^{12}\,, \tag{1.64}
$$

which means that the CMB photons come from 10^{12} causally disconnected regions.

The horizon problem is the observation that, since we observe the CMB at all directions of the sky with the same average temperature of 2.76K, that must mean that all regions started with the same temperature, since they had not enough time to enter into causal contact and thermalize. This is a problem of initial conditions: the initial temperature of 10^{12} regions must be calibrated by hand to be consistent with observations.

1.4.3 Origin of the Perturbations

It is clear from observations that, although the universe is highly homogeneous and isotropic in cosmological scales, that is not the case in smaller ones, where we have structures such as galaxies and clusters that contribute to the geometry of spacetime. To describe the evolution of the universe as a whole, one should consider perturbations around a FLRW background geometry, that is

$$
g_{ab} \approx g_{ab}^{\text{FLRW}} + \delta g_{ab}\,, \tag{1.65}
$$

with[28] small δg_{ab} or, in terms of the conformal line element (1.22):

$$
\mathrm{d}s^2 = -a^2(\eta)\left[-(1+2A)\,\mathrm{d}\eta^2 + 2B_i \mathrm{d}x^i\mathrm{d}\eta + (h_{ij}+\gamma_{ij})\,\mathrm{d}x^i\mathrm{d}x^j\right]\,, \tag{1.66}
$$

[28]This condition will be rigorously defined when we discuss perturbation theory in the next chapter.

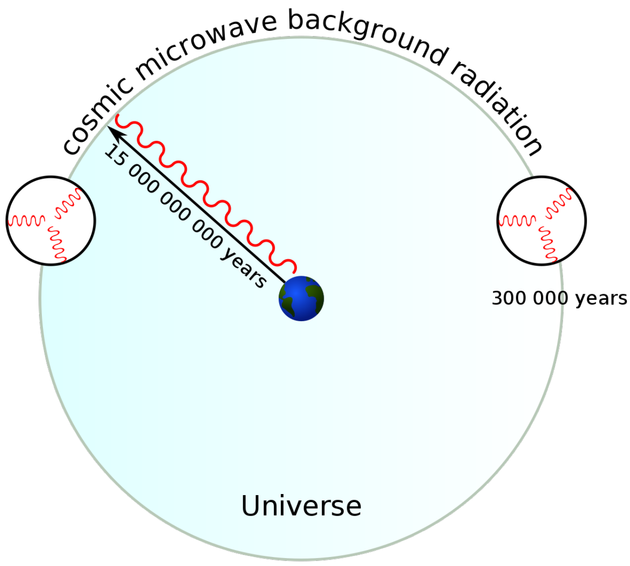

Figure 1.9: Illustration of the horizon problem. Photons from opposite sides of the sky have the same temperature, but never had causal contact. Source: PNGwing

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which will be thoroughly discussed on chapter 2.

When we analyze the perturbed Einstein equations for an FLRW background, we see that the perturbations Fourier modes $\delta\phi_k$ decouple between themselves. To see this, let's consider an specific model. Consider that the universe is filled by a scalar field $\phi(\vec{x},t) = \phi_0(t) + \delta\phi(\vec{x},t)$, where $\phi_0(t)$ is the unperturbed background field, and $\delta\phi(\vec{x},t)$ is its perturbation. The dynamics in physical time t for each Fourier mode $\delta\phi_k$ are then described by equations of the form [9]

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left(\frac{k}{a}\right)^2 \delta\phi_k = 0\,, \tag{1.67}$$

and one can see that the expansion of the universe, which enter the dynamics as H, acts as a damping term proportional to the Hubble function.

Equations of the form (1.67) have two important limits depending on the relation of the wavelength $\lambda_k = k/a$ of the perturbation to the Hubble radius $R_H = 1/H$:

1. Sub-Hubble: $\lambda_k \ll R_H$,

$$\delta\ddot{\phi}_k + \frac{1}{\lambda_k^2}\delta\phi_k = 0\,,$$
$$\implies \delta\phi_k = \frac{C_k}{a}\exp\left\{\pm i\int\frac{dt}{\lambda_k}\right\};$$

2. Super-Hubble: $\lambda \gg R_H$

$$\delta\ddot{\phi}_k + \frac{3}{R_H}\delta\dot{\phi}_k = 0\,,$$
$$\implies \delta\phi_k = A_k + B_k\int\frac{dt}{a^3(t)};$$

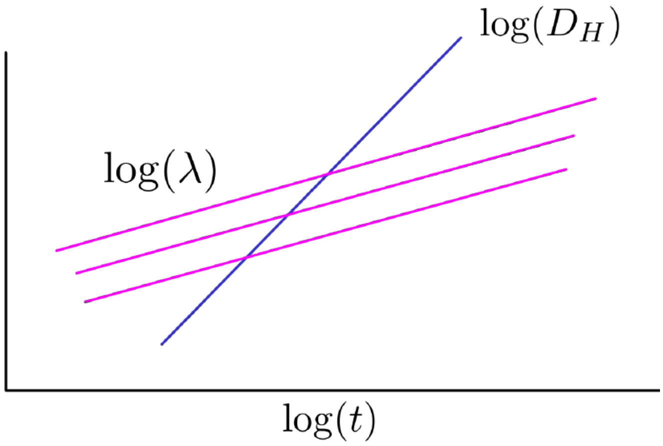

Figure 1.10: Evolution of perturbations and the Hubble radius [9]. Note that today's observable scales started well outside the Hubble horizon.

and we see that sub-Hubble perturbations tend to oscillate and decay with the scale factor, while super-Hubble perturbations rapidly approach a constant value, staying "frozen" until reentering the Hubble horizon. it is important to note that such sub-Hubble and super-Hubble behavior is quite general for the perturbations amplitudes.

We can also compare the evolution of the perturbations wavelength with respect to the Hubble radius R_H:

$$\lambda(t) = \lambda_0 a(t); \tag{1.68a}$$

$$R_H(t) = \frac{a(t)}{\dot{a}(t)}; \tag{1.68b}$$

and we see that, for a decelerating universe, since $\dot{a}(t)$ tends to get smaller, the Hubble radius expands faster than the perturbations wavelength. Hence, we see that perturbations that are inside the Hubble horizon decay with the scale factor, while those that are out of the Hubble horizon stay "frozen" until they enter the horizon and start to decay [43].

A more quantitative analysis may be made by assuming $a(t) = a_0 t^n, n \in \mathbb{R}$, which is valid for universes dominated by a single perfect fluid. By taking the natural logarithm:

$$\ln(\lambda) = \ln(\lambda_0) + n\ln(t)\,, \tag{1.69a}$$

$$\ln(R_H) = -\ln(n) + \ln(t)\,, \tag{1.69b}$$

which can be seen in Figure 1.10. It was assumed that $n = \frac{2}{3}\frac{1}{1+w} < 1 \Leftrightarrow w > -1/3$, which is valid for all types of ordinary matter.

Analyzing the behavior (1.69) one sees that physical scales λ grow faster than the Hubble radius in this class of models. Hence, most of the physical scales λ that we observe nowadays must have been in the super-Hubble regime in the past, which makes it impossible to give a dynamic explanation for the observed spectrum. That is, this model does not provide a causal structure for the origin of perturbations around a homogeneous and isotropic background.

It is also an observable fact that the temperature anisotropy spectrum of the CMB is well described with very high precision by the temperature fluctuations resulting from the evolution of initial curvature perturbations with an almost scale invariant gaussian spectrum with a small red tilt [9]. Any theory that attempts to explain the origin of perturbations should also describe its observed statistical properties. In particular, this means that the initial conditions A_k, B_k must be calibrated in order to generate a gaussian spectrum, which can be done, but demands a large fine tuning. Therefore, one sees that perturbations in the Standard Model cannot satisfactorily describe structure formation in the universe, nor the fluctuations of order 10^{-5} in the CMB power spectrum.

1.4.4 The Singularity Problem

Note that, for the single fluid solution (1.34), we have

$$\begin{aligned} \lim_{t\to 0} a(t) &= \lim_{t\to 0} \left(\frac{t}{t_0}\right)^{\frac{2}{3}\frac{1}{(w+1)}} , \\ &= 0 . \end{aligned}$$

which means that a homogeneous and isotropic universe filled by a perfect fluid admits an instant of time $t = 0$ where, in a certain sense "the whole space collapses down to a point", at a past time t_0 (Equation 1.35) as registered by clocks that are stationary with the Hubble flow. This is satisfied for every fluid with $w > -1/3$, condition that is valid by all known ordinary matter[29].

This concept is known as a singularity: spacetime itself is not defined at $t = 0$. In general, we state that a spacetime has singularities if timelike (or lightlike) curves cannot be extended up to a finite value of its affine parameter (like t_0 for the Hubble flow observers) [29]. This is associated with pathological behaviour, such as the divergence of observable properties of space time. For instance, the Ricci scalar of an FLRW metric (1.21) is given by:

$$R = \frac{6}{a^2}\left(\ddot{a}a + \dot{a}^2 + \mathcal{K}\right) , \tag{1.70}$$

and one can see that it diverges when $a \to 0$, if one assumes that the universe is filled by a barotropic perfect fluid with scale factor (1.34). The same happens for other observables such as the redshift z, which diverges for $a \to 0$, as can be seen in (1.46c). In practice, this means that General Relativity cannot make testable predictions on singularities, for spacetime itself becomes ill-defined. More in-depth discussions can be found in [13], [29] and [44].

Of course, a single perfect fluid universe is a highly idealized model. However, it can be shown that a singularity occurs in more realistic models, like the Λ-CDM. Rewriting the first Friedmann

[29]The condition is violated by a cosmological constant, but its contribution in the early universe was very small.

equation (1.23a) in terms of the present relative densities:

$$\left(\frac{H}{H_0}\right)^2 = \sum_{i=1}^{n} \frac{\kappa}{3H_0^2}\rho_i - \frac{\mathcal{K}}{a^2 H_0^2}\,,$$
$$= \sum_{i=1}^{n} \Omega_{i0} a^{-3(1+w_i)} + \Omega_{\mathcal{K}0} a^{-2}\,.$$

Considering contributions from matter, radiation, a cosmological constant, and also using $H = \frac{1}{a}\frac{da}{dt}$, we can express:

$$dt = \frac{1}{H_0} \frac{da}{\sqrt{\Omega^{a}_{\Lambda 0} + \Omega_m a^{-1} + \Omega_{r0} a^{-2} + \Omega_{\mathcal{K}0}}}\,. \tag{1.71}$$

Integrating from $a = 0$ to $a = 1$, we obtain the time t_0 since the universe was in a singular state:

$$t_0 = \int_0^{t_0} dt = \frac{1}{H_0} \int_0^1 \frac{da}{\sqrt{\Omega_{\Lambda 0} a^2 + \Omega_{m0} a^{-1} + \Omega_{r0} a^{-2} + \Omega_{\mathcal{K}0}}}\,. \tag{1.72}$$

Using the approximated values for the Λ-CDM model (Table 1.2), we find that

$$t_0 \approx 13.7\,\text{billion years}\,, \tag{1.73}$$

which means that a singularity is predicted at a finite time in the past, as measured by the fundamental observers. Hence, the Standard Cosmological model cannot make predictions for the universe at earlier times.

One could argue that this effect occurs due to the great simplifications assumed by Standard Cosmology, like the Cosmological Principle, as discussed in section 1.3. However, it was shown by Hawking [29][44], that a past singularity is predicted by General Relativity if, for all timelike vector field ξ^a, matter satisfies the Strong Energy Condition (SEC)

$$R_{ab}\xi^a\xi^b \geq 0\,,$$
$$\implies T_{ab}\xi^a\xi^b \geq -\frac{1}{2}T\,,$$
$$\implies \rho + 3p \geq 0\,,$$
$$\implies w \geq -\frac{1}{3} \text{ if } p(\rho) = w\rho\,.$$

This means that a singularity is a generic prediction of General Relativity coupled to ordinary matter[30], and not restrained to high symmetry solutions. Hence, one needs to extend the Standard Model by including non-ordinary types of matter or modifying General Relativity to deal with the Singularity Problem.

[30] By "ordinary", we mean matter and radiation.

2 EXTENSIONS OF THE STANDARD MODEL

In this chapter, we explore some extensions of the Standard Model of Cosmology. We begin by relaxing the Cosmological Principle disconsidering isotropy and considering only homogeneous cosmologies in section 2.1. The next modifications are introduced in order to solve the Standard Model puzzles, which are the Inflationary Paradigm in section 2.2 and the Bouncing Paradigm in section 2.3. We conclude with Cosmological Perturbation Theory in section 2.4, which relax the homogeneity hypothesis by considering small perturbations around a FLRW background. In the next chapters, we shall combine such modifications in order to consider quantum perturbations of inflationary/bouncing models in a Bianchi I background.

2.1 Homogenenous Cosmologies

In section 1.1, we saw that isotropy at all points imply homogeneity, so isotropy is a much more restrictive condition on space-time. In this section, we relax the isotropy condition by demanding that space-time is only homogeneous. In subsection 2.1.1, we implement the homogeneity symmetry using Lie Groups and classify[1] homogeneous cosmologies, while their dynamics are explored in subsection 2.1.2.

2.1.1 Bianchi Classification

A Lie Group is a differentiable manifold G equipped with an internal operation that enables one to take products between the manifold points [45]. More formally, a Lie Group is a differentiable manifold equipped with maps

$$\begin{aligned} \mu : G \times G &\to G\,, \\ (g_1, g_2) &\mapsto g_1 g_2\,, \end{aligned}$$

and

$$\begin{aligned} l : G &\to G\,, \\ g &\mapsto g^{-1}\,, \end{aligned}$$

that are smooth with respect to G's differential structure, and the map μ must satisfy the usual group axioms[2].

Lie Groups can then be used to model symmetries on a manifold $\mathcal{M}$. This is done by defining the concept of action of a group on a manifold. We define the *left action* of a group on a manifold as

[1]If the reader is interested only in the applications to Bianchi I universes, he/she can skip directly to subsection 2.1.1, more precisely, to Equation 2.9.

[2]The group operation must be internal, associative, admit a neutral element e and an inverse element g^{-1} .

a map between the group G and the group Perm($\mathcal{M}$) of bijections of $\mathcal{M}$ onto itself:

$$\begin{aligned} \gamma : G &\to \text{Perm}(\mathcal{M})\,, \\ g &\mapsto \gamma_g\,, \end{aligned}$$

that is also a group homomorphism, that is

1. $\gamma_e(p) = p,\ \forall p \in \mathcal{M}$;
2. $\gamma_{g_1} \circ \gamma_{g_2} = \gamma_{g_1 g_2}\ \ \forall g_1, g_2 \in G$;

We now restrict ourselves to the case of actions such that, for all $p, q \in \mathcal{M}$, there exists one and only one $g \in G$ such that $g(p) = q$. If such condition is satisfied, we say that the group *acts transitively* on $\mathcal{M}$. In this case, by choosing an arbitrary $p \in \mathcal{M}$ we can construct a map $I : G \to \mathcal{M}$ such that

$$\begin{aligned} I_p : G &\to \mathcal{M}\,, \\ g &\mapsto gp\,, \end{aligned}$$

which can be shown to be a bijection. Since the isometries of space-times with spatial homogeneity can be described by Lie Groups, we then have that $\mathcal{M} \cong G$, that is: the spatial sections Σ_t of homogeneous cosmologies admit one to one maps to Lie Groups [29].

One can then classify 3-dimensional homogeneous cosmologies using Lie Groups, since they are equivalent. To accomplish such task, we start by defining, for each $h \in G$, a map L_h such that

$$\begin{aligned} L_h : G &\to G\,, \\ g &\mapsto hg\,, \end{aligned}$$

which translates the element $g \in G$ by h and is hence called a left translation[3]. Such maps induce a pushforward L_{h*} on $T_g G$. We then say that a vector field v^a is *left invariant* if it is preserved by left translations, that is

$$L_{h*}\left(v^a|_g\right) = v^a|_{L_h(g)}\ \forall\ h, g \in G\,.$$

An interesting property of left invariant vector fields is that, since $L_h(e) = h$, they can be completely defined in the entire manifold by their respective vector at the identity element $v^a|_e$:

$$v^a|_h = L_{h*}\left(v^a|_e\right)\,, \tag{2.1}$$

hence, the tangent space at the identity element e, $T_e G$, completely characterizes all left invariant vector fields on G due to the class of bijections L_{h*}.

[3] One can also define right translations R_h in a totaly analogous way.

We also have that the commutator $[v, w]$ between two left invariant vector fields is preserved by left translations, that is:

$$L_{h*}[v, w] = [L_{h*}(v), L_{h*}(w)]\,,$$

so that the commutator $[,]$ between left invariant vector fields v, w can be used to define an analogous operation on T_eG:

$$[v|_e, w|_e] \equiv L_{h^{-1}_*}\left([v, h]|_h\right)\,, h \in G\,,$$

which is bilinear, antisymmetric and satisfies the usual Jacobi Identity. Hence, the vector space T_eG is a Lie Algebra. We then say that the space T_eG is the Lie Algebra[4] associated to G.

The advantage of analyzing the Lie Algebra of G instead of G itself is twofold. First, a vector space has more structure than a manifold. Second, there exists an interesting result that expresses that a Lie Algebra completely characterizes a Lie Group, provided that the group is simply connected. This is a reasonable assumption[5] for physical space-time, and also exhausts[6] the possibilities in three dimensions.

Now, consider two left invariant vectors fields v, w. Since $[v, w]$ is also a left invariant vector field, and the commutator $[,]$ is a bilinear map, we can represent the commutator in terms of a tensor as

$$[v, w]^a = C^a_{\ bc} v^b w^c\,, \tag{2.2}$$

and, in the case of Lie Group, $C^a_{\ bc}$ is called the *structure constant tensor* of the Lie Group. It is a well known result that if two Lie Algebras have the same structure constant tensor, then they are isomorphic. So, the 3-dimensional Lie Algebras can be completely classified if we classify their possible structure constant tensors.

From Equation 2.2 and the Jacobi identity, it follows directly that

1. $C^a_{\ bc} = C^a_{\ [cb]}$;
2. $C^e_{\ d[a} C^d_{\ bc]} = 0$;

we can then define the following tensors

$$A_a \equiv C_{ab}^{\ \ a}\,, \tag{2.3a}$$

$$M^{ab} \equiv \frac{1}{2}\epsilon^{acd}\left(C_{cd}^{\ \ b} - \delta_c^{\ b} A_d\right)\,, \tag{2.3b}$$

and it can be readily seen that $M^{[ab]} = 0$. By direct substitution of this parametrization, we have that

$$C_{ab}^{\ \ c} = M^{cd}\epsilon_{dab} + \delta^c_{[a} A_{b]}\,, \tag{2.4}$$

[4]Such Lie Algebra is also the space of translational Killing fields of spacetime.

[5]It is reasonable in the sense that we have yet no observational evidence for non-trivial topologies for space-time.

[6]The only exception is the Kantowski-Sachs space-time, which is homogeneous and has the topology $\mathbb{S}^2 \times \mathbb{R}$ (a "three cylinder) and Lie Group $SO(3) \times \mathbb{R}$, which is not simply connnected and does not act transitively on the manifold.

and one can see that a tensor that satisfies $C_{[ab]}{}^{c} = 0$ can be parametrized by a choice of a vector A_b and a symmetric matrix M^{ab}. One can then obtain all possible structure constant tensors by studying the possible eigenvalues of M^{ab} and the vector A_b. Since one can always adapt a coordinate system to A_b and/or the eigenvectors, then one sees that the absolute values of the eigenvalues are irrelevant: only the signature of M^{ab} is of interest.[7] The Bianchi Classification is completed by analyzing the above possibilities, which are:

Class A ($A_b = 0$)

Bianchi I: $(0, 0, 0)$

Bianchi II: $(+, 0, 0)$

Bianchi VI$_0$: $(+, -, 0)$

Bianchi VII$_0$: $(+, +, 0)$

Bianchi VIII: $(+, +, -)$

Bianchi IX: $(+, +, +)$

Class B ($A_b \neq 0$)

Bianchi III: $(+, -, 0)$

Bianchi IV: $(+, 0, 0)$

Bianchi V: $(+, +, +)$

Bianchi VI$_h$: $(+, -, 0)$

Bianchi VII$_h$: $(+, +, 0)$

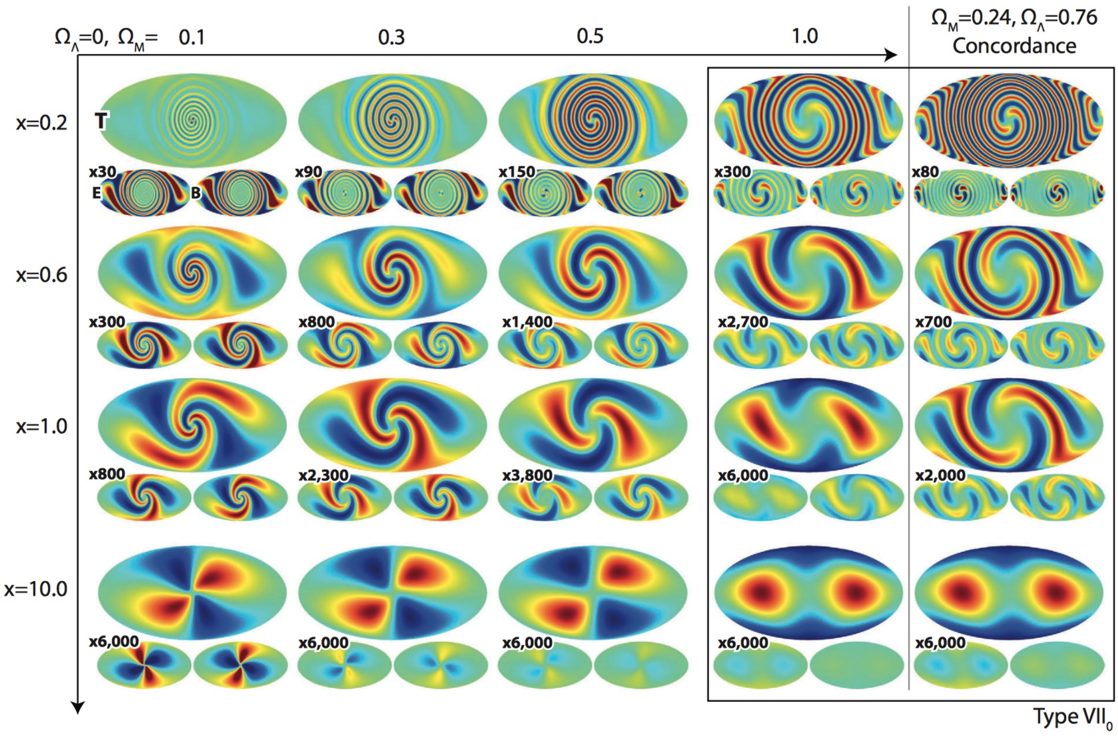

Figure 2.1: The temperature patterns in the cosmic microwave background expected in a Bianchi VII universe which was close to isotropy at early times. Source [46].

[7]There are two exceptions: the Bianchi VI$_h$ and Bianchi VII$_h$, which are one parameter families of Bianchi cosmologies. This happens because a scalar $h \in \mathbb{R}$ is determined by the formula $A_e A_f = h M^{ac} M^{bd} \epsilon_{cde} \epsilon_{abf}$ [29].

Here an important commentary is in order. Of all the Bianchi types, there are only five that admit an isotropic limit. They are the Bianchi I and VII_0, Bianchi IX, Bianchi V and Bianchi VII_h, whose spatial sections have the topology of $\mathbb{R}^3$ (I and VII_0), $\mathbb{S}^3$ (IX), $\mathbb{H}^3$ (V and VII_h), respectively. Hence, if one is interested in models of homogeneous (but anisotropic) cosmologies in the past, one needs to consider only those three types, for they are the only ones that can evolve to an isotropic universe. Furthermore, Planck Data [38] seems to disfavour a universe of type Bianchi VII_h, which can only fit the CMB spectrum with parameters that are incompatible with other observations[8].

Due to the above reasons, in this work we shall focus on the Bianchi I model since it is one of the only two Bianchi types which has both an isotropic limit and is compatible with Planck data in the sense of having flat and simply connected spatial sections, and also has the simplest (trivial) Lie Algebra: $C_{ab}^{\ \ c} = 0$. Now that we have completed our characterization of homogeneous cosmologies, in the next section, we shall proceed to analyze their dynamics using Einstein's field equations.

2.1.2 Dynamics of Homogeneous Cosmologies

To analyze the time evolution of the Bianchi universes, we start by facing a problem. Due to the homogeneity symmetry, we have three spacelike Killing vector fields $(\xi^i)_a$. However, due to the Lie Algebras commutation relations (2.2), they do not commute in general. So, we cannot use our Killing fields to construct a coordinate system adapted to the symmetries of such space-times [29].

Another option would be to write the metric spatial metric of Σ_0 in terms of a basis of left invariant 1−forms $\{(\sigma^i)_a\}$ associated to the Killing vector fields by $(\sigma^i)_a \equiv g_{ab}\,(\xi^i)^a$. In such basis, the metric is given by

$$h_{ab} = \sum_{i,j=1}^{3} h_{ij}\left(\sigma^i\right)_a\left(\sigma^j\right)_b\,,$$

where the h_{ij} are numerical coefficients. Such expression can then be extended to the whole spacetime using "Lie Transport". This is implemented as follows. Take $p \in \Sigma_0$, with n^a being the unit normal to Σ_0. We then demand that

$$\mathscr{L}_n\left(\sigma^i\right)_a = 0\,,$$

then, we define the 1−form in the whole space-time using the pullback:

$$\left(\sigma^i\right)_a(t) \equiv \phi_t^*\left[\left(\sigma^i\right)_a(0)\right]\,,$$

where ϕ_t denotes the one-parameter family of diffeomorphisms generated by n^a. Finally, we can express the space-time metric as

$$g_{ab} = -n_a n_b + \sum_{i,j=1}^{3} h_{ij}(t)\left(\sigma^i\right)_a\left(\sigma^j\right)_b \tag{2.5}$$

[8]It is interesting to point out that simulations of a Bianchi VII_h generate a spiral-like pattern in the CMB temperature fluctuations map, which is due to its anisotropic behavior. See Figure 2.1.

where the coefficients $h_{ij}(t)$ depend only on time due to homogeneity, and represent the metric on each spatial hypersurface Σ_t.

To obtain the explicit form of $h_{ij}(t)$, one must solve the Einstein field equations with the appropriate T_{ab} and boundary conditions. However, one must first obtain the explicit form of the curvature tensors, which is a difficult task for all Bianchi types due to the fact that we cannot use a coordinate system adapted to the Killing Vector Fields. However, this can be done by using a tetrad basis[9]. A possible tetrad basis is

$$(e_0)_a \equiv n_a \,, \tag{2.6}$$

$$(e_i)_a \equiv \sum_{j=1}^{3} B_{ij}(t)(\sigma^j)_a \qquad (i,j = 1,2,3)\,, \tag{2.7}$$

which can be used to obtain explicit formulas for the Riemann and Ricci tensor in terms of B_{ij} and the structure constant tensor $C^c{}_{ab}$. For our purposes, we shall show the final result for the Einstein equations, but omit the intermediary calculations. More details can be found in [29]. The relevant Einstein and Ricci tensor components are:

$$2G_{ab}(e_0)^a(e_0)^b = K^2 - K_{ab}K^{ab} - C^a{}_{ab}C^c{}_c{}^b - \frac{1}{2}C^a{}_{cb}C^c{}_a{}^b - \frac{1}{4}C_{abc}C^{abc}\,, \tag{2.8a}$$

$$G_{bc}h^b_a(e_0)^c = K^b_c C^c_{ba} + K^b_a C^c_{bc}\,, \tag{2.8b}$$

$$h^c_a h^d_b R_{cd} = \mathscr{L}_n K_{ab} + K^c_c K_{ab} - 2K_{ac}K^c_b + \frac{1}{4}C_{acd}C_b^{cd}\,, \tag{2.8c}$$

where $K_{ab} \equiv \frac{1}{2}\mathscr{L}_n h_{ab}$ is the extrinsic curvature of the spatial sections Σ_t, and all indices were lowered or raised using the spatial metric h_{ab}. Equation 2.8 then enables one to study the dynamics of the Bianchi universes simply by inputting the structure constant tensor $C^c{}_{ab}$ for the chosen Bianchi type.

We now focus on the Bianchi I model. In this case, since the Killing fields commute, we consider a set of three mutually orthogonal Killing fields $\{\xi_x, \xi_y, \xi_z\}$ to construct a global coordinate system on the space-like hypersurface $\Sigma = \mathbb{R}^3$. In such coordinate system, the metric is given by:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + X^2(t)\mathrm{d}x^2 + Y^2(t)\mathrm{d}y^2 + Z^2(t)\mathrm{d}z^2\,, \tag{2.9}$$

and one cand readily see that such space-time is homogeneous, but anisotropic, since it expands at different rates $X_i(t)$ in each direction x_i. We can also introduce quantities that are generalizations of the Hubble parameter. To do so, we first define the mean scale expansion rate

$$S(t) \equiv [X(t)Y(t)Z(t)]^{1/3}\,,$$

[9]A tetrad basis is a basis of vector fields $\{(e_\alpha)\}_a$ such that $(e_\alpha)_a(e_\beta)^a = \eta_{\alpha\beta}$, but do not necessary commute, $[e_\alpha, e_\beta] \neq 0$, that is, in such basis the metric coefficients are the same as the Minkowski metric, but it is not a coordinate basis.

from which we can rewrite the line element as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + S^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \,,$$

where

$$\gamma_{ij} \equiv \exp\left[2\beta_i(t)\right]\delta_{ij} \,, \tag{2.10}$$

and the β_i satisfy the constraint $\sum_{i=1}^{3}\beta_i = 0$. We then define the directional scale factors as

$$a_i(t) \equiv e^{\beta_i(t)} \tag{2.11}$$

and the original directional expansion rates can be recovered using $X_i(t) = a_i(t)S(t)$. The generalizations of the Hubble function then are defined by

$$H \equiv \frac{\dot{S}}{S} \,, \tag{2.12a}$$
$$h_i \equiv \frac{\dot{X}_i}{X_i} \,, \tag{2.12b}$$
$$\dot{\beta}_i \equiv \frac{\dot{a}_i}{a_i} \,, \tag{2.12c}$$

and we recover the usual Hubble function in the isotropic limit $a_x(t) = a_y(t) = a_z(t) = a(t)$. It is also useful to introduce a quantity that measures the degree of anisotropy of our universe. We then define the shear tensor as

$$\tilde{\sigma}_{ij}(t) \equiv \frac{1}{2}\frac{d\gamma_{ij}}{dt} \,, \tag{2.13}$$

and we can see that, in the isotropic limit, since $S(t) \to a(t)$, $\sigma_{ij} \to 0$.

The dynamics of a Bianchi I universe can then be studied using Einstein's equations. This can be done both by using standard coordinate methods with (2.9) or inputting $C^c{}_{ab} = 0$ in (2.8). We now couple the geometry with a generic fluid[10] stress energy tensor.

$$T_{ab} = (\rho + p)\, u_a u_b + p g_{ab} + \pi_{ab} \,, \tag{2.14}$$

where π_{ab} is the anisotropic stress tensor such that $\pi_{ab}u^a = 0, \pi_a{}^a = 0$. The Einstein equations then imply

$$3H^2 = \kappa\rho + \frac{1}{2}\tilde{\sigma}^2 \,, \tag{2.15a}$$
$$\frac{\ddot{S}}{S} = -\frac{\kappa}{6}\left(\rho + 3p\right) - \frac{1}{3}\tilde{\sigma}^2 \,, \tag{2.15b}$$
$$\frac{d}{dt}\tilde{\sigma}^i{}_j = -3H\tilde{\sigma}^i{}_j + \kappa\tilde{\pi}^i{}_j \,, \tag{2.15c}$$

[10] We consider a generic fluid because the background is anisotropic. Hence, we should also consider anisotropic contributions in T_{ab}.

where $\pi_{ij} \equiv S^2\tilde{\pi}_{ij}$. The usual Friedmann equations (1.23) are then recovered in the isotropic limit, for which $\tilde{\sigma}_{ij} \to 0 \Rightarrow \pi_{ij} \to= 0$. They can also be expressed in terms of conformal time η (1.22) as [23]

$$3\mathcal{H}^2 = \kappa a^2 \rho + \frac{1}{2}\sigma^2 \,, \tag{2.16a}$$

$$\mathcal{H}' = -\frac{\kappa a^2}{6} a^2 \left(\rho + 3p\right) - \frac{1}{3}\sigma^2 \,, \tag{2.16b}$$

$$\frac{d}{d\eta}\sigma^i{}_j = -2\mathcal{H}\sigma^i{}_j + \kappa a^2 \tilde{\pi}^i{}_j \,, \tag{2.16c}$$

where $\mathcal{H} \equiv S'/S$ and $\sigma_{ij} \equiv \gamma_{ij}/2$ is the conformal shear tensor.

As a final remark, we would like to point out that, since our universe presently satisfies the Cosmological Principle with great accuracy, the universe *is not* described by a Bianchi mode nowadays. What one can ask, however, is if, by considering an initial Bianchi model, the universe can be isotropized to acquire the state that we observe nowadays. If this is implemented, this can relax our imposition of initial conditions, since we would not need to impose by hand that the universe "started" at a highly isotropic state in the past. Of course, if one assumes anisotropic initial conditions, then one must need a mechanism to isotropize the universe. This can be achieved both in inflationary models and in bouncing models, as we shall discuss in the next sections.

2.2 Inflationary Paradigm

By considering the problems of the Standard Model section 1.4, one can see that they are associated to the variation of the Hubble radius with respect to the scale factor [1]:

$$\frac{dR_H}{da} = \frac{1}{\dot{a}}\left(1 - \frac{a\ddot{a}}{\dot{a}^3}\right) \tag{2.17}$$

which, for a decelerating expanding universe with $\ddot{a} < 0$, $\dot{a} > 0$, always satisfies $dR_H/da > 0$. Therefore, the Hubble radius grows faster than the scale factor and hence the physical scales of interest.

The main idea of inflation is that, by considering a period of accelerated expansion in the past such that $\ddot{a} > 0, \dot{a} > 0 \implies dR_H/da < 0$, then such fine-tuning problems shall be alleviated. This is implemented by considering a period of approximately exponential expansion

$$a(t) \approx e^{Ht} \,,$$

which is achieved by adding a primordial field ϕ known as inflaton coupled to GR. The duration of inflation is parametrized by the e-fold number

$$N \equiv \ln\left(\frac{a_f}{a_i}\right) = \int_{t_i}^{t_f} H(t)dt \,, \tag{2.18}$$

which measures how much the universe expanded during such period. Usually, one fixes the duration of inflation by demanding that the horizon and flatness problems are properly solved.

In general, one uses the *slow roll* approximation, which is characterized by a regime where the field ϕ evolves slowly with respect to its potential $V(\phi)$ and leads naturally to $H \approx$ constant. For a single scalar field model, such approximation is quantified by demanding that the slow roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \tag{2.19a}$$

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{2.19b}$$

are first order quantities. Although generally one adds unknown fields to implement inflation, it has the advantage that is able to make predictions that are quite independent of the inflationary mechanism details, and are parametrized only by the slow roll parameters (2.19). For an interesting discussion, see [47].

We now proceed to analyze how such considerations solve the problems of the Standard Model in a qualitative way, then, we also introduce a quantitative example.

2.2.1 Qualitative Solution

1. **Flatness Problem**

   If we turn to Equation 1.59, we see that

   $$\frac{d}{da}\Delta(t) = -2\frac{\ddot{a}}{\dot{a}^3}$$

   where $\Delta(t) \equiv |\Omega - 1|$ measures the deviation from a flat spatial section. Inflation then forces such derivative to be negative, which implies that

   $$\Delta(t) \to 0\,,$$

   that is, the flat universe becomes an attractor, and does not need an ad hoc initial condition to be described. This alleviates the before mentioned problem of initial conditions is cosmology, since arbitrary initial conditions on Σ_t lead to the same prediction: a flat universe.

   A more intuitive picture can be imagined as follows. Suppose that that we live in a curved universe, but that physical scales underwent a massive quasi-exponential expansion. Then, the radius of curvature would be so large that we would need to observe incredibly large scales to probe it. Hence, we effectively see a flat universe.

2. **Horizon Problem**

   The Horizon Problem is characterized by the fact that apparently causally disconnected regions of the universe are in thermal equilibrium. However, if we consider a period of quasi-exponential expansion, then the scales that we observe nowadays would be just a tiny causally connected region of our universe in the era before inflation. This solves the puzzle, for such regions would have been in thermal equilibrium before the inflationary era.

3. **Origin of Perturbations**

   For an inflationary period, we have that

$$\frac{dR_H}{da} < 0 \,,$$

   which means that physical scales grow faster than the Hubble radius. Hence, initially sub-Hubble[11] perturbations are amplified by the inflationary period and cross the Hubble horizon, where they stay "frozen" until they reenter the horizon much after inflation ends. The evolution of the universe then changes into the usual decelerated expansion phase.

   This is a feature of inflationary models that is very important, so it must be emphasized. The inflaton is coupled to geometry, so that vacuum fluctuations of the inflaton generate small perturbations on the metric, which are then amplied to become large scale structure and leave its imprint on the CMB spectrum. However, this is only possible because such physical scales started sub-Hubble, then were amplified until they crossed the horizon. **The perturbative modes freeze when they cross the horizon**, and would decay otherwise, and this allows for a quantitative description of the CMB power spectrum [9].

   It is also important to point out that, in addition to predicting the CMB power spectrum, it also offers a *causal explanation* for the origin of perturbations: they were generated by vacuum fluctuations of the metric, which were then inflated to cosmological scales due to the quasi-exponential expansion. We shall discuss more about this when we quantize the inflaton.

4. **Singularity Problem**

   In general, inflationary models do not address the singularity problem. Although quantum scalar fields violate the weak energy condition, this is not enough to avoid a singularity, as shown by Guth [48]. In general, classical models that violate energy conditions are plagued by instabilities that lead to more singularities [1][6].

2.2.2 Quantitative Example: Quadratic Inflation $V(\phi) = \frac{1}{2}m^2\phi^2$

In this section, we consider the model of Quadratic Inflation, which consists of a homogeneous and isotropic Klein-Gordon field $\phi(t)$ in a FLRW background. In order to implement inflation and derive the proper spectrum of the CMB, one needs to consider perturbations around such homogeneous state. This shall be done after we develop cosmological perturbation theory in section 2.4. For now, this example gives a taste of the usual calculations on inflationary cosmology.

[11] In inflation, the scales that we observe nowadays start on the sub-Hubble regime and then are amplified, in contrast to the Standard Model. This happens because initially super-Hubble scales are amplified during inflation, becoming larger than the observable universe and hence unobservable.

We start with the action of a minimally coupled scalar field [9]

$$S_\phi = -\int \sqrt{-g}\left(\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi + V(\phi)\right)d^4x, \tag{2.20}$$

and we couple such field to the usual field equations through the Einstein-Hilbert action. Considering a homogeneous and isotropic universe, we have that $\phi = \phi(t)$. Its stress-energy tensor is then given by

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \left(\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi + V\right)g_{\mu\nu}.$$

The energy density and pressure of the field are then given by, respectively

$$\rho_\phi = \frac{1}{2}\dot\phi^2 + V(\phi)\,, \tag{2.21a}$$

$$p_\phi = \frac{1}{2}\dot\phi^2 - V(\phi)\,, \tag{2.21b}$$

and, as we have seen in section 1.2, an equation of state $w = -1$ (cosmological constant) generates an exponential expansion. In order to implement Inflation, we then demand that such condition should be approximately satisfied, that is

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi}\,, \tag{2.22}$$

$$= \frac{\dot\phi^2 - 2V(\phi)}{\dot\phi^2 + 2V(\phi)}\,, \tag{2.23}$$

$$\approx -1\,, \tag{2.24}$$

which is implemented if the condition

$$\frac{\dot\phi^2}{V(\phi)} \ll 1\,, \tag{2.25}$$

is satisfied. This means that the field "rolls slowly" through its potential, that it is, the potential energy is dominant in the inflationary phase.

The Friedmann equations become, in terms of the Hubble factor H:

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right)\,, \tag{2.26a}$$

$$\dot H + H^2 = -\frac{4\pi G}{3}\left(\frac{1}{2}\dot\phi^2 - V(\phi)\right)\,, \tag{2.26b}$$

which shows that condition (2.25) is equivalent to the slow roll approximation (2.19). The continuity equation[12] takes the form

$$\ddot\phi + 3H\dot\phi + m^2\frac{dV}{d\phi} = 0\,. \tag{2.27}$$

Dividing the first Friedmann equation by the second, we see that (2.25) combined with the

[12]This is also the field equation for the scalar field ϕ;

Friedmann equations just gives the slow parameter ϵ condition (2.19). Applying such conditions to the first Friedmann equation and the continuity equation, we get

$$H^2 \approx \frac{8\pi G}{3} V(\phi) \,, \tag{2.28}$$

$$-3H\dot{\phi} \approx \frac{dV}{d\phi} \,, \tag{2.29}$$

where we used the slow roll condition (2.19), that is, $\delta \ll 1$.

We now focus in the Quadratic Inflation case, defined by $V(\phi) = m^2\phi^2/2$. The Friedmann equations then assume the form

$$H \approx \frac{1}{\sqrt{6}} \frac{m}{M_{pl}} \phi \,, \tag{2.30}$$

$$-3H\dot{\phi} \approx m^2 \phi \,, \tag{2.31}$$

where we defined the reduced Planck mass, $M_{pl} = 1/\sqrt{8\pi G}$. Since $H \propto \phi$, we can integrate the second equation to obtain

$$\phi(t) \approx \phi_i - m M_{pl} \sqrt{\frac{2}{3}} t \,,$$

where ϕ_i is an integration constant corresponding to the initial value of the field. Now, since $H = \dot{a}/a$, we obtain

$$a(t) \approx a_i \exp \left\{ \frac{\phi_i^2 - \phi^2(t)}{4M_{pl}^2} \right\} ,$$

which is an approximately exponential expansion, and solves the puzzles of the Standard Model according to the previous arguments.

We now proceed to quantize the test field in order to introduce some concepts of later importance and show how it can derive the CMB spectrum. For simplicity, we consider the approximation in which the field does not influence the dynamics of the background. Hence, we consider the action (3.73) in a De Sitter background, with $a(t) = e^{Ht}$, $H =$ constant. The metric in conformal time is then given explicitly by

$$ds^2 = a^2(\eta) \left(-d\eta^2 + dl^2 \right) \,, \tag{2.32}$$

where $a(\eta) = -(H\eta)^{-1}$, $\eta \in (-\infty, 0)$ and $\sqrt{-g} = a^4(\eta)$.

Integrating (3.73) by parts and introducing

$$v(\eta) \equiv a\phi(\eta) \,, \tag{2.33}$$

$$m_{ef}^2 \equiv \frac{(2H^2 - m^2)}{H^2 \eta^2} \tag{2.34}$$

the action can be rewritten in canonical form:

$$S = \frac{1}{2}\int \left[(v')^2 - m_{ef}^2 v^2\right] , \tag{2.35}$$

which has the following equation of motion in Fourier space:

$$v_k'' + \omega_k^2(\eta) v_k = 0 , \tag{2.36}$$

where $\omega(\eta) \equiv k^2 - m_{ef}^2$ is the time-dependent frequency of the k-mode. This equation can be solved explicitly. First, we introduce

$$s \equiv k|\eta| , \tag{2.37}$$

$$\nu^2 \equiv \frac{9}{4} - \frac{m^2}{H^2} , \tag{2.38}$$

$$f \equiv \frac{v}{\sqrt{s}} , \tag{2.39}$$

and the mode equation (2.36) takes the form

$$s^2 \frac{d^2 f}{ds^2} + s\frac{df}{ds} + \left(s^2 - \nu^2\right) f = 0 , \tag{2.40}$$

which can be readily solved in terms of Hankel functions. The general solution in terms of the original variable can then be expressed by

$$v_k(\eta) = \sqrt{|\eta|}\left[c_1 H_\nu^{(1)}(-k\eta) + c_2 H_\nu^{(2)}(-k\eta)\right] . \tag{2.41}$$

Now, we need to fix the initial conditions. We do so by demanding that, in the low wavelength limit $k \to \infty$, the solution tends to the usual plane waves $e^{-ik\eta}/\sqrt{2k}$ of Minkowski space. Using the asymptotic limit of the Hankel functions

$$H_\nu^{(1,2)}(x \gg 1) \approx \sqrt{\frac{2}{\pi x}} \exp\left[\pm i\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right] , \tag{2.42}$$

we then see that our condition is implemented if we set the boundary conditions

$$c_1 = \frac{\pi}{4} e^{i(\nu+1/2)\frac{\pi}{2}} , \qquad c_2 = 0 , \tag{2.43}$$

which leads to the general normalized solution

$$v_k(\eta) = \frac{\sqrt{-\pi\eta}}{2} e^{i(\nu+1/2)} H_\nu^{(1)}(-k\eta) . \tag{2.44}$$

With the general solution (3.27) we can quantize our field. By analyzing the action as $S =$

$\int Ld^4x$ (2.35), we see that the canonical momentum of the field is given by

$$\Pi\left(\eta,\vec{x}\right) \equiv \frac{\partial L}{\partial v'} = v'(\eta,\vec{x})\,, \tag{2.45}$$

so that the field can be quantized by promoting the field and its momentum to operators that satisfy the usual algebra

$$[\hat{v}(\eta,\vec{x}),\hat{\Pi}(\eta,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{2.46}$$

and, if we express the quantized field and its momentum in Fourier space,

$$\hat{v}(\eta,\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int\left[v_k(\eta)\hat{a}_k e^{i\vec{k}\cdot\vec{x}} + v_k^*(\eta)\hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\right]d^3\vec{k}\,,$$
$$\hat{\Pi}(\eta,\vec{x}) = \frac{1}{(2\pi)^3}\int\left[v_k'(\eta)\hat{a}_k e^{i\vec{k}\cdot\vec{x}} + v_k^{*\prime}(\eta)\hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\right]d^3\vec{k}\,,$$

one can see that the canonical commutation relation (2.46) implies the usual creation and anihilation algebra

$$[\hat{a}_k,\hat{a}_{k'}^\dagger] = (2\pi)^3\delta(\vec{k}-\vec{k}')\,. \tag{2.47}$$

We can now proceed to calculate correlation functions using our quantized field:

$$\langle\hat{v}\left(\vec{x},\eta\right)\hat{v}\left(\vec{y},\eta'\right)\rangle \equiv \langle u|\hat{v}\left(\vec{x},\eta\right)\hat{v}\left(\vec{y},\eta'\right)|u\rangle \tag{2.48}$$

in the quantum state $|u\rangle$. Here, one needs to answer an important question: in which quantum state $|u\rangle$ were the fields at the end of inflation?

The answer to this question is not definitive in literature. However, one may use some heuristic arguments to select a particular state. Since inflation leads to an almost exponential period of expansion, and the temperature falls as $T \propto 1/a(t)$[13], then

$$T \propto e^{-Ht}\,, \tag{2.49}$$

which means that the temperature of the universe decreases by an exponential factor during the inflationary period. If we analyze the Bose-Einstein or Fermi-Dirac statistics, we see that the dominant contribution for low temperatures is given by the vacuum[14] (the peak is close to 0). This means that the quantum state of the field is almost a vacuum state at the end of inflation. Hence, we shall evaluate the correlation function in the vacuum state $|0\rangle$ such that

$$\hat{a}_{\vec{k}}\,|0\rangle = 0\,,\quad \forall\,\vec{k}\,.$$

[13]This relation can be obtained by applying Kinetic Theory in curved spacetime [49].
[14]Intuitively, the dominant probability is that one finds no particles for such states.

Actually, it is more useful to work with the Fourier transform of the correlation function, that is

$$\langle 0|\hat{v}\,(\vec{x},\eta)\,\hat{v}\,(\vec{y},\eta')\,|0\rangle = \int G\,(k,\eta,\eta')\,e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\frac{d^3\vec{k}}{(2\pi)^3}\,, \tag{2.50}$$

where $G\,(k,\eta,\eta') \equiv v_k(\eta)v_k(\eta')$. We then define the power spectrum $\mathcal{P}_v(k)$ in terms of the green function at equal times $\eta = \eta'$:

$$\frac{2\pi^2}{k^3}\mathcal{P}_v(k,\eta) \equiv \lim_{\eta\to\eta'} G\,(k,\eta,\eta') = |v_k(\eta)|^2\,, \tag{2.51}$$

where η^* is the conformal time instant for which the modes cross the horizon.

Finally, we can evaluate the power spectrum for our simple model of a scalar field in De Sitter spacetime. We shall use the modes $\{v_k(\eta)\}$ obtained from the field equation with the Minkowski boundary condition (2.43). Since inflation increases the perturbations exponentially, we are interested only in large wavelengths $k \to 0$. Using the asymptotic limit of the Hankel functions $H_\nu^{(1,2)}(x)$:

$$H_\nu^{(j)}(x \ll 1) \approx (-1)^j i\frac{2^{\nu-1}}{\pi}\frac{\Gamma(\nu)}{\Gamma(3/2)}x^{-\nu}\,,$$

we then get the following power spectrum:

$$\mathcal{P}_\phi(k) = 2^{2\nu-3}\left[\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2\left(\frac{H}{2\pi}\right)^2(-k\eta)^{3-2\nu}\,, \tag{2.52}$$

which has the form of a power-law:

$$\mathcal{P}_\phi(k) = A_S k^{n_s-1}\,, \tag{2.53}$$

where n_s is called the *spectral index*, which measures the deviation of the spectrum from a constant one. If $n_s < 1$, we say that the spectrum has a red tilt, and a blue tilt if $n_s > 1$, with the observed CMB powerspectrum having a slight red tilt. In our case, $n_s - 1 = 3 - 2\nu$ and, if we use the condition that $m/H \ll 1$, we get

$$n_s - 1 \approx \frac{m^2}{3H^2}\,. \tag{2.54}$$

so that a non-constant spectrum is a direct consequence of the mass of the field.

Equation 2.52 is a generic prediction of slow roll inflation, and also its greatest success. From first principles, one can derive the power law (2.52) which was a prediction confirmed by CMB observations.

But not everything works as it should: the observed spectrum of the CMB has a red tilt, while (2.52) has a blue tilt. Furthermore, the CMB spectrum is associated to perturbations in the FLRW geometry, which is a much longer calculation that we postpone to the section about Cosmological Perturbation Theory. After developing such apparatus, one may calculate the power spectrum for both scalar and tensor perturbations on a FLRW universe, which agrees to great accuracy with observations of the CMB for a variety of inflationary models. Such observations can then be used to exclude models or constrain their parameters [50], as is illustrated in 2.2.

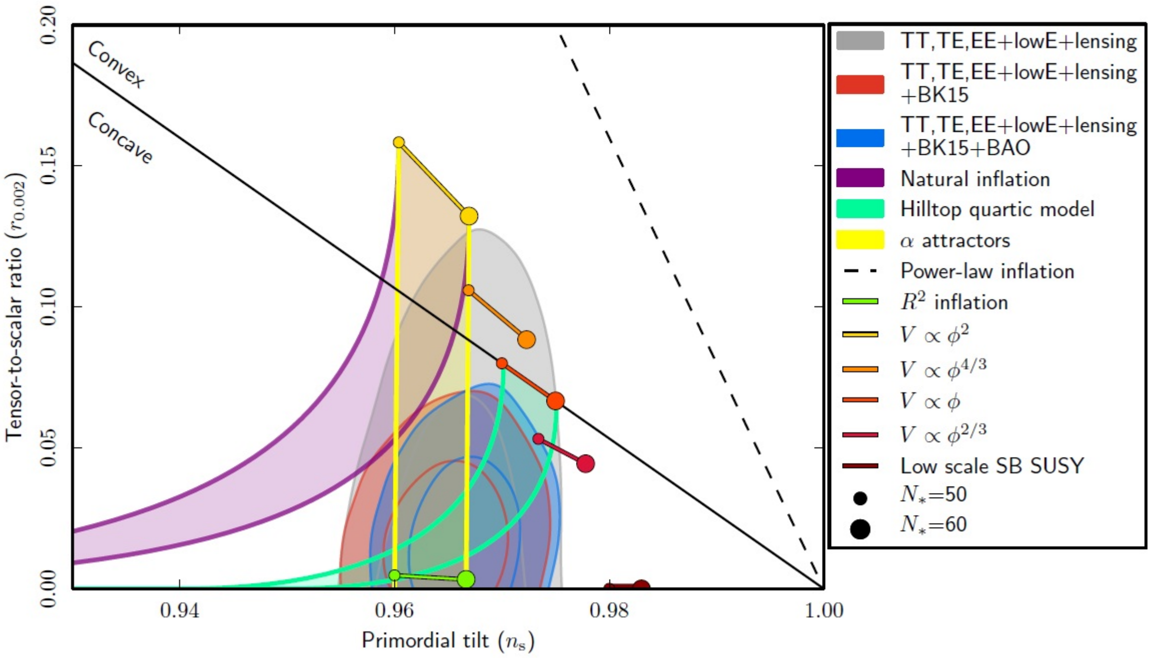

Figure 2.2: Planck constraints on Inflation. The illustrated circular areas define regions of 68% and 95% confidence level in parameter space, respectively. Source: Planck Collaboration.

Before we conclude this section, we shall do a brief recap of the main advantages and disadvantages of inflation. Firstly, it solves the fine tuning problems of the Standard Model by assuming a phase of exponential expansion in the early universe. This also leads to a causal explanation for structure formation and predictions about the CMB power spectrum. It should be emphasized that inflation was the first explanation from first principles to such spectrum, and is quite independent on the inflationary period's details.

However, inflation demands one to include an unobserved quantum field (usually a scalar one), to generate an almost exponential expansion and it also does not solve (does not even address) the singularity problem. After inflation, it is customary to assume that the universe undergoes a period known as **reheating**, where the energy of the inflaton is converted into particles of the Standard Model of Particle Physics. However, the details of such period are not quite fully understood, and they are difficult to constrain observationally [51] [52].

Another, maybe even more serious problem, is known as the **trans-planckian problem**. It consists of the following: if we consider that inflation occured, then physical scales were much smaller in the past: scales corresponding to galaxies were of the order of quantum fluctuations. This can be visualized in Figure 2.3. The problem is that, since inflation amplifies such scales, some of the scales that we observe nowadays were smaller than the Planck length $\ell_{pl} = \sqrt{\hbar G/c^3} \sim 10^{-35}$m in the past. This is a serious problem because, in such scales, one would expect quantum gravity effects, for which we have no precise description yet. So, a large part of the physical scales that we observe were in a kind of "region of ignorance" in the past, where quantum gravity effects should be dominant.

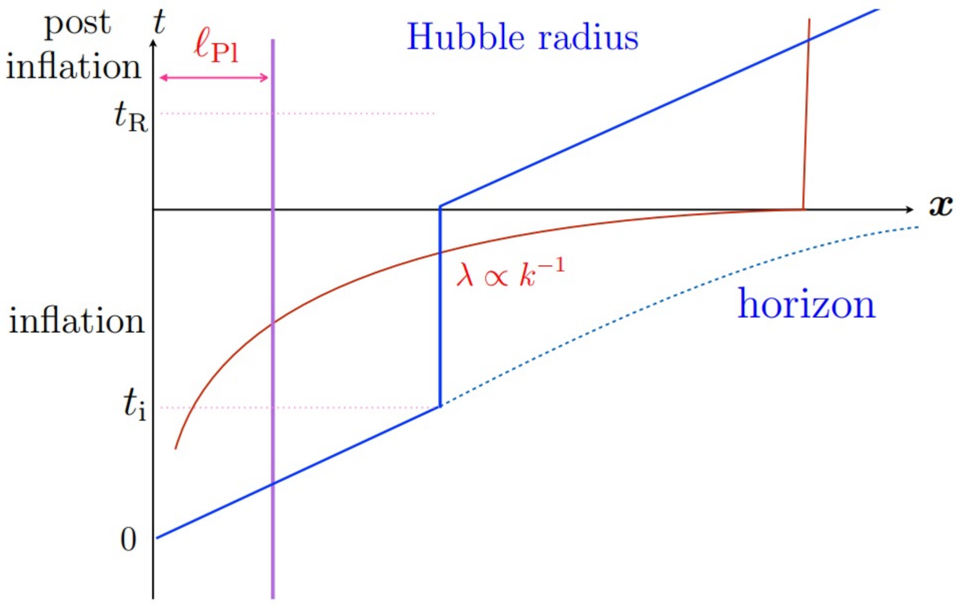

Figure 2.3: Problem of transplanckian frequencies. The vertical axis represents time, and the horizontal one represents physical scales. As one can see, physical scales of interest nowadays were amplied due to inflation, and some of them started in the "region of ignorance" $x < \ell_{pl}$. Note that the Hubble radius stays approximately constant during inflation. Source: [14].

In the next section, we shall analyze the bouncing paradigm and how it solves the problems of the cosmological Standard Model, while also offering an explanation for the CMB power spectrum and large scale structure of the universe, and has advantages and disadvantages in comparison with the inflationary paradigm.

2.3 Bouncing Paradigm

As already mentioned in section 1.4, classical General Relativity coupled to ordinary matter leads to a primordial singularity. Bouncing models were then studied in response to this problem[15], and their main idea is avoiding such singularity by introducing a previous contracting phase to our universe. Such phase would end when the scale factor reaches a minimum (See Figure 2.4), from where the usual expansion phase starts; it bounces. Due to the singularity theorems, this can only be achieved either by modifying matter or the dynamics of gravity. For a pedagogical review, see [13].

The class of theories that suppose new kinds of matter or fields but maintain classical GR are called classical bounces. As we have seen, such models do not satisfy the weak or the null energy condition. Due to this, they generally exhibit unstable behavior, which in most cases leads to singularities. A complete critical review can be found in [6].

Another option would be to modify GR itself. This may be achieved by considering classical modified theories of gravity - such as the $f(R)$ class - or by quantizing gravity itself. As mentioned in [6], most classical bounces have stability problems, so, in this section, we shall focus our attention to quantum bounces.

[15]It should be emphasized that bouncing models do solve the other problems, but are constructed to attack the singularity one.

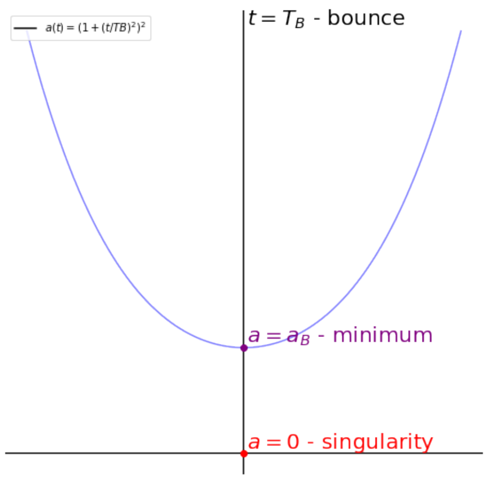

Figure 2.4: Example of a bouncing scale factor. It is explicitly form shall be derived later, when we analyze an example. The graph was made by the author using Python.

2.3.1 Qualitative Solution

As already mentioned in the inflationary case, the puzzles of the Standard Model are associated to the derivative of the Hubble radius R_H with respect to the scale factor a:

$$\frac{dR_H}{da} = \frac{1}{\dot{a}}\left(1 - \frac{a\ddot{a}}{\dot{a}^3}\right), \tag{2.55}$$

and the fact that $dR_H/da > 0$ in Standard Cosmology. While inflation solves this problem by considering an exponential expansion with $\ddot{a} > 0$, $\dot{a} > 0$, bouncing models consider contracting phases with $\ddot{a} > 0$, $\dot{a} < 0$, and obtain analogous results.

1. **Flatness Problem**

   As in the inflationary case, we have that

$$\frac{d}{da}\Delta(t) = -2\frac{\ddot{a}}{\dot{a}^3},$$

   and, as in the inflationary case, the bounce predicts such derivative to be negative, hence

$$\Delta(t) \to 0,$$

   and the flat universe becomes an attractor, and does not need ad hoc initial conditions to be achieved.

2. **Horizon Problem**

   In the bouncing paradigm, a pre-Big Bang contracting phase is considered (see Figure 2.5). Hence, the universe has existed for much more time than the 13.8 billions years predicted by the Standard Model. This means that regions of the sky that are in thermal equilibrium nowadays

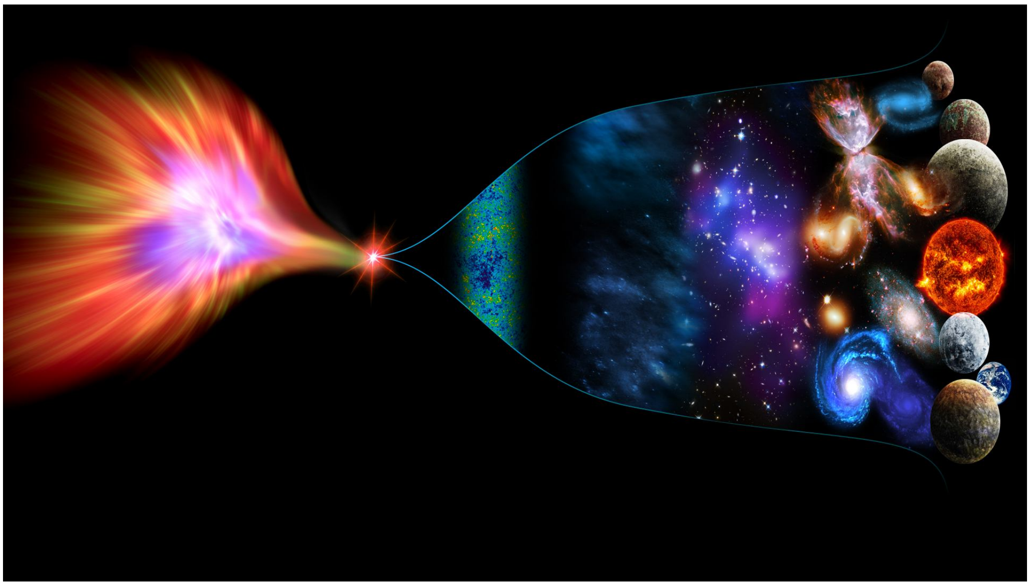

Figure 2.5: In Bouncing Models, a contracting pre-Big Bang phase is considered.

have had much more time to get into causal contact and achieve such state, which solves the horizon problem. This is also associated to the Origin of Perturbations solution.

3. **Origin of Perturbations**

   As already mentioned, the condition

$$\frac{dR_H}{da} < 0\,,$$

   is satisfied. This means that physical scales of interest become larger than the Hubble radius as the universe evolves. This leads to the same behavior as in inflation: scales of physical interest start inside the Hubble horizon, cross it where they stay "frozen", until they reenter the horizon to generate structure in the post bounce phase.

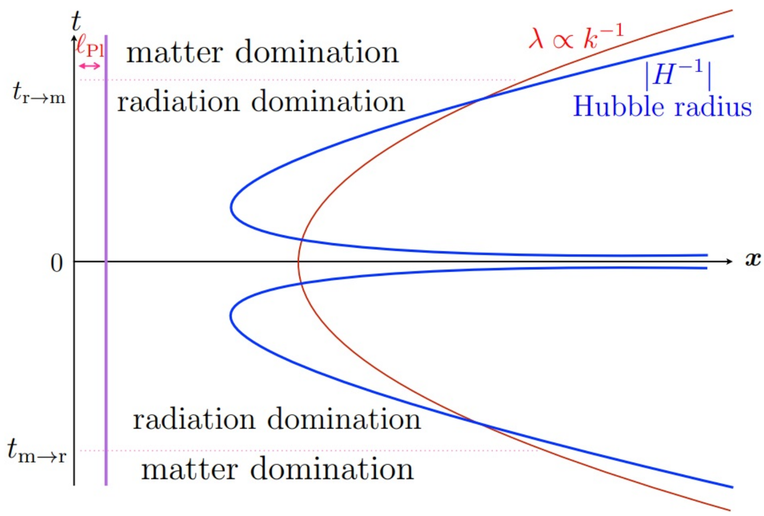

Figure 2.6: Evolution of the Hubble radius in a specific bouncing cosmology. Note that physical scales were not generated in the region of ignorance. Source: [14].

However, the picture is qualitatively different. Right in the bounce $t = T_B$, where the scale factor achieves the minimum value $a(T_B) = a_B$, we have that $\dot{a}(T_B) = 0$, which means that the Hubble radius diverges:

$$\lim_{t \to T_B} R_H = \lim_{t \to T_B} \frac{\dot{a}}{a} = \infty \,.$$

The evolution of the Hubble radius is illustrated in Figure 2.6. This means that, at the bounce, all scales of physical interest are inside the Hubble radius, which then begins to shrink after the bounce, allowing the perturbations to reenter the Hubble horizon and generate structure at later times[16].

As in inflation, the origin of perturbations is quantum vacuum fluctuations. However, instead of imposing vacuum boundary conditions in the bounce, one generally imposes such conditions in the far past (extreme left of Figure 2.4), when the universe presented a very low temperature and could be described by a vacuum state.

4. **Singularity Problem**

   Bouncing models solve the singularity problem by construction, for they substitue the primordial singularity with a bounce where the scale factor achieves a minimum value $a(T_B) = a_B$. This also enables one to make predictions for the universe at times earlier than 13.8 billion years ago, which is not possible in the Standard Model of Cosmology nor in Inflationary Cosmology.

2.3.2 Quantitative Example: Quantum Cosmology of a Single Fluid

As mentioned, since classical bounces have problems with instabilities and singularities, we shall consider an example of a quantum bounce, where General Relativity itself is quantized. In this section, we shall use Planck units for the sake of clarity.

Since there is yet no definitive theory of quantum gravity, there is no preferred quantization method. For this reason, in this section we shall focus on the canonical quantization of GR, which is the most conservative one in the sense that it just applies the usual quantization techniques of constrained systems to GR and does not modify the theory itself. However, bouncing cosmologies were also considered in Loop Quantum Gravity (Loop Quantum Cosmology)[17] and String Theory (String Gas Cosmology), for which we refer the reader to [14].

To use canonical methods, the first step is to obtain the Hamiltonian of the theory. Therefore, we assume space-time to be globally hyperbolic with topology $\mathcal{M} = \mathbb{R} \times \Sigma$ and foliate it using a normal vector n^a such that $n^a n_a = -1$. We now introduce a time vector field t^a defined by

$$t^a = N n^a + N_i \left(X^i\right)^a \,,$$

[16] To be more precise, the dynamic scale in bouncing models is not always given by R_H, but by another quantity, that depends on the type of perturbation. Such quantities are defined in a geometric way, so that they do not diverge on the bounce.

[17] More material can also be found in [17] and [6].

where $\{(X^i)^a\}$ denotes a set of 3 linearly independent spacelike vector fields parallel to Σ. Here, the $i = 1,2,3$ index labels the spatial vector, while a is an abstract notation index. Using a coordinate system with coordinate vector fields $\{t^a, (X^i)^a\}$, the metric is then given by

$$\mathrm{d}s^2 = \left(-N^2 + N_i N^i\right)\mathrm{d}t^2 + 2N_i \mathrm{d}x^i \mathrm{d}t + h_{ij}\mathrm{d}x^i \mathrm{d}x^j \,. \tag{2.56}$$

Here, N and N_i are the lapse and shift function, respectively. They correspond to Lagrange multipliers and enforce constraints on the theory, while the true dynamical variable is the spatial metric h_{ij} on Σ_t.

By direct substitution of (2.56) in the Einstein-Hilbert Lagrangian and performing a Legendre transform[18] after removing boundary terms, one obtains the total Hamiltonian [53]

$$H_T = \int \left(N H_0 + N_i H^i\right) d^3x \,, \tag{2.57}$$

where Π^{ij} is the momentum canonically conjugate to h_{ij} and

$$H_0 \equiv G_{ijkl}\Pi^{ij}\Pi^{kl} + \sqrt{h}\,{}^{(3)}R \,, \tag{2.58a}$$

$$H^i \equiv -2\partial_j \Pi^{ij} \,, \tag{2.58b}$$

and

$$G_{ijkl} \equiv \frac{1}{2}\sqrt{h}\left(h_{ik}h_{jl} + h_{il}h_j - h_{ij}h_{kl}\right) \,,$$

is known as the Wheeler-DeWitt metric.

Variation of the total Hamiltonian (2.57) with respect to the lapse N and shift functions N_i then leads to the constraints

$$H_0 = 0 \,, \tag{2.59a}$$

$$H^i = 0 \,. \tag{2.59b}$$

Here, the second constraint can be eliminated by a redefinition of our phase space[19] [29], but the same does not apply to the first. Hence, the effective Hamiltonian is given by $H_T = \int N H_0 d^3x$, which is constrained due to the lapse N, which acts as a Lagrange multiplier.

We now perform canonical quantization using such Hamiltonian. To do so, we begin by promoting our variables to operators that satisfy the canonical commutation relation

$$h_{ij} \to \hat{h}_{ij} \,, \tag{2.60a}$$

$$\Pi^{ij} \to \hat{\Pi}^{ij} \,, \tag{2.60b}$$

[18] Since GR is a constrained theory, one should be careful when taking Legendre transforms and analyzing its phase space. This happens even before quantization.

[19] More precisely, one takes the phase space of the theory to be the space of equivalence classes of riemannian spatial metrics that differ by a diffeomorphism, which is known as superspace.

$$[\hat{h}_{ij}, \hat{\Pi}^{kl}] = i\delta_i{}^k\delta_j{}^l\hat{\mathbb{1}}\,. \tag{2.60c}$$

Now, following the Dirac quantization procedure [53], constraints $\Phi(q^i, p_i) = 0$ on the classical variables are promoted to operatorial identities on states $|u\rangle$, which constrains the Hilbert space:

$$\Phi(\hat{q}^i, \hat{p}_i)\,|u\rangle = 0\,,$$

implementation of the Hamiltonian constraint (2.58) then leads to the Wheeler-DeWitt equation

$$\hat{H}_0\,|u\rangle = 0\,, \tag{2.61}$$

where $|u\rangle$ is the quantum state of our gravitational system. If we choose the representation

$$\Psi_u[h_{ij}(x^\mu)] \equiv \langle h_{ij}(x^{\mu)}|u\rangle\;, \tag{2.62a}$$

$$\hat{h}_{ij}(x^\mu)\Psi_u[h_{mn}(x^\mu)] = h_{ij}(x^\mu)\Psi_u[h_{ij}(x^\mu)]\,, \tag{2.62b}$$

$$\hat{\Pi}_{ij}(x^\mu)\Psi_u[h_{mn}(x^\mu)] = -i\frac{\delta}{\delta h_{ij}}\Psi_u[h_{mn}(x^\mu)]\,, \tag{2.62c}$$

where $\delta/\delta h_{ij}$ denotes a functional derivative with respect to the spatial metric h_{ij} the Wheeler-DeWitt equation reads, explicitly[20] [54]

$$\left(-\hat{G}_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} - \sqrt{\hat{h}}\hat{R}\right)\Psi[h_{ij}(x^{\mu)}] = 0\,, \tag{2.63}$$

which is just the quantum version of the constraint (2.59). This equation is just a formal one, for its solutions are given in superspace, the space of all three-metrics on the hypersurfaces Σ[21]. Such space is very poorly understood, due to lots of problems: it is infinite dimensional, and the signature of the Wheeler-DeWitt metric G_{ijkl} makes it a tough task to define a Cauchy Problem in such space.

There is yet another problem: due to the Wheeler-DeWitt equation, if one tries to write a usual Schrödinger equation to study the dynamics of a quantum state $|u\rangle$

$$\hat{H}_0\,|u\rangle = i\frac{\partial}{\partial\tau}\,|u\rangle\;,$$

where τ would be a time parameter, one immediately sees that, $\partial\,|u\rangle\,/\partial\tau = 0$: there is no time evolution due to the constraint (2.61). This happens because, classically, one has the freedom to choose an arbitrary time parameter which, after quantization, translates that the Hamiltonian gives a trivial time evolution [54].

Even if one puts such problems aside, there are other puzzles related to solutions of the Wheeler-

[20] Since both $\hat{h}_{ij}$ and $\hat{\Pi}_{ij}$ are operators, there are ordering ambiguities. Here, we choose the trivial order, for the sake of illustration.

[21] More formally, it is the space of equivalence classes $[h_{ij}]$ of spatial metrics that differ only by a spatial diffeomorfism. This definition also "solves" the second constraint in (2.59). Such consideration makes it hard to even define a notion of functional derivative, since one needs to consider a derivative with respect to a class of equivalence of metrics, $\delta/\delta[h_{ij}]$.

DeWitt equation: they are wavefunctionals of the metric[22], $\Psi[h_{ij}(x^\mu)]$, and it is not clear how to interpret them. In the case of cosmology, for instance, the usual Copenhagen interpretation does not apply, for there is no classical observer to collapse the wavefunction of the universe [55]. This means that, when constructing a theory of quantum cosmology, one also needs to propose an interpretation of the state of the universe, or a protocol for which predictions may be derived [54][53].

As one shall see, just like in any attempt to quantize gravity, canonical quantum gravity has severe problems. However, one may try to deal with them by making additional hypothesis. Observations then shall test if those are reasonable assumptions. Even if they fail, at least one would probe theories of quantum gravity, which may help to construct a satisfactory theory in the future.

To deal with the problem of the superspace in the Wheeler-DeWitt equation, we shall take a **mini-superspace quantization**. That is, instead of quantizing the space of all riemannian geometries (Σ, h_{ij}), we shall consider only the space of geometries with a certain symmetry[23]. In the cosmological case, this would be the class of homogeneous and isotropic cosmologies. This greatly simplifies calculations, for now one does not quantize infinite degrees of freedom, but only one: the scale factor $a(t)$. This solves the problem of representing the operators, and also degenerates the functional derivatives to partial derivatives.

The fact that the Hamiltonian vanishes due to the time constraint is known as the **Problem of Time** in quantum gravity. To circunvent this problem and define a non-trivial propagator, we shall use an intrinsic variable of the system whose classical evolution is monotonic. Then, we shall demand that the classical notion of time is recovered from such variable in the appropriate classical limit.

As for the problem of interpreting the wavefunction, we shall adopt the **Bohmian interpretation of quantum mechanics**. In such interpretation, there is no wavefunction collapse and quantum mechanics becomes deterministic, but with non-local effects [55]. Such interpretation consists on parametrizing the wavefunction by

$$\Psi = Ae^{iS},$$

where A, S are both functions. Direct substitution of such form in the Schrödinger equation leads to the usual conservation of probability density, and to a modification of the Hamilton-Jacobi equation with an extra quantum potential [1]. The deterministic Bohmian trajectories then are obtained by identifying the S function with the classical action. This then leads to the guidance relations

$$p_i = \frac{\partial S}{\partial q_i}, \tag{2.64}$$

which are differential equations that give the evolution of the modified trajectories $q_i(t)$. An interesting introduction to the Bohmian interpretation and its application to quantum cosmology can be found in [55]. Of course, one could also consider other approaches, such as Many Worlds interpretation, or

[22] In Standard Quantum mechanics, the position $x(t)$ and momentum $p(t)$ become operators $\hat{x}, \hat{p}$, while the time t is still a parameter. When quantizying GR, the spatial metric $h_{ij}(x^\mu)$ and its momentum $\Pi^{ij}(x^\mu)$ become operators $\hat{h}_{ij}, \hat{\Pi}^{ij}$, while the coordinates x^μ are treated as parameters.

[23] Here, a word of caution is in order. In the classical case, such symmetries are preserved by the Cauchy problem of General Relativity. However, there is no guarantee that this will hold in the final theory of Quantum Gravity.

Consistent Histories Approach, which are also discussed in [55].

So, to do a brief recap, we shall consider the following hypothesis:

1. **Mini-superspace** : to deal with the mathematical problems of the Wheeler-DeWitt equation;
2. **Internal time** : to deal with the problem of time in quantum gravity;
3. **Bohmian mechanics** : to deal with the problem of interpreting the wavefunction of the universe;

We shall now apply such considerations to a flat homogeneous and isotropic universe that contains a single perfect fluid with equation of state (1.30) and constant w. Imposing such conditions on the metric, we have that $N_i = 0$ due to isotropy, and $N = N(t)$, $h_{ij} = a^2(t)\delta_{ij}$ due to homogeneity and flatness.The ADM metric (2.56) then simplifies to

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,, \tag{2.65}$$

since the spatial sections satisfy $\Sigma = \mathbb{R}^3$, we have that ${}^{(3)}R = 0$. We now couple geometry to a single perfect fluid[24], by adding the following matter Hamiltonian

$$H_M = N\left(c\frac{p_\phi^{1+w}}{a^{3w}}\right),$$

where p_ϕ is the momentum conjugate to ϕ, and $c \equiv 1/\left(w\sqrt{2}^{1+3w}n^{1+w}\right)$ is a constant, for one assumes that the fluid satisfies the equation of state $p(\rho) = w\rho$, and $n = \frac{1}{2}(1 + 1/w)$.

We also introduce the following canonical pair of variables, in terms of a canonical transformation:

$$T \equiv \frac{1}{c(1+w)}\frac{\phi}{p_\phi^w},$$
$$\Pi_T \equiv c p_\phi^{1+w}.$$

The total Hamiltonian becomes, explicitly:

$$H_T = H_0 + H_M\,, \tag{2.66}$$
$$H_T = N\left(-\frac{\Pi_a^2}{4a} + \frac{\Pi_T}{a^{3w}}\right), \tag{2.67}$$

where

$$\Pi_a \equiv \frac{\partial L}{\partial \dot{a}}\,, \tag{2.68}$$

[24] To couple the fluid to the metric in the Hamiltonian formalism of GR, one first needs an action principle to describe fluids. This is explored in [1] which presents a method to describe a perfect fluid in terms of a scalar field ϕ.

is the canonical moment associated to the scale factor. Note that Π_T appears linearly in the Hamiltonian, and that its classical equation of motion is given by

$$\dot{T} = \frac{N}{a^{3w}} ,$$

so that $\dot{T} > 0$ which means that T is a monotonic function of classical time. Due to such reasons, we can consider T to be a time variable, and write a Schrödinger equation using it as a parameter. Here, an important note is in order: although the variable T is not connected to our intuitive notion of time, one should bear in mind that time is essentially an always increasing quantity that measures successions of events[25]. Hence, this variable fills the same role, although it does not cope with our intuition[26].

Performing canonical quantization, we begin by promoting the classical variables to operators that satisfy the canonical commutation relations:

$$[\hat{a}, \hat{\Pi}_a] = i\hat{\mathbb{1}} ,$$
$$[\hat{T}, \hat{\Pi}_T] = i\hat{\mathbb{1}} ,$$

we then follow by choosing the representation[27]

$$\hat{a}\Psi(a,T) = a\Psi(a,T), \qquad \hat{\Pi}_a\Psi(a,T) = -i\frac{\partial}{\partial a}\Psi(a,T) ,$$
$$\hat{T}\Psi(a,T) = T\Psi(a,T), \qquad \hat{\Pi}_T\Psi(a,T) = -i\frac{\partial}{\partial T}\Psi(a,T) ,$$

and we obtain the following Wheeler-DeWitt equation for the wavefunction of the universe $\Psi(a,T)$:

$$i\frac{\partial}{\partial T}\Psi(a,T) = \frac{1}{4}\left\{a^{(3w-1)}\frac{\partial}{\partial a}\left[a^{(3w-1)}\frac{\partial}{\partial a}\right]\right\}\Psi(a,T) , \tag{2.69}$$

where a specific operator factor ordering was chosen[28]. By performing the change of variable

$$q \equiv \frac{2a^{\frac{3}{2}(1-w)}}{3(1-w)} , \tag{2.70}$$

we obtain a simpler form

$$i\frac{\partial}{\partial T}\Psi(q,T) = \frac{1}{4}\frac{\partial^2}{\partial q^2}\Psi(q,T) , \tag{2.71}$$

which has the form of a time reversed free particle Schrödinger equation.

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[25] Or, more precisely, coincidences of events. For instance, if one says that an apple fell at 9 AM, then it just means that the apple fell exactly when the pointers of the clock pointed to 9 AM.

[26] It should also be pointed out that this is not the only choice of time, since the Problem of Time is not yet solved.

[27] Here, it is important to emphasize that, while we do choose this representation, T is treated as *parameter*, and *not an observable*. This means that the wavefunction will be normalized with respect to a, but not with respect to T, which will define the unitary evolution.

[28] The choice was based on preserving the symmetries of the classical system in the quantum one.

Here an important observation should be made. Since $a > 0$ at all times, the Hilbert Space associated to this problem is the space of square integrable functions defined on the semi line $\{x \in \mathbb{R} | x > 0\}$. To ensure that the operator $\hat{\Pi}_a$ is Hermitian when acting in such Hilbert Space, the condition

$$\left(\Psi^* \frac{\partial \Psi}{\partial q} - \Psi \frac{\partial \Psi^*}{\partial q} \right) \Bigg|_q = 0 \,,$$

must be satisfied[29]. Such condition is fulfilled by the gaussian state

$$\Psi_0(q, 0) = \left(\frac{8}{\pi \tau_B} \right)^{1/4} \exp\left\{ -\frac{q^2}{\tau_B} \right\} \,, \tag{2.72}$$

which has the advantage that its functional form is preserved by the associated propagator. Therefore, we shall take (2.72) as our initial condition. Applying the propagator, the wave function at an arbitrary instant T is given by:

$$\Psi(a, T) = \left[\frac{8 T_{\mathrm{B}}}{\pi \left(T^2 + T_{\mathrm{B}}^2 \right)} \right]^{1/4} \exp\left[-\frac{4}{9} \frac{T_{\mathrm{B}} a^{3(1-\omega)}}{\left(T^2 + T_b^2 \right) (1-\omega)^2} \right] \tag{2.73}$$

$$\times \exp\left\{ -i \left[-\frac{4}{9} \frac{T_{\mathrm{B}} a^{3(1-\omega)}}{\left(T^2 + T_B^2 \right) (1-\omega)^2} + \frac{1}{2} \arctan\left(\frac{T_{\mathrm{B}}}{T} \right) - \frac{\pi}{4} \right] \right\} . \tag{2.74}$$

Using the Bohmian interpretation of quantum mechanics, one can extract trajectories from the phase of the wavefunction. To do so, one needs to solve the guidance relation (2.64)

$$\frac{da}{dT} = -\frac{a^{(3w-1)}}{2} \frac{\partial S}{\partial a} \,, \tag{2.75}$$

which was obtained by choosing the gauge $N = a^{3w} \implies \dot{T} = 1$. The solution is the associated scale factor

$$a(t) = a_B \left[1 + \left(\frac{T}{T_B} \right)^2 \right]^{\frac{1}{3} \frac{1}{(1-w)}} \,, \tag{2.76}$$

where a_B appears as an integration constant and represents the minimum value for the scale factor. Note that $a(t) \neq 0$ for all values of t. Therefore, this model is non-singular and represents an eternal universe.

The integration constant a_B is a free parameter of the theory and must be fixed by observations. It also fixes the energy scale of the bounce, which is usually assumed to be higher than the Planck energy, where more complex quantization techniques may be needed. In comparison to scalar field inflation, this theory has less freedom in the following sense. In the former case one has the freedom to select an arbitrary function $V(\phi)$ of the field[30], while there is only an indeterminate constant a_B in the latter. However, this model evokes quantum gravity and other assumptions, so that it deals with

[29] Such condition follows from discarding a boundary term when analyzing the hermitean condition $\langle u | \hat{\Pi}_q v \rangle = \langle u \hat{\Pi}_q^{\dagger} | v \rangle$.

[30] To be more precise, in slow roll inflation, the predictions depend only on the slow roll parameters ϵ, δ, and not on the specific form of the potential [47].

physics that are poorly understood in comparison to inflation, which is based on usual Quantum Field Theory applied to a scalar field.

We conclude this example by pointing that, as in inflation, one may consider perturbations around the background and derive the spectral power index of such perturbations. For a fluid with equation of state $p(\rho) = wp$, it is given by [1]

$$n_s = 1 + \frac{12w}{1 + 3w} , \tag{2.77}$$

and one can see that an nearly scale invariant spectrum is obtained in the limit $w \to 0$, that is, an approximate dust equation of state. To obtain such power spectrum, one imposes boundary conditions in the distant past, when the physical scales of the universe were much larger. In such regime, the universe had a very low temperature, which enables one to consider a vacuum state as the initial state for perturbations[31]. Again, we see that quantum cosmological perturbations[32] demand an initial vacuum state to be described and to obtain their associated two point functions.

Now, a brief recap: bouncing cosmologies solve the fine tuning problems of the cosmological standard model by assuming a contracting period for the universe, which also leads to a causal explanation and predictions for the structure formation and the CMB power spectrum. It also solves the singularity problem by construction but, while inflation usually demands one to consider unobserved fields, bouncing cosmologies need to consider exotic types of matter and/or new Physics such as quantum gravity, which gives one a lot of freedom.

2.4 Cosmological Perturbation Theory

In this section, we develop the theory of cosmological perturbations based on the usual Bardeen approach [56]. As we have seen, one needs to consider perturbations around a homogeneous and isotropic universe to explain the growth of large scale structure and the temperature fluctuations of the CMB. However, Standard Cosmology cannot appropriately do so due to the before mentioned Problem of Origin of Perturbations (subsection 1.4.3). Hence, we choose to first develop the inflationary and bouncing paradigms before perturbation theory. Then, by combining such paradigms with perturbations, one is can make definite predictions about the CMB power spectrum.

We start by formally defining what we mean by perturbations and also their gauge invariance. To do so, first remember that a spacetime is a manifold $\mathcal{M}$ with a pseudo-riemannian metric g_{ab} defined on it. When we consider perturbations, we fix the manifold, but change the metric so that we are dealing with two space-times: the "true" physical space-time $(\mathcal{M}, g_{ab})$, and we compare it with a fiducial unperturbed space-time $(\mathcal{M}, \bar{g}_{ab})$. The perturbations δg_{ab} are then defined by

$$\delta g_{ab} \equiv g_{ab} - \bar{g}_{ab} ,$$

[31] In this paradigm, quantum vacuum fluctuations emerged in the distant past, and all scales were shrank to such small scales.

[32] It is also interesting to point out that the presented bohmian interpretation also explains why we see a classical universe nowadays while the perturbations started in the quantum regime, which is an unsolved problem in inflation [1].

and can effectively be treated as a rank 2 tensor field defined on the space-time $(\mathcal{M}, g_{ab})$. We also assume that the perturbations are small $\delta g/g \ll 1$, which is more formally explored in [57]. Note that, since we introduced the fiducial spacetime $(\mathcal{M}, \bar{g}_{ab})$, our definition of perturbation δg_{ab} contains an intrinsic arbitrariness, which is discussed below.

Since the manifold $\mathcal{M}$ is the same for both spacetimes, they should preserve their structure under a diffeomorphism. However, note that there is a difference in applying such diffeomorphism on the physical spacetime and on the fiducial one. In particular, consider two tensors $\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}$ and $T^{a_1\ldots a_k}{}_{b_1\ldots b_l}$, defined on the fiducial spacetime and the physical one, respectively. Then, by applying a local, infinitesimal diffeomorphism parametrized by a vector field $\bar{\xi}^a$ on the background, such tensors transform as

$$\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} \rightarrow \bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \mathscr{L}_{\bar{\xi}}\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}\,, \tag{2.78a}$$

$$T^{a_1\ldots a_k}{}_{b_1\ldots b_l} \rightarrow T^{a_1\ldots a_k}{}_{b_1\ldots b_l}\,, \tag{2.78b}$$

where we considered only the first contribution of the expansion of $\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}$. From here on, we will consider only the first order contributions. Note that, since the diffeomorphism was applied on the background, the physical tensor does not transform. If, however, one applies a small general coordinate transformation generated by a vector field ξ on the physical spacetime $(\mathcal{M}, g_{ab})$, one gets

$$\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} \rightarrow \bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \mathscr{L}_{\xi}\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}\,, \tag{2.79a}$$

$$T^{a_1\ldots a_k}{}_{b_1\ldots b_l} \rightarrow T^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \mathscr{L}_{\xi}T^{a_1\ldots a_k}{}_{b_1\ldots b_l}\,, \tag{2.79b}$$

so that both tensors are transformed. Then, one can also combine both transformations to yield:

$$\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} \rightarrow \bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \mathscr{L}_{\bar{\xi}+\xi}\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}\,, \tag{2.80a}$$

$$T^{a_1\ldots a_k}{}_{b_1\ldots b_l} \rightarrow T^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \mathscr{L}_{\xi}T^{a_1\ldots a_k}{}_{b_1\ldots b_l}\,. \tag{2.80b}$$

Now, consider a perturbation on such tensor, $\delta T^{a_1\ldots a_k}{}_{b_1\ldots b_l} \equiv T^{a_1\ldots a_k}{}_{b_1\ldots b_l} - \bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}$. The perturbation then transforms as

$$\delta T^{a_1\ldots a_k}{}_{b_1\ldots b_l} \rightarrow \left(T^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \mathscr{L}_{\xi}T^{a_1\ldots a_k}{}_{b_1\ldots b_l}\right) - \left(\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \mathscr{L}_{\bar{\xi}+\xi}\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}\right)\,, \tag{2.81}$$

$$\approx \left(T^{a_1\ldots a_k}{}_{b_1\ldots b_l} - \bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}\right) - \mathscr{L}_{\bar{\xi}}\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}\,, \tag{2.82}$$

$$\approx \delta T^{a_1\ldots a_k}{}_{b_1\ldots b_l} - \mathscr{L}_{\bar{\xi}}\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}\,. \tag{2.83}$$

Here, it is interesting to note that the same result could be obtained by considering $\bar{\xi} = -\xi$ directly from the beginning, yielding:

$$\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} \rightarrow \bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}\,,$$

$$\delta T^{a_1\ldots a_k}{}_{b_1\ldots b_l} \rightarrow \delta T^{a_1\ldots a_k}{}_{b_1\ldots b_l} - \mathscr{L}_{\bar{\xi}}\bar{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}\,.$$

This is a very convenient choice, for the background quantity maintains its funcional form when such transformation is applied. Hence, we *define* a gauge transformation as the combination (diffeomorphism $\oplus$ coordinate) for which $\xi = -\bar{\xi}$. This transformation has different effects on the background and physical objects, and is identified with an approximate symmetry group induced by the presence of an extra strucutre, namely the fiducial spacetime $(\mathcal{M}, \bar{g}_{ab})$ which is compared to the physical spacetime [58].

We now focus on perturbations on the metric tensor g_{ab}. Due to (2.81), it transforms as

$$g_{ab} \to g_{ab} + \mathscr{L}_\xi g_{ab} \,. \tag{2.85}$$

One then sees that the perturbations on the metric are equivalent up to a Lie derivative. Hence, one must define gauge invariant objects to study observables. Otherwise, one may find objects that are coordinate/gauge dependent.

We now avail the explicit gauge transformation for g_{ab} in the case of a FLRW background. Since the diffeomorphism is parametrized by ξ^a, the μ-th coordinate transforms as

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu \,. \tag{2.86}$$

Using $g_{ab} = \bar{g}_{ab} + \delta g_{ab}$ for the case of $\bar{g}_{ab} = g_{ab}^{\text{FLRW}}$, the perturbed line element in conformal time can be expressed by

$$\mathrm{d}s^2 = a^2(\eta)\left[(-(1+2A)\mathrm{d}\eta^2 + 2B_i \mathrm{d}\eta \mathrm{d}x^i + (\gamma_{ij} + h_{ij})\,\mathrm{d}x^i \mathrm{d}x^j\right] \tag{2.87}$$

where due to the background isotropy, we parametrized the metric using

$$A \equiv -\frac{1}{2a^2}\delta g_{00} \,, \tag{2.88a}$$

$$B_i \equiv \frac{1}{2a^2}\delta g_{0i} \tag{2.88b}$$

$$h_{ij} \equiv \frac{1}{a^2}\delta g_{ij} \,, \tag{2.88c}$$

which transform as a scalar, vector and rank 2 tensor under spatial rotations, respectively. This parametrization is useful because the variables are defined in terms of representations of the background rotation group. However, this will not be the case for an anisotropic background.

We now focus on the flat FLRW case, for which $\gamma_{ij} = \delta_{ij}$. Generally, one imposes that perturbations decay at infinity. With such condition, one can parametrize the perturbation modes as

$$B_i = \partial_i B + \bar{B}_i \,, \tag{2.89a}$$

$$h_{ij} = 2C\delta_{ij} + 2\partial_i\partial_j E + 2\partial_{(i}E_{j)} + 2E_{ij} \,, \tag{2.89b}$$

if the following constraints are satisfied [58]

$$\partial^i \bar{B}_i = 0 \,, \tag{2.90a}$$

$$\partial^i E_i = 0\,, \tag{2.90b}$$

$$\partial^i E_{ij} = 0\,. \tag{2.90c}$$

With such parametrization, we have the following types of SVT (scalar-vector-tensor) modes:

1. Scalar Modes: A, B, C, E ;
2. Vector Modes: $\bar{B}_i, E_i$;
3. Tensor Modes: E_{ij} ;

and each type of mode carries:

1. Scalar Modes: 1 degree of freedom per mode, with a total of 4 (4 modes);
2. Vector Modes: 2 degrees of freedom per mode (they are divergence free), with a total of 4 (2 modes);
3. Tensor Modes: 2 degrees of freedom per mode (divergence free and null trace), with a total of 2 (1 mode).

So, we have a total of 10 degrees of freedom, which shows that this parametrization is indeed general. However, due to the gauge invariance (2.85), not all modes are physical: some of them are coordinate dependent. We now proceed to construct the true degrees of freedom of our system: those that do not depend on the gauge.

We begin by considering an infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$. We then see that the components $h_{\mu\nu}$ are not gauge independent. Using the Lie derivative $\mathscr{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$, and decomposing the components ξ^μ as

$$\xi^0 = T(\vec{x}, \eta)\,, \tag{2.91a}$$

$$\xi^i = \partial^i L(\vec{x}, \eta) + L^i(\vec{x}, \eta)\,, \tag{2.91b}$$

we obtain the following set of transformations for each mode:

1. Scalar Modes

$$A \to \tilde{A} = A + \frac{(aT)'}{a}\,, \tag{2.92a}$$

$$B \to \tilde{B} = B - T + L'\,, \tag{2.92b}$$

$$C \to \tilde{C} = C + \mathcal{H}T\,, \tag{2.92c}$$

$$E \to \tilde{E} = E + L\,, \tag{2.92d}$$

$$\tag{2.92e}$$

2. Vector Modes

$$\bar{B}^i \to \tilde{\bar{B}}^i = \bar{B}^i + L^{i\prime}\,, \tag{2.93a}$$

$$E^i \to \tilde{E}^i = E^i + L^i\,, \tag{2.93b}$$

3. Tensor Mode

$$E_{ij} \to \tilde{E}_{ij} = E_{ij}\,, \tag{2.94}$$

and we see that, apart from the unique tensor mode, the perturbation modes are gauge dependent.

At this point, one can see that, by appropriate linear combinations of the SVT modes, one may define quantities that do not transform when a gauge transformation is applied. Hence, from the SVT modes, we define gauge invariant variables known as the **Bardeen Variables** as[33]

1. Scalar Modes

$$\Phi(\vec{x},\eta) \equiv A + \frac{1}{a}\left[a\left(B - E\right)\right]'\,, \tag{2.95a}$$

$$\Psi(\vec{x},\eta) \equiv -C - \mathcal{H}\left(B - E'\right)\,, \tag{2.95b}$$

2. Vector Modes

$$\Phi^i(\vec{x},\eta) \equiv \bar{B}^i - E^{i\prime}\,. \tag{2.96}$$

Such variables are gauge independent. In practice, one selects a particular gauge and performs calculations on it. However, one can always transform between gauges by the use of the above transformations. A gauge is then defined by imposing conditions on the SVT modes or on the matter variables, and some common options are:

1. **Newtonian Gauge**

This gauge is implemented by the following choice of vector field ξ in terms of the modes,

$$T = B - E', \qquad L = -E, \qquad L^{i\prime} = -\bar{B}^i\,, \tag{2.97}$$

which implies the following constraints for the transformed modes

$$B \to B = 0, \qquad E \to E = 0, \qquad B^i \to B^i = 0\,, \tag{2.98}$$

so that the line element takes the form

$$ds^2 = a^2(\eta)\left\{(-(1+2A)d\eta^2 + \left[(1+2C)\,\delta_{ij} + 2\partial_{(i}E_{j)} + 2E_{ij}\right]dx^i dx^j\right\}\,, \tag{2.99}$$

[33] There is no need to introduce a Bardeen Variable for the tensor mode, which is already gauge independent by construction.

and the Bardeen Variables are simplified to

$$\Phi = A, \qquad \Psi = -C, \qquad \Phi^i = E^{i\prime} \qquad ; \tag{2.100}$$

2. **Synchronous Gauge**

This gauge is implemented by the following choice of vector field ξ in terms of the modes,

$$(aT)' = -aA, \qquad L' = T - B, \qquad , \tag{2.101}$$

which implies the following constraints for the transformed modes

$$A \to A = 0, \qquad B \to B = 0, \qquad \bar{B}^i \to \bar{B}^i = 0, \tag{2.102}$$

so that the line element takes the form

$$ds^2 = a^2(\eta)\left\{-d\eta^2 + \left[(1+2C)\,\delta_{ij} + 2\partial_i\partial_j E + 2\partial_{(i}E_{j)} + 2E_{ij}\right]dx^i dx^j\right\}, \tag{2.103}$$

and the Bardeen Variables are simplified to

$$\Phi = -\frac{(aE)'}{a}, \qquad \Psi = -C + \mathcal{H}E', \qquad \Phi^i = -E^{i\prime} \qquad ; \tag{2.104}$$

It is important to emphasize that, although the Bardeen Variables assume a definite form in each gauge, they do not depend on the gauge itself. After transforming from one to another, they retain their exact form, as can be shown by explicitly evaluating their transformation law using the relations (2.92) and (2.93).

Now that we have defined gauge invariant quantities, we can proceed to analyze dynamics using the perturbed Einstein equations, which we shall do using the Newtonian Gauge[34]. To simplify our discussion, we shall only consider the classical Einstein equations for the moment, and we shall couple the geometry to a homogeneous scalar field $\phi(t)$ with inflation in mind. Such field is then perturbed to

$$\phi(\eta) \to \phi(\eta) + \delta\phi(\vec{x},\eta),$$

from which we introduce yet another gauge invariant quantity

$$\chi \equiv \delta\phi + (B - E')\,\phi',$$

which is also gauge invariant. After a lengthy calculation, we get the following tensor components:

[34] We choose this gauge instead of the synchronous because, while the newtonian gauge is completely determined, one still has the freedom in introducing an arbitrary function in the synchronous gauge. However, the newtonian gauge is not free of issues, since it is not a valid gauge in some bouncing models, as discussed in [57], but those shall not be of concern in this section.

1. Einstein Tensor

$$a^2\delta G^0_{\ 0} = 6\mathcal{H}^2\Phi + 6\mathcal{H}\Psi' - 2\nabla^2\Psi\,, \tag{2.105a}$$

$$a^2\delta G^0_{\ i} = -2\partial_i\left(\Psi' + \mathcal{H}\Phi\right) + \frac{1}{2}\nabla^2\Phi_i\,, \tag{2.105b}$$

$$a^2\delta G^i_{\ j} = \partial^i\partial_j\left(\Psi - \Phi\right) + \delta^i_{\ j}\left[2\Psi'' + \left(2\mathcal{H}^2 + 4\mathcal{H}'\right)\Phi - \nabla^2\left(\Psi - \Phi\right) + 4\mathcal{H}\Psi' + 2\mathcal{H}\Phi'\right]$$
$$+ \delta^{ik}\partial_{(k}\Phi'_{j)} + 2\mathcal{H}\delta^{ik}\partial_{(k}\Phi_{j)} + E^i_{\ j}{}'' + 2\mathcal{H}E^i_{\ j}{}' - \nabla^2 E^i_{\ j}\,; \tag{2.105c}$$

2. Stress Energy-Tensor

$$a^2\delta T^0_{\ 0} = -\phi'\chi' - a^2 V_{,\phi}\chi + \Phi\phi'^2\,, \tag{2.106a}$$

$$a^2\delta T^0_{\ i} = -\partial_i\left(\phi'\chi\right)\,, \tag{2.106b}$$

$$a^2\delta T^i_{\ j} = -\left(\phi'^2\Phi + a^2 V_{,\phi}\chi - \phi'\chi'\right)\delta^i_{\ j}\,. \tag{2.106c}$$

Now, one can write the Einstein equations for the perturbations. For an isotropic background, the SVT modes decouple between themselves, that is: scalars are only coupled to scalars, vectors to vectors, and tensors to tensors.

Before analyzing the equations for each mode, note that the (i,j) component of $\delta T^i_{\ j}$ is diagonal, and depends only on scalars. The scalar mode contribution of the tensor $\delta G^i_{\ j}$ has only one non-diagonal part contribution, which is $\partial^i\partial_j\left(\Psi - \Phi\right)$. This means that we have a constraint

$$\partial^i\partial_j\left(\Psi - \Phi\right) = 0\,, \tag{2.107}$$

which, if combined with the boundary condition that the perturbations decay at infinity, forces the constraint $\Phi = \Psi$, which simplifies calculations. We now proceed to briefly discuss the dynamics of each type of mode separately.

1. **Scalar Modes**

   Scalar perturbations generate scalars in the late universe, such as temperature perturbations on the CMB and galaxies. Their equations of motion are given by

$$\nabla^2\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Phi\right) = \frac{\kappa}{2}\left(\phi'\chi' + a^2 V_{,\phi}\chi - \Phi\phi'^2\right)\,, \tag{2.108a}$$

$$\frac{\left(a\Phi\right)'}{a} = \frac{\kappa}{2}\phi'\chi\,, \tag{2.108b}$$

$$\Phi'' + 3\mathcal{H}\Phi' + \left(\mathcal{H}^2 + 2\mathcal{H}'\right)\Phi = \frac{\kappa}{2}\left(\phi'\chi' - \phi'^2\Phi - a^2 V_{,\phi}\chi\right)\,, \tag{2.108c}$$

   and one can see that (2.108b) is essentially a constraint between χ and Φ, so that we only have one degree of freedom: $\Phi(\vec{x}, t)$. We also have an additional equation, which is the perturbed

Klein-Gordon equation

$$\chi'' + 2\mathcal{H}\chi' - \nabla^2\chi + a^2 V_{,\phi\phi}\chi = 2\frac{(a^2\phi')'}{a^2}\Phi + 4\phi'\Phi' . \tag{2.109}$$

By combining (2.108a) with (2.108c), and utilizing the background Klein-Gordon equation in conformal time, one obtains a dynamical equation for the remaining degree of freedom:

$$\Phi'' + 2\left(\mathcal{H} - \frac{\phi''}{\phi}\right)\Phi' + \left[2\left(\mathcal{H}' - \mathcal{H}\frac{\phi''}{\phi}\right) - \nabla^2\right]\Phi = 0 . \tag{2.110}$$

2. **Vector Modes**

   Vector perturbations generate vectors in the late universe, such as velocity perturbations on the motion of galaxies. Their equations of motion are given by

$$\nabla^2\Phi_i = 0 , \tag{2.111a}$$

$$\Phi_i' + 2\mathcal{H}\Phi_i = 0 , \tag{2.111b}$$

   and, analyzing (2.111b), one sees that vector modes decay in an expanding universe. More precisely, direct integration leads to

$$\Phi\left(\vec{x}, \eta\right) = C_i\left(\vec{x}\right) a^{-2} ,$$

   where $C_i(\vec{x})$ is an integration function that depends on the chosen boundary conditions. This means that one needs to suppose large initial conditions for such modes to be detected nowadays. In fact, (2.111a) puts a much stronger constraint: the only solution compatible with the boundary condition $\lim\limits_{\vec{x}\to\infty} C_i(\vec{x}) = 0$ is the trivial solution $C_i(\vec{x}) = 0 \; \forall \; \vec{x}$, so that we shall not discuss vector modes any further[35].

3. **Tensor Modes**

   Tensor perturbations essentially generate tensors in the late universe, such as gravitational waves. Their equation of motion is given by

$$E_{ij}'' + 2\mathcal{H}E_{ij}' - \nabla E_{ij} = 0 , \tag{2.112}$$

   and it is usual to denote $E_{ij}(\vec{x}, \eta)$ as describing primordial gravitational waves.

The perturbative equations can then be solved using standard techniques, and the fields can be expanded in terms of a complete basis $\{v_k\}$ of functions. The next step in order to describe the spectrum of the CMB in terms of vacuum fluctuations is to quantize such perturbations. This could be achieved by applying canonical quantization to the fields expanded in the basis $\{v_k\}$ as shown in

[35] Of course, the same would not apply to bouncing universes.

subsection 2.2.2. However, to do so, we also need the canonically conjugated momentum of the field, which in turn can be obtained from the action.

At this point, one can try to guess the action in terms of the equations of motion. However, this would lead to an incorrect action, and hence a wrong momentum [9]. What one should do is the laborious calculation of expanding the Einstein-Hilbert action up to second order[36] in the perturbations. Only then one can properly identify the momentum and quantize the system [59].

It is also important to emphasize that, in inflationary models, one can just consider the classical Einstein-Hilbert action, and some terms vanish due to the classical equations of motion. This was analyzed in detail in [59]. However in quantum bouncing models, one cannot use the classical equations of motion, which makes such calculation even harder. This was also analyzed for FLRW backgrounds, and the main results can be found in [60].

To simplify our discussion, we skip the calculations and present the second order action for a classical FLRW metric coupled to a single scalar field. In future sections, we shall elaborate more on such action. Initially, we introduce the variables

$$v \equiv z\mathcal{R}\,, \tag{2.113a}$$
$$z \equiv \frac{a\phi'}{\mathcal{H}}\,, \tag{2.113b}$$
$$\mu_{ij} \equiv \sqrt{\frac{M_{pl}^2}{4}}aE_{ij}\,, \tag{2.113c}$$

which are known as the Mukhanov-Sasaki variables, and $\mathcal{R}$ is given by

$$\mathcal{R} \equiv \Phi + \frac{2\mathcal{H}}{\kappa\phi'^2}\left(\Phi' + \mathcal{H}\Phi\right)\,. \tag{2.114}$$

We also decompose the tensor μ_{ij} in terms of the $\times$ and $+$ polarizations of gravitational waves:

$$\mu_{ij} = \sum_{\lambda=\times,+}\mu_\lambda\varepsilon^\lambda_{ij}\,, \qquad \varepsilon^\lambda_{ij}\varepsilon^{ij}_{\lambda'} = \delta_{\lambda\lambda'}\,,$$

and the second order action becomes [9]

$$S^{(2)} = \frac{1}{2}\int\left\{\left[(v')^2 - \partial_i v\partial^i v + \frac{z''}{z}v^2\right] + \sum_{\lambda=\times,+}\left[(\mu'_\lambda)^2 - \partial_l\mu_\lambda\partial^l\mu_\lambda + \frac{a''}{a}\mu_\lambda^2\right]\right\}d^3\vec{x}d\eta\,. \tag{2.115}$$

Variation of (2.115) provides the following equations of motion in Fourier space

$$v''_k + \left(k^2 - \frac{z''}{z}\right)v_k = 0\,, \tag{2.116a}$$
$$\mu''_{\lambda,k} + \left(k^2 - \frac{a''}{a}\right)\mu_{\lambda,k} = 0\,. \tag{2.116b}$$

[36] This happens because the Euler-Lagrange equations lower the order of the Lagrangian by 1, so that, to get linear equations of motion, one needs a quadratic Lagrangian. E.g. : harmonic oscillator.

Such equations present the interesting property that they are decoupled, that is: there is no interaction between modes of different wavenumber, nor between scalar modes and tensor modes. However, this property will not hold in the anisotropic case.

The next logical step is to quantize such perturbations and compute its power spectrum to generate the CMB spectrum. However, as already mentioned, to do this, one needs to define a vacuum state for the field in order to impose vacuum boundary conditions.

In the previous case of a homogeneous field evolving in De Sitter space analyzed in subsection 2.2.2, one could define a vacuum by demanding that the analytic solutions recovered the usual plane waves of a Klein-Gordon field in Minkowski space. However, this cannot be done in the general case, for analytic solutions may not be available, or the small wavelength limit of the modes $\{v_k\}$ may be non-trivial, or even undefined.

Due to the aforementioned reasons, we delay the quantization of the perturbations for the moment. Before we do so, we shall analyze how to define a vacuum state in curved space-time, which demands knowledge of Quantum Field Theory in Curved Space-Time and is a very non-trivial task, for one does not have an unique prescription to define a vacuum state in general space-times. After we develop such formalism, we shall return to the problem of quantizing cosmological perturbations and compute its associated power spectrum.

3 QUANTUM FIELD THEORY IN CURVED SPACE-TIME

In this chapter, we discuss Quantum Field Theory in Curved Spacetime in order to properly treat the quantization of cosmological perturbations, which can be treated as quantum fields defined on an expanding background spacetime. We start by discussing the usual Canonical Quantization procedure in section 3.1, where we appoint the vacuum determination problem. We then proceed by analyzing the vacuum determination of quantum fields in Minkowski spacetime in section 3.2 and then FLRW spacetime in section 3.3, where we emphasize that there are multiple vacuum choices available for each case.

3.1 Canonical Quantization

As is well known, we have yet no definitive theory of quantum gravity. Quantum field theory in curved spacetime was developed as a response to such problem. It is characterized by considering the semiclassical approximation where all other fundamental fields are quantized, but the background gravitational field/spacetime geometry is classical [61], and is currently the most solid framework to treat (semiclassical) gravitational effects on quantum fields [62].

Such considerations were able to predict new effects that are expected from a theory of quantum gravity, such as cosmological particle creation, first considered by Parker (1969) [63], black hole radiation, also known as Hawking radiation, first considered by Hawking (1975) [64], and also the Fulling-Davies-Unruh effect, discovered by Fulling (1973), Davies (1975) and Unruh (1976) [65–67].

To develop our formalism, we consider the quantization of a free scalar field, which is described by the action

$$S = -\frac{1}{2}\int \sqrt{-g}\left(\nabla^a\phi\nabla_a\phi + m^2\phi^2\right) d^4x, \tag{3.1}$$

and satisfies the covariant Klein-Gordon equation

$$\Box\phi = 0\,,$$

where $\Box \equiv \nabla^a\nabla_a$.[1] To perform calculations, this equation can be represented in a coordinate system and assumes the form

$$\frac{1}{\sqrt{-g}}\partial^\mu\left(\sqrt{-g}\partial_\mu\phi\right) + m^2\phi = 0\,, \tag{3.2}$$

where g is the determinant of g_{ab}, and such expression follows by direct application of the operator $\nabla^\mu\nabla_\mu$.

One of the main problems of quantum field theory in curved spacetime is the vacuum determination problem, or representation problem. In order to understand it, it is interesting to consider the quantization of a simple quantum system: a harmonic oscillator. In Canonical Quantization, one

[1] We emphasize that the presented results can be generalized to other types of fields: vectors, spinors, tensors, etc.

usually takes the following steps [53]:

1. **Classical Dynamics**: write the classical Hamiltonian

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\,;$$

2. **Quantum Observables**: promote the canonical variables x, p to operators that will be the observables of the quantum theory, and map the classical Poisson brackets to commutators:

$$x, p \to \hat{x}, \hat{p}\,,$$
$$\{x, p\} \to \frac{1}{i\hbar}[\hat{x}, \hat{p}]\,,$$

which ensures that $[\hat{x}, \hat{p}] = i\hbar\hat{\mathbb{1}}$, an identity that is known as the canonical commutation relation;

3. **Representation**: the above algebra does *not* inform one how the operators act on quantum states $|u\rangle \in \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space, and hence how to calculate observables. Therefore, one now needs to define a representation for such algebra i.e. the way the operators act on physical states. In quantum mechanics one usually chooses the representation

$$\hat{x}\psi_u(x) = x\psi_u(x)\,,$$
$$\hat{p}\psi_u(x) = -i\hbar\frac{\partial}{\partial x}\psi_u(x)\,,$$

where $\psi_u(x) \equiv \langle x|u\rangle$ is the systems wavefunction and $|x\rangle$ is an eigenstate of the position operator $\hat{x}$. Hence, the observables are well defined as eigenvalues of the operators $\hat{x}, \hat{p}$ and one can extract physical predictions;

4. **Dynamics**: using the choosen representation, one can define dynamics either through the Schrödinger picture, where the state $|u\rangle \in \mathcal{H}$ evolves according to the Schrödinger Equation:

$$i\hbar\frac{\partial}{\partial t}|u\rangle = \hat{H}|u\rangle\,,$$

where $\hat{H} = \hat{H}(\hat{x}, \hat{p})$, or using the Heisenberg picture, where the operators $\hat{A}$ evolve according to the Heisenberg equation:

$$\frac{d\hat{A}}{dt} = \frac{1}{i\hbar}[\hat{A}, \hat{H}] + \frac{\partial\hat{A}}{dt}\,. \tag{3.3}$$

The above construction is the usual one taught at undergraduate level quantum mechanics. However, note that the quantum states $|u\rangle$ belong to a Hilbert space $\mathcal{H}$, which was never explicit in the presented construction. One then is lead to the question: how does one construct the Hilbert space associated to a quantum theory, starting from its classical counterpart?

The answer lies in one imporant theorem: the Stone Von-Neumann theorem [68]. It guarantees that, provided the classical theory has a **finite** number of degrees of freedom, and the quantum operators follow the canonical commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \mathbb{1}\,, \tag{3.4}$$

then **all representations of such algebra are equivalent up to a unitary transformation**. In other words, the choice of Hilbert space $\mathcal{H}$ is irrelevant: one can choose an arbitrary space for which (3.4) is valid.[2]

In the case of the simple harmonic oscilator, there is an interesting choice of Hilbert Space. Start by defining the usual creation and annihilation operators by

$$\hat{a} \equiv \frac{m\omega}{2\hbar}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right), \tag{3.5a}$$

$$\hat{a}^\dagger \equiv \frac{m\omega}{2\hbar}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right), \tag{3.5b}$$

which satisfy $[\hat{a}, \hat{a}^\dagger] = \mathbb{1}$ due to the canonical commutation relation (3.4). One can then show that, when such operators act on energy eigenstates $|n\rangle$, we get:

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{3.6a}$$

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{3.6b}$$

hence, an arbitrary energy eigenstate $|n\rangle$ can be constructed by sucessive applications of the creation operator $\hat{a}^\dagger$ on the vacuum state $|0\rangle$:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle\,, \tag{3.7}$$

and, since the Hamiltonian $\hat{H}$ is assumed to be Hermitian, the $\{|n\rangle\}$ form a complete basis of the Hilbert space $\mathcal{H}$.

To *construct* the Hilbert space, one can then proceed as follows. Start by *postulating* the existence of the vacuum state $|0\rangle$ such that $\hat{a}\,|0\rangle = 0$, for which the energy eigenstates can be defined by (3.7), and then *define* $\mathcal{H}$ as the space generated by $\{|n\rangle\}$. This is a valid Hilbert space, and all other choices are equivalent up to a unitary transformation due to the Stone Von-Neumann theorem. Another, more formal construction can also be found in [68].

The discussion can be straightforwardly generalized to systems with N degrees of freedom, labeled by x_i. In this case, each oscillator will have an associated creation and annihilation pair $\hat{a}_i, \hat{a}_i^\dagger$, with each of them annihilating the vacuum

$$\hat{a}_i\,|0\rangle = 0 \;\; \forall\, i\,, \tag{3.8}$$

[2]It is irrelevant in the sense that the physical predictions are the same, no matter the choice of $\mathcal{H}$. Different choices will differ only by an unitary transformation, which preserves both the eigenvalues and probabilities.

and the energy eigenstates are defined by

$$|n_1, n_2, ..., n_N\rangle \equiv \prod_{j=1}^{N} \left[\frac{\left(\hat{a}^{\dagger}_{k_j}\right)^{n_j}}{\sqrt{n_j!}} \right] |0\rangle \,, \tag{3.9}$$

with the Hilbert space being defined as the space spawned by such states.

In quantum field theory, the situation is very different: since in field theory one deals with systems that have an infinite number of degrees of freedom , the Stone Von Neumann theorem no longer applies.[3] Hence, distinct and **non-unitary equivalent** representations of the canonical commutation relation can exist.[4] For the case of harmonic oscillators, one can still define a representation by choosing a vacuum state and construct the energy eigenstates (3.9). However, such vacuum choice is inherently ambiguous, as we will show in the next section by treating a simple case: the quantization of a free scalar field in flat spacetime.

3.2 Minkowski Spacetime

The Minkowski spacetime is a rather simple spacetime, which is defined by the pair $(\mathbb{R}^4, \eta_{ab})$, where η_{ab} is the Minkowski metric. Using inertial coordinates[5], the line element can be expressed by

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,,$$

and one readily sees that, in this coordinate system, $g = -1$, and the Klein-Gordon equation (3.2) reduces to

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi + m^2\phi = 0 \,, \tag{3.10}$$

where the field is a function of both space and time, $\phi = \phi(x, y, z, t)$, and $\nabla^2 \equiv \partial^i\partial_i$.

In the Minkowski spacetime one has a rather large set of symmetries that connects inertial observers, which are described in terms of the Poincaré Group. Therefore, one is able to define a vacuum state using that class of privileged observers and hence a Hilbert space that is preserved by the spacetime symmetries. However, as we shall show, this is not the only vacuum choice available for Minkowski spacetime, and one demands physical arguments to select a preferred vacuum state and hence a representation of the quantum theory.

3.2.1 Mode Expansion and Bogoliubov Transformations

In this section, we shall present an alternative quantization procedure that makes explicit the vacuum choice of a theory. However, before we quantize our field, we need to solve the classical field

[3] This happens because, since fields are functions of the position $\vec{x}$, one may decompose them in terms of (countable) infinite basis of functions, with one degree of freedom per basis function.

[4] If the representations do not differ by a unitary transformation, physical predictions will in general be different.

[5] This choice of coordinates is valid because the Minkowski spacetime is maximally symmetric, and admits a set of four global commuting Killing vectors fields ξ_a.

equation (3.10). The calculations are simplified by applying a Fourier transform, where one writes the field as

$$\phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int \left(\phi_k(t) e^{i\vec{k}\cdot\vec{r}}\right) d^3\vec{k}\,, \tag{3.11}$$

and the fact that the field is real implies $(\phi_k)^* = \phi_{-k}$.[6]

Substituting in (3.10), and using that in Fourier space $\nabla^2 \to -k^2$, one obtains

$$\frac{d^2}{dt^2}\phi_k(t) + \left(k^2 + m^2\right)\phi_k(t) = 0\,, \tag{3.12}$$

and the original field equation is now decoupled into a set of infinite differential equations of the form

$$\frac{d^2}{dt^2}\phi_k(t) + \omega_k^2 \phi_k(t) = 0\,, \tag{3.13}$$

where $\omega_k^2 \equiv k^2 + m^2$. One then sees that a free relativistic scalar field is equivalent to a set of infinite harmonic oscillators: one for each $\vec{k}$. Hence, the Stone Von-Neumann theorem need not apply, and there is an arbitrariness in defining a representation of the canonical commutation relation, which is equivalent to choosing a vacuum state. Also, we emphasize that the previous discussion was carried by analyzing the field in inertial coordinates, which are preferred in Minkowski spacetime.

We now proceed to quantize the field and define a vacuum state as perceived by the inertial observers. Since our field can be described by a set of infinite harmonic oscillators represented by the variables $\phi_k(t)$, we shall proceed to quantize each oscillator by following the same steps presented in section 3.1.

1. **Classical Dynamics**: we first express our Hamiltonian in terms of the variables ϕ_k. By direct substitution of the Fourier transform (3.11) in the action, one obtains

$$S = \frac{1}{2}\int \left(\dot{\phi}_k\dot{\phi}_{-k} - \omega_k^2\phi_k\phi_{-k}\right) d^3\vec{k}dt\,,$$

from where one identifies that the total Lagrangian such that $S = \int L dt$ is given by the sum of each oscillators Lagrangian:

$$L = \int \mathcal{L}_k d^3\vec{k}\,,$$
$$\mathcal{L}_k \equiv \frac{1}{2}\left(\dot{\phi}_k\dot{\phi}_{-k} - \omega_k^2\phi_k\phi_{-k}\right)$$

the associated momenta are then obtained as in standard classical mechanics:

$$\pi_k \equiv \frac{\partial \mathcal{L}_k}{\partial \dot{\phi}_k} = \dot{\phi}_{-k}\,, \tag{3.14a}$$
$$\pi_{-k} \equiv \frac{\partial \mathcal{L}_k}{\partial \dot{\phi}_{-k}} = \dot{\phi}_k\,, \tag{3.14b}$$

[6] When a Fourier transform is performed in $\mathbb{R}^3$, the respective Fourier space is also given by $\mathbb{R}^3$, and each mode must be identified by a vector $\vec{k} \in \mathbb{R}^3$. However, to simplify our notation, we omit the arrow in the indexes: $\phi_{\vec{k}} \to \phi_k$.

and the total Hamiltonian H is given by a sum of each oscillator's Hamiltonian

$$H = \int H_k d^3\vec{k} \,,$$

where the H_k are obtained in terms of a usual Legendre transform:

$$H_k = \pi_k \dot{\phi}_k - \mathcal{L}_k \tag{3.15}$$

$$= \pi_k \dot{\phi}_k - \frac{1}{2}\left(\dot{\phi}_k \dot{\phi}_{-k} - \omega_k^2 \phi_k \phi_{-k}\right) \tag{3.16}$$

$$= \frac{1}{2}\left(\pi_k \pi_{-k} + \omega_k^2 \phi_k \phi_{-k}\right) \,; \tag{3.17}$$

2. **Quantum Observables**: we now promote each ϕ_k, π_k to quantum operators $\hat{\phi}_k, \hat{\pi}_k$, which represent our physical observables and satisfy the commution relations:

$$\phi_k, \pi_k \to \hat{\phi}_k, \hat{\pi}_k \,, \tag{3.18a}$$

$$[\hat{\phi}_k, \hat{\pi}_{k'}] = \frac{1}{i\hbar}\delta(\vec{k} + \vec{k}') \,, \tag{3.18b}$$

$$[\hat{\phi}_k, \hat{\phi}_{k'}] = [\hat{\pi}_k, \hat{\pi}_{k'}] = 0 \,, \tag{3.18c}$$

where the δ function is the continuous analogue of δ_{ij}, and its argument is $\vec{k} + \vec{k}'$ due to the fact that the momentum canonically conjugate to ϕ_k is $\pi_k = \dot{\phi}_{-k}$, as can be seen in (3.14);

3. **Representation**: we shall now construct a Hilbert space in a similar fashion as the one presented for the simple harmonic oscillator. To do so, we first need to define a vacuum state. Hence, we introduce creation and annihilation operator pairs $\hat{a}_k^\dagger$, $\hat{a}_k$ for each oscillator ϕ_k with momentum π_k in the standard way:

$$\hat{a}_k \equiv \sqrt{\frac{\omega_k}{2\hbar}}\left(\hat{\phi}_k + \frac{i}{\omega_k}\hat{\pi}_k\right) \,, \tag{3.19a}$$

$$\hat{a}_k^\dagger \equiv \sqrt{\frac{\omega_k}{2\hbar}}\left(\hat{\phi}_k - \frac{i}{\omega_k}\hat{\pi}_k\right) \,. \tag{3.19b}$$

Now, we postulate the existence of the vacuum state $|0\rangle$ and demand that it is annihilated by all annihilation operators $\hat{a}_k$:

$$\hat{a}_k |0\rangle = 0 \;\; \forall \, \vec{k} \,. \tag{3.20}$$

The energy eigenstates are then constructed by successive applications of the creation operators $\hat{a}_k^\dagger$ as in (3.9):

$$|n_1, n_2, \ldots\rangle \equiv \prod_{j=1}^{\infty}\left[\frac{\left(\hat{a}_{k_j}^\dagger\right)^{n_j}}{\sqrt{n_j!}}\right]|0\rangle \tag{3.21}$$

and the Hilbert space is defined as the space spawned by the kets $|n_1, n_2, \ldots\rangle$;

4. **Dynamics**: dynamics can be defined both in the Heisenberg Picture or the Schrödinger Picture, but the Heisenberg one is the most commonly used for quantum field theory, and hence is the one we shall focus [69].

   Our main objective is to obtain the time evolution of quantum field $\hat{\phi}$. To do so, it is useful to first obtain the time evolution of the creation and annihilation operators, which we do by applying the Heisenberg equations. Note that each oscillators Hamiltonian can be expressed by

$$\hat{H}_k = \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\mathbb{1} \right) , \tag{3.22}$$

   which follows by substituting (3.19) in the Hamiltonian (3.15). Therefore, the Heisenberg equations become, explicitly

$$\begin{aligned} \frac{d\hat{a}_k^\dagger}{dt} &= +i\omega_k \hat{a}_k^\dagger , \\ \frac{d\hat{a}_k}{dt} &= -i\omega_k \hat{a}_k , \end{aligned}$$

   which are operatorial equations, but can be solved in terms of ordinary differential equations by representing the operators $\hat{a}_k, \hat{a}_k^\dagger$ on a basis. The general solution is then given by

$$\begin{aligned} \hat{a}_k^\dagger(t) &= e^{+i\omega_k t}\hat{a}_k^+ , \\ \hat{a}_k(t) &= e^{-i\omega_k t}\hat{a}_k^- , \end{aligned}$$

   where the $\hat{a}_k^\pm$ are time-independent operators that act as initial conditions.

   Finally, note that, by inverting (3.19), one obtains

$$\hat{\phi}_k = \sqrt{\frac{\hbar}{2\omega_k}} \left(\hat{a}_k + \hat{a}_k^\dagger \right) , \tag{3.25a}$$

$$\hat{\pi}_k = \frac{1}{i}\sqrt{\frac{\hbar}{2\omega_k}} \left(\hat{a}_k - \hat{a}_k^\dagger \right) , \tag{3.25b}$$

   so that one can reexpress the quantum version of the Fourier transform (3.11) as

$$\hat{\phi}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int \sqrt{\frac{\hbar}{2\omega_k}} \left(e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_k t}\hat{a}_k^- + e^{-i\vec{k}\cdot\vec{r}} e^{i\omega_k t}\hat{a}_k^+ \right) d^3\vec{k} , \tag{3.26}$$

   which fixes the field evolution and concludes the quantization procedure.

In the above construction, one sees that an appropriate Hilbert space was constructed in terms of the vacuum state $|0\rangle$, which in turn was specified in terms of the creation and annihilation pair $\hat{a}_k^\dagger, \hat{a}_k$. However, an important assumption was not explicit in such construction.

By direct analysis of (3.26), one sees that it is essentially the general solution of the classical field equation (3.10) expanded in terms of plane waves $e^{i\omega_k t}/\sqrt{\omega_k}$. However, one could also expand the general solution in terms of other functions. For instance, consider an arbitrary solution of the classical field equations

$$v_q(t) = \sum\!\!\!\!\!\!\int_k \left(A_{qk} e^{i\omega_k t} + B_{qk} e^{-i\omega_k t}\right) , \tag{3.27}$$

where the notation $\sum\!\!\!\!\!\int_k$ is schematic since the set $\left\{e^{i\omega_k t}/\sqrt{\omega_k}\right\}$ can be either discrete or continuous, depending on the fields boundary conditions.[7] Now, let $\{v_q\}$ denote a set of solutions such that the normalization condition

$$\frac{dv_q}{dt} v_q^*(t) - v_q(t) \frac{dv_q^*}{dt} = 2i \tag{3.28}$$

is satisfied. A set with such properties is also a valid complete basis of solutions, and functions that belong to $\{v_q\}$ are called **modes**. In particular, substituting the expansion (3.27) in (3.28), we see that the coefficients A_{qk}, B_{qk} must satisfy

$$|A_{qk}|^2 - |B_{qk}|^2 = \frac{1}{\omega_k}$$

and this motivates the introduction of

$$\alpha_{qk} \equiv \sqrt{\omega_k} A_{qk} , \tag{3.29a}$$

$$\beta_{qk} \equiv \sqrt{\omega_k} B_{qk} , \tag{3.29b}$$

which must satisfy the normalization condition

$$|\alpha_{qk}|^2 - |\beta_{qk}|^2 = 1 . \tag{3.30}$$

Now, since one can use any mode basis $\{v_q\}$ to express solutions of (3.13), one can invert the system

$$v_q = \sum\!\!\!\!\!\!\int \left(A_{qk} e^{+i\omega_k t} + B_{qk} e^{-i\omega_k t}\right) , \tag{3.31a}$$

$$v_q^* = \sum\!\!\!\!\!\!\int \left(B_{qk}^* e^{+i\omega_k t} + A_{qk}^* e^{-i\omega_k t}\right) , \tag{3.31b}$$

to express the plane waves in the mode basis $\{v_k\}$:

$$e^{+i\omega_k t} = \sum\!\!\!\!\!\!\int \left(A_{qk}^* v_q - B_{qk} v_k^*\right) , \tag{3.32a}$$

$$e^{-i\omega_k t} = \sum\!\!\!\!\!\!\int \left(-B_{qk}^* v_q + A_{qk} v_q^*\right) , \tag{3.32b}$$

and, substituting in the field expansion (3.26), we obtain:

$$\hat{\phi}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int \sqrt{\frac{\hbar}{2\omega_k}} \left(e^{i\vec{k}\cdot\vec{r}} v_k(t) \hat{b}_{\vec{k}}^- + e^{-i\vec{k}\cdot\vec{r}} v_k^*(t) \hat{b}_{\vec{k}}^+\right) d^3\vec{k} , \tag{3.33}$$

[7] For instance, if the field is quantized in a box of size L with periodic boundary conditions, the functions $e^{i\omega_k t}/\sqrt{\omega_k}$ will form a discrete set.

where we have introduced the operators

$$\hat{b}_l^+ \equiv \sum\!\!\!\!\!\!\int \left(\alpha_{qk}^* \hat{a}_k^+ - \beta_{qk}^* \hat{a}_{-k}^-\right) , \tag{3.34a}$$

$$\hat{b}_l^- \equiv \sum\!\!\!\!\!\!\int \left(\alpha_{qk} \hat{a}_k^- - \beta_{qk} \hat{a}_{-k}^+\right) . \tag{3.34b}$$

Relation (3.34) defines new creation and annihilation operators $\hat{b}_k^\pm$ in terms of the old ones $\hat{a}_k^\pm$. Note that they satisfy the same algebra by construction:

$$[\hat{b}_q^-, \hat{b}_{q'}^+] = \delta(\vec{q} + \vec{q}\,') , \tag{3.35a}$$

$$[\hat{b}_q^-, \hat{b}_{q'}^-] = 0 , \tag{3.35b}$$

$$[\hat{b}_q^+, \hat{b}_{q'}^+] = 0 , \tag{3.35c}$$

as can be shown by direct substitution. Hence, one can construct another valid Hilbert space $\mathcal{H}_b$ by postulating the existence of the vacuum state

$$\hat{b}_q^- |0_b\rangle = 0 \;\; \forall \, \vec{q} ,$$

and defining the energy eigenstates using the analogue of (3.21).

Relations (3.34) are called **Bogoliubov transformations** and connect two different choices of creation and annihilation operators, while the α_{qk}, β_{qk} are called **Bogoliubov coefficients**. Since the introduction of the new operators $\hat{b}_k^\pm$ occurred due to a change of basis (3.32) in solution space, one can see that the vacuum choice and hence the notion of particles is intrinsically linked with a choice of modes $\{v_k\}$. This formalizes the idea that "particles are modes of vibration of fields". Therefore, different observers will have different mode choices, a different notion of "vibration", and hence different notions of what a "particle" is. This can be made explicitly by evaluating the b-vacuum particle number operator

$$\hat{N}_b \equiv \sum\!\!\!\!\!\!\int \hat{b}_k^+ \hat{b}_k^- ,$$

in the vacuum determined by the operators $\hat{a}_k^\pm$, now denoted $|0_a\rangle$ to distinguish it from $|0_b\rangle$. Using the Bogoliubov transformation (3.34), the mean particle number in the a-vacuum is then given by

$$\begin{aligned} \langle \hat{N}_b \rangle_a &= \langle 0_a | \left\{ \sum\!\!\!\!\!\!\int \left[\alpha_{qk} \hat{a}_k^- - \beta_{qk} \hat{a}_k^+\right] \times \sum\!\!\!\!\!\!\int \left[\alpha_{q'k} \hat{a}_k^- - \beta_{q'k} \hat{a}_k^+\right] \right\} |0_a\rangle , \\ &= \sum\!\!\!\!\!\!\int |\beta_{qk}|^2 , \end{aligned}$$

which shows explicitly the a-vacuum associated to the operators $\hat{a}_k^\pm$ contains particles associated to the b-vacuum associated to $\hat{b}_q^\pm$, provided $\beta_{qk} \neq 0$.

Equation (3.33) has one more interesting use. It enables one to perform an alternative quantization procedure without ever using the operators $\hat{\phi}_k, \hat{\pi}_k$. To do so, start directly with the general mode expansion (3.33) and postulate the commutation relations (3.35) for the creation and annihilation operators. This automatically implies the canonical commutation relations for the field and its momentum (3.18), which means that it is an equivalent quantization procedure, and we shall adopt it in our following discussions.

Now that we have completed our quantization construction in terms of modes $\{v_k\}$ in Minkowski spacetime, we shall analyze two different vacuum choices:

1. Minkowski Vacuum $|0_M\rangle$: vacuum state defined by inertial observers;
2. Rindler Vacuum $|0_R\rangle$: vacuum state defined by uniformly accelerated observers (with respect to an inertial frame);

3.2.2 Minkowski Vacuum

In this section we discuss the Minkowski vacuum state, already implicitly defined in the previous section. It is the vacuum state perceived by inertial observers and is invariant with respect to Lorentz transformations, that is: **all inertial observers agree that the Minkowski vacuum contains no particles**, which is this state's main feature [61].

First, consider the Hamiltonian density (3.15), and substitute the field and momentum Fourier modes (3.25). One obtains a diagonal form

$$\hat{H} = \int \hbar\omega_k \left(\hat{a}_k^+ \hat{a}_k^- + \frac{1}{2}\mathbb{1}\right) d^3\vec{k}\,, \tag{3.36}$$

which is the Hamiltonian in terms of the implicit mode choice $v_k(t) = e^{ikt}/\sqrt{\omega_k}$. We shall now *derive* this consideration from more general arguments, which will also serve as an interesting illustration of how a mode choice may be performed.

First, consider the inverse Bogoliubov transformation obtained by inverting (3.34):

$$\hat{a}_k^+ \equiv \sumint \left(\alpha_{qk}\hat{b}_q^+ + \beta_{qk}^*\hat{b}_{-l}^-\right)\,, \tag{3.37a}$$

$$\hat{a}_k^- \equiv \sumint \left(\alpha_{qk}^*\hat{b}_q^- + \beta_{qk}\hat{b}_{-l}^+\right)\,, \tag{3.37b}$$

and substitute in the Hamiltonian. One then obtains:

$$\hat{H} = \int \hbar\omega_k \left[\alpha_k^*\beta_k^*\hat{b}_k^-\hat{b}_{-k}^- + \alpha_k\beta_k\hat{b}_k^+\hat{b}_{-k}^+ + \left(\hat{b}_k^+\hat{b}_k^- + \frac{1}{2}\mathbb{1}\right)\right] d^3\vec{k}\,, \tag{3.38}$$

where we have used $\alpha_k = \alpha_{-k}$ and $\beta_k = \beta_{-k}$. Note that this expression contains cross terms $\hat{b}_k^-\hat{b}_{-k}^-$, $\hat{b}_k^+\hat{b}_{-k}^+$.

Now, assume the existence of a vacuum state $|0_M\rangle$ annihilated by $\hat{b}_k^-$ such that

$$\hat{b}_k^- |0_M\rangle = 0 \;\; \forall\, \vec{k}\,, \tag{3.39}$$

that is also an eigenstate of the Hamiltonian, that is:

$$\hat{H}\,|0_M\rangle = E_0\,|0_M\rangle\,, \tag{3.40}$$

where E_0 is called its **vacuum energy**. By direct substitution in (3.38), due to (3.39) one obtains:

$$\hat{H}\left|0_M\right\rangle = \int \hbar\omega_k \left\{\left[\alpha_k\beta_k\left(\hat{b}_k^{+}\hat{b}_{-k}^{+}\left|0_M\right\rangle\right)\right] + \frac{1}{2}\left|0_M\right\rangle\right\} d^3\vec{k}\,. \tag{3.41}$$

Note that, since $\hat{b}_k^{+}\hat{b}_{-k}^{+}\left|0_M\right\rangle \neq \left|0_M\right\rangle$ by construction, the proposed state $\left|0_M\right\rangle$ can only be a Hamiltonian eigenstate if the condition

$$\alpha_{qk}\beta_{qk} = 0\,,$$

is satisfied. However, due to the normalization condition (3.30), can only be achieved if

$$\begin{aligned} |\alpha_{qk}| = 1 &\implies \alpha_{qk} = e^{i\delta_{qk}}\,, \\ |\beta_{qk}| = 0 &\implies \beta_{qk} = 0\,, \end{aligned}$$

where $\delta_k \in \mathbb{R}$ is an irrelevant phase factor. Substituting in the Bogoliubov transformation (3.34), we reobtain the mode choice

$$\phi_k(t) = \frac{1}{\sqrt{\omega_k}}e^{i\omega_k t}\,, \tag{3.42}$$

which defines our vacuum state and Hilbert space due to the standard construction (3.21). We then see that the Minkowski vacuum is the vacuum choice consistent with minimization of the vacuum energy. This criterium implies a choice of plane wave modes, which leads to a vacuum state that is invariant with respect to Lorentz transformations and hence is the same for all inertial observers.

It is also relevant to point out that the vacuum energy becomes, after imposing $\beta_k = 0$ in (3.41):

$$\begin{aligned} \hat{H}\left|0_M\right\rangle &= \left(\frac{1}{2}\int \hbar\omega_k d^3\vec{k}\right)\left|0_M\right\rangle\,, \\ \implies E_0 &= \frac{1}{2}\int \hbar\omega_k d^3\vec{k}\,, \end{aligned}$$

which is divergent, since it is given by the sum of the non-trivial vacuum energy of all oscillators. However, when coupling to gravity is not important, only energy differences are physically relevant, and we can then subtract the vacuum energy from all energy values, which essentially amounts to changing our zero-point energy to E_0. Hence, we redefine

$$E_0 = 0 \tag{3.43}$$

which makes our calculations finite.

However, as already mentioned, the obtained vacuum state $\left|0_M\right\rangle$ is not unique, and was constructed by demanding minimization of energy, even though other criteria could be used to select the modes $\{\phi_k\}$. Hence, we shall now present another vacuum prescription, associated with non-inertial observers, and then compare each vacuum choice to decide what should be the quantum theory Hilbert space.

3.2.3 Rindler Vacuum

In the previous construction, we quantized the field in terms of inertial observers, which are a set of privilleged observers in the Minkowski spacetime, and are associated to foliations $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ induced by the inertial observer's four velocity u^a. However, one can ask the question: how would be the quantization of a field in the frame of a non-inertial observer?

To answer such question, we consider the case of an observer of constant acceleration. First, due to the normalization of the four velocity

$$u^a u_a = -1\,, \tag{3.44}$$

direct differentiation leads to

$$a^a u_a = 0\,, \tag{3.45}$$

where $a^a = u^b \nabla_b u^a$ is its four-acceleration, which we assume to be constant, that is

$$a^a a_a = a^2 \tag{3.46}$$

Hence, one sees that the four-acceleration and four-velocity are always orthogonal. To simplify our discussion, let's consider the case of a quantum field in $1+1$ spacetime. In this case, in the comoving frame of an observer with four-velocity u^a, this vector has the coordinate representation

$$u^\mu \doteq (1,0)$$

hence, since relation (3.45) is geometric, one sees that, in order for it to be satisfied, the four acceleration in this frame is represented by

$$a^\mu \doteq (0,a)\,.$$

We now proceed to quantize the field in the reference frame of the non-inertial observer. To do so, we shall follow the following steps:

1. **Trajectory** : determine the trajectory of the accelerated observer as seem from the inertial frame;
2. **Accelerated Frame**: using the obtained trajectory, we shall construct the accelerated comoving frame;
3. **Modes**: we shall solve the wave equation and compare the modes and hence the vacuum state in both frames.

Some calculations will be simplified if we first introduce **lightcone coordinates** as

$$u(x,t) = t - x\,,$$
$$v(x,t) = t + x\,,$$

which describe left and right moving light rays, respectively. In this coordinate system, the metric is then given by

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v\,,$$

which is invariant under the following transformations:

$$u \to \tilde{u} = au\,, \tag{3.47a}$$
$$v \to \tilde{v} = \frac{v}{a}\,. \tag{3.47b}$$

1. **Trajectory**

   Here, we shall explicitly obtain the accelerated observer's trajectory using lightcone coordinates. To do so, note that its trajectory in such coordinate system can is given by

$$x^\sigma(\tau) \doteq (u(\tau), v(\tau)) \tag{3.48}$$

   which, if substituted in the four-velocity normalization (3.44) and the acceleration normalization (3.46) leads to the conditions

$$\dot{u}(\tau)\dot{v}(\tau) = -1\,, \tag{3.49a}$$
$$\ddot{u}(\tau)\ddot{v}(\tau) = a^2\,. \tag{3.49b}$$

   From the first equation (3.49a), it follows that

$$\dot{u}(\tau) = \frac{1}{v(\tau)}\,,$$
$$\implies \ddot{u} = -\frac{\ddot{v}}{\dot{v}^2}$$

   and, substituting in the second (3.49b), we obtain

$$\left(\frac{\ddot{v}}{\dot{v}}\right)^2 = a^2\,, \tag{3.51}$$

   which can be readily integrated, and leads to the following parametrization for the observer's trajectory:

$$v(\tau) = \frac{A}{a}e^{a\tau} + B\,,$$
$$u(\tau) = -\frac{A}{a}e^{-a\tau} + C\,,$$

   where A, B, C are integration constants. The one constant A can be set to 1 by an appropriate coordinate reescaling and, by changing the coordinate system origin, one can always make the

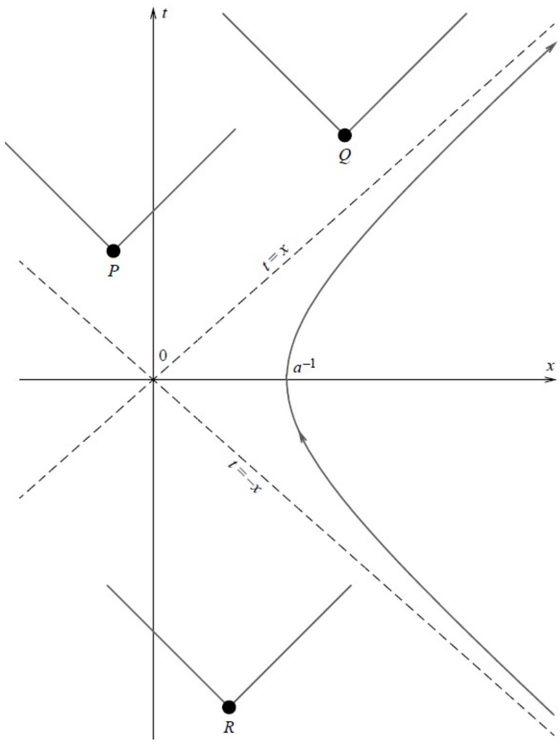

Figure 3.1: Accelerated observer's trajectory. Source: [69].

constants B, C vanish, so that one gets the simpler form

$$v(\tau) = \frac{1}{a}e^{a\tau}\,,$$
$$u(\tau) = -\frac{1}{a}e^{-a\tau}\,,$$

now, returning to the original inertial coordinates t, x:

$$t(\tau) = \frac{1}{2}\left(u(\tau) - v(\tau)\right) = \frac{1}{a}\sinh(a\tau)\,,$$
$$x(\tau) = \frac{1}{2}\left(u(\tau) + v(\tau)\right) = \frac{1}{a}\cosh(a\tau)\,.$$

This parametrization shows that the accelerated observer's trajectory is the branch of a hyperbola, as shown in Figure 3.1, with the lightcone being its asymptotes. In fact one can show that signals from the other side of the lightcone can never get into causal contact with the accelerated observer, so that the lightcone acts as an effective event horizon for the accelerated observer [70].

2. **Accelerated Frame**

Now that we have obtained the observer's trajectory, let's construct an associated accelerated frame of reference. To do so, we shall choose a set of coordinates such that the time coordinate coincides with the proper time τ of our observer (because the observer feels himself to be at rest), and another spacelike coordinate ξ. We may express the metric in our accelerated frame of reference in conformally flat form, that is

$$\mathrm{d}s^2 = \Omega^2(\tau, \xi) \left[-\mathrm{d}\tau^2 + \mathrm{d}\xi^2 \right] .$$

Now, we introduce the accelerated observer's lightcone coordinates in an analogue way:

$$\tilde{u}(\tau, \xi) \equiv \tau - \xi \, ,$$
$$\tilde{v}(\tau, \xi) \equiv \tau + \xi \, ,$$

and, in such coordinate system, the accelerated observer's worldline is given by

$$\tilde{u}(\tau) = \tilde{v}(\tau) = \tau \, ,$$

because, $\xi(\tau) = 0$, for the observer perceives itself as being at rest. Applying such consideration to the line element in lightcone coordinates, we obtain that, along the worldline of the accelerated observer

$$\Omega(\tilde{u}(\tau) = \tau, \tilde{v}(\tau) = \tau) = 1 \, ,$$

since $\Delta s^2 = -\Delta\tau^2$.

We now turn to the task of obtaining the inertial observer lightcone coordinates in terms of the non-inertial ones, that is

$$u = u(\tilde{u}, \tilde{v}) \, ,$$
$$v = v(\tilde{u}, \tilde{v}) \, .$$

However, such functions can only depend on one of the two arguments. If they depended on both, quadratic terms like $d\tilde{u}^2, d\tilde{v}^2$ would appear on the metric. Hence, we may choose

$$u = u(\tilde{u}) \, ,$$
$$v = v(\tilde{v}) \, .$$

Now, using the chain rule

$$\frac{d}{d\tau} u(\tau) = \frac{du}{d\tilde{u}} \frac{d\tilde{u}(\tau)}{d\tau} \, ,$$

and, since

$$\frac{d\tilde{u}(\tau)}{d\tau} = 1 \, ,$$
$$\frac{d}{d\tau} u(\tau) = -a u(\tau) \, ,$$

it follows that $u(\tilde{u})$ satisfies the differential equation

$$\frac{du}{d\tilde{u}} = -au$$

with general solution $u(\tilde{u}) = C_1 e^{-a\tilde{u}}$, where C_1 is an integration constant. By similar methods, one can obtain $v(\tilde{v}) = C_2 e^{a\tilde{v}}$, where C_2 is another integration constant. Due to the normalization of the four-velocity (3.49a), one obtains the following constraint for the integration constants:

$$a^2 C_1 C_2 = -1\,,$$

so that one can take $C_2 = -C_1$ without losing generality. Hence, we obtain the explicit coordinate transformation

$$u(\tilde{u}) = -\frac{1}{a}e^{-a\tilde{u}}\,, \tag{3.52a}$$

$$v(\tilde{v}) = \frac{1}{a}e^{a\tilde{v}}\,. \tag{3.52b}$$

And, in this coordinate system, the conformal factor is found to be

$$\Omega^2\left(\tilde{u},\tilde{v}\right) = \exp\left\{2a\left(\tilde{u}-\tilde{v}\right)\right\}$$

hence, the transformed line element becomes

$$\mathrm{d}s^2 = \exp\left\{2a\xi\right\}\left[-\mathrm{d}\tau^2 + \mathrm{d}\xi^2\right]\,.$$

Here, it is important to note that this coordinate system does **not** cover the entire space-time. This can be seen by analyzing the coordinate transformation (3.52), which is undefined in the lightcone, since $v = 0$ for the inertial observer. The covered region corresponds to the right region of Figure 3.2. This region is can be shown to be a spacetime of its own, known as Rindler spacetime which, since it is a subset of the Minkowski spacetime, is locally flat.

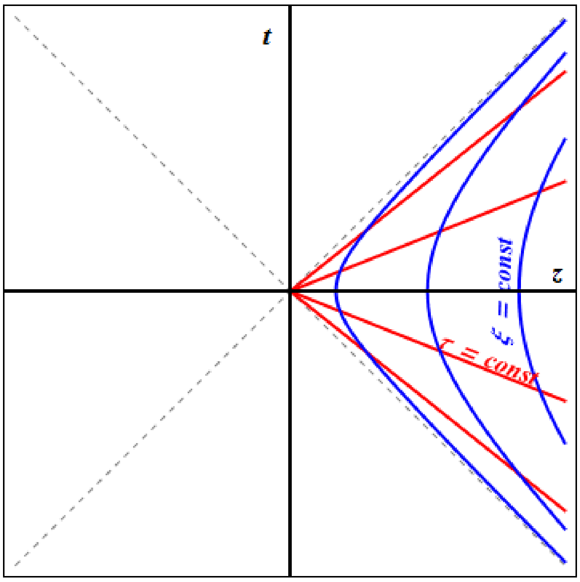

Figure 3.2: Covered region of the coordinates τ, ξ, known as Rindler Spacetime. Source: [70].

3. **Modes**

   Having constructed the accelerated observer's frame of reference, we can now write the Klein-Gordon equation in such and quantize our field. In what follows, every quantity with a tilde ~ refers to the analogue of the non inertial observer.

   In lightcone coordinates, apart from a global multiplicative factor, the field equation (3.2) takes the form

$$\frac{\partial^2}{\partial u \partial v}\phi(u,v) = 0\,,$$
$$\frac{\partial^2}{\partial \tilde{u} \partial \tilde{v}}\phi(\tilde{u},\tilde{v}) = 0\,,$$

   and the general solution can be written as

$$\phi(u,v) = f(u) + g(v)\,,$$
$$\phi(\tilde{u},\tilde{v}) = \tilde{f}(\tilde{u}) + \tilde{g}(\tilde{v})\,,$$

   for which a particular solution is a right-moving plane wave:

$$\phi(u) \propto e^{-i\omega u} = e^{-i\omega(t-x)}\,,$$
$$\phi(\tilde{u}) \propto e^{-i\tilde{\omega}\tilde{u}} = e^{-i\Omega(\tau-\xi)}\,,$$

   with the respective left-moving counterparts being given by

$$\phi(v) \propto e^{-i\omega v} = e^{-i\omega(t+x)}\,,$$
$$\phi(\tilde{v}) \propto e^{-i\tilde{\omega}\tilde{u}} = e^{-i\Omega(\tau+\xi)}\,.$$

We now proceed to write the field mode expansion (3.33) in both frames:

$$\hat{\phi}(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \sqrt{\frac{\hbar}{2\omega}}\left(e^{+i\omega u}\hat{a}^+_\omega + e^{-i\omega u}\hat{a}^-_\omega\right)d\omega \; + \text{left-moving}, \tag{3.53a}$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \sqrt{\frac{\hbar}{2\Omega}}\left(e^{+i\Omega\tilde{u}}\hat{b}^+_\Omega + e^{-i\Omega\tilde{u}}\hat{b}^-_\Omega\right)d\Omega \; + \text{left-moving}, \tag{3.53b}$$

and one sees that the field is the same for both observers: what changes is the way they decompose the fields into modes, and hence their particle definitions. In particular, while the Minkowski vacuum $|0_M\rangle$ is annihilated by the operators $\hat{a}^-_\omega$, the operators $\hat{b}^-_\Omega$ define another vacuum state:

$$\hat{b}^-_\Omega\,|0_R\rangle = 0\,, \tag{3.54}$$

which is known as *Rindler vacuum*, and is the accelerated observer's vacuum state.

We can now use the Bogoliubov transformation (3.34) to compare the notion of particles of both observers, for which we write

$$\hat{b}^+_\Omega = \int_0^\infty \left[\alpha^*_{\Omega,\omega}\hat{a}^+_\omega - \beta^*_{\Omega,\omega}\hat{a}^-_\omega\right]d\omega\,, \tag{3.55a}$$

$$\hat{b}^-_\Omega = \int_0^\infty \left[\alpha_{\Omega,\omega}\hat{a}^-_\omega - \beta_{\Omega,\omega}\hat{a}^+_\omega\right]d\omega\,. \tag{3.55b}$$

Note that, since the accelerated observer's coordinates are incomplete, the inverse Bogoliubov transformation is not defined.

We now shal proceed to obtain the explicit form of the Bogoliubov coefficients $\alpha_{\Omega,\omega}, \beta_{\Omega,\omega}$. We start by noting that, compatibility of (3.55) with the commutation relation

$$[\hat{b}^-_\Omega, \hat{b}^+_{\Omega'}] = \delta(\Omega - \Omega')\,,$$

leads to the constraint

$$\delta(\Omega - \Omega') = \int_0^\infty \left[\alpha_{\Omega,\omega}\alpha^*_{\Omega',\omega} - \beta_{\Omega,\omega}\beta^*_{\Omega',\omega}\right]d\omega\,, \tag{3.56}$$

which is the generalization the normalization condition (3.30). By direct substitution of the Bogoliubov transformation (3.55) in the mode expansion (3.53), we obtain

$$\hat{\phi}(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \left[\left(\sqrt{\frac{\hbar}{2\omega}}e^{+i\omega u}\right)\hat{a}^+_\omega + \left(\sqrt{\frac{\hbar}{2\omega}}e^{-i\omega u}\right)\hat{a}^-_\omega\right]d\omega \; + \text{left-moving}, \tag{3.57a}$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \sqrt{\frac{\hbar}{2\Omega}}\left\{e^{+i\Omega\tilde{u}}\left[\int_0^\infty \left(\alpha^*_{\Omega,\omega}\hat{a}^+_\omega - \beta^*_{\Omega,\omega}\hat{a}^-_\omega\right)d\omega\right]\right. \tag{3.57b}$$

$$\left. + e^{-i\Omega\tilde{u}}\left[\int_0^\infty \left(\alpha_{\Omega,\omega}\hat{a}^-_\omega - \beta_{\Omega,\omega}\hat{a}^+_\omega\right)d\omega\right]\right\}d\Omega\text{left-moving} \tag{3.57c}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \sqrt{\frac{\hbar}{2\Omega}} \left\{ \left[\int_0^\infty \left(e^{+i\Omega\tilde{u}} \alpha^*_{\Omega,\omega} - e^{-i\Omega\tilde{u}} \beta_{\Omega,\omega} \right) d\Omega \right] \hat{a}^+_\omega \right. \tag{3.57d}$$

$$\left. + \left[\int_0^\infty \left(e^{-i\Omega\tilde{u}} \alpha_{\Omega,\omega} - e^{+i\Omega\tilde{u}} \beta^*_{\Omega,\omega} \right) d\Omega \right] \hat{a}^-_\omega \right\} d\omega \; + \text{left-moving} \,, \tag{3.57e}$$

and, by comparing such expressions, we see that

$$\frac{1}{\sqrt{\omega}} e^{+i\omega u} = \int_0^\infty \frac{1}{\Omega} \left(e^{+i\Omega\tilde{u}} \alpha^*_{\Omega,\omega} - e^{-i\Omega\tilde{u}} \beta_{\Omega,\omega} \right) d\Omega \,, \tag{3.58a}$$

$$\frac{1}{\sqrt{\omega}} e^{-i\omega u} = \int_0^\infty \frac{1}{\Omega} \left(e^{-i\Omega\tilde{u}} \alpha_{\Omega,\omega} - e^{+i\Omega\tilde{u}} \beta^*_{\Omega,\omega} \right) d\Omega \,. \tag{3.58b}$$

Multiplying both sides by $e^{\pm i\Omega\tilde{u}}$, and using the integral representation of the δ function:

$$\delta(\Omega - \Omega') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(\Omega-\Omega')\tilde{u}} d\tilde{u}$$

and evaluating explictly the integral over $\tilde{u}$:

$$\alpha_{\Omega,\omega} = \int_{-\infty}^{+\infty} e^{-\omega u + i\Omega\tilde{u}} d\tilde{u} = +\frac{1}{2\pi} \sqrt{\frac{\Omega}{\omega}}_{-\infty}^{0} (-au)^{-\frac{i\Omega}{a}-1} e^{-\omega u} du \tag{3.59a}$$

$$\beta_{\Omega,\omega} = \int_{-\infty}^{+\infty} e^{+\omega u + i\Omega\tilde{u}} d\tilde{u} = -\frac{1}{2\pi} \sqrt{\frac{\Omega}{\omega}}_{-\infty}^{0} (-au)^{-\frac{i\Omega}{a}-1} e^{+\omega u} du \,, \tag{3.59b}$$

identifying the integrals on the left as Γ functions

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

it follows that

$$\alpha_{\Omega,\omega} = +\frac{1}{2\pi a} \sqrt{\frac{\Omega}{\omega}} e^{+\frac{\pi\Omega}{2a}} \exp\left[\frac{i\Omega}{a} \ln\left(\frac{\omega}{a}\right) \right] \Gamma\left(-\frac{i\Omega}{a} \right) , \tag{3.60a}$$

$$\beta_{\Omega,\omega} = -\frac{1}{2\pi a} \sqrt{\frac{\Omega}{\omega}} e^{-\frac{\pi\Omega}{2a}} \exp\left[\frac{i\Omega}{a} \ln\left(\frac{\omega}{a}\right) \right] \Gamma\left(-\frac{i\Omega}{a} \right) , \tag{3.60b}$$

so that, after taking the squared norm

$$|\alpha_{\Omega,\omega}|^2 = \frac{1}{(2\pi a)^2} \frac{\Omega}{\omega} e^{\frac{\Omega\pi}{a}} \,, \tag{3.61a}$$

$$|\beta_{\Omega,\omega}|^2 = \frac{1}{(2\pi a)^2} \frac{\Omega}{\omega} e^{-\frac{\Omega\pi}{a}} \,, \tag{3.61b}$$

from which follows the relation

$$|\alpha_{\Omega,\omega}|^2 = e^{\frac{2\pi\Omega}{a}} |\beta_{\Omega,\omega}|^2 \,. \tag{3.62}$$

Now applying the normalization condition (3.56) for $\Omega = \Omega'$, we obtain

$$\int_0^\infty \left(|\alpha_{\Omega,\omega}|^2 - |\beta_{\Omega,\omega}|^2\right) d\omega = \delta(0)\,,$$

which, if combined with (3.62) leads to

$$\int_0^\infty |\beta_{\Omega,\omega}|^2 d\omega = \left(e^{\frac{2\pi\Omega}{a}} - 1\right)^{-1} \delta(0)\,. \tag{3.63}$$

Finally, we are able to calculate the mean Unruh particle number $\hat{N}_\Omega$ in the Minkowski vacuum state:

$$\begin{aligned}\langle \hat{N}_\Omega \rangle_M &\equiv \langle 0_M \hat{N}_\Omega | 0_M \rangle\,,\\ &= \langle 0_M | \hat{b}_\Omega \hat{b}^\dagger_\Omega | 0_M \rangle\,,\end{aligned}$$

substituting the Bogoliubov transformation (3.55), we get

$$\begin{aligned}\langle \hat{N}_\Omega \rangle_M &= \langle 0_M | \int_0^\infty (\alpha_{\Omega,\omega} \hat{a}^-_\omega - \beta_{\Omega,\omega} \hat{a}^+_\omega)\, d\omega \times \int_0^\infty (\alpha_{\Omega,\omega'} \hat{a}^-_{\omega'} - \beta_{\Omega,\omega'} \hat{a}^+_{\omega'})\, d\omega' | 0_M \rangle\,,\\ &= \int_0^\infty |\beta_\omega|^2 d\omega\,,\end{aligned}$$

which, due to (3.63) becomes

$$\begin{aligned}\langle \hat{N}_\Omega \rangle_M &= \int_0^\infty |\beta_\omega|^2 d\omega\\ &= \left(e^{\frac{2\pi\Omega}{a}} - 1\right)^{-1} \delta(0)\,,\end{aligned}$$

where the $\delta(0)$ factor appears because we are considering the mean particle number over infinite space. If one quantizes the field in a box of volume V, the divergent factor would be replaced by V ($\delta(0)$ can be thought as the volume of infinite space). Hence, the mean particle density n_Ω is finite, and is given by

$$\begin{aligned}n_\Omega &\equiv \frac{\langle \hat{N}_{\Omega M} \rangle}{V}\,,\\ &= \left[\exp\left(\frac{2\pi\Omega}{a}\right) - 1\right]^{-1}\,.\end{aligned}$$

Note that, by comparing such result to a Bose-Einstein distribution, one obtains the same form, provided that one defines the temperature as

$$T_U \equiv \frac{\hbar a}{2\pi c k_B}\,, \tag{3.64}$$

which is known as **Unruh Temperature**, and we have restored the constants. Note that, by analyzying the constants, one would need a very large acceleration to generate relevant temperatures. For instance, to obtain a temperature of 1K, one would need an acceleration of order 10^{20}m/s^2 [70].

Since the Unruh Effect depends only on the constants $c, \hbar, k_B$, this is a prediction from quantum field theory alone, that is, it does not depend on the chosen gravitational theory. Hence, if one accepts usual quantum field theory, he/she should also accept the Unruh effect, even if we cannot yet provide relevant accelerations for a direct observation . In fact, the Unruh effect is *necessary* to maintain the consistency between different frames of reference, as discussed in [71, 72]. An even more complete discussion can be found in [73].

One then sees that an accelerated observer "sees" particles in the Minkowski vacuum state. In particular, the associated particles follow a thermal distribution with temperature (3.64). This clearly shows the ambiguity in defining vacuum states in quantum field theory, even for flat Minkowski spacetime.

It should be noted that, in principle, there is no preferred vacuum state. The Minkowski vacuum was defined in terms of inertial observers, while the Rindler one was defined in term of non-inertial ones, which are both valid observers. However, one can use physical arguments to select a particular vacuum state (and hence representation) in this case.

Consider the Rindler modes in the lightcone, which acts as an apparent horizon for the accelerated observer. One then see that the coordinates (3.52) are undefined due to a divergent factor, which means that the vacuum state $|0_R\rangle$ is itself undefined in the horizon. Hence, one selects the Minkowski vacuum as the "true" vacuum in this case, since it is regular for all spacetime.[8]

We then see that, although one can construct different vacuum states even for Minkowski spacetime, it is still possible to use physical arguments to select a preferred vacuum state. However, the same may not apply for general, curved spacetimes, as we shall see.

3.3 FLRW Spacetime

In this section we shall consider the quantization of a free scalar field in FLRW spacetime. To simplify our discussion, we shall restrict ourselves to the case of flat spatial sections with $\mathcal{K} = 0$. Using cartesian coordinates adapted to the fundamental observers, the FLRW line element (1.21) reduces to

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) . \tag{3.65}$$

However, as we shall see, it is useful to use the conformal time η, for which the line element reads

$$\mathrm{d}s^2 = a^2(t)\left(-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) , \tag{3.66}$$

[8] It can also be shown that the Rindler vacuum state demands an infinite energy to be created, and hence is unphysical [69].

and it is clear that, in this coordinate system, $g = -a^8$, so that the Klein-Gordon equation (3.2) reduces to

$$\chi'' - \nabla^2\chi + m_{\text{eff}}^2\chi = 0\,, \tag{3.67}$$

where we have introduced

$$\chi \equiv a\phi\,,$$
$$m_{\text{eff}}^2(\eta) \equiv m^2a^2 - \frac{a''}{a}\,,$$

and one can see that, in terms of the auxiliar variable χ, the field equation has the same form of a free scalar field in Minkowski spacetime with a time-dependent mass, which accounts for the interaction with the gravitational field [69].

As in the Minkowski case, one can expand the field in Fourier modes $\chi_k(\eta)$:

$$\chi(\vec{x},\eta) = \frac{1}{(2\pi)^{3/2}}\int \chi_k(\eta)e^{i\vec{k}\cdot x}d^3\vec{k}\,,$$

which satisfy the differential equations

$$\chi_k'' + \omega_k^2(\eta)\chi_k = 0\,, \tag{3.68}$$

where

$$\omega_k^2(\eta) \equiv k^2 + m_{\text{eff}}^2\,,$$
$$= k^2 + m^2a^2 - \frac{a''}{a}\,.$$

Here, the exact form of the Fourier modes must be determined by solving the differential equation (3.68), which may be done explicitly only after one has fixed the background dynamics and hence $a(\eta)$. However, one can always express the general solution as

$$\chi_k(\eta) = \frac{1}{\sqrt{2}}\left(a_k^+v_k(\eta) + a_k^-v_k^*(\eta)\right)\,, \tag{3.69}$$

where the $v_k(\eta), v_k^*(\eta)$ are two linearly independent solutions of (3.68), and the $a_k^\pm$ are integration constants.

We now proceed to quantize the field ϕ by using the mode expansion, in the same fashion as was done for the Minkowski case (3.33). We then get:

$$\hat{\chi}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\sqrt{\frac{\hbar}{2\omega_k}}\left(e^{-i\vec{k}\cdot\vec{r}}v_k(\eta)\hat{a}_k^+ + e^{+i\vec{k}\cdot\vec{r}}v_k^*(\eta)\hat{a}_k^-\right)d^3\vec{k}\,, \tag{3.70}$$

where the operators $\hat{a}_k^{\pm}$ do not evolve in time and satisfy the commutation relations

$$[\hat{a}_k^-, \hat{a}_k^+] = \delta(\vec{k} + \vec{k}')\,, \tag{3.71}$$

$$[\hat{a}_k^+, \hat{a}_k^+] = [\hat{a}_k^-, \hat{a}_k^-] = 0\,. \tag{3.72}$$

the Hilbert space is then constructed by sucessive applications of the creation operator $\hat{a}_k^+$ on the vacuum state, which is annihilated by the anihhilation operators

$$\hat{a}_k^- \left|0\right\rangle = 0\,,$$

however, as in the Minkowski case, the vacuum is determined only after we select a particular set $\{v_k\}$ of modes. We shall now presented two prescriptions to define vacuum states in the flat FLRW spacetime.

3.3.1 Minimum Energy Vacuum Prescription

In Minkowski spacetime, the Minkowski vacuum $|0_M\rangle$ is also the state of lowest energy, for it is a state where all of the oscillators are in the ground state. Hence, it is reasonable to define a vacuum state in curved spacetime as the state that minimizes energy. However, since in the FLRW spacetime there is no time translation symmetry, energy is not conserved. Hence, if we consider time evolution[9], the field shall be excited, and the initial vacuum state will present particles. This is the phenomenon of gravitational particle creation. Nevertheless one can still consider a vacuum state that *instantly minimizes the energy* at a particular time $\eta = \eta_0$, which we shall now develop.

We start by explicitly writing the Hamiltonian in terms of the field and its conjugated momentum. First, consider the Klein-Gordon action (3.73) for a free field in flat FLRW spacetime, expressed in conformal time:

$$S = -\frac{1}{2}\int a^2\left[-\phi'^2 + (\nabla\phi)\cdot(\nabla\phi) + m^2\phi^2\right]d^3\vec{x}dt, \tag{3.73}$$

in terms of the auxiliary field $\chi \equiv a\phi$ and integrating by parts, one gets

$$S = -\frac{1}{2}\int\left[(\chi')^2 - (\nabla\chi)\cdot(\nabla\chi) - m_{\text{eff}}\chi^2\right]d^3xdt, \tag{3.74}$$

the Lagrangian density is then given by

$$\mathcal{L} = \frac{1}{2}\left[(\chi')^2 - (\nabla\chi)\cdot(\nabla\chi) - m_{\text{eff}}(\eta)\chi^2\right] \tag{3.75}$$

hence, the conjugated momentum becomes:

$$\pi \equiv \frac{\partial\mathcal{L}}{\partial\chi'} = \chi'\,. \tag{3.76}$$

[9] Since we are mainly using the Heisenberg picture, one should note that, although the vacuum state does not evolve, the creation and annihilation operators evolve, and hence the number of particles.

The total classical Hamiltonian can be obtained by means of a Legendre transform, and becomes

$$H = \frac{1}{2}\int \left[\pi^2 + (\nabla\chi)\cdot(\nabla\chi) + m_{\text{eff}}^2(\eta)\chi^2\right] d^4x\,. \tag{3.77}$$

We now proceed to substitute the mode expansion (3.70) to obtain the quantum Hamiltonian $\hat{H}$ as a function of the creation and annihilation operators:

$$\hat{H}(\eta) = \frac{1}{4}\int \left[F_k^*(\eta)\hat{a}_k^-\hat{a}_{-k}^- + F_k(\eta)\hat{a}_k^+\hat{a}_{-k}^+ + \left(2\hat{a}_k^+\hat{a}_{-k}^- + \delta^3(0)\right)E_k(\eta)\hat{\mathbb{1}}\right] d^3\vec{k}\,, \tag{3.78}$$

where we have introduced the functions

$$\begin{aligned} E_k(\eta) &\equiv |v_k'|^2 + \omega_k(\eta)|v_k|^2\,, \\ F_k(\eta) &\equiv v_k'^2 + \omega_k(\eta)v_k^2\,. \end{aligned}$$

Now, consider the vacuum state $|0_v\rangle$ associated to the still unkown modes $\{v_k\}$. Assuming it to be a Hamiltonian eigenstate at time $\eta = \eta_0$, the vacuum energy is then given by

$$E_0(\eta_0) = \langle 0_v|\hat{H}(\eta_0)|0_v\rangle = \frac{1}{4}\delta^{(3)}(0)\int E_k(\eta_0)d^3\vec{k}\,, \tag{3.79}$$

where the factor $\delta^{(3)}(0)$ appears because we are considering the field energy over the whole space-time, which is infinite. However, if we consider the respective energy density, it is is given by

$$\begin{aligned} \varepsilon(\eta_0) &\equiv \frac{1}{4}\int E_k(\eta_0)d^3\vec{k}\,, \\ &= \frac{1}{4}\int \left(|v_k'|^2 + \omega_k^2(\eta_0)|v_k(\eta_0)|^2\right) d^3\vec{k}\,, \end{aligned}$$

and we now have the task of determining the mode $\{v_k\}$ that minimize such energy.[10] Note that the total energy density is a sum of each oscillator's contribution, which is given by

$$|v_k'(\eta_0)|^2 + \omega_k^2(\eta_0)|v_k(\eta_0)|^2\,, \tag{3.80}$$

which, assuming $\omega_k^2 > 0$, is positive definite. Hence, we can proceed to minimize each oscillator's contribution, which will minimize the total energy. Since in Minkowski space the modes were given by plane waves $\left\{e^{-\omega_k t}/\sqrt{\omega_k}\right\}$, we may suppose a plane wave ansatz:

$$\phi_k = r_k e^{i\alpha_k}\,, \tag{3.81}$$

[10] This can be formalized by quantizing the field in a box of volume V with periodic boundary conditions. The volume V would then replace the infinite factor $\delta^{(3)}(0)$.

which, if we substitute in the normalization condition (3.30), implies the constraint:

$$1 = \alpha'_k r_k^2 \,,$$

and the energy density at time η_0 then becomes:

$$E_k(\eta_0) = r_k'^2 + \frac{1}{r_k^2} + \omega_k^2 r_k^2 \,.$$

Since $\omega_k(\eta_0)$ is a constant (it does not depend on the mode choice), we have that $E_k = E_k(r_k, r'_k)$. The energy is extremized considering $dE = 0$, which implies the conditions

$$\frac{\partial E_k}{\partial r_k} = 0 \implies r'_k = 0 \,, \tag{3.82a}$$

$$\frac{\partial E_k}{\partial r'_k} = 0 \implies -\frac{2}{r_k^3} + 2\omega_k^2 r_k = 0 \,, \tag{3.82b}$$

whose solution is given by

$$r_k(\omega_k) = \frac{1}{\sqrt{\omega_k}} \,,$$
$$r'_k(\omega_k) = 0 \,.$$

We now proceed to analyze the second derivatives, which are given by

$$\frac{\partial^2 E_k}{\partial r_k^2} = 2\omega_k^2 \,, \tag{3.83a}$$

$$\frac{\partial^2 E_k}{\partial r_k \partial r'_k} = 0, \tag{3.83b}$$

$$\frac{\partial^2 E_k}{\partial r_k'^2} = 2 \,, \tag{3.83c}$$

from which follows

$$\frac{\partial^2 E_k}{\partial r_k^2}\frac{\partial^2 E_k}{\partial r_k'^2} - \left(\frac{\partial^2 E_k}{\partial r_k r'_k}\right)^2 = 4\omega_k^2 \,, \tag{3.84}$$

and we see that this function presents a minimum only if $\omega_k^2 > 0$.

Substituting the results (3.83) in our ansatz, the mode functions that minimize energy are then found to be

$$\phi_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}} e^{i\alpha_k(\eta_0)} \tag{3.85a}$$

which then determine a vacuum state $|0_{\text{min}}\rangle$ that is an eigenstate of the Hamiltonian operator $\hat{H}$ with lowest possible eigenvalue at time $\eta = \eta_0$.

It is important to emphasize that, as we have shown for Minkowski spacetime, this is not the only possible vacuum choice. In the latter case, the minimum energy prescription yeilded a vacuum state that is preferred by the inertial observers, but this is not the case for this vacuum state. This happens because, due to time evolution, the chosen state will present particle creation, so that a observer that sees such vacuum at time $\eta = \eta_1$ will see particles in the same state at time $\eta = \eta_2$.

It is also of fundamental importance to note that our prescription was obtained with the assumption that $\omega_k^2 > 0$. However, since

$$\omega_k^2 = m_{\text{eff}} - k^2 \,, \tag{3.86}$$

$$= m^2 a^2 - \frac{a''}{a} - k^2 \,, \tag{3.87}$$

we see that, depending on the background dynamics, the effective frequency can be negative $\omega_k^2 < 0$, which means that the oscillator's contributions are not positive definite in general[11]. Worse yet: since the energy can even become negative, it is not bounded from below, and a minimum is not even defined. Due to such problems, we now proceed to present another vacuum prescription, in order complement the minimum energy choice.

3.3.2 Adiabatic Vacuum Prescription

In flat Minkowski spacetime, we have chosen the modes as plane waves $v_k \propto e^{ikt}$, which defined a particle notion associated with inertial observers. However, in curved spacetimes, such plane waves may not be good approximations to the modes. More precisely, if we consider a wavepacket with momentum spread Δk, the particle momentum is only defined if $\Delta k \ll k$, that is, if the fluctuations are small with respect to the mean value k. Since the characteristic scale λ of the wavepacket is inversely proportional to Δk, $\lambda \sim 1/\Delta k$, we have that the notion of a particle is well defined only if $\lambda \gg 1/k$. Hence, if spacetime geometry varies significantly across a region of size λ, then the plane waves are not a good approximation to the wave equation (3.1) solutions at such region and the usual Minkowski spacetime definition of particles fails [64, 69].

Nevertheless, one could also consider an *approximate* notion of particle, determined by a specific choice of modes. To do so, we consider that the modes are given by approximate plane waves. This is implemented by assuming the ansatz

$$v_k(\eta) = \frac{1}{\sqrt{W_k(\eta)}} \exp\left[i \int_{\eta_0}^{\eta} W_k(\eta) d\eta\right] , \tag{3.88}$$

which is essentially the usual plane wave expansion $\phi_k(t) = e^{i\omega_k t}/\sqrt{\omega_k}$, but with the frequency ω_k replaced by a function, $W_k(\eta)$. Since we want such prescription to describe approximate particles, the function $W_k(\eta)$ must vary slowly, which can be visualized in Figure 3.3.

[11] In this case, the modes do not even present oscillatory behavior, which makes the usual Minkowski particle definition not applicable.

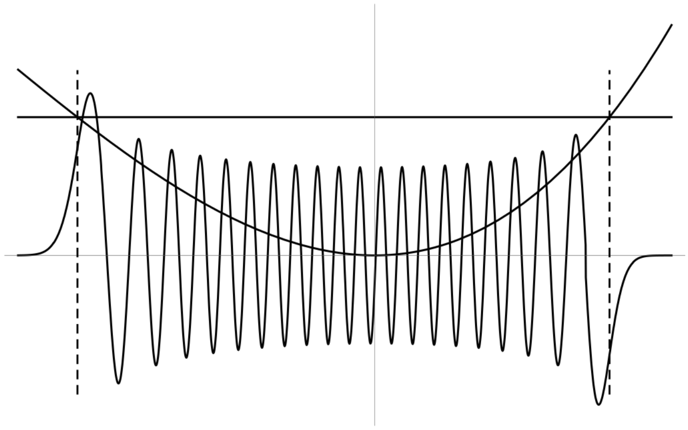

Figure 3.3: Illustration of the WKB approximation in quantum mechanics. Note that the frequency (number of oscillations) is approximately constant in the whole interval. Source: Wikipedia.

We now proceed to substitute such ansatz in the mode evolution equation (3.68) to impose conditions on the unkown function $W_k(\eta)$. Explicit evaluation leads to

$$v_k(\eta)' = \exp\left[i\int_{\eta_0}^{\eta} W_k(\eta)d\eta\right]\left\{-\frac{1}{2}W_k^{-3/2}W_k' + iW_k^{1/2}\right\} ,$$
$$\implies v_k(\eta)'' = \frac{1}{\sqrt{W_k(\eta)}}\exp\left[i\int_{\eta_0}^{\eta} W_k(\eta)d\eta\right]\left\{-W_k^2 + \Theta_k\right\} ,$$

where we introduced

$$Q_k(\eta) \equiv \frac{1}{2}\left[\frac{3}{2}\left(\frac{W_k'}{W_k}\right) - \frac{W_k''}{W_k}\right] \tag{3.89}$$

and one can see clearly that $v_k''(\eta) = \left(-W_k^2 + Q_k\right)v_k$. The mode evolution equation (3.68) is then satisfied by the ansatz only if the condition

$$W_k^2 = \omega_k^2\left(1 - \frac{Q_k}{\omega_k^2}\right) , \tag{3.90}$$

is satisfied. The assumption that the solution $v_k(\eta)$ is an approximate plane wave is implemented if one considers that $W_k \approx \omega_k$ and that it varies slowly, which is satisfied if one assumes that W_k and all its time derivatives change substantially only for large intervals of time $T \gg 1/\omega_k$. For such approximation to be valid, the condition

$$\left|\frac{Q_k}{\omega_k^2}\right| \ll 1 \tag{3.91}$$

must be satisfied. Hence, we consider an expansion of the form

$$W_k(\eta) = {}^{(0)}W_k(\eta) + \frac{1}{T\omega_k}{}^{(1)}W_k(\eta) + \frac{1}{(T\omega_k)^2}{}^{(2)}W_k(\eta) + \ldots$$

Substituting such consideration in the condition (3.91) and grouping terms of same order leads to a recurrence relation for the expansion of W_k. In particular, the first two contributions are given by

$$
\begin{aligned}
{}^{(0)}W_k(\eta) &= \omega_k \,, \\
{}^{(2)}W_k(\eta) &= \omega_k \left(1 - \frac{1}{4}\frac{\omega_k''}{\omega_k^3} + \frac{3}{8}\frac{\omega_k'^2}{\omega_k^4}\right) .
\end{aligned}
$$

It should be noted that, although in principle one could obtain ${}^{(n)}W_k$ to arbitrary order, the series is only asymptotic: the approximation reaches an optimum value at a particular N and becomes worse at larger orders [69]. It is also interesting to point out that, for the case of flat Minkowski spacetime, since the ω_k are constants, one obtains the trivial solution $W_k = \omega_k$ recovering the usual Minkowski vacuum state.

Now that we have developed vacuum prescriptions for quantum fields in FLRW spacetimes, we may proceed to quantize cosmological perturbations and compute their associated power spectrum, which may be used to compute the CMB spectrum and make predictions about the universes large scale structure.

4 BIANCHI I PERTURBATION THEORY

In this chapter, we combine the previous concepts to study perturbed cosmological models defined on Bianchi I backgrounds. We start by defining perturbations and gauge invariant quantities in section 4.1, in a way that *does not* depend on the background dynamics. In section 4.2, we proceed by analyzing a bouncing model defined by quantizing a Bianchi I background coupled to a barotropic fluid, in similar fashion to subsection 2.3.2. We then proceed to analyze an inflationary model characterized by the potential $V(\phi) = m\phi^2/2$ in section 4.3 and analyze its physical predictions for the CMB spectrum.

4.1 Perturbative Parametrization

In this section we analyze perturbations on general Bianchi I backgrounds. We follow mainly [23]. We emphasize that this decomposition is kinematical in the sense that it does not involves dynamics: the same perturbative parametrization can be used for gravitational theories that do not rely on the Einstein equations — such as the $f(R)$ class — and even for backgrounds that follow quantum dynamics.

We start by defining projection operators, which are used to extract the scalar, vector and tensor modes from a given perturbation. The same operators could be used to extract the SVT modes in the FLRW backgrounds studied in section 2.4, but do not need to be explicitly defined. However, as we shall see, those will be essential to study the mode dynamics for Bianchi I backgrounds in General Relativity. We then proceed by considering perturbations on the Bianchi metric (2.9) in a similar fashion to the FLRW case. In particular, we shall perform an SVT decomposition and then define the analogous Bardeen variables for a Bianchi I background.

4.1.1 Projection Operators

It is convenient to define the projection operators in Fourier space. We then define the Fourier transform of a function $f(x^j, \eta)$ as:

$$f(x^j, \eta) = \frac{1}{(2\pi)^{3/2}} \int \tilde{f}(k_i, \eta) d^3 dk_i \,,$$

where we highlighted the components of the wavevector k_i due to anisotropy. It should be noted that, due to said anisotropy, there will occur non-trivial differences with respect to the isotropic case. For instance, if we consider such vectors to be time independent

$$\frac{dk_i}{d\eta} = 0 \,,$$

it follows that their "contravariant" form $k^i \equiv \gamma^{ij}k_j$ where γ^{ij} is the spatial metric, becomes a time dependent quantity, since

$$\begin{aligned}\frac{dk^i}{d\eta} &= \frac{d(\gamma^{ij}k_j)}{d\eta}\,,\\ &= -2\sigma^{ij}k_j\,,\end{aligned}$$

where, due to the parametrization defined on (4.104), $\gamma_{ij} = e^{2\beta_i}\delta_{ij} \implies \gamma^{ij} = e^{-2\beta_i}\delta_{ij}$ and $\sigma_{ij} = \gamma_{ij'}/2 \implies \sigma^{ij} = -\gamma^{ij\prime}/2$. This also implies that the modulus k of the wavevector itself will become time dependent:

$$\begin{aligned}k^2 \equiv k^i k_i &= \gamma^{ij}k_i k_j\,,\\ \implies \frac{1}{k}\frac{dk}{d\eta} &= -\sigma^{ij}\hat{k}_i\hat{k}_j\,,\end{aligned}$$

where we introduced the unit vector $\hat{k}_i \equiv k_i/k$, which evolves as

$$\left(\hat{k}^i\right)' = \left(\sigma^{jl}\hat{k}_j\hat{k}_l\right)\hat{k}^i - 2\sigma^{ij}\hat{k}_j\,.$$

We are now ready to define our projection operators. Recall that we can decompose an arbitrary 3 dimensional euclidean vector field $\vec{v}$ components v_i as

$$v_i = \partial_i v + \bar{v}_i, \quad \text{with} \quad \partial^i\bar{v}_i = 0\,, \tag{4.1}$$

provided it decays to 0 at infinity.[1] However, this decomposition can be better understood in Fourier space. Applying a Fourier transform, $\partial_i \to k_i$, and the decomposition becomes:

$$\tilde{v}_i = k_i\tilde{v} + \tilde{\bar{v}}_i \quad \text{with} \quad k^i\tilde{\bar{v}}_i = 0\,.$$

The condition $k^i\tilde{\bar{v}}_i = 0$ is particularly interesting, since it shows that, in Fourier space, $\tilde{\bar{v}}_i$ lives in the subspace that is orthogonal to the wavevector k^i, $V_\perp$, which is 2-dimensional by construction. Now, consider a orthonormal base $\{e^1, e^2\}$, $e^a \in V_\perp$, which satisfies,

$$\begin{aligned}e^a{}_i k_j\gamma^{ij} &= 0\,,\\ e^a{}_i e^b{}_i\gamma^{ij} &= 0\,, \quad \text{for} \quad a \neq b\,,\end{aligned}$$

where $a, b = 1, 2$, and, in this chapter only, we denote latin indexes from the start of the alphabet as denoting labels rather than abstract indexes. This is due to the fact that we are dealing with spatial quantities that have labels.

[1] To use such decomposition, we have chosen coordinates adapted to the Killing vector fields ξ on the spatial sections $\mathbb{R}^3$, which are orthogonal but are *not* Cartesian due to space anisotropy [23]. Such coordinate system has the advantage that the spatial Christoffel symbols vanish due to $\partial_k\gamma_{ij} = 0$.

Since $\tilde{\tilde{v}}_i \in V_\perp$ we may expand it in our orthogonal basis as

$$\tilde{\tilde{v}}\left(k_i,\eta\right) = \sum_{a=1,2} v_a\left(\hat{k}_i,\eta\right) e^a{}_i\left(\hat{k}_i\right) , \tag{4.2}$$

which defines two degrees of freedom v_a, which are the coordinates associated to such basis. Such coordinates can be extracted by defining a projector operator analogous to (A.34) by

$$P_{ij} \equiv \gamma_{ij} - \hat{k}_i\hat{k}_j = e^1{}_i e^1{}_j + e^2{}_i e^2{}_j , \tag{4.3}$$

which satisfies the properties

$$P^i_{\ j}P^j_{\ k} = P^i_{\ k} , \tag{4.4a}$$
$$P^i_{\ j}k^j = 0 , \tag{4.4b}$$
$$P^{ij}\gamma_{ij} = 2 . \tag{4.4c}$$

The previous operator then extracts the coordinates of the transverse mode of the vector $\vec{v}$ in Fourier space. One may then represent a general vector field $\vec{v}$ in Fourier space by

$$\tilde{v}_i = \left(\hat{k}^j\tilde{v}_j\right)k_i + P_i{}^j\tilde{v}_j , \tag{4.5a}$$

which shows explicitly that an arbitrary vector field is decomposed into 1 scalar mode associated to its projection on the direction of $\hat{k}_i$ and two transverse vector modes associated to projections on the orthogonal subspace $V_\perp$.

We now proceed to define similar projection operators for tensor modes. As we already saw in section 2.4, any rank 2 symmetric tensor field h_{ij} can be decomposed as

$$h_{ij} = 2C\gamma_{ij} + 2\partial_i\partial_j S + 2\partial_{(i}\bar{E}_{j)} + 2\bar{E}_{ij} ,$$

provided the field decays at infinity and the conditions

$$\partial^i E_i = 0 , \tag{4.6a}$$
$$\partial^i E_{ij} = 0 , \tag{4.6b}$$

are satisfied, with the tensor E_{ij} being also traceless: $\gamma^{ij}E_{ij} = E^i_{\ i} = 0$. One may also decompose the field in the more compact form

$$T_{ij} = T\gamma_{ij} + \mathcal{D}_{ij}S + 2\partial_{(i}E_{j)} + 2E_{ij} , \tag{4.7}$$

where we introduced $T \equiv 2S$ and $\mathcal{D}_{ij} \equiv \partial_i\partial_j - \frac{\gamma_{ij}}{3}\nabla^2$.

Note that the 3 dimensional symmetric and trace free rank 2 tensors contain only 2 independent degrees of freedom. Such space is then 2-dimensional and admits a basis $\left\{\varepsilon^\lambda_{ij}\right\}$, with $\lambda = 1,2$. A possible basis may be constructed in terms of the previous vectors basis $\{e^a{}_i\}$ by defining

$$\varepsilon^{+}_{ij} = \frac{e^1{}_i e^1{}_j - e^2{}_i e^2{}_j}{\sqrt{2}}\,, \tag{4.8a}$$

$$\varepsilon^{\times}_{ij} = \frac{e^1{}_i e^2{}_j + e^2{}_i e^1{}_j}{\sqrt{2}}\,, \tag{4.8b}$$

which are known as polarization tensors. It can be readily checked that they satisfy

$$\varepsilon^{\lambda}_{ij}\gamma^{ij} = 0 \quad - \text{ traceless}\,, \tag{4.9a}$$

$$\varepsilon^{\lambda}_{ij}k^{i} = 0 \quad - \text{ transversality}\,, \tag{4.9b}$$

$$\varepsilon^{\lambda}_{ij}\varepsilon^{ij}_{\mu} = \delta^{\lambda}_{\mu} \quad - \text{orthogonality}\,. \tag{4.9c}$$

And the tensor transverse Fourier mode can be expressed in such basis as

$$\tilde{E}_{ij}\left(k_i,\eta\right) = \sum_{\lambda=+,\times} \tilde{E}_{\lambda}\left(k^i,\eta\right)\varepsilon^{\lambda}_{ij}\left(\hat{k}_i\right)$$

with the respective projection being extracted by the operator

$$\Lambda^{ab}_{ij} \equiv P^a_i P^b_j - \frac{1}{2}P_{ij}P^{ab}\,. \tag{4.10}$$

Similarly, one may introduce a "trace extracting operator" by

$$\mathcal{T}_i^{\,j} \equiv \hat{k}_i\hat{k}^j - \frac{1}{3}\delta_i^{\,j}\,. \tag{4.11}$$

With such operators at hand, one may decompose a general symmetric rank 2 tensor by

$$T_{ij} = \left(\frac{1}{3}T_{ab}\gamma^{ab}\right)\gamma_{ij} + \left(\frac{3}{2}T_{ab}\mathcal{T}^{ab}\right)\mathcal{T}_{ij} + 2\hat{k}_{(i}\left(P^a_{j)}\hat{k}^b T_{ab}\right) + \Lambda^{ab}_{ij}T_{ab}\,. \tag{4.12}$$

In particular, the background shear tensor σ_{ij} will be of fundamental importance in our discussion. Hence, we shall analyze the shear decomposition, which is written as

$$\sigma_{ij} = \frac{3}{2}\sigma\mathcal{T}_{ij} + 2\sum_{a=1,2}\sigma_{\mathrm{V}a}\hat{k}_{(i}e^a_{j)} + \sum_{\lambda=+,\times}\sigma_{\mathrm{T}\lambda}\varepsilon^{\lambda}_{ij}\,. \tag{4.13}$$

Since the shear is a symmetric traceless tensor field, it contains 5 degrees of freedom, organized as

$$\sigma \quad - \quad \text{1 scalar degree of freedom}\,; \tag{4.14a}$$

$$\sigma_{Va} \quad - \quad \text{2 vector degrees of freedom}\,; \tag{4.14b}$$

$$\sigma_{T\lambda} \quad - \quad \text{2 tensor degrees of freedom}\,. \tag{4.14c}$$

(4.14d)

The decomposition (4.13) leads directly to the identities

$$\gamma^{ij}\sigma_{ij} = 0\,, \tag{4.15a}$$
$$\hat{k}^i\sigma_{ij} = \sigma_\parallel + \sum_a \sigma_{Va} e^a_{\ i}\,, \tag{4.15b}$$
$$\hat{k}^i\hat{k}^j\sigma_{ij} = \sigma_\parallel\,, \tag{4.15c}$$

and it can be shown that the scalar shear $\sigma \equiv \sigma_{ij}\sigma^{ij}$ leads to the constraint

$$\sigma^2 = \sigma_{ij}\sigma^{ij} = \frac{3}{2}\sigma_\parallel^2 + 2\sum_a \sigma_{\mathrm{v}a}^2 + \sum_\lambda \sigma_{\mathrm{T}\lambda}^2\,. \tag{4.16}$$

4.1.2 Gauge Invariant Quantities

Our starting point is to consider a perturbed Bianchi I metric. This can be done in analogous way to the FLRW case by using the appropriate variables. To do so, recall that, in a convenient coordinate system, the Bianchi I line element can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + X^2(t)\mathrm{d}x^2 + Y^2(t)\mathrm{d}y^2 + Z^2(t)\mathrm{d}z^2\,, \tag{4.17}$$

and, introducing the mean scale factor as

$$S(t) \equiv [X(t)Y(t)Z(t)]^{1/3}\,,$$

we may rewrite the line element as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + S^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\,,$$

finally, by introducing conformal time $\mathrm{d}\eta \equiv \mathrm{d}t/S(t)$, the metric becomes

$$\mathrm{d}s^2 = S^2(t)\left(-\mathrm{d}\eta^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right)\,,$$

which is very similar to the FLRW conformal metric (1.22) with $a(t) \to S(t), \delta_{ij} \to \gamma_{ij}$.

We now proceed in analogy to section 2.4. First, consider a perturbed metric $g_{ab} = \bar{g}_{ab} + \delta g_{ab}$ for the case of $\bar{g}_{ab} = g_{ab}^{\text{Bianchi I}}$. The perturbed line element in conformal time then becomes

$$\mathrm{d}s^2 = S^2(\eta)\left[(-(1+2A)\mathrm{d}\eta^2 + 2B_i\mathrm{d}\eta\mathrm{d}x^i + (\gamma_{ij} + h_{ij})\,\mathrm{d}x^i\mathrm{d}x^j\right]$$

where we parametrized the perturbations as

$$A \equiv -\frac{1}{2S^2}\delta g_{00}\,, \tag{4.18a}$$
$$B_i \equiv \frac{1}{2S^2}\delta g_{0i} \tag{4.18b}$$

$$h_{ij} \equiv \frac{1}{S^2}\delta g_{ij}\,. \tag{4.18c}$$

Here, an important note is in order. In the FLRW case, the background was isotropic, that is, it was invariant under the action of the rotation group. Therefore, it was convenient to parametrize the perturbations in terms of representations of such group: scalars, vectors and tensors. In the Bianchi I case, since the background itself is now anisotropic, there is no such motivation to parametrize perturbations in terms of SVT variables. One then is lead to the question: "is there a more convenient choice to parametrize perturbations for an anisotropic background?" Which remains to be answered.

Since, to our knowledge, there is yet no alternative successful parametrization choice, we shall follow the literature and analyze perturbations in terms of the usual SVT modes [9, 22, 23]. While this parametrization does not have the same mathematical advantages as in the isotropic case, at least it will provide a simple way to check the results consistency by analyzing the isotropic limit characterized by $\sigma_{ij} \to 0$, which should coincide with the usual perturbation theory developed in section 2.4.

We now proceed to define the SVT modes in analogous way to the FLRW case. By imposing that perturbations decay at infinity, one may parametrize the perturbation modes as

$$B_i = \partial_i B + \bar{B}_i\,, \tag{4.19a}$$

$$h_{ij} = 2C\left(\delta_{ij} + \frac{\sigma_{ij}}{\mathcal{H}}\right) + 2\partial_i\partial_j E + 2\partial_{(i}E_{j)} + 2E_{ij}\,, \tag{4.19b}$$

if the following constraints are satisfied [23]

$$\partial^i \bar{B}_i = 0\,, \tag{4.20a}$$

$$\partial^i E_i = 0\,, \tag{4.20b}$$

$$\partial^i E_{ij} = 0\,. \tag{4.20c}$$

where $\mathcal{H} \equiv S'/S$, as in the isotropic case, and $E^i_{\ i} = 0$. Note that this decomposition has a non-trivial difference in comparison to the FLRW case: the presence of the shear tensor σ_{ij} in the tensor mode. This is included in order for simple gauge invariant quantities to be later defined. If such term is not included in the decomposition, the gauge invariant quantities are rather hard to define when compared to the isotropic case, due to the fact that the SVT modes have more complicated transformation laws [9].

Now, as in the FLRW case, one may apply an infinitesimal coordinate shift $x^\mu \to x^\mu + \xi^\mu$ to analyze how the SVT modes transform under a gauge transformation $g_{ab} \to g_{ab} + \mathscr{L}_\xi g_{ab}$. Using the Lie derivative (A.24), the metric transforms as $\mathscr{L}_\xi g_{ab} = \nabla_a\xi_b + \nabla_b\xi_a$. Decomposing the components ξ^μ as

$$\xi^0 = T(\vec{x},\eta)\,, \tag{4.21a}$$

$$\xi^i = \partial^i L(\vec{x},\eta) + L^i(\vec{x},\eta)\,, \tag{4.21b}$$

we obtain, explicitly

$$\mathcal{L}_\xi \bar{g}_{00} = -2S^2 \left(T' + \mathcal{H}T\right) , \tag{4.22a}$$
$$\mathcal{L}_\xi \bar{g}_{0i} = S^2 \left(\xi_i' - \partial_i T - 2\sigma_{ji}\xi^j\right) , \tag{4.22b}$$
$$\mathcal{L}_\xi \bar{g}_{ij} = S^2 \left[2\partial_{(i}\xi_{j)} + 2\mathcal{H}T\gamma_{ij} + 2T\sigma_{ij}\right] . \tag{4.22c}$$

Using (4.22) and the projector operators, we find that the SVT modes transform as

1. **Scalar Modes**

$$A \to \tilde{A} = A + \frac{(ST)'}{S} , \tag{4.23a}$$
$$B \to \tilde{B} = B - T + \frac{(k^2 L)'}{k^2} , \tag{4.23b}$$
$$C \to \tilde{C} = C + \mathcal{H}T , \tag{4.23c}$$
$$E \to \tilde{E} = E + L , \tag{4.23d}$$
$$\tag{4.23e}$$

2. **Vector Modes**

$$\bar{B}^i \to \tilde{\bar{B}}^i = \bar{B}^i + L^{i\prime} - 2ik^j\sigma_{lj}P^{li}L , \tag{4.24a}$$
$$E^i \to \tilde{E}^i = E^i + L^i , \tag{4.24b}$$

3. **Tensor Mode**

$$E_{ij} \to \tilde{E}_{ij} = E_{ij} , \tag{4.25}$$

and, as in the FLRW case, apart from the unique tensor mode, the perturbation modes are gauge dependent.[2]

Now, by combinations of the SVT modes, we may construct gauge invariant objects that generalize the Bardeen Variables as

1. **Scalar Modes**

$$\Phi(\vec{x},\eta) \equiv A + \frac{1}{S}\left\{S\left[B - \frac{(k^2E)'}{k^2}\right]\right\}' , \tag{4.26a}$$
$$\Psi(\vec{x},\eta) \equiv -C - \mathcal{H}\left[B - \frac{(k^2E)'}{k^2}\right] ; \tag{4.26b}$$

2. **Vector Modes**

$$\Phi^i(\vec{x},\eta) \equiv \bar{B}^i - \gamma^{ij}(E_j)' + 2ik^j\sigma_{lj}P^{li}E . \tag{4.27}$$

[2]This would not be the case if the shear tensor σ_{ij} was not included in the decomposition (4.19): even the tensor mode would be gauge dependent.

Now that we have defined gauge invariant quantities, we may choose a gauge to perform calculations and express geometrical quantities in terms of them. As in the isotropic case, we define the Newtonian Gauge by the conditions

$$A = \Phi \tag{4.28a}$$
$$C = \Psi \tag{4.28b}$$
$$(E^i)' = -\Phi^i\,, \tag{4.28c}$$

with the later condition being equivalent to $\Phi_i = -E_i' + 2\sigma_{ij}E^j$ [9].

To conclude this section, we use the Newtonian gauge to express the Einstein tensor in terms of our gauge invariant quantities for later use:

$$S^2\delta G^0_{\ 0} = -2\nabla^2\Psi + 6\mathcal{H}\Psi' + 2\sigma^2\Psi - \left(\frac{\Psi}{\mathcal{H}}\right)'\sigma^2 + \frac{\sigma^{ij}}{\mathcal{H}}\partial_i\partial_j\Psi \tag{4.29a}$$
$$- \sigma^{ij}\partial_i\Phi_j + \left(E^i_{\ j}\right)'\sigma^j_{\ i} + \left(6\mathcal{H}^2 - \sigma^2\right)\Phi\,, \tag{4.29b}$$
$$S^2\delta G^0_{\ i} = -\sigma^2\frac{\partial_i\Psi}{\mathcal{H}} + \sigma^j_{\ i}\partial_j\left[\Phi + \Psi + \left(\frac{\Psi}{\mathcal{H}}\right)'\right] - 2\partial_i\left(\Psi' + \mathcal{H}\Phi\right) \tag{4.29c}$$
$$+ \frac{1}{2}\nabla^2\Phi_i - 2\sigma^{jk}\partial_j E_{ik} + \sigma^{jk}\partial_i E_{jk}\,, \tag{4.29d}$$
$$S^2\delta G^i_{\ j} = \delta^i_j\left[2\Psi'' + \left(2\mathcal{H}^2 + 4\mathcal{H}'\right)\Phi + \nabla^2(\Phi - \Psi) + 2\mathcal{H}\Phi' + 4\mathcal{H}\Psi'\right] \tag{4.29e}$$
$$+ \partial^i\partial_j(\Psi - \Phi) - \frac{2}{\mathcal{H}}\gamma^{il}\sigma_{k(l}\partial_{j)}\partial^k\Psi + \sigma^i_{\ j}\left[-\mathcal{H}\left(\frac{\Psi'}{\mathcal{H}^2}\right)' + \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)'\Psi + \frac{\nabla^2\Psi}{\mathcal{H}} - \Phi' - \Psi'\right] \tag{4.29f}$$
$$+ \delta^i_{\ j}\left[\sigma^2\left(\Phi + \left(\frac{\Psi}{\mathcal{H}}\right)' - 2\Psi\right) + \frac{\sigma^{kl}}{\mathcal{H}}\partial_k\partial_l\Psi\right] \tag{4.29g}$$
$$+ \left(E^i_{\ j}\right)'' + 2\mathcal{H}\left(E^i_{\ j}\right)' - \nabla^2 E^i_{\ j} + 2\left[\sigma^i_{\ k}\left(E^k_j\right)' - \sigma^k_j\left(E^i_{\ k}\right)'\right] - \left[\left(E^k_l\right)'\sigma^l_{\ k}\right]\delta^i_{\ j} \tag{4.29h}$$
$$+ \delta^i_{\ j}\sigma^{kl}\partial_k\Phi_l - \gamma^{ik}\left[\partial_{(k}\left(\Phi_{j)}\right)' + 2\mathcal{H}\partial_{(k}\Phi_{j)} - 2\sigma^l_{\ (k}\partial_{|l|}\Phi_{j)}\right]. \tag{4.29i}$$

Again, we emphasize that the presented perturbative decomposition does not depend on the background dynamics, which may not be given by the classical Einstein equations. Hence, it can be used both in inflationary and bouncing models, classical or quantum.

4.2 Quantum Cosmology in the Bianchi I Minisuperspace

In this section, we consider a Bianchi I minisuperspace quantum cosmological model in canonical quantum gravity coupled to a single barotropic fluid, in similar fashion to the case explored in section 2.3. A similar model with the matter sector being described by a scalar field has also been considered in [55, 74], where it was shown that non-singular solutions exist, and that the universe isotropizes, recovering the FLRW in the asymptotic future. Another quantum bouncing scalar field model was also considered in [75] in the context of Loop Quantum Gravity, which demanded the construction of a Bianchi I perturbation theory in Hamiltonian formalism in a companion paper [76].[3] A Hamiltonian formalism was also considered in [77] and such Hamiltonian methods may prove useful for applications in quantum bounces in the future.

As in the FLRW case, we couple matter to geometry by considering a Hamiltonian

$$H_T = H_G + H_M \,, \tag{4.30}$$

however, now we consider the minisuperspace of Bianchi I metrics, which is more general than the flat FLRW one. Therefore, the main difference shall occur in the gravitational section represented by H_G.

To obtain the new gravitational Hamiltonian, we start from its associated Lagrangian and perform a Legendre transform. Recall that the gravitational Lagrangian in GR is given by

$$L_G = \frac{1}{2\kappa} R \,, \tag{4.31}$$

where R is the Ricci scalar. To compute such Lagrangian, we first express the Bianchi I metric in the ADM form (2.56) [78]:

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + X^2(t)\mathrm{d}x^2 + Y^2(t)\mathrm{d}y^2 + Z^2(t)\mathrm{d}z^2 \,, \tag{4.32}$$

where the shift vector N_a is trivial due to homogeneity. It will prove useful to express the expansion rates $X_i(t)$ in the following parametrization [55]:

$$X(t) = e^{\beta_0 + \beta_+ + \sqrt{3}\beta_-} \,, \tag{4.33a}$$

$$Y(t) = e^{\beta_0 + \beta_+ - \sqrt{3}\beta_-} \,, \tag{4.33b}$$

$$Z(t) = e^{\beta_0 - 2\beta_+} \,, \tag{4.33c}$$

with the isotropic limit being characterized by $\beta_+ = \beta_- = 0$ and the mean scale factor being given by $S(t) = e^{\beta_0}$. Direct computation then leads to the lagrangian

$$L_G = -6\frac{e^{\beta_0}}{N}\left(\dot{\beta}_0^2 - \dot{\beta}_+^2 - \dot{\beta}_-^2\right) \,, \tag{4.34}$$

where we discarded boundary terms [74]. We proceed by computing the canonical momenta:

$$\Pi_0 \equiv \frac{\partial L_H}{\partial \dot{\beta}_0} = -12\frac{e^{3\beta_0}}{N}\dot{\beta}_0 \,, \tag{4.35a}$$

$$\Pi_+ \equiv \frac{\partial L_H}{\partial \dot{\beta}_+} = +12\frac{e^{3\beta_0}}{N}\dot{\beta}_+ \,, \tag{4.35b}$$

[3] Of course, such alternative models suffer from the same "problem" as inflationary models, in the sense that they invoke an yet unobserved scalar field ϕ. Worse yet, [75] considers a bounce followed by an inflationary phase, having the benefits of both paradigms and being less simple in this sense.

$$\Pi_- \equiv \frac{\partial L_H}{\partial \dot{\beta}_-} = +12\frac{e^{3\beta_0}}{N}\dot{\beta}_- \,, \tag{4.35c}$$

and, after performing a Legendre transformation, the final Hamiltonian is given by

$$H_T = Ne^{-3\beta_0}\left[-\frac{1}{24}\left(\Pi_0^2 - \Pi_+^2 - \Pi_-^2\right) + e^{3(1-w)\beta_0}p_T\right] , \tag{4.36}$$

where w is the fluid equation of state constant parameter $p = w\rho$. This Hamiltonian is totally analogous to (2.66). As pointed out in [79] a particular feature of this Hamiltonian is the hyperbolic signature of its kinetic term, which is not present in the isotropic FLRW case and will lead to non-trivial differences. Also, simlarly to the isotropic case, we impose the condition

$$\Psi'|_{\beta_i \to \pm\infty} = \alpha\Psi|_{\beta_i \to \pm\infty} \,, \tag{4.37}$$

$\alpha \in (-\infty, +\infty]$ in order for the Hamiltonian to be Hermitian [74, 78]. It should be noted that, although the obtained Hamiltonian is Hermitian, it *is not* self-adjoint, which will lead to non-trivial consequences [78].

We now proceed to perform canonical quantization analogously to section 2.3. In particular, we choose the representation

$$\hat{\beta}_i\Psi(\beta_i, T) = \beta_i\Psi(\beta_i, T), \qquad \hat{\Pi}_i\Psi(\beta_i, T) = -i\frac{\partial}{\partial \beta_i}\Psi(\beta_i, T) \,, \tag{4.38}$$

$$\hat{T}\Psi(\beta_i, T) = T\Psi(\beta_i, T), \qquad \hat{\Pi}_T\Psi(\beta_i, T) = -i\frac{\partial}{\partial T}\Psi(\beta_i, T) \,, \tag{4.39}$$

and consider T as a time parameter due to the fact that it appears linearly in the Hamiltonian. It should also be pointed out that, although we defined a notion of time associated to the matter sector, in the Bianchi I case this could also be done using the gravitational degrees of freedom only, as was pointed out in [80], which explores possible time choices in a vacuum Bianchi I minisuperspace model.

In the chosen representation, we obtain the following Wheeler-De Witt equation

$$\left(\frac{\partial^2}{\partial \beta_0^2} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2}\right)\Psi(\beta_i, T) = -24e^{3(1-w)\beta_0}\frac{\partial}{\partial T}\Psi(\beta_i, T) \,, \tag{4.40}$$

which can be solved through the separation of variables method. First, we split the wavefunction as

$$\Psi(\beta_i, T) = \psi(\beta_i, T)e^{-iET} \,, \tag{4.41}$$

which leads to the equation

$$\left(\frac{\partial^2}{\partial^2 \beta_0^2} - \frac{\partial^2}{\partial^2 \beta_+^2} - \frac{\partial^2}{\partial^2 \beta_-^2}\right)\psi(\beta_i, T) = -24Ee^{3(1-w)\beta_0}\psi(\beta_i, T) \,, \tag{4.42}$$

we proceed by expressing the wave function as

$$\psi(\beta_i) = \Upsilon_0(\beta_0)\Upsilon_+(\beta_+)\Upsilon_-(\beta_-)\,, \tag{4.43}$$

which leads to

$$\frac{1}{\Upsilon_0}\frac{\partial^2\Upsilon_0}{\partial\beta_0^2} + 24Ee^{(1-w)\beta_0} - \frac{1}{\Upsilon_+}\frac{\partial^2\Upsilon_+}{\partial\beta_+^2} - \frac{1}{\Upsilon_-}\frac{\partial^2\Upsilon_-}{\partial\beta_-^2} = 0\,, \tag{4.44}$$

and can be split in the set of equations

$$\frac{d^2\Upsilon_+}{d\beta_+^2} = -k_+^2\Upsilon_+\,, \tag{4.45a}$$

$$\frac{d^2\Upsilon_-}{d\beta_-^2} = -k_-^2\Upsilon_-\,, \tag{4.45b}$$

$$\frac{d^2\Upsilon_0}{d\beta_0^2} = -\left(24Ee^{3(1-w)\beta_0}k_+^2 + k_-^2\right)\Upsilon_0\,, \tag{4.45c}$$

where the $k_\pm$ are separation parameters. The $\pm$ solutions are trivially given by plane waves

$$\Upsilon_\pm = A_\pm e^{+ik_\pm\beta_\pm} + B_\pm e^{-ik_\pm\beta_\pm}\,, \tag{4.46}$$

while the remaining equation can be rewritten as a Bessel equation

$$\frac{d^2\Upsilon_0}{dy^2} + \frac{1}{y}\frac{d\Upsilon_0}{dy} + \left(\frac{24E}{r^2} + \frac{k^2}{r^2}\frac{1}{y^2}\right)\Upsilon_0 = 0\,, \tag{4.47}$$

where we introduced

$$a \equiv e^{\beta_0}\,, \quad y \equiv a^4\,, \quad r \equiv \frac{3}{2}(1-w)\,, \quad k^2 \equiv k_+^2 + k_-^2\,. \tag{4.48}$$

The general solution for Υ_0 is then given by

$$\Upsilon_0 = C_1 J_\nu\left(\frac{\sqrt{24E}}{r}a^r\right) + C_2 J_{-\nu}\left(\frac{\sqrt{24E}}{r}a^r\right)\,, \tag{4.49}$$

where $\nu \equiv ik/r$, and $C_{1,2}$ are integration constants. This fixes the energy eigenfunctions, which can be used to express the general solution by superposition.

We shall now analyze a simple solution in order to illustrate some consequences of the fact that the obtained Hamiltonian (4.40) is not self-adjoint. Following [79], we now consider a gaussian wavepacket with $k_- = 0$ for simplicity:

$$\Psi = \int_{-\infty}^{+\infty}\int_0^\infty A(k_+,q)\,e^{ik_+\beta_+}J_v(qa^r)\,e^{-iq^2T}dk_+dq\,, \tag{4.50}$$

where $q \equiv \sqrt{24E}/r$ and $A(k_+,q) \equiv e^{-\gamma k_+^2}q^{\nu+1}e^{-\lambda q^2}$. In this case, the integrals can be explicitly computed, and lead to the following closed expression

$$\Psi = \frac{1}{B}\sqrt{\frac{\pi}{\gamma}}\exp\left[-\frac{a^{2r}}{4B} - \frac{(\beta_+ + C(a,B))^2}{4\gamma}\right] \tag{4.51}$$

where we introduced

$$B = \lambda + isT, \quad C(a,B) = \ln a - \frac{2}{3(1-w)}\ln 2B, \quad s = -\frac{3(1-w)^2}{32}.$$

The proposed wavepacket has a rather interesting property: its norm is time dependent. Explicitly, it is given by

$$\int_0^\infty \int_{-\infty}^\infty a^{2-3w}\Psi^*\Psi da d\beta_+ = \frac{\sqrt{2\gamma\pi}}{3(1-w)}\frac{2}{\lambda}F(T) \tag{4.52}$$

where $F(T) = \exp\left(\frac{C_I^2}{2\gamma}\right)$, $C(a,B) = C_R + iC_I$ and

$$C_R = \ln a - \frac{1}{3(1-w)}\ln 4B^*B, \quad C_I = \frac{-2}{3(1-w)}\arctan\left(\frac{sT}{\lambda}\right).$$

Note that the obtained norm depends explicitly on time T through the function $F(T)$. Hence, probabilities are not conserved, and the evolution is not unitary.

The loss of unitarity happens mainly due to the previously mentioned Hamiltonian's hyperbolic signature. Recall that, after defining the T as the time variable, the effective Hamiltonian was given by

$$\hat{H}_{\text{eff}} = \left(\frac{\partial^2}{\partial\beta_0^2} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2}\right),$$

which was Hermitian due to condition (4.37), but is not self-adjoint. In fact the result is even stronger: such operator does not admit any self-adjoint extension, as was discussed in [78]. Its signature also leads to the existence of negative energies and possible instabilities.

The non-unitary evolution leads to problems in interpreting the state of the universe. As stated in [78], one may consider creations of universes, but that concept remains to be formally defined.[4] Worse yet: since time evolution is not unitary, different interpretations of quantum mechanics lead to different predictions. To see this, we present the expectation values of two possible interpretations: many worlds and De Broglie-Bohm.

1. **Many Worlds Interpretation**

   In this interpretation, given an observable $\hat{A}$, its expected value is given by

$$\langle \hat{A} \rangle = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{3(1-w)\beta_0}\Psi^*\hat{A}\Psi d\beta_0 d\beta_+}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{3(1-w)\beta_0}\Psi^*\Psi d\beta_0 d\beta_+}.$$

[4] Creations of universes out of "nothing" in quantum cosmology were discussed by Vilenkin in [81, 82], but in a quite different context.

we now compute the expected value of the mean scale factor $S(t)$. Since $S(t) = e^{\beta_0}$, we first obtain

$$\langle\beta_0\rangle = \frac{1}{3(1-w)}\left\{\ln\left(\frac{2|B|^2}{\lambda}\right) + n\right\},$$

which leads to [78]

$$e^{\langle\beta_0\rangle} = a_B\left[1 + \left(\frac{T}{T_B}\right)^2\right]^{\frac{1}{3(1-w)}},$$

where a_B, T_B are constants. Such scale factor describes a bouncing universe, and is *exactly* the one obtained in the isotropic case, (2.76). Therefore, in the many worlds interpretation of quantum mechanics, if one considers an ensemble of universes, the expected value is that its evolution is isotropic, which is a trivial result compared to the isotropic case.

2. **De Broglie-Bohm Interpretation**

To compute the expected values, we first obtain the Bohmian trajectories and then consider an ensemble of them in order to extract a expected value. We start by noting that the phase of our wavepacket (4.51) is given by

$$S\left(\beta_0, \beta_+, T\right) = -\arctan\left(\frac{sT}{\lambda}\right) + \frac{sTa^{3(1-w)}}{4B^*B} - \frac{C_I}{2\gamma}\left(\beta_+ + C_R\right),$$

where it was used that $C = C_R + iC_I$ and $\arctan(x) = \ln((1+x)/(1-x))/2$. The Bohmian trajectories can then be obtained by solving the guidance relations (2.64):

$$\Pi_+ = \frac{\partial S}{\partial \beta_+}, \tag{4.53a}$$

$$\Pi_0 = \frac{\partial S}{\partial \beta_0}, \tag{4.53b}$$

Since we are using T as our time variable, $\mathrm{d}t = e^{3w\beta_0}\mathrm{d}T$, and the conjugate momenta become

$$\Pi_0 = -12S^{2-3w}\dot{\beta}_0, \tag{4.54a}$$

$$\Pi_+ = +12S^{3(1-w)}\dot{\beta}_+. \tag{4.54b}$$

The Bohmian trajectories are then obtained by solving

$$-12a^{2-3w}\dot{S} = 3(1-w)\frac{sT}{4B^*B}a^{3(1-w)} - \frac{C_I}{2\gamma}, \tag{4.55a}$$

$$12a^{3(1-w)}\dot{\beta}_+ = -\frac{C_I}{2\gamma}, \tag{4.55b}$$

with the final result being given by

$$S(T) = \left(\frac{-1}{24s\lambda\gamma}\right)^{\frac{1}{3(1-w)}}\left[\lambda^2 + s^2T^2\right]^{\frac{1}{3(1-w)}}\left[\arctan^2\left(\frac{sT}{\lambda}\right) + E\right]^{\frac{1}{3(1-w)}}, \tag{4.56a}$$

$$\beta_+(T) = -\frac{1}{3(1-w)} \ln\left\{\arctan^2\left(\frac{sT}{\lambda}\right) + E\right\} + \ln\left\{[-24s\lambda\gamma]^{\frac{1}{3(1-w)}}\right\} + \ln D, \tag{4.56b}$$

which present a bounce as in the previous case, but are anisotropic. However, to compare such results with the many worlds case, we need to compute its expected values using an ensemble of Bohmian trajectories. In the De Broglie-Bohm interpretation, the expected value of a trajectory $x(t)$ is obtained by

$$\bar{x} = \int_{-\infty}^{+\infty} \mathcal{R}_0 x(t) dx_0\,, \tag{4.57}$$

where $\mathcal{R}_0$ corresponds to the module of the wavefunction at $T = 0$. Explicitly, our expected values then read

$$\overline{\beta_0}(T) = \frac{1}{9(1-w)^2} \frac{\Psi_0^2}{-24s\gamma\lambda} \int_0^\infty \int_0^\infty \exp\left(\frac{y}{48s\gamma}\right) \exp\left(-\frac{(\ln x)^2}{2\gamma}\right) \tag{4.58a}$$

$$\times \ln\left\{\left(\lambda^2 + s^2T^2\right)\left[(\arctan(sT/\lambda))^2 + y\right]\left(-\frac{1}{48s\gamma\lambda}\right)\right\} \frac{dxdy}{x}\,, \tag{4.58b}$$

$$\bar{\beta}_+(\bar{T}) = -\frac{1}{9(1-w)^2} \frac{\Psi_0^2}{-24\gamma s\lambda} \int_0^\infty \int_0^\infty \exp\left(\frac{y}{48s\gamma}\right) \exp\left(-\frac{(\ln x)^2}{2\gamma}\right) \tag{4.58c}$$

$$\times \ln\left\{\frac{[\arctan(sT/\lambda)]^2 + y}{-96\gamma s\lambda x^{3(1-w)}}\right\} \frac{dydx}{x}\,, \tag{4.58d}$$

in this case, the expected values *do not* coincide with the isotropic limit: there is a time dependent anisotropy represent by β_+.

One then sees that, due to the fact that the time evolution is not unitary, different interpretations lead to different predictions [78], which is a problem, since one would need to arbitrarily consider one particular interpretation over the other.

It has been argued that the non-unitary evolution a general feature of anisotropic quantum cosmological models [15], but this is still a matter of debate. In particular, in [83] a Bianchi I model with unitary evolution was constructed by considering a different operator ordering, with similar models being considered for the Bianchi types V and IX in [84]. Therefore, to construct an appropriate minisuperspace model, one then needs to consider particular operator orderings, which must be motivated by reasonable assumptions, a discussion that is beyond the scope of this work.[5]

Due to this difficulties in implementing quantum cosmological models in the Bianchi I minisuperspace at background level, we shall focus on inflationary scalar field models, with an appropriate perturbative Bianchi I quantum cosmological model being left to a future work.

4.3 Bianchi I Inflationary Model

In this section, we explore a generalization of the inflationary model presented in section 2.2 in an anisotropic background. We start by analyzing its background dynamics and then the dynamics

[5] Possible criteria are simplicity and symmetry, as also unitary evolution. However, there is no guarantee that such assumptions will hold in the quantum gravity realm.

of perturbations and its physical predictions, for which we follow references [9, 22, 23]. This is also considered in [21], which developed an alternative perturbation theory applied to an inflationary model, but within a less general background model.[6]

4.3.1 Background Dynamics

As in the FLRW case, we shall consider the analogous of the Friedmann equations on a Bianchi I universe in cosmic time t, (2.15), which are given by

$$
\begin{aligned}
3H^2 &= \kappa\rho + \frac{1}{2}\tilde{\sigma}^2\,, \\
\frac{\ddot{S}}{S} &= -\frac{\kappa}{6}\left(\rho + 3p\right) - \frac{1}{3}\tilde{\sigma}^2\,, \\
\frac{d}{dt}\tilde{\sigma}^i{}_j &= -3H\tilde{\sigma}^i{}_j + \kappa\tilde{\pi}^i{}_j\,,
\end{aligned}
$$

coupled to a homogeneous scalar field $\phi(t)$. Its stress-energy tensor components are then given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi)\right)g_{\mu\nu},$$

so that its energy density and pressure are then given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{4.60a}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,, \tag{4.60b}$$

with the anisotropic stress being trivial for a scalar field, $\pi_{\mu\nu} = 0$. The Friedmann equations then become

$$3H^2 = \kappa\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] + \frac{1}{2}\tilde{\sigma}^2, \tag{4.61a}$$

$$\frac{\ddot{S}}{S} = -\frac{\kappa}{3}\left[\dot{\phi}^2 - V(\phi)\right] - \frac{1}{3}\tilde{\sigma}^2, \tag{4.61b}$$

$$\frac{d}{dt}\tilde{\sigma}^i{}_j = -3H\tilde{\sigma}^i{}_j, \tag{4.61c}$$

with the continuity equation $\nabla_a T^{ab} = 0$ maintaining its formal form:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial\phi} = 0\,. \tag{4.62}$$

[6] In particular, [21] considered the Bianchi I line element $\mathrm{d}s^2 = -dt^2 + a^2(t)\mathrm{d}x^2 + b^2(t)\left(\mathrm{d}y^2 + \mathrm{d}z^2\right)$, which contains two equal directional scale factors, and is less general than the line element (2.9) considered in [22, 23] and in this work.

Equation (4.61c) can be easily integrated. Its general solution is explicitly given by [23]

$$\tilde{\sigma}^i{}_j(t) = \frac{\mathcal{S}^i{}_j}{S^3(t)}\,, \tag{4.63}$$

where $\mathcal{S}^i{}_j$ is a constant valued tensor the acts as an integration constant. Hence,

$$\tilde{\sigma}^2(t) = \tilde{\sigma}^i{}_j\tilde{\sigma}^j{}_i = \left(\frac{S_*}{S(t)}\right)^6\,, \tag{4.64}$$

where we introduced $S_* \equiv \mathcal{S}^{1/3} = \left(\mathcal{S}^i{}_j\mathcal{S}^j{}_i\right)^{1/6}$.

We now specialize in the $V(\phi) = m\phi^2/2$ case. By introducing the variables $\tau \equiv mt$, $h \equiv H/m$ and reescaling the field by $\phi \to \sqrt{\kappa}\phi$, the first Friedmann equation and the continuity equation become, respectively:

$$h^2 = \frac{1}{6}\left(\frac{1}{2}\dot{\phi}^2 + \phi^2 + \left(\frac{S_*}{S}\right)^6\right)\,, \tag{4.65a}$$

$$\ddot{\phi} + 3h\dot{\phi} + m\phi = 0, \tag{4.65b}$$

where we used τ as a time variable, so that $\dot{\phi} \equiv d\phi/d\tau$. Note that the shear essentially contributes as an extra energy density, and that such equations recover the usual FLRW case in the limit $S_* \to 0$.

The solutions of the system (4.65) are completely specified by the initial conditions $\left(S_0, \phi_0, \dot{\phi}_0, S_*\right)$. A numerical solution for $S(\tau)$ and $\phi(\tau)$ can be seem in Figure 4.1, while the system's slow roll attractor can be visualized in Figure 4.2.

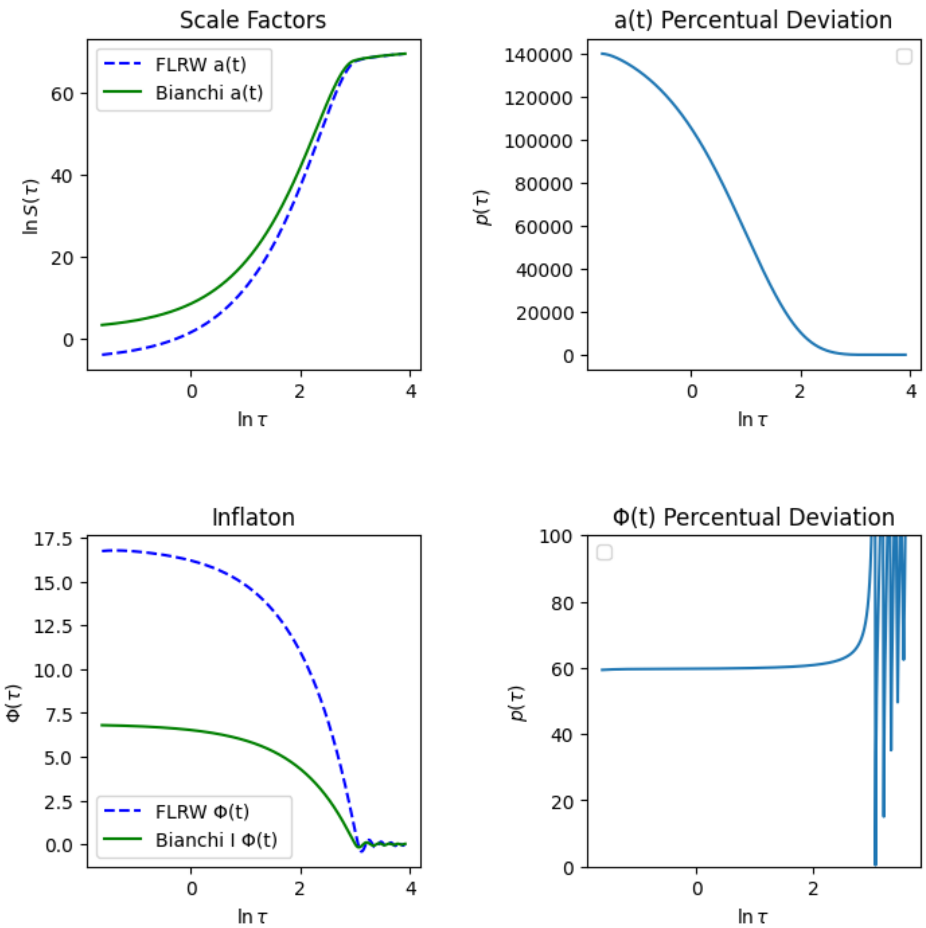

Figure 4.1: Numerical solutions for the mean scale factor $S(\tau)$ and inflaton $\phi(\tau)$ in both FLRW and Bianchi I spacetimes, with their percentual deviation $p(\tau)$ on the right. The solutions are characterized by the initial conditions $\phi_0 = 16$, $\dot{\phi}_0 = 20$, $S_0 = 0.01$, $S_* = 1000$. The solutions were rescaled so that they have the same value at the end of inflation. Note that the solutions are similar in the isotropic limit, but differ in the primordial limit of shear domination. In particular, the inflaton presents very different dynamics, with the percentual deviation $p(\tau)$ becoming very large near the turning points where $\phi_{\text{FLRW}} = 0$.

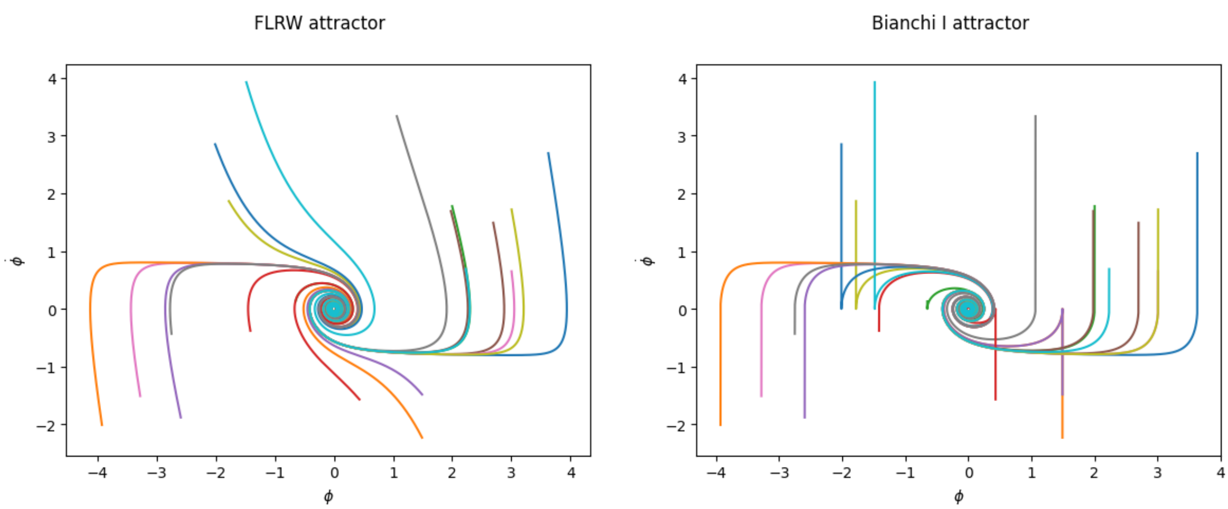

Figure 4.2: Phase portrait $\phi \times \dot{\phi}$ for a FLRW and a Bianchi I spacetime with $S_0 = 0.01$, $S_* = 1000$, and random initial conditions for $(\phi_0, \dot{\phi}_0)$. Solutions with same initial conditions have the same colors in both graphs. Note in both cases the solutions tend to the central region, which corresponds to an attractor associated to the slow roll regime (2.19). In a sense, this implements the idea that general initial conditions present similar regions, and hence alleviates the initial conditions problems of the Standard Model.

4.3.2 Perturbative Dynamics

We now proceed to analyze perturbations in this inflationary model using the decomposition presented in section 4.1 through the perturbed Einstein equations. However, to do so we first need to consider perturbations on the matter sector by $\phi \to \phi + \delta\phi$ and construct appropriate gauge invariant quantities.

Under the diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ the scalar field ϕ transforms as

$$\phi \to \phi + \mathscr{L}_\xi \phi$$

which, using (4.22), means that its perturbations transform as

$$\delta\phi \to \delta\phi + \phi' T.$$

Hence, by analyzying (4.23), we may construct two new gauge invariant quantities associated to $\delta\phi$:

$$\mathcal{R} \equiv \delta\phi - \frac{C}{\mathcal{H}}\phi' , \tag{4.66a}$$

$$\chi \equiv \delta\phi + \left[B - \frac{(k^2 E)'}{k^2} \right] \phi' , \tag{4.66b}$$

which are related by

$$\mathcal{R} = \chi + \frac{\Psi}{\mathcal{H}}\phi' .$$

By considering perturbations $\phi \to \phi + \delta\phi$ on the Klein-Gordon equation (4.62), one obtains

$$\chi'' + 2\mathcal{H}\chi' - \gamma^{ij}\partial_i\partial_j\chi + S^2 V_{\phi\phi}\chi = 2\left(\phi'' + 2\mathcal{H}\phi'\right)\Phi + \phi'\left(\Phi' + 3\Psi'\right) , \tag{4.67}$$

which has the same form as in the FLRW case, which happens due to the fact that the background Klein-Gordon equation itself had the same form as in the FLRW case, due to the vanishing of the anisotropic stress $\pi_{\mu\nu}$.

To conclude the matter sector, we note that the perturbations on the stress energy tensor $T_{\mu\nu}$ in terms of our gauge invariant quantities are given by

$$S^2 \delta T^0_{\ 0} = \phi'^2 \Phi - \phi'\chi' - V_\phi S^2 \chi, \tag{4.68a}$$

$$S^2 \delta T^0_{\ i} = -\partial_i \left[\phi'\chi\right] , \tag{4.68b}$$

$$S^2 \delta T^j_{\ i} = -\delta^i_{\ j} \left[\phi'^2 \Phi - \phi'\chi' + V_\phi S^2 \chi\right] , \tag{4.68c}$$

where the Newtonian gauge was assumed.

We are now ready to analyze dynamics through the perturbed Einstein equations

$$\delta G^\nu_{\ \mu} = \kappa \delta T^\nu_{\ \mu} . \tag{4.69}$$

It is at this point that the projection operators defined on subsection 4.1.1 shall come in hand. Note that, by comparing the Bianchi I Einstein tensor (4.29) with its isotropic counterpart (2.105), there is not a natural procedure to isolate the scalar, vector and tensor dynamical equations: they are coupled through the shear tensor σ_{ij}. Hence, we shall use our projection operators to project each of the Einstein equations in its respective mode equations, which is performed directly in Fourier space. This is a rather lengthy procedure, and we shall focus mainly on the results [23].

To better organize the projection process, note that we may split the Einstein equations in scalar, vector and tensor parts as we did to the metric. Then, each equation can be projected, which will lead to scalar, vector and tensor equations as in the SVT decomposition:

$$\delta G^0_{\ 0} = \kappa\delta T^0_{\ 0} \implies \text{1 scalar equation} \tag{4.70a}$$
$$\delta G^0_{\ i} = \kappa\delta T^0_{\ i} \implies \text{1 scalar equation and 1 vector equation} \tag{4.70b}$$
$$\delta G^i_{\ j} = \kappa\delta T^i_{\ j} \implies \text{2 scalar equations, 1 vector equation and 1 tensor equation} \tag{4.70c}$$

where the vector and tensor equations have trivial divergence and also trivial trace in the tensor case, which amounts to 16 dynamical equations. We now proceed to project the Einstein equations and obtain each one of them.

1. **Scalar Equations**

   (a) The first scalar equation is essentially the 00 Einstein equation $\delta G^0_{\ 0} = \kappa\delta T^0_{\ 0}$, and is given by

$$k^2\Psi + 3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) - \frac{\kappa}{2}\left(\phi'^2\Phi - \phi'\chi' - V_\phi S^2\chi\right) = \frac{1}{2}\sigma^2[X - 3\Psi] + \frac{1}{2}\frac{k^2}{\mathcal{H}}\sigma_\parallel\Psi - \frac{1}{2}k^2\sum_a\tilde{\sigma}_{\mathrm{V}a}\Phi_a - \frac{1}{2}\sum_\lambda\left[\sigma_{\mathrm{T}\lambda}E'_\lambda + \left(\sigma'_{\mathrm{T}\lambda} + 2\mathcal{H}\sigma_{\mathrm{T}\lambda}\right)E_\lambda\right], \tag{4.71}$$

   where we introduced the auxiliar variable

$$X \equiv \Phi + \Psi + \left(\frac{\Psi}{\mathcal{H}}\right)',$$

   and $\sigma_{Va} \equiv ik\tilde{V}a$;

   (b) The second scalar equation is obtained from the scalar projection of the $0i$ Einstein equation $\hat{k}^i\left[\delta G^0_{\ i} = \kappa\delta T^0_{\ i}\right]$:

$$\Psi' + \mathcal{H}\Phi - \frac{\kappa}{2}\phi'\chi = -\frac{1}{2\mathcal{H}}\sigma^2\Psi + \frac{1}{2}\sigma_\parallel X + \frac{1}{2}\sum_\lambda\sigma_{\mathrm{T}\lambda}E_\lambda; \tag{4.72}$$

(c) The third scalar equation is obtained from the trace of the ij Einstein equation $\delta^j_i\left[\delta G^i{}_j = \kappa \delta T^i{}_j\right]$:

$$\Psi'' + 2\mathcal{H}\Psi' + \mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi - \frac{1}{3}k^2(\Phi - \Psi) + \frac{\kappa}{2}\left[\phi'^2\Phi - \phi'\chi' + V_\phi S^2\chi\right] = \tag{4.73}$$
$$-\frac{1}{2}\sigma^2[X - 3\Psi] + \frac{1}{6}\frac{k^2}{\mathcal{H}}\sigma_\parallel\Psi + \frac{1}{2}k^2\sum_a \tilde{\sigma}_{\mathrm{v}a}\Phi_a + \frac{1}{2}\sum_\lambda\left[\sigma_{\mathrm{T}\lambda}E'_\lambda + \left(\sigma'_{\mathrm{T}\lambda} + 2\mathcal{H}\sigma_{\mathrm{T}\lambda}\right)E_\lambda\right];$$

(d) The final scalar equation is obtained by extracting the trace of the ij Einstein equation $\mathcal{T}^j_i\left[\delta G^i{}_j = \kappa\delta T^i{}_j\right]$:

$$\frac{2}{3}k^2(\Phi - \Psi) = \sigma_\parallel\left[X' - \frac{k^2\Psi}{3\mathcal{H}}\right] + 4k^2\sum_{\lambda,a,b}\mathcal{M}^\lambda_{ab}\tilde{\sigma}^a_{\mathrm{V}}\tilde{\sigma}^b_{\mathrm{V}}E_\lambda - 2k^2\sum_a\tilde{\sigma}_{\mathrm{v}a}\Phi_a; \tag{4.74}$$

2. **Vector Equations**

(a) The first vector equation is obtained from the $i0$ Einstein equation by applying the projector $e^i_a\left[\delta G^0_{\ i} = \kappa\delta T^0_{\ i}\right]$:

$$\Phi_a = -2\tilde{\sigma}_{\mathrm{v}a}X + 4\sum_{b,\lambda}\mathcal{M}^\lambda_{ab}\tilde{\sigma}_{\mathrm{v}b}E_\lambda\,. \tag{4.75}$$

Note that this equation gives a closed expression for the vector mode Φ_a in terms of the other modes, which will be very useful to simplify the dynamical equations.

(b) The second vector equation is obtained from the ij Einstein equation by applying the projector $k_i e^j{}_a\left[\delta G^i{}_j = \kappa\delta\delta T^i{}_j\right]$:

$$\Phi'_a + 2\mathcal{H}\Phi_a - \frac{5}{2}\sigma_\parallel\Phi_a + \sum_{b\lambda}\mathcal{M}^\lambda_{ab}\sigma_{T\lambda}\Phi_b = +2\tilde{\sigma}_{\mathrm{v}a}X' + 4\sum_{b,\lambda}\mathcal{M}^\lambda_{ab}\tilde{\sigma}_{\mathrm{v}b}E'_\lambda$$
$$+ 4\sum_{b\lambda}\mathcal{N}_{ab}\tilde{\sigma}_{\mathrm{v}b}\left(\sigma_{\mathrm{T}+}\delta^\times_\lambda - \sigma_{\mathrm{T}\times}\delta^+_\lambda\right)E_\lambda\,, \tag{4.76}$$

where we introduced

$$\mathcal{M}^\lambda_{ab} \equiv \varepsilon^\lambda_j e^{\,i}_a e^{\,ij}_b\,, \tag{4.77a}$$
$$\mathcal{N}_{ab} \equiv Q_{ij}e^{\,i}_a e^{\,j}_b\,, \tag{4.77b}$$

and $Q_{ij} \equiv e^1_{\ i}e^2_{\ j} - e^2_{\ i}e^1_{\ j}$.

As in the previous case, note that, by analyzing the first term Φ_a, one sees that such equation fixes the vector mode's time derivative in terms of the other SVT modes. This will also be useful to simplify the dynamical equations.

3. **Tensor Equation**

There is only one tensor equation, which is obtained from the ij Einstein equation by applying the projector $\varepsilon_i^{j\lambda}\left[\delta G^i{}_j = \kappa\delta T^i{}_j\right]$, and is given by

$$E_\lambda'' + 2\mathcal{H}E_\lambda' + k^2 E_\lambda = \sigma_{\mathrm{T}\lambda}\left[k^2\left(\frac{\Psi}{\mathcal{H}}\right) + X'\right] + 2k^2\sum_{a,b}\mathcal{M}_{ab}^\lambda\tilde{\sigma}_{\bar{v}a}\Phi_b$$
$$- 2k^2\sum_a\tilde{\sigma}_{\mathrm{v}a}^2 E_\lambda - 2\sigma_{\mathrm{T}\times}\sigma_{\mathrm{T}}E_{(1-\lambda)} + 2\sigma_{T(1-\lambda)}^2 E_\lambda, \tag{4.78}$$

where, following [23], we introduce the convention that $(1-\lambda)$ denotes the inverse polarization of λ, that is

$$\lambda = + \implies (1-\lambda) = \times\,,$$
$$\lambda = \times \implies (1-\lambda) = +\,.$$

It can be readily seem that, even by projecting the Einstein equations into its SVT components, the dynamical equations are still strongly coupled through the projections of the shear tensor. However, before directly solving this system, it can be shown that it can be further simplified [23]. We start by defining scalar and tensor Mukhanov-Sasaki variables, as in the isotropic case (2.113):

$$v \equiv S\mathcal{R}\,, \tag{4.79a}$$
$$\sqrt{\kappa}\mu_\lambda \equiv SE_\lambda\,. \tag{4.79b}$$

With thei introduction of the Mukhanov-Sasaki variables, the dynamical equations can be expressed in more compact form, as we shall see.

1. **Scalar Modes**

   To begin, note that the second scalar equation (4.72) can now be rewritten as:

$$\left(2\mathcal{H} - \sigma_\parallel\right)X = \frac{\kappa}{S}\phi' v + \sum_\lambda \sigma_{\mathrm{T}\lambda}E_\lambda \tag{4.80}$$

   Now, combining the first scalar equation (4.71) with the third scalar equation (4.73) and eliminating the vector mode using the first vector equation (4.75), we obtain the simpler form

$$\mathcal{H}X' + 2\left(\mathcal{H}' + 2\mathcal{H}^2\right)X + \kappa SV_\phi v + k^2\Psi = \frac{k^2}{3}(\Phi - \Psi) + \frac{2}{3}\frac{k^2}{\mathcal{H}}\sigma_\parallel\Psi\,, \tag{4.81}$$

   Analogously, it is possible to simplify the fourth scalar equation (4.74) by eliminating the vector mode with the constraint (4.75) and the background equations (2.15):

$$\left(2\mathcal{H} - \sigma_\parallel\right)\left(X' + \frac{k^2}{\mathcal{H}}\Psi\right) + 4\kappa S^2 VX + 2\kappa aV_\phi v = 4k^2\left(\sum_a\tilde{\sigma}_{VS}^2 X - \sum_{a,b,\lambda}\mathcal{M}_{ab}^\lambda\tilde{\sigma}_{Va}\tilde{\sigma}_{Vb}E_\lambda\right). \tag{4.82}$$

   Continuing, we may express the Klein-Gordon equation (4.67) in terms of the variable $\mathcal{R}$. In the process, some terms vanish due to its background version (4.62), and we obtain

$$\mathcal{R}'' + 2\mathcal{H}\mathcal{R} + k^2\mathcal{R} + S^2V_{\phi\phi}\mathcal{R} + 2S^2V_{\phi}X - \phi'\left(X' + \frac{k^2}{\mathcal{H}}\Psi\right) = 0. \tag{4.83}$$

After some lengthy manipulations, we can combine the scalar modes into one single dynamical equation for the Mukhanov-Sasaki variable v:

$$v'' + \left(k^2 - \frac{S''}{S} + S^2V_{,\phi\phi}\right)v = \frac{1}{S^2}\left(\frac{2S^2\phi'^2}{2\mathcal{H} - \sigma_{\|}}\right)'\kappa v + \sum_{\nu}\frac{1}{S^2}\left(\frac{2S^2\phi'\sigma_{T\nu}}{2\mathcal{H} - \sigma_{\|}}\right)'\sqrt{\kappa\mu_{\nu}} \tag{4.84}$$

2. **Tensor Modes**

   As for the tensor mode, we may eliminate the scalar contribution using (4.82) and the vector mode by using the constraint (4.75), to obtain

$$\begin{aligned}\mu_{\lambda}'' + \left(k^2 - \frac{S''}{S}\right)\mu_{\lambda} = &- 2\mu_{(1-\lambda)}\sigma_{T+}\sigma_{T\times} + 2\mu_{\lambda}\sigma^2_{T(1-\lambda)} + \frac{1}{S^2}\left(\frac{2S^2\phi'\sigma_{\mathrm{T}\lambda}}{2\mathcal{H} - \sigma_{\|}}\right)'\sqrt{\kappa}v \\ &+ \sum_{\nu}\frac{1}{S^2}\left(\frac{2S^2\sigma_{\mathrm{T}\nu}\sigma_{\mathrm{T}\lambda}}{2\mathcal{H} - \sigma_{\|}}\right)'\mu_{\nu} + \frac{(S^2\sigma_{\|})'}{S^2}\mu_{\lambda}\,,\end{aligned} \tag{4.85}$$

   which is one single dynamical equation for the gauge invariant tensor mode.

We now present a simple summary of the developed perturbation theory [22].

PERTURBATION THEORY SUMMARY

After reduction, the perturbative equations are given by

$$v'' + \omega_v^2 (k_i, \eta) v = \sum_\lambda \aleph_\lambda (k_i, \eta) \mu_\lambda, \tag{4.86a}$$

$$\mu_\lambda'' + \omega_\lambda^2 (k_i, \eta) \mu_\lambda = \aleph_\lambda (k_i, \eta) v + \beth (k_i, \eta) \mu_{(1-\lambda)}, \tag{4.86b}$$

where the time dependent frequencies are given by

$$\omega_v^2 (k_i, \eta) \equiv k^2 - \frac{z_\mathrm{s}''}{z_\mathrm{s}}, \tag{4.87a}$$

$$\omega_\lambda^2 (k_i, \eta) \equiv k^2 - \frac{z_\lambda''}{z_\lambda} \tag{4.87b}$$

and the z functions are given by

$$\frac{z_\mathrm{s}''}{z_\mathrm{s}} (\eta, k_i) \equiv \frac{S''}{S} - S^2 V_{,\phi\phi} + \frac{1}{S^2} \left(\frac{2S^2 \kappa \phi'^2}{2\mathcal{H} - \sigma_\parallel} \right)', \tag{4.88a}$$

$$\frac{z_\lambda''}{z_\lambda} (\eta, k_i) \equiv \frac{S''}{S} + 2\sigma_{\mathrm{T}(1-\lambda)}^2 + \frac{1}{S^2} \left(S^2 \sigma_\parallel \right)' + \frac{1}{S^2} \left(\frac{2S^2 \sigma_{\mathrm{T}\lambda}^2}{2\mathcal{H} - \sigma_\parallel} \right)' \tag{4.88b}$$

finally, the coupling terms are given by

$$\aleph_\lambda (\eta, k_i) \equiv \frac{1}{S^2} \sqrt{\kappa} \left(\frac{2S^2 \phi' \sigma_{\mathrm{T}\lambda}}{2\mathcal{H} - \sigma_\parallel} \right)', \tag{4.89a}$$

$$\beth (\eta, k_i) \equiv \frac{1}{S^2} \left(\frac{2S^2 \sigma_{\mathrm{T}\times} \sigma_{\mathrm{T}+}}{2\mathcal{H} - \sigma_\parallel} \right)' - 2\sigma_{\mathrm{T}\times} \sigma_{\mathrm{T}+} \tag{4.89b}$$

note that, in the limit of vanishing shear $\sigma_{ij} \to 0$, the coupling terms vanish, and the SVT modes decouple, as in the FLRW case.

The perturbative equations can also be written schematically as

$$V'' + \omega^2 V = \Upsilon V \tag{4.90}$$

where the ω^2 matrix represents the frequencies and Υ the interactions. They are explicitly given by

$$\omega^2 = \begin{pmatrix} k^2 - \frac{z_s''}{z_s} & 0 & 0 \\ 0 & k^2 - \frac{z_+''}{z_+} & 0 \\ \bar{0} & \bar{0} & k^2 - \frac{z_\times''}{z_\times} \end{pmatrix} \quad , \quad \Upsilon = \begin{pmatrix} 0 & \aleph_+ & \aleph_\times \\ \aleph_+ & 0 & \beth \\ \aleph_\times & \beth & 0 \end{pmatrix}.$$

By direct comparison of the obtained perturbative equations (4.86) to the isotropic ones (2.116), one is already able to note some qualitative non-trivial differences to the FLRW case, which are:

1. Due to anisotropy, the predicted power spectra will depend on the wavevector $\vec{k}$ and not just on its modulus k: $\mathcal{P}_v(k_i)$ and $\mathcal{P}_\lambda(k_i)$[7];
2. Due to the coupling between scalar and tensor modes, there will be non-trivial cross-correlations between the scalar and tensor power spectra, represented by the two point functions $\langle \hat{v}\hat{\mu}_\lambda \rangle_0$;
3. The two gravitational wave polarizations will presented different power spectra: $\mathcal{P}_+(\vec{k}) \neq \mathcal{P}_\times(\vec{k})$ due to the fact that they present different dynamics;

we now proceed our discussion by quantizing the perturbations in order to make some more quantitative statements.

4.3.3 Quantization of Perturbations and Predictions

As in the isotropic case, before quantization, one needs to identify the theory's canonical variables, which can be done by analyzying the action. Therefore, one needs to expand the Einstein-Hilbert action coupled to a scalar field up to second order on the perturbations. The calculation is straightforward, where one essentially needs to group similar terms and eliminate total derivatives - but rather lengthy. Therefore, we shall only present the results. A more complete discussion can be found in [23].

The total action S_T is then split into zeroth order, first order and second order as

$$S_T = S_0 + S_1 + S_2\,, \tag{4.91}$$

where, respectively:

$$\begin{aligned}
S_0 &= \frac{1}{2\kappa}\int \mathrm{d}t\,\mathrm{d}^3x \left[S^3\left(-6H^2 + \tilde{\sigma}^2 - 2\kappa V + \kappa\dot{\phi}^2\right)\right] \\
&= \frac{1}{2\kappa}\int \mathrm{d}t\,\mathrm{d}^3x \left[-4\frac{\mathrm{d}}{\mathrm{d}t}\left(S^3H\right)\right], && \text{(4.92a)}\\
S_1 &= \frac{1}{2\kappa}\int \mathrm{d}t\,\mathrm{d}^3x S^3\Big[R_1^{(3)} + \tilde{\sigma}^{ij}\dot{h}_{ij} - 2\tilde{\sigma}_{ij}\tilde{\sigma}^i_l h^{jl} + 12H\dot{\Psi} + 3\Psi\left(6H^2 - \tilde{\sigma}^2 + 2\kappa V - \kappa\dot{\phi}^2\right) \\
&\quad + \Phi\left(6H^2 - \tilde{\sigma}^2 - 2\kappa V - \kappa\dot{\phi}^2\right) - 2\kappa V_\phi \delta\phi + 2\kappa\dot{\phi}\dot{\delta\phi}\Big] \\
&= \frac{1}{2\kappa}\int \mathrm{d}t\,\mathrm{d}^3x \left\{\partial_i\left[\partial^i(4S\Psi) - \partial^i\left(\frac{S\Psi}{H}\right)^{\cdot}\right]\right. \\
&\quad \left. + \frac{\mathrm{d}}{\mathrm{d}t}\left[\nabla^2\left(\frac{S\Psi}{H}\right) + S^3\tilde{\sigma}^{ij}h_{ij} + 12S^3H\Psi + 2S^3\kappa\dot{\phi}\delta\phi\right]\right\}, && \text{(4.92b)}\\
S_2 &= \frac{1}{2\kappa}\int \mathrm{d}t\,\mathrm{d}^3x S^3\left[R_2^{(3)} + N_1 R_1^{(3)} + \frac{1}{2}hR_1^{(3)} + \mathcal{K}_2 + \frac{1}{2}h\mathcal{K}_1 + \frac{1}{8}h^2\mathcal{K}_0\right.
\end{aligned}$$

[7] To be more precise, even in the isotropic FLRW case, this kind of dependence could occur. However, in such case, if one assumes isotropic initial conditions, dynamics preserves such property, while in the Bianchi I case, even if one starts with isotropic initial conditions, they will in general evolve to anisotropic ones.

$$- \frac{1}{4}h^i_j h^j_i \mathcal{K}_0 - N_1\mathcal{K}_1 - \frac{1}{2}N_1 h\mathcal{K}_0 + N_1^2\mathcal{K}_0 + \kappa\left(-S^{-2}\partial_i\delta\phi\partial^i\delta\phi - V_{\phi\phi}\delta\phi^2\right.$$
$$- 2N_1V_\phi\delta\phi - hV_\phi\delta\phi - hN_1V - \frac{1}{4}h^2V + \frac{1}{2}h^i_jh^j_iV + \delta\dot{\phi}^2 - 2N_1\dot{\phi}\delta\dot{\phi}$$
$$\left.\left. + N_1^2\dot{\phi}^2 + h\dot{\phi}\delta\dot{\phi} - \frac{1}{2}hN_1\dot{\phi}^2 + \frac{1}{8}h^2\dot{\phi}^2 - \frac{1}{4}h^i_jh^j_i\dot{\phi}^2\right)\right], \tag{4.92c}$$

where we introduced

$$S^2R_1^{(3)} = 4\left(\nabla^2 - \frac{\tilde{\sigma}^{ij}\partial_i\partial_j}{2H}\right)\Psi, \tag{4.93a}$$
$$S^2R_2^{(3)} = -\partial_l h^{lj}\partial_i h^i_j - 2h^{jl}\partial_j\partial_i h^i_l - 9\partial_i\Psi\partial^i\Psi - \frac{1}{4}\partial_l h^{ij}\partial^l h_{ij} - \frac{1}{2}\partial_l h_{ij}\partial^i h^{lj} \tag{4.93b}$$
$$- 6\partial_i\left(h^{ji}\partial_j\Psi\right) + \frac{1}{2}\partial^i\partial_i\left(h^{jl}h_{jl}\right), \tag{4.93c}$$
$$\mathcal{K}_0 = -6H^2 + \tilde{\sigma}^2, \tag{4.93d}$$
$$\mathcal{K}_1 = -2H\dot{h} + \tilde{\sigma}^{ij}\dot{h}_{ij} - 2\tilde{\sigma}_{ij}\tilde{\sigma}^j_l h^{li}, \tag{4.93e}$$
$$\mathcal{K}_2 = 2H\dot{h}_{ij}h^{ij} - 4H\tilde{\sigma}_{ij}h^{il}h^j_l - 2\tilde{\sigma}^l_i h^{im}\dot{h}_{ml} + 2\tilde{\sigma}_{ij}\tilde{\sigma}^j_l h^{im}h^l_m + \frac{1}{4}\dot{h}^{ij}\dot{h}_{ij}, \tag{4.93f}$$
$$+ \tilde{\sigma}_{ij}\tilde{\sigma}_{lm}h^{im}h^{jl} - \frac{1}{4}\dot{h}^2. \tag{4.93g}$$

In (4.92), the background field equations were used to rewrite the zero and first order contributions as total derivatives, which explicitly shows that any non-trivial contributions to the perturbations come from the second order action S_2. It should also be noted that this procedure cannot be used for quantum bounces, where the background itself is quantized. However, the second order action for a general background was obtained in [43, 85] without ever applying classical equations of motion. Such result may prove useful to construct a perturbed Bianchi I minisuperspace quantum cosmological model in the future.

The obtained action may be further simplified by eliminating the vector constraint and expressing it in Fourier space. Using such protocol and eliminating boundary terms, the action in terms of the remaining degrees of freedom is then given by:

$$S_2 = \frac{1}{2}\int \mathrm{d}\eta\mathrm{d}^3k\left\{v'v'^* + \left(\frac{z_s''}{z_s} - k^2\right)vv^* + \sum_\nu \frac{1}{S^2}\left(\frac{2S^2\sqrt{\kappa}\phi'\sigma_{\mathrm{T}\nu}}{2\mathcal{H} - \sigma_\parallel}\right)'(v^*\mu_\nu + v\mu_\nu^*)\right. \tag{5.20}$$
$$\left.\sum_\lambda\left[\mu_\lambda'\mu_\lambda'^* + \left(\frac{z_\lambda''}{z_\lambda} - k^2\right)\mu_\lambda\mu_\lambda^* + \left[-2\sigma_{\mathrm{T}\times}\sigma_{\mathrm{T}+} + \frac{1}{S^2}\left(\frac{2S^2\sigma_{\mathrm{T}\times}\sigma_{\mathrm{T}+}}{2\mathcal{H} - \sigma_\parallel}\right)'\right]\mu_{(1-\lambda)}\mu_\lambda^*\right]\right\}, \tag{4.94}$$

which can be shown to yield the obtain the scalar and tensor dynamical equations (4.84) and (4.85). It also leads to the canonical conjugate momenta

$$\pi_v = \frac{\partial L_2}{\partial \dot{v}} = v'^*, \tag{4.95}$$
$$\pi_\lambda = \frac{\partial L_2}{\partial \dot{\mu}_\lambda} = \mu_\lambda'^*, \tag{4.96}$$

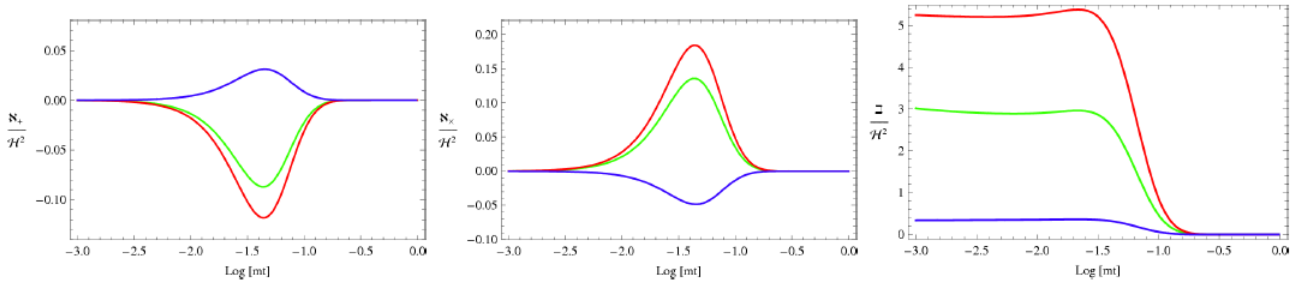

Figure 4.3: Time evolution of the couplings: scalar-tensor $\aleph_+$ on the left, scalar-tensor $\aleph_\times$ on the center and tensor-tensor $\sqsupset$ on the right, for three different modes, each aligned with one of the three orthogonal directions defined by the Killing vector fields. Note that the scalar-tensor couplings $\aleph_\lambda$ decay both in the future, as the universe isotropizes, as in the past. However, the tensor-tensor coupling $\sqsupset$ tends to a constant value, which shows that the tensor modes are more strongly coupled than the scalar to tensor. This graph was reproduced from [22]. All graphs from this section are from such reference, unless otherwise explicitly stated.

where L_2 is the Lagrangian density such that $S_2 = \int L_2 d\eta d^3\vec{x}$.

To proceed with quantization, one must first note a very non-trivial fact: as was emphasized, the system (4.86) is coupled, which in turn means that we are dealing not only with a free theory, but with an interacting 3 field theory. This makes quantization a lot harder, since most of quantum field theory in curved spacetime results assume a free theory in a classical background [69].

Since the quantization of interacting field theories is problematic, a reasonable assumption would be to look for a regime where the interaction terms go to zero and the modes decouple. As discussed in [23], in the sub-Hubble limit, the modes decouple and behave as harmonic oscillators. In particular, it is possible to show that the scalar-tensor coupling is bounded by

$$|\aleph_\lambda| < Z\mathcal{H}^2 \,, \tag{4.97}$$

where Z is a finite constant, with similar results for the tensor-tensor coupling $\sqsupset(k_i, \eta)$. Therefore, in the sub-Hubble limit the modes decouple and behaves as a set of harmonic oscillators, which means that the system (4.86) becomes, schematically:

$$V'' + \omega^2 V = 0 \,. \tag{4.98}$$

Such result can also be visualized by numerical implementations, as in Figure 4.3. Note that, while the scalar-tensor couplings $\aleph_\lambda$ decay in the (sub-Hubble) past, the same does not happen for the tensor-tensor coupling $\sqsupset$, which does not decay in general.

Now, one may proceed to quantize perturbations in the regime where they can effectively be treated as free fields evolving in an expanding spacetime. It should be noted that the decoupling considerations are valid only for a certain range of scales, which makes it harder to quantize perturbations outside a certain k_{ref} range.

To proceed with our quantization procedure, as in the usual case, we promote our fields to quantum operators through the mode expansion (3.70) satisfying the canonical commutation relations (3.71). Then, as previously discussed in chapter 3, one now needs to define a vacuum state and hence a representation for the states.

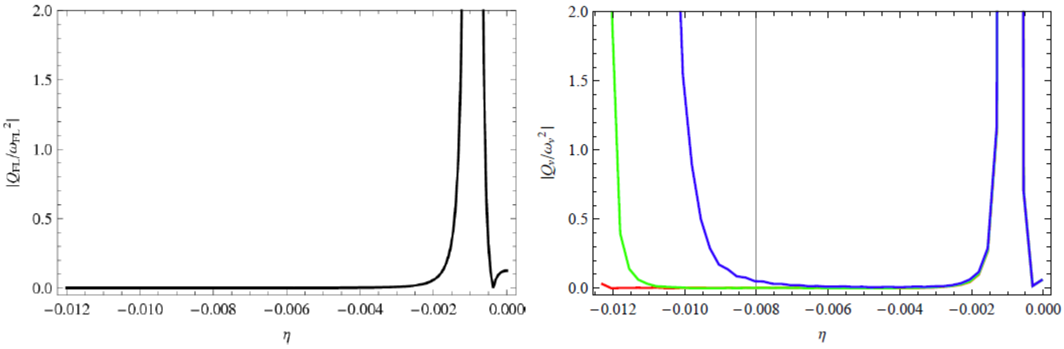

Figure 4.4: Evolution of the ratio $|Q/\omega^2|$ for scalar modes of a given k. On the left is the evolution of such quantity in the FLRW case. Note that the ratio approaches 0 further and further in the past, allowing an adiabatic vacuum prescription. On the right is the analogous for the Bianchi I case, for modes aligned with each one of the orthogonal directions. Note that the ratio $|Q/\omega^2|$ gets small in the past and then starts to grow again, allowing a WKB approximation only for a limite region.

As can be seem by inspection, the frequencies (4.87) have a quite complicated expression (4.88), and are not positive define in general, with their signal depending on the background dynamics. Therefore, the minimization of energy prescription discussed in subsection 3.3.1 is not applicable in the general case.

One then is lead to consider the adiabatic prescription discussed in subsection 3.3.2, which relies on the WKB approximation. Recall that, since our perturbative equations are schematically given by

$$V'' + \omega^2 V = 0 \tag{4.99}$$

where we neglected the couplings Υ, the WKB approximation is a good approximation for the solutions if the condition

$$\left|\frac{Q}{\omega^2}\right| \ll 1\,, \tag{4.100}$$

$$Q \equiv \frac{1}{2}\left[\frac{3}{2}\left(\frac{W_k'}{W_k}\right) - \frac{W_k''}{W_k}\right], \tag{4.101}$$

is satisfied. For consistency, before applying an adiabatic vacuum prescription, one should then study the validity of the WKB approximation, which was analyzed in detail in [22] using numerical methods. However, a striking surprise was found: as one can see in Figure 4.4 and Figure 4.5, the WKB regime is violated in the past for some modes, which forbids one to define an adiabatic vacuum state in the usual way. To illustrate such failure, a comparison between a WKB solution and an exact (numerical) solution can be seem in Figure 4.6.

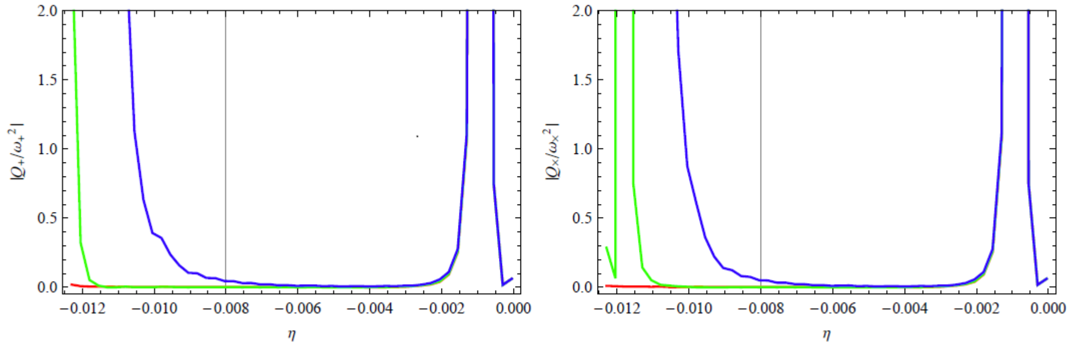

Figure 4.5: Evolution of the ratio $|Q/\omega^2|$ for tensor modes of a given k in a Bianchi I universe. The left picture represents the + polarization, with the right one representing the × one. The interpretation is analogous to the scalar one: the WKB approximation is valid onlid for a limited region, and is not applicable for the asymptotic past.

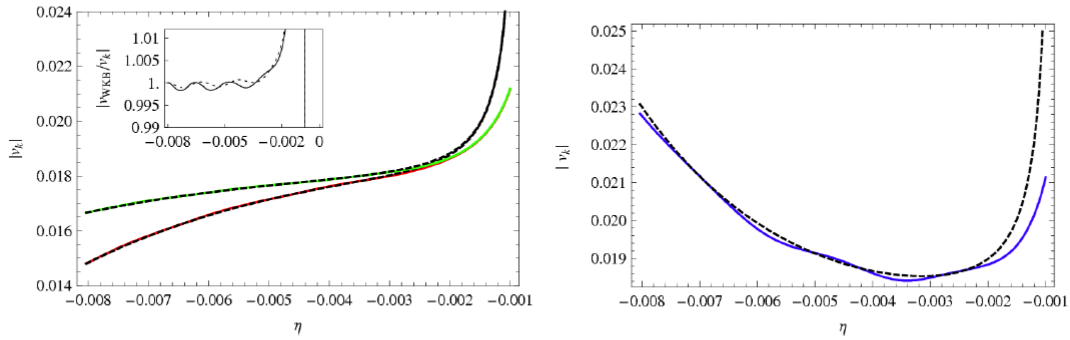

Figure 4.6: Comparison between the WKB approximated solution and the exact (numerical) solution. The left figure shows two modes for which the WKB approximation is valid, while the right one shows one mode for which it isn't. The inner left figure shows the ratio $v_k^{\rm WKB}/v_k$ between the WKB and the numerical solution.

The main reason for the failure of the WKB approximation is due to the background's anisotropic dynamics. In the isotropic FLRW case, one may analyze if one given mode is in its sub-Hubble or super-Hubble regime by comparing its given wavelength

$$\lambda_k^{-1}(t) = \frac{k}{a(t)}, \tag{4.102}$$

with the Hubble radius $R_H(t) = 1/H = a/\dot{a}$. However, in the Bianchi I case, since the background is anisotropic, a given mode has wavelengths aligned with each direction evolving as

$$\lambda_{k_i}^{-1}(t) = \frac{k_i}{a_i(t)}, \tag{4.103}$$

which can be visualized in Figure 4.7. Therefore, a given mode may be sub-Hubble in one direction and super-Hubble in the other due to anisotropy.[8] Of course, such difference between directions is negligible in the isotropic post-inflation limit, but since the background shear evolves as $\sigma \sim 1/S^6$, the shear dominates in the past, and anisotropic effects become relevant.

[8]In the Bianchi I case, the Hubble radius is defined as $R_H(t) \equiv 1/H = S/\dot{S}$ and recovers the usual definition in the FLRW case.

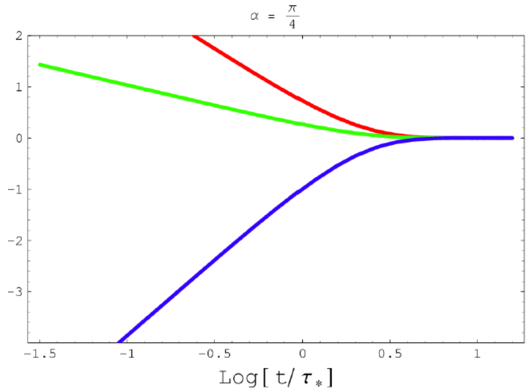

Figure 4.7: Evolution of the logarithm of the ratio k/k_{ref}, where k_{ref} is the reference scale. Note that, in the past, the different directions evolve in very different ways due to the presence of shear σ_{ij}.

In fact, remembering the parametrization introduced in (2.11), that is:

$$a_i(t) \equiv e^{\beta_i(t)} \tag{4.104}$$

with the β_i satisfying the constraint

$$\sum_{i=1}^{3} \beta_i = 0\,, \tag{4.105}$$

direct differentiation then leads to

$$\sum_{i=1}^{3} \dot{\beta}_i = 0\,, \tag{4.106}$$

one sees that, in order for (4.105) and (4.106) to be satisfied, $\beta_i < 0$ and $\dot{\beta}_i < 0$ for at least one of the β_is/$\dot{\beta}_i$s. This means that, in a generic Bianchi I, at least one of the three orthogonal directions (defined by the Killing vector fields) is contracting, and this is *independent of dynamics*. Since the β_i are functions of time, this means that one of the directions always performs a bounce.[9]

In practice, this property means that the directional Hubble rates

$$H_i = \frac{\dot{a}_i}{a_i}\,, \tag{4.107}$$

may be negative [9]. Hence, differently of the usual isotropic inflationary case, where a given sub-Hubble mode was always sub-Hubble before the end of inflation, a given sub-Hubble mode may have had a super-Hubble period in the past, which makes the WKB approximation inapplicable. In fact, the situation is even worse: as was shown in [22]: *all modes* had a super-Hubble regime in the past, and there are modes for which the WKB approximation is not valid for any period of time. For such modes, an adiabatic vacuum state cannot be defined, and one is then unable to impose initial conditions for them, and hence to make definite predictions about the CMB spectrum.

However, [22] has offered an interesting alternative procedure to impose initial conditions. It is based on considering the initial conditions only for the modes that enter a WKB regime and at the

[9] Note that, although the Bianchi I universes allow bounces in one direction coupled to ordinary matter and GR, it does not violate the Singularity Theorems and contain past singularities.

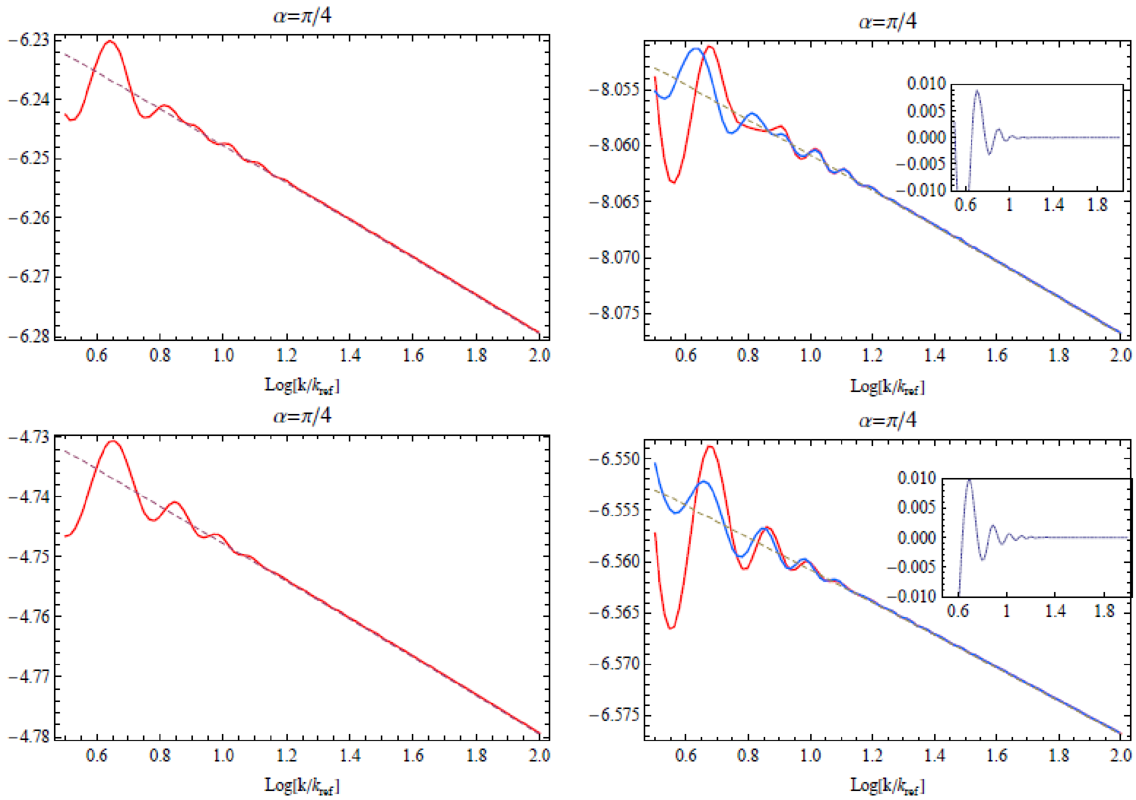

Figure 4.8: The angular averaged spectra $\log\left[f_{\mathcal{R}}(k)\right]$ and $\log\left[f_{\lambda}(k)\right]$ for the curvature perturbations $\mathcal{R}$ and the two gravitational wave polarizations $+, \times$ as functions of $\log\left(k/k_{\text{ ref}}\right)$, on the left and right, respectively. The inner right figures depict the relative difference between the two polarizations, while the dashed line represents the FLRW case. It can be seem that, for scales that are smaller than the reference scale, where $\log\left(k/k_{\text{ref}}\right) > 0.8$, isotropy is recovery and the spectra are praticaly identical, with non-trivial differences being present only for larger scales.

exact time $t_i(\vec{k})$ for which the ratio $|Q/\omega^2|$ is minimal. While this prescription does not enable one to make predictions for all scales, it does make define predictions for a certain range of scales up to a reference scale k_{ref}, which we shall now briefly discuss.

First, we define the averaged over directions power spectrum $f_X(k)$ by

$$f_X(k) \equiv \frac{1}{4\pi}\int \mathcal{P}_X(\vec{k})d^2\hat{k}\,, \tag{4.108}$$

where $X = \mathcal{R}, +, \times$ is an index that represents the each possible mode type. A plot of $f_X(k)$ for each type of mode can be seem in Figure 4.8, while a Mollweide projection for $\mathcal{P}_{\mathcal{R}}(\vec{k})$ can be seem in Figure 4.9, which provides visual intuition for isotropization on small scales [22].

As noted in [22], the evolution of perturbations can be solved numerically, but such process is quite complicated, so that we merely state the results. More details can be found in appendix A of [22]. The main results are then given by

1. Analyzying the evolution of the direction averaged power spectra $f_X(k)$, the obtained scalar spectral index is given by $n_s - 1 \approx -0.032 \implies ns \approx 0.968$, which is essentially the same as predicted by usual inflationary models [50, 86];

2. For primordial gravitational waves, the predicted tensor spectral index n_T is given by $n_T \approx -0.016$. Also, as previously noted the two gravitational wave polarizations present different

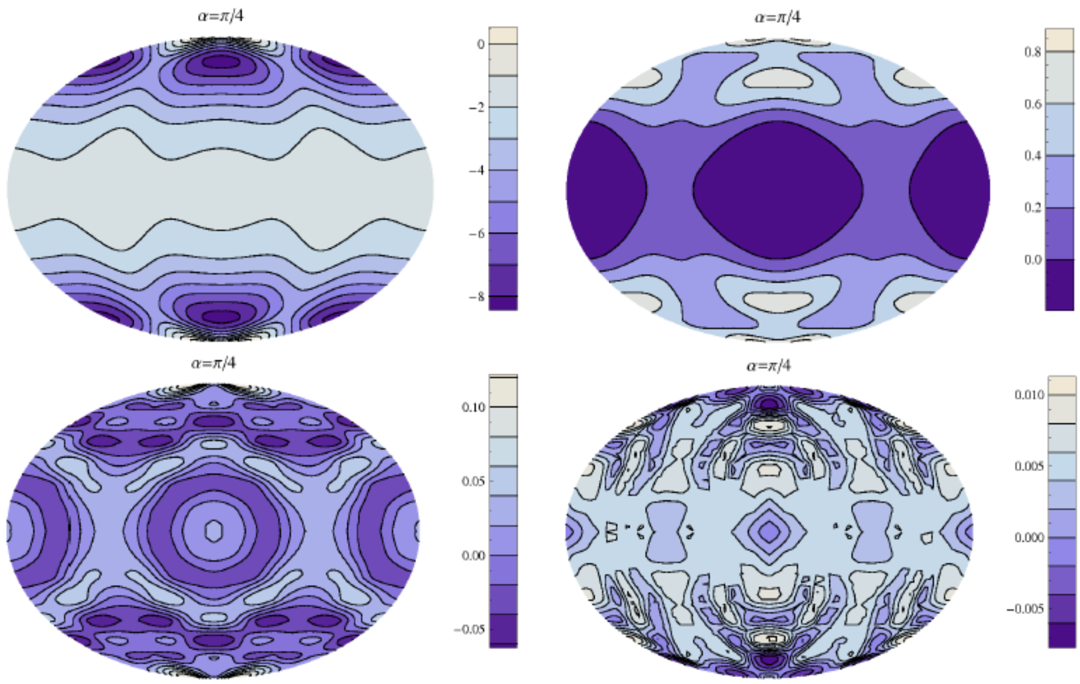

Figure 4.9: Mollweide projection of the percentual ratio between the $\mathcal{P}_{\mathcal{R}}(\vec{k})$ obtained in the Bianchi I model and its value on the FLRW case. They are plotted for $\log(k/k_{\rm ref}) = 1/2, 1, 3/2, 2$ from left to right and top to bottom [22].

dynamics, as can be seem in Figure 4.8 ;

As for the modes with unknown initial conditions it can be shown [22] that tsuch modes are still outside the Hubble radius nowadays, and hence are unobservable, provided inflation has lasted for at least $N \approx 60$ e-folds. This prescription then has the problem that, while it does indeed make predictions, they are limited to modes that are closer to isotropy, so that their differences with respect to the FLRW case are somewhat limited, and also demands a certain fine tuning of the initial anisotropy of the universe in a way that only the largest modes that are observable were affected by the primordial anisotropic phase [9]. Such fine tuning is unsatisfactory in the sense that it returns to the class of problems of initial conditions, which inflation was proposed to solve.

To conclude this chapter, we shall briefly compare such predictions with other works, in order to try to identify model independent aspects of Bianchi I perturbed models. However, it should be noted that there are but a few works on the subject, since perturbation theory on Bianchi I spacetimes has not been developed until recent years [9, 76].

In comparison to the presented model, the previously mentioned Loop Quantum Gravity scalar field bouncing model [75] presents one interesting non-trivial difference: due to the bounce, one can properly define an adiabatic vacuum space in the pre-bounce era, allowing one to make definite predictions, such as:

1. Scalar-tensor and tensor-tensor cross correlations, which suggests such property to be model independent for perturbations defined on Bianchi I backgrounds;

2. Anisotropic features in all the CMB angular correlations, with a particular quadrupolar modulation that is suggested to account for a similar feature observed by the Planck mission.

Furthermore, [75] also reports that the cross correlations feature is manifest in CMB temperature-polarization correlations at low multipoles, with a future detection of B-modes being able to test such prediction.

As for the before mentioned inflationary model defined in a Bianchi I background with line element [21]

$$\mathrm{d}s^2 = -dt^2 + a^2(t)\mathrm{d}x^2 + b^2(t)\left(\mathrm{d}y^2 + \mathrm{d}z^2\right) ,$$

scalar-tensor and tensor-tensor correlations are also reported, with the detection of B-modes in the CMB being again suggested as a way to test such prediction. Works [22] and [21] use different perturbation schemes, but yield the same perturbation evolution according to [21]. It should also be noted that [21] provides yet another way to define initial conditions for the perturbations by analyzing its Kasner-line behavior at early times, but is still not able to consider initial conditions for all modes.[10]

[10]The Kasner metric is the Bianchi I vacuum solution of the Einstein field equations, being explicitly given by $\mathrm{d}s^2 = -dt^2 + t^{2p_1}\mathrm{d}x^2 + t^{2p_2}\mathrm{d}y^2 + t^{2p_3}\mathrm{d}z^2$ with the t_i satisfying the constraints $\sum_{i=1}^{3} p_i = 1$, $\sum_{i=1}^{3} p_i^2 = 1$. For further reading, see [29].

5 CONCLUSIONS AND FUTURE PERSPECTIVES

In this dissertation, we did a self-contained and comprehensive literature review on cosmological perturbations in Bianchi I backgrounds and their quantization process, which we hope will be useful to students or researchers with interest in such models. In particular, it became evident that, by relaxing only one background hypothesis — isotropy — , we obtained much richer models, both at background and perturbative level, and non-trivial problems at the quantization process.

As an example of such difficulties, it was shown that minisuperspace Bianchi I models constructed in canonical quantum gravity may have a non-unitary evolution depending on a chosen operator orderering, which leads to inequivalent interpretations about its state [15, 78]. This leads to quantization problems right at background level, which are crucial to address before perturbations are included.

In the case of inflationary models on Bianchi I backgrounds, ambiguities at the background level are not present since gravity is treated classically. However, at perturbation level, they present another kind of difficulty: the arduous computational steps involved in the analysis of the perturbed classical Einstein field equations, where one needs to perform projections using projection operators [23].

Be as it may be, the difficulties of inflationary models at classical level can be properly solved, but the true challenges occur at the quantization process. This is due to the fact that the obtained dynamical system of equations (4.86) presents 3 interacting fields, a kind of system for which quantization techniques in curved space-times are more scarce [9, 69].

Even in regimes where the couplings can be neglected, the quantization process is still non-trivial due to the fact that, as discussed in section 4.3, the WKB approximation is violated in the primordial shear dominated phase, which does not allow one to define the usual adiabatic vacuum states and hence set initial conditions unambiguously [22]. While one can still make predictions for a range of scales, they are very similar to the isotropic case, and demand an initial shear fine tuning, returning to a problem of initial conditions in cosmology [9].

The aforementioned problems are highly non-trivial and lack a definite solution in literature. Thus, this work may serve as an interesting starting point for students and researchers seeking to address these challenges. It should also be noted that any new results must be compared with the already obtained in literature, in particular references [22, 23, 78, 83]. One might also consider extensions to the analyzed models, such as: 1 developing perturbation theory for the other Bianchi types, in particular the class with isotropic limits (Bianchi types V, VII_0, VII_h, IX); 2. considering other potentials instead of $V(\phi) = m\phi^2/2$ for the inflationary case.

However, before considering such extensions, it would be interesting to solve the problems of the already developed models. To that end, a set of promising new theoretical techniques provide hopes of handling the mentioned difficulties, which we shall now briefly discuss.

In the case of the canonical quantum gravity Bianchi I minisuperspace model, a direct solution would be to look for a physically reasonable operator ordering that allows unitary evolution, and then to consider perturbations in a way similar to the isotropic case done in [87]. Another option would be to consider different notions of time, which would modify the theory's effective Hamiltonian. For instance, a recent new proposal was to consider time on a Bianchi I universe defined only in relation to the gravitational degrees of freedom [80], which remains to be studied in other Bianchi types. It would then be interesting to consider perturbations with such alternative time choices and compare their predictions, which may even shed light on the Problem of Time in quantum cosmology.

As for the inflationary case, an interesting method to deal with the quantization of coupled perturbations was presented in [88], which is based on the diagonalization of the hamiltonian tensor that defines the perturbative equations. One can then define appropriate variables for which a vacuum state can be properly defined and extract predictions. In particular, during the course of this work, we started working on a perturbed two barotropic fluid flat FLRW minisuperspace model. This model is a natural extension of the one presented in section 2.3 and, at perturbative level, one finds two perturbative coupled variables that demand such methods to be applied. This model can be seen as a preliminary step before addressing the three coupled Bianchi I perturbations in future research.

Nonetheless, the WKB approximation problem persists. A possible way to circumvent such challenge would be to consider alternative prescriptions for constructing vacuum states and setting initial conditions. In particular, a quite recent work [89] has introduced a new prescription to define vacuum states for scalar fields which is also valid in the non-adiabatic regime. It is mainly based on the notion of vacuum stability, which is equivalent to particle creation minimization.[1] Such new prescription was created to attack the problem of bouncing models with a positive cosmological constant [89]. This happens due to the fact that, in the asymptotic past of such models, the adiabatic regime is also violated. Such new prescription was able to define a vacuum state for a single scalar field in such case, and it was also shown to recover the usual Bunch-Davies vacuum from considerations on the super-Hubble limit. It would then be interesting to apply similar considerations for inflationary Bianchi I models, but that would require extending the prescription to interacting fields, which is clearly way beyond the scope of this work.

Additionally, another question was called attention during the course of this work: would it possible to define more convenient variables for perturbations on Bianchi I models?

This is due to the fact that, while in the isotropic FLRW case one performs a SVT decomposition in order to describe perturbations in terms of representations of the rotation group, such symmetry is lost in the anisotropy case. Hence, a SVT decomposition on the Bianchi I case does not lead to the same advantages, as was discussed in section 4.3: it lead to laborious calculations at the classical level, but ended with only two dynamical equations for coupled scalar and tensor modes. By properly defining perturbative variables, one would then hope that such lengthy calculations could be avoided, and a simple method to obtain such dynamical equations would appear.

A procedure to obtain such variables was proposed by my advisor. It consists on analyzying the

[1]In the language of chapter 3, this would amount to minimizing the Bogoliubov coefficients (3.34) that compare a chosen vacuum state at time η_0 which its evolved version at time η_1.

full second order perturbed Einstein-Hilbert action — obtained for a general background in [43] — for the Bianchi I case, and eliminating the constraints to isolate the true degrees of freedom without ever making an SVT decomposition. While this procedure seems promising, it has not yet provided enough results to be included in this work, and remains a topic for future research.

Although the quantization of perturbations in a Bianchi I background poses substantial challenges that cannot be overcome using conventional methods, the application of the mentioned novel approaches offers hope for resolving the quantization procedure. This, in turn, may lead to new predictions for the primordial universe and even tighter constraints on its anisotropy and hence the initial conditions of our universe.

A DIFFERENTIAL GEOMETRY

In this appendix we shall introduce the main Differential Geometry concepts of importance for the main text. We start by giving some general definitions, and then we specialize in (pseudo) Riemannian Geometry, which is relevant for General Relativity.

A.1 General Definitions

In this section we introduce the main definitions that are going to be of use in the main text. In particular, we follow Wald's [29] chapter 2 closely. We assume a basic knowledge of topological spaces, and refer the reader to [45] for a more detailed account.

A n-dimensional real **manifold** $\mathcal{M}$ is a topological space with a collection of open sets $\{O_i\}$ such that

1. $\bigcup_i O_i = \mathcal{M}$;
2. For each i there is a homeomorphism $\psi_i : O_i \subset \mathcal{M} \to U_i \subset \mathbb{R}^n$;

in this context, $\{O\}_i$ is an atlas of the manifold, and the pairs (O_i, ψ_i) are coordinate charts in $\mathcal{M}$.

We say that $\mathcal{M}$ admits a **differential structure** if, when there is an intersection $O_i \cap O_j \neq \phi$ between two charts (O_i, ψ_i), (O_j, ψ_j), the composition map $\psi_j \circ \psi_i^{-1} : U_i \subset \mathbb{R}^n \to U_j \subset \mathbb{R}^n$ is smooth, that is, it is a C^∞ function. If $\mathcal{M}$ admits a differential structure, we call $\mathcal{M}$ a differentiable C^∞ manifold. We denote $C^\infty(\mathcal{M})$ the space of all functions of class C^∞ with respect to the differential structure of $\mathcal{M}$.

We define a **derivation** at a point $p \in \mathcal{M}$ as a map $v : C^\infty(\mathcal{M}) \to \mathbb{R}$ which satisfies

1. Linearity:
$$v(af + bg) = av(f) + bv(g)\,;$$
2. Leibniz Rule:
$$v(fg) = v(f)g + fv(g)\,;$$

we then denote as $T_p\mathcal{M}$ as the set of all derivations at $p \in \mathcal{M}$. It follows that $T_p\mathcal{M}$ is a vector space with $\dim T_p\mathcal{M} = \dim\mathcal{M}$, and is known as the **tangent space at** p in $\mathcal{M}$, with the derivations at p being denoted **tangent vectors** at p. If we choose a coordinate chart (O, ψ), then we say that the set $\{X_\mu\}$ of derivations such that
$$X_\mu(f) = \frac{\partial}{\partial x^\mu}\left(f \circ \psi^{-1}\right)\Big|_{\psi(p)}$$
is the coordinate basis associated to the chart (O, ψ), which can be shown to be a basis of $T_p\mathcal{M}$. It is usual to use the notation
$$X_\mu = \frac{\partial}{\partial x^\mu}\,,$$

for a coordinate basis, or $X_\mu = \partial_\mu$, and we shall alter between those interchangeably [45].[1]

A **curve** λ in $\mathcal{M}$ is a C^∞ map $\lambda : I \subset \mathbb{R} \to \mathcal{M}$. A curve can also act on functions $f \in C^\infty(\mathcal{M})$ as derivations at each point $p \in \mathcal{M}$ by the expression

$$\lambda(f)\bigg|_p \equiv \frac{d}{dt}(f \circ \lambda)\bigg|_p ,$$

which can be shown to be a derivation. Here $t \in I$ is the parameter of the curve and $f \circ \lambda : \mathbb{R} \to \mathbb{R}$. At each point $p \in \mathcal{M}$, we can map a curve to a vector $v \in T_p\mathcal{M}$. Using a coordinate chart, the curves components are given by

$$\lambda^\mu \equiv \frac{dx^\mu}{dt} ,$$

where x^μ is the μ-nth coordinate of $\phi \circ \lambda|_t \in \mathbb{R}^n$.

A function $\phi : \mathcal{M} \to \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifolds, induces a function between curves on $\mathcal{M}$ and $\mathcal{N}$, which is known as a **push-forward** ϕ_*. It is defined by the expression:

$$\phi_*(\lambda) \equiv \phi \circ \lambda , \tag{A.1}$$

and, since curves can be mapped to derivations, it follows that ϕ_* maps tangent vectors in $\mathcal{M}$ to $\mathcal{N}$, that is, $\phi_* : T_p\mathcal{M} \to T_{\phi(p)}\mathcal{N}$, which can also shown to be a linear map.

A **vector field** X is a smooth assignment of a vector $v \in T_p\mathcal{M}$ for each $p \in \mathcal{M}$. Here "smooth" means that given a smooth function $f \in C^\infty(\mathcal{M})$, then $X(f)$ is also a smooth function defined for all $p \in \mathcal{M}$. We define the **integral curve** of a vector field X that passes through p by the differential equation

$$\frac{dx^\mu}{dt} = X^\mu(x^1, x^2, ..., x^n) . \tag{A.2}$$

We also define another operation between vector fields: the **commutator** $[,]$. It is a bilinear map that maps two vector fields X and Y to another vector field $[X, Y]$

$$[X, Y](f) = X(Y(f)) - Y(X(f)) , \tag{A.3}$$

and it can be shown that this map is linear, anti-commutative and satisfies the usual Jacobi identity.

We denote $T_p^*\mathcal{M}$ as the **dual space** of $T_p\mathcal{M}$, that is, the space of linear functionals $\omega : T_p\mathcal{M} \to \mathbb{R}$. To a coordinate basis $\{\partial_\mu\}$, we associate the dual basis as being the set of functionals $\{\mathrm{d}x^\nu\}$ that satisfies

$$\mathrm{d}x^\nu\left(\frac{\partial}{\partial x^\mu}\right) = \delta_\mu{}^\nu . \tag{A.4}$$

Note that this definition is basis dependent. However, when we introduce a metric tensor, there will be a natural, basis independent correspondence between $T_p\mathcal{M}$ and $T_p^*\mathcal{M}$, as we shall se.

We define **tensors** in $\mathcal{M}$ as multilinear applications that maps vectors and covectors to numbers.

[1] A more precise notation would be $X_\mu = \left(\frac{\partial}{\partial x^\mu}\right)_p$, which identifies the point p. However, we shall not use such notation, since it would be cumbersome to carry the p subindex.

In a more precise way, we say that an application

$$T : \underbrace{T_p^*\mathcal{M} \times \ldots \times T_p^*\mathcal{M}}_{k} \times \underbrace{T_p\mathcal{M} \times \times T_p\mathcal{M}}_{l} \to \mathbb{R}\,, \tag{A.5}$$

that is linear in each slot is a tensor of rank (k, l), which **tensor fields** being defined in an analog way. In this work, we use Wald's **abstract index notation** [68] to differentiate between geometric objects and coordinate dependent ones. In this notation, latin indexes are integral parts of the tensors, and **do not** represent coordinate components, which are represented by greek indexes. For example, in this notation the application of a tensor map $T : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ on a pair of vectors $u, v \in T_p\mathcal{M}$ is given by:

$$T(v, u) = T_{ab}u^a v^b\,, \tag{A.6}$$

where the indexes represent the "slots" of the multilinear application. This notation is useful to make sure that our expressions do not depend on the choice of coordinates in the manifold, while also presenting the advantage that such expressions look very similar to coordinate ones.

In terms of the abstract index notation, we also define the **symmetrization** and **anti-symmetrization** of rank $(0, 2)$ tensors as follows:

$$T_{(ab)} \equiv \frac{1}{2}\left(T_{ab} + T_{ba}\right)\,, \tag{A.7}$$

$$T_{[ab]} \equiv \frac{1}{2}\left(T_{ab} - T_{ba}\right)\,, \tag{A.8}$$

and in the general case, as follows:

$$T_{(a_1 \ldots a_l)} \equiv \frac{1}{l!}\sum_{\pi} T_{a_{\pi(1)} \ldots a_{\pi(l)}}\,, \tag{A.9}$$

$$T_{[a_1 \ldots a_l]} \equiv \frac{1}{l!}\sum_{\pi} \delta_\pi T_{a_{\pi(1)} \ldots a_{\pi(l)}}\,, \tag{A.10}$$

where the sum is taken over all permutations π of $1, \ldots, l$ with $\delta_\pi = +1$ for even permutations and $\delta_\pi = -1$ for odd permutations.

A totally antisymmetric tensor field $T_{a_1 \ldots a_k}$ of type $(0, k)$, that is

$$T_{a_1 \ldots a_l} = T_{[a_1 \ldots a_k]}\,, \tag{A.11}$$

is called a **differential k-form**.

The **Lie Derivative** of a tensor field T with respect to the vector field X is defined by

$$\mathscr{L}_X T\Big|_p \equiv \lim_{\epsilon \to 0} \frac{\phi_{-\epsilon}^X T(\phi_\epsilon^X(p)) - T(p)}{\epsilon} \tag{A.12}$$

where ϕ_ϵ^X denotes the push-forward induced by the integral curve of X that passes through p, which is parametrized by ϵ. The Lie derivative maps tensors of rank (k, l) to tensors of the same rank, and is a geometric notion of derivative, which can be introduced in any differentiable manifold.

The field of differential geometry studies manifolds, and we introduced structures that can be defined on essentially any manifold. Now we specialize to more specific structures, which are of importance to Einstein's General Relativity.

A.2 (PSEUDO) RIEMANNIAN GEOMETRY

In (pseudo) Riemannian Geometry, we equip $\mathcal{M}$ with a **metric tensor** g_{ab}, that is, a bilinear tensor of rank $(0, 2)$ which satisfies

1. Symmetry: $g_{ab} = g_{ba}$;
2. Non-degeneracy: $g_{ab}X^a = 0 \implies X^a = 0$.

Given a metric tensor g_{ab}, at all points $p \in \mathcal{M}$ we can always find an orthonormal basis $\{(v_\mu)^a\}$ defined on its respective tangent space such that

1. $g_{ab}(v_\mu)^a (v_\nu)^b = 0$ if $\mu \neq \nu$;
2. $g_{ab}(v_\mu)^a (v_\nu)^b = \pm 1$ for $\mu = \nu$;

however, it can be shown that number of basis vectors such that $g_{ab}(v_\mu)^a (v_\nu)^b = +1$ and the number of basis vectors such that $g_{ab}(v_\mu)^a (v_\nu)^b = -1$ is *basis independent*. Such numbers define what we call the **signature** of the metric tensor g_{ab}. We shall work mainly with two possible metric signatures:

1. $+ + ++$: Riemannian Metrics;
2. $- + ++$: Lorentzian Metrics.

With a metric tensor, it is possible to associate each vector $v^a \in T_p\mathcal{M}$ to a covector $v_a \in T_p^*(\mathcal{M})$ by defining the operation of "lowering indexes":

$$v_a \equiv g_{ab}v^b \,, \tag{A.13}$$

we have also that, since g_{ab} is non degenerate, it has an inverse application g^{ab} such that

$$g_{ab}g^{bc} = \delta_a{}^b \,, \tag{A.14}$$

from which we can "raise indexes", that is, for each covector $u_a \in T_p^*(\mathcal{M})$, we define $u^a \in T_p\mathcal{M}$ by

$$u^a \equiv g^{ab}u_b \,, \tag{A.15}$$

with analogous definitions for indexes of arbitrary tensors. A metric tensor then enables one to connect the tangent space to the cotangent space. In particular, for a coordinate basis $\{\partial_\mu\}$, we have

$$dx^\mu_a \equiv g_{ab}\left(\frac{\partial}{\partial x^\mu}\right)^b . \tag{A.16}$$

Hence, in a coordinate basis, we can express the metric tensor as

$$g_{ab} = g_{\mu\nu}\left(dx^\mu\right)_a \otimes \left(dx^\nu\right)_b , \tag{A.17}$$

where

$$g_{\mu\nu} \equiv g_{ab}\left(\frac{\partial}{\partial x^\mu}\right)^a \left(\frac{\partial}{\partial x^\nu}\right)^b \tag{A.18}$$

are the components of the metric tensor. it is also usual to express A.17 in the simplified notation

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu , \tag{A.19}$$

which is similar to an "infinitesimal square distance" [29]. This form is know as the "line element" of a manifold, in the chosen coordinate system.

A **Killing vector field** ξ is a vector field such that the respective Lie Derivative of the metric vanishes:

$$\mathscr{L}_\xi g = 0 \tag{A.20}$$

and they generate isometries, that is, metric doesn't vary along the integral curves of a Killing vector field.

We also equip $\mathcal{M}$ with a **connection** ∇, which is a map that takes each tensor of rank (k,l) to a tensor of rank $(k,l+1)$. We denote the tensor field resulting from the action of ∇ on T by $\nabla_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l}$. It is often notationally convenient to attach an index direclty to the derivative operator and write is as ∇_a. However, it should be noted that this is an abuse of the index notation, since ∇_a *is not* a dual vector. We then demand that the connection satisfies the following conditions, for all tensor fields A, B :

1. Linearity:

$$\nabla_c(\alpha A^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \beta B^{c_1\ldots c_k}{}_{d_1\ldots d_l}) = \alpha\nabla_c(A^{a_1\ldots a_k}{}_{b_1\ldots b_l}) + \beta\nabla_c(B^{c_1\ldots c_k}{}_{d_1\ldots d_l})) ;$$

2. Leibniz rule:

$$\nabla_c(A^{a_1\ldots a_k}{}_{b_1\ldots b_l} B^{c_1\ldots c_k}{}_{d_1\ldots d_l}) = \nabla_c(A^{a_1\ldots a_k}{}_{b_1\ldots b_l}) B^{c_1\ldots c_k}{}_{d_1\ldots d_l} + A^{a_1\ldots a_k}{}_{b_1\ldots b_l}\nabla(B^{c_1\ldots c_k}{}_{d_1\ldots d_l}) ;$$

3. Commutativity with contraction:

$$\nabla_c(A^{a_1\ldots c\ldots a_k}{}_{b_1\ldots c\ldots b_l}) = \nabla_c A^{a_1\ldots c\ldots a_k}{}_{b_1\ldots c\ldots b_l} ;$$

4. Consistency with directional derivatives:

$$t(f) = t^a \nabla_a f$$

5. Torsion free: for all $f \in C^\infty \mathcal{M}$

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f\,;$$

The connection is a geometric structure that, in general, is independent of the metric. However, if we demand

$$\nabla_c g_{ab} = 0\,, \tag{A.21}$$

we say that the connection ∇ is compatible with the metric g_{ab}. This is enough to fix the connection, which in this case is called the **Levi-Civita connection**. In this case, for a general tensor of rank (k,l), we have

$$\nabla_\sigma T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l} = \frac{\partial}{\partial x^\sigma} T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l} + \sum_{i=1}^{k} \Gamma^{\mu_i}{}_{\sigma\rho} T^{\mu_1\ldots\rho\ldots\mu_k}{}_{\nu_1\ldots\nu_l} - \sum_{i=1}^{k} \Gamma^{\rho}{}_{\sigma\nu_i} T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\rho\ldots\nu_l}\,, \tag{A.22}$$

where the $\Gamma^\sigma{}_{\mu\nu}$ are known as the **Christoffel symbols** and are given by [29]

$$\Gamma^\sigma{}_{\mu\nu} \equiv \frac{1}{2} g^{\sigma\rho} \left(\frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\rho\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right)\,. \tag{A.23}$$

In terms of an arbitrary connection, the Lie Derivative, when applied to an arbitrary tensor field of rank (k,l) is given by:

$$\mathscr{L}_v T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = v^c \nabla_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l} - \sum_{i=1}^{k} T^{a_1\ldots c\ldots a_k}{}_{b_1\ldots b_l} \nabla_c v^{a_i} + \sum_{i=1}^{l} T^{a_1\ldots a_k}{}_{b_1\ldots c\ldots b_l} \nabla_{b_i} v^c\,. \tag{A.24}$$

To a connection ∇, the associated **Riemann tensor** is implicitly defined by

$$\nabla_{[a} \nabla_{b]} \omega_c = R_{abc}{}^d \omega_d\,, \tag{A.25}$$

which defines a notion of curvature on the manifold. In a coordinate basis, the riemann tensor components are given by

$$R_{abc}{}^d = R_{\sigma\mu\nu}{}^\rho \mathrm{d}x^\sigma \otimes \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \partial_\rho\,, \tag{A.26}$$

$$R_{\mu\nu\sigma}{}^\rho \equiv \frac{\partial}{\partial x^\nu} \Gamma^\rho_{\sigma\mu} - \frac{\partial}{\partial x^\mu} \Gamma^\rho_{\sigma\nu} + \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma}\,. \tag{A.27}$$

It also presents some important properties, which are

1. Anti-symmetry on the first pair: $R_{abc}{}^d = -R_{bac}{}^d$;

2. Anti-symmetry on the second pair: $R_{abcd} = -R_{abdc}$;

3. First Bianchi Identity: $R_{[abc]}{}^{d} = 0$;

4. Second Bianchi Identity: $\nabla_{[a}R_{bc]d} = 0$

it can be shown that the Bianchi identitites are equivalent to $\nabla_a G^{ab} = 0$, where

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab}\,, \tag{A.28}$$

is the Einstein tensor and

$$R_{ab} \equiv \delta^{c}{}_{d}R_{abc}{}^{d}\,, \tag{A.29}$$

$$R \equiv g^{ab}R_{ab}\,, \tag{A.30}$$

are the Ricci tensor and Ricci scalar, respectively.

We define spacetime as a manifold $\mathcal{M}$ equipped with a Lorentzian metric tensor g_{ab}, being represented by the pair $(\mathcal{M}, g_{ab})$. **General Relativity** is a theory that considers the geometry of spacetime to be dynamic due to an interaction with matter. Such interaction is implemented by assuming that the gravitational field is described by the metric tensor g_{ab} and that it follows the dynamical equations

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}\,, \tag{A.31}$$

where T_{ab} is the matter stress energy tensor, and the coupling constant κ must satisfy $\kappa = 8\pi G/c^4$ to recover the newtonian limit. The system A.31 is called Einstein equations. We also assume that free particles follow geodesics, that is, curves whose tangent vectors v^a obey

$$v^a\nabla_a v^b = 0\,, \tag{A.32}$$

whose coordinate representation is given by

$$\frac{dv^\sigma}{d\tau} + \Gamma^{\sigma}{}_{\mu\nu}v^\mu v^\nu = 0\,, \tag{A.33}$$

where τ is an affine parameter.

To end this section, we point out that the relevant spacetimes for cosmology have an important property: they can be foliated. This means that we express can express their manifold $\mathcal{M}$ using a family of spacelike hypersurfaces $\{\Sigma_t\}$ and a timelike vector field $n^a n_a = -1$ such that

1. $\bigcup_t \Sigma_t = \mathcal{M}$;

2. $\Sigma_{t_1} \bigcap \Sigma_{t_2} = \varnothing$ if $t_1 \neq t_2$;

3. $n^a v_a = 0$ for all v_a tangent to Σ_t;

and t is the parameter associated to the integral curves of n^a.

We can then induce a riemannian metric h_{ab} in each Σ_t by defining

$$h_{ab} \equiv g_{ab} + n_a n_b \,, \tag{A.34}$$

which naturally defines a induced connection and a notion of curvature on Σ_t.

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