Universidade Estadual de Londrina
Centro de Ciências Exatas
Departamento de Física

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Generalized symmetries and their anomalies

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Orientador: Prof. Dr. Pedro Rogério Sérgi Gomes
Coorientador: Profa. Dra. Paula Fernanda Bienzobaz

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Londrina, 17 de julho de 2023

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"When creating a character in a D&D game, we have to decide if his personality is good, neutral or evil. Symmetries also have personalities. A good symmetry gives conserved quantities and selection rules. A neutral symmetry break spontaneously and gives Goldstone bosons. An evil symmetry has an anomaly associated with it and sometimes destroy the whole theory."

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Resumo

É inegável que as simetrias desempenham um papel crucial em todos os sistemas físicos. Uma vez encontradas, suas restrições cinemáticas fornecem informações valiosas na descrição dos fenômenos físicos que estamos tentando descrever. Neste trabalho desenvolvemos sobre o tema de *simetrias generalizadas*, mais especificamente *simetrias de formas superiores* [1], e a anomalia de 't Hooft presente nas teorias $SU(N)$ de Yang-Mills [2]. Para tal, fornecemos uma revisão sobre anomalias e suas consequências na física de baixas energias, bem como, uma discussão sobre operadores de linha em teorias de gauge não-abelianas.

Palavras-Chave: 1. Teoria Quântica de Campos. 2. Teorias de calibre não-abelianas. 3. Anomlias. 4. Simetrias generalizadas. 5. Simetrias de formas superiores.

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Abstract

It is undeniable that symmetries play a crucial role in all physical systems. Once they are found, their kinematic constraints provide a lot of information about the physical phenomena we are trying to describe it. In this work we develop on the subject of *generalized symmetries*, more specifically *higher-forms symmetries* [1], and the 't Hooft anomaly present in $SU(N)$ Yang-Mills theories [2]. To accomplish this, we provide a review on anomalies and their consequences on the low energy physics, as well as, a discussion about line operators in both Abelian and non-Abelian gauge theories.

Key-words: 1. Quantum Field Theory. 2. Non-abelian gauge theories. 3. Anomalies. 4. Generalized symmetries. 5. Higher-forms symmetries.

Lista de ilustrações

Lista de Siglas

QFT	*Quantum Field Theory*
CFT	*Conformal Field Theory*
IR	*InfraRed*
UV	*UltraViolet*
QED	*Quantum ElectroDynamics*
QCD	*Quantum ChromoDynamics*

Sumário

1 Introduction

The application of symmetry principles in theoretical physics can be traced back to Noether work [3], and have been since then one of the leading foundations in a wide variety of subjects, such as, Quantum Field Theory, Condensed Matter, String Theory and General Relativity. Nonetheless, the story goes far beyond what Noether was able to conceive, it is in the possible paths a symmetry can follow where its true power lies in.

A generic continuous global symmetry gives a Noether current and organize the spectrum into quantum numbers. Conversely, in QFT they establish several Ward identities, and those in turn connect different correlation functions [4]. Another possibility would be to consider that the symmetry in question is spontaneously broken. As a consequence, Goldstone bosons arise in the spectrum and, by following Landau paradigm, the phases of the theory can be characterized whether the symmetry is broken or not. Finally, these global symmetries can be gauged. As a result, a completely new theory with distinct behavior arises. However, logically speaking, we must first determine if those global symmetries are non-anomalous [5, 6, 7]. Otherwise, the quantum theory becomes ill-defined after gauging, which by itself can lead to physical consequences.

It is by employing these analyses that we can better comprehend the underlying nature of the theory. As a result, the recent discovery of higher-forms symmetries by Gaiotto, Kasputin, Seiberg, and Willett [1] represents a significant breakthrough in modern physics, given that all those methods mentioned above also generalize to these new type of symmetries. For those interested in further exploration, additional resources and different perspectives on the subject can be found in [8, 9, 10, 11, 12].

The principal feature of higher-forms symmetries is rooted in the fact that their charged objects are extended operators [1], e.g. Wilson and 't Hooft lines [13, 14, 15]. Essentially, this follows from the observation that the symmetry global parameter is now generalized to a closed *q-form*, as opposed to the usual numbers (*0-form*) in the case of ordinary symmetries. Consequently, we are able to rephrase conservation laws in terms of topological operators.

However, these are not mere generalizations. On the contrary, their consideration can lead to valuable consequences. For example, by incorporating higher-symmetries we are capable to reconcile the existence of topological phases of matter [16], e.g. topological insulators and fractional quantum Hall states, with Landau paradigm, as discussed in [17].

Furthermore, once the relation between conversation laws and topological operators has been established, it becomes clear that the conversely may not necessarily be true.

[1] Although these higher-forms symmetries act on extended objects, they differ from string theory in the sense that we are not dealing with dynamical quantities.

In this case, we have a set of topological operators generating a *non-invertible symmetry*. Typically, the presence of these operators can lead to constraints in the theory spectrum [18, 19, 20, 21], thereby providing a tool that is on an equal footing with 't Hooft anomalies.

The objective of this work is to find and characterize the several 't Hooft anomalies associated with different higher-forms symmetries. The main result being the mixed 't Hooft anomaly within $SU(N)$ Yang-Mills theory [2] at $\theta = \pi$. In chapter 2 we focus on the topic of anomalies by first introducing the Chiral anomaly, and then presenting a general discussion about 't Hooft anomalies. Chapter 3 is devoted to line operators in both Abelian and non-Abelian gauge theories. In chapter 4 we finally define higher-forms symmetries and identify two examples within QED and Chern-Simons theories. In chapter 5 we derive the mixed 't Hooft anomaly between the electric $\mathbb{Z}_N$ *1-form* symmetry and time-reversal. Appendix A offers an overview about representation theory, while Appendix B briefly discuss the Wess-Zumino consistency condition. Appendix C summarize some proprieties of differential forms and in Appendix D we rephrase the Chiral anomaly as an non-invertible symmetry.

2 Anomalies

Although certain symmetries may exist in the classical theory, there is no guarantee that they will survive the quantization process. Formally speaking, the Ward identities may have additional terms that violate the symmetry. When this happens, we say that the theory has an anomaly. For the case of 't Hooft anomalies, interesting constraints are imposed on the spectrum of the theory, due to the anomaly matching condition [22], making, therefore, anomalies a powerful tool in the analysis of non-perturbative theories.

2.1 Chiral Anomaly

Historically, the first appearance of such a phenomena was the Chiral anomaly [23, 24], in which the symmetry

$$\psi \to \psi' = e^{i\gamma_5\theta}\psi, \;\; \bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{i\gamma^5\theta} \tag{2.1}$$

was found to be broken by the quantum theory of charged massless fermions. This chiral anomaly can be stated as a violation of the conservation law $\partial_\mu j_5^\mu = 0$, namely,

$$\partial_\mu \langle j_5^\mu(x)\rangle = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho}, \tag{2.2}$$

where

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi,$$

is the Noether current associated with (2.1) and

$$\gamma_5 \equiv \frac{i}{4!}\epsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho.$$

It is worth emphasizing that the anomaly (2.2) is not a pure theoretical problem. In the case where the matter fields are coupled to a background electromagnetic field, the anomaly has to be considered when calculating physical quantities. As an example, the chiral anomaly completely determines the decay rate $\pi^0 \to \gamma\gamma$ [5, 6]. However, if the symmetry is made local and its associated gauge field is treated as dynamical, then we have a gauge anomaly. The *S-matrix* ceases to be unitary and renormalizability is destroyed [5].

The difficult here lies in the fact that we must regularize our theory from possible infinities and still preserve the chiral symmetry. We already can see that dimensional regularization is not a good choice, once γ_5 relies on the Levi-Civita symbol, which is dimensional dependent. Alternatively, Pauli-Villars is not a good choice neither, since adding a mass term would violate the chiral symmetry from the principle.

Furthermore, differently from other derivations, Fujikawa [25] was able to trace back the chiral anomaly to a very comprehensive concept, that is, the symmetry (2.1) does not leave the path integral measure

$$Z \equiv \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS[\psi,\bar{\psi}]}$$

invariant, because of a non-trivial Jacobian. As a consequence, additional terms are found in the Ward identities [4, 5] and the conservation law is violated in the quantum theory.

2.1.1 Anomalous Ward identities

Let us consider the local version of (2.1), i.e. $\theta \to \theta(x)$, and that the Jacobian of this transformation can be written as

$$J = \exp\left(i \int \theta(x) \mathcal{A}(x) d^4x\right). \tag{2.3}$$

Consequently, the action transforms as

$$S[\psi, \bar{\psi}] \to S'[\psi', \bar{\psi}'] = S[\psi, \bar{\psi}] + i \int d^4x j_5^\mu(x) \partial_\mu \theta(x), \tag{2.4}$$

which is the only scalar that we can construct from the objects available and that gives zero when θ is a global parameter. From (2.3) and (2.4) we see that the partition function changes as

$$\begin{aligned} Z' &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS[\psi,\bar{\psi}]} \left(1 - i \int \theta(x) \left(\partial_\mu j_5^\mu(x) - \mathcal{A}(x)\right) d^4x + \mathcal{O}(\theta^2)\right) \\ \delta_\theta Z &= -i \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS[\psi,\bar{\psi}]} \int \theta(x) \left(\partial_\mu j_5^\mu(x) - \mathcal{A}(x)\right) d^4x, \end{aligned}$$

where we have performed an integration by parts. Using now the fact that the fields are dummy variables, we must impose $\delta_\theta Z = 0$, thus finding

$$\begin{aligned} \int d^4x \theta(x) \left(\partial_\mu \langle j_5^\mu(x) \rangle - \mathcal{A}(x)\right) = 0 \\ \Rightarrow \partial_\mu \langle j_5^\mu(x) \rangle = \mathcal{A}(x). \end{aligned} \tag{2.5}$$

Therefore, when a symmetry transformation generates a Jacobian, we must conclude that the conservation law is violated and, as a consequence, it does not represent a symmetry in the quantum theory.

Expression (2.5) is one of the infinity many Ward identities that we could deduce. The others can be found by considering possible insertions in the partition function. If the operator inserted transforms as

$$\mathcal{O}(x_1) \to \mathcal{O}(x_1) + \theta(x_1) \delta\mathcal{O}(x_1),$$

then we would find that

$$0 = -i \int \theta(x) \left(\partial_\mu \langle j_5^\mu \mathcal{O}(x_1)\rangle - \mathcal{A}(x)\right) d^4x + \theta(x_1) \langle \delta\mathcal{O}(x_1)\rangle$$
$$\Rightarrow \partial_\mu \langle j_5^\mu(x)\mathcal{O}(x_1)\rangle = \mathcal{A}(x) + i\delta^{(4)}(x - x_1) \langle \delta\mathcal{O}(x_1)\rangle .$$

The term with a Dirac delta is said to be a contact term, which is stating the fact that there is a charged object at x_1, implying that the conservation law is violated at this point. A generic Ward identity can be written as

$$\partial_\mu \langle j_5^\mu(x)\mathcal{O}(x_1)...\rangle = \mathcal{A}(x) + i\sum_i \delta^{(4)}(x - x_i) \langle \mathcal{O}(x_1)...\delta\mathcal{O}(x_i)...\rangle . \tag{2.6}$$

For a non-anomalous symmetry we have $\mathcal{A}(x) = 0$ and the current is conserved for all $x \neq x_i$.

2.1.2 Anomaly in $(1 + 1)d$ pertubation theory

Here we explore another perspective for the chiral anomaly and explicit calculate $\mathcal{A}(x)$ in 1+1 dimensions. We insist in gauging the symmetry (2.1) and study the effective action $\Gamma[V, A]$, defined by

$$\begin{aligned} Z[V, A] &\equiv e^{i\Gamma[V,A]} \\ &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(-\int d^2x\bar{\psi}\left(\not{\partial} - \not{V} - \gamma_5\not{A}\right)\psi\right), \end{aligned} \tag{2.7}$$

where V_μ and A_μ are the vector and the axial background gauge fields, respectively. In this point of view, the quantum theory is determined by $\Gamma[V, A]$ and, from a consistent condition, it must be gauge invariant under both gauge transformations

$$U_V(1) : \begin{cases} \delta_\theta\psi = i\theta(x)\psi \\ \delta_\theta V_\mu = \partial_\mu\theta(x) \end{cases} , \tag{2.8}$$

$$U_A(1) : \begin{cases} \delta_\phi\psi = i\gamma_5\phi(x)\psi \\ \delta_\phi A_\mu = -\partial_\mu\phi(x) \end{cases} . \tag{2.9}$$

If this is not the case, then the quantum theory does not have such symmetries and hence they cannot be simultaneously gauged.

In $d = 1 + 1$ spacetime, the gamma matrices are

$$\gamma^0 = \sigma^1,\ \gamma^1 = i\sigma^2\ \ \gamma_5 = i\gamma^0\gamma^1 = \sigma^3,$$

and they respect

$$\gamma^\mu\gamma_5 = i\epsilon^{\mu\nu}\gamma_\nu. \tag{2.10}$$

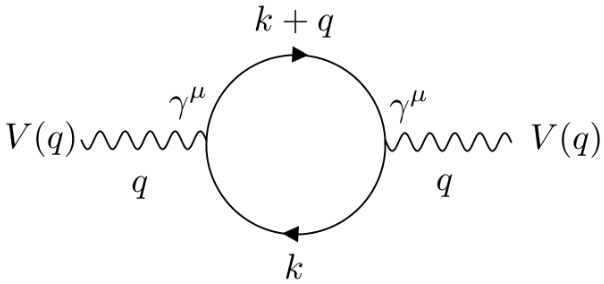

Figura 1 – Fermion loop containing two vectors fields.

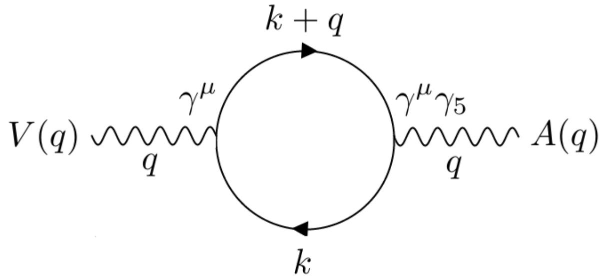

Figura 2 – Fermion loop containing an vector field and a axial field.

In our present case, the anomaly lives in the two-point functions, associated with the Feynman diagrams presented in figures 1 and 2 [1]. In the momentum space they are given by

$$\Gamma_{VV} = \int \frac{d^2q}{(2\pi)^2} V_\mu(q)\mathcal{M}^{\mu\nu}(q)V_\nu(-q),$$
$$\Gamma_{VA} = \int \frac{d^2q}{(2\pi)^2} V_\mu(q)\mathcal{M}_5^{\mu\nu}(q)A_\nu(-q),$$

where

$$\mathcal{M}^{\mu\nu}(q) = \int \frac{d^2k}{(2\pi)^2} Tr\left(\gamma^\mu \frac{i}{\not k}\gamma^\nu \frac{i}{\not k + \not q}\right), \tag{2.11}$$
$$\mathcal{M}_5^{\mu\nu}(q) = \int \frac{d^2k}{(2\pi)^2} Tr\left(\gamma^\mu \frac{i}{\not k}\gamma_5\gamma^\nu \frac{i}{\not k + \not q}\right). \tag{2.12}$$

Note that, (2.11) and (2.12) have an logarithmic divergence and, therefore, both integrals must be regularized. However, because of (2.10) it follows that

$$\mathcal{M}_5^{\mu\nu} = i\epsilon^{\nu\sigma}\mathcal{M}^\mu{}_\sigma, \tag{2.13}$$

thus, implying that regularizing one of them automatically eliminates the divergence of the other.

But before that, let us study the gauge invariance of the two-point function Γ_{VV}, which under a vectorial gauge transformation changes as

$$\delta_\theta\Gamma_{VV} = 2i\int \frac{d^2q}{(2\pi)^2} q_\mu\theta(q)V_\nu(-q)\mathcal{M}^{\mu\nu}(q),$$

[1] The Feynman diagram with two axial fields gives automatically zero, as a consequence of the Furry theorem [4].

where we have used $\delta_\theta V_\mu(q) = iq_\mu\theta(q)$ and $\mathcal{M}^{\mu\nu}(q) = \mathcal{M}^{\nu\mu}(-q)$. Thus, vectorial gauge invariance implies that

$$q_\mu \mathcal{M}^{\mu\nu}(q) = 0. \tag{2.14}$$

Conversely, an axial gauge transformation changes Γ_{VA} as

$$\delta_\phi \Gamma_{VA} = -2i \int \frac{d^2q}{(2\pi)^2} \phi(q) V_\mu(q) q_\nu \mathcal{M}_5^{\mu\nu}(q),$$

which then implies

$$q_\nu \mathcal{M}_5^{\mu\nu}(q) = 0. \tag{2.15}$$

Equantions (2.14) and (2.15) represent two different Ward identities in the momentum space. If the effective action $\Gamma[V, A]$ is in fact gauge invariant under both gauge transformations, then (2.14) and (2.15) must be respected simultaneously.

However, this is not the case. A genereal Lorentz decomposition of $\mathcal{M}^{\mu\nu}(q)$ is given by

$$\mathcal{M}^{\mu\nu}(q) = \left(\zeta g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) T(q^2), \tag{2.16}$$

where $T(q^2)$ is a function yet to be determined by the regularization process. Equation (2.14) is satisfied by (2.16) provide that $\zeta = 1$,

$$q_\mu \mathcal{M}^{\mu\nu}(q) = \left(q^\nu - \frac{q^2}{q^2} q^\nu\right) T(q^2) = 0.$$

However, because of (2.13), (2.15) is only satisfied with $\zeta = 0$

$$q_\nu \mathcal{M}_5^{\mu\nu}(q) = -iq_\nu \epsilon^{\nu\sigma} \frac{q^\mu q_\sigma}{q^2} T(q^2) = 0.$$

Therefore, we must conclude that the effective action is not gauge invariant under both (2.8) and (2.9), since the requirements on ζ cannot be simultaneously satisfied. Different choices of ζ, represent different regularization schemes.

Let us now determine the function $T(q^2)$ by employing the dimensional regularization scheme [26, 27] [2]. For that, we must first take the trace in the gamma matrices, which respect the following properties in $2d$

$$Tr(\gamma^\mu \gamma^\nu) = 2g^{\mu\nu}, \tag{2.17}$$

$$Tr(\gamma^\mu \gamma^\nu \gamma^\sigma) = 0, \tag{2.18}$$

$$Tr(\gamma^\mu \gamma^\sigma \gamma^\nu \gamma^\rho) = 2\left(g^{\mu\sigma} g^{\nu\rho} - g^{\mu\nu} g^{\sigma\rho} + g^{\mu\rho} g^{\nu\sigma}\right). \tag{2.19}$$

[2] Those interested in the historical context surrounding these two articles are encouraged to consult [28].

We also have to deal with IR divergences, since our fermions are massless. This can be easily overcome by giving mass for the fermions and once our calculations are done, we take the limit of $m^2 \to 0$. As a result, the matrix element $\mathcal{M}^{\mu\nu}(q)$ becomes

$$\mathcal{M}^{\mu\nu}(q) = \int \frac{d^dk}{(2\pi)^d} \frac{i^2}{k^2 - m^2} \frac{Tr\left(\gamma^\mu \left(\not{k} + m\right) \gamma^\nu \left(\not{k} + \not{q} + m\right)\right)}{(k+q)^2 - m^2}. \tag{2.20}$$

It is enough for us to consider the trace of (2.20), since this is related with $T(q^2)$ as

$$\mathcal{M}^\mu{}_\mu(q) = \left(g^\mu{}_\mu - \frac{q^2}{q^2}\right) T(q^2) = (d-1)T(q^2),$$

where we have taken $\zeta = 1$ and used $g^{\mu\nu} g_{\mu\nu} = d$. Applying (2.17), (2.18) and (2.19) in (2.20) we have

$$M^\mu{}_\mu(q) = (d-1)T(q^2) = -2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m^2} \frac{(2-d)\left(k^2 + kq\right) + dm^2}{(k+q)^2 - m^2}.$$

By employing Feynman parameter integral

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{(ax + b(1-x))^2},$$

we find after some algebra

$$T(q^2) = -\frac{2}{d-1} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{(2-d)(k^2 + kq) + dm^2}{(k^2 + 2kqx + q^2x - m^2)^2},$$

which upon completing squares and defining $\Delta = m^2 - q^2x(1-x)$ becomes

$$T(q^2) = -\frac{2}{d-1} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(2-d)(k^2 + kq) + dm^2}{([k + qx]^2 - \Delta)^2}.$$

Making the shift $k \to k - qx$ and eliminating any odd term in k, we arrive at

$$T(q^2) = -\frac{2}{d-1} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(2-d)(k^2 - q^2x(1-x)) + dm^2}{(k^2 - \Delta)^2}.$$

Using now the integrals [4]

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^2} = \frac{i}{(4\pi)^{d/2}} \Delta^{d/2-2} \Gamma\left(\frac{4-d}{2}\right),$$
$$\int \frac{d^dk}{(2\pi)^d} \frac{k^2}{(k^2 - \Delta)^2} = -\frac{d}{2} \frac{i}{(4\pi)^{d/2}} \Delta^{d/2-1} \Gamma\left(\frac{2-d}{2}\right),$$

we finally find

$$T(q^2) = -\frac{2}{d-1} \frac{i}{(4\pi)^{d/2}} \int_0^1 dx \Delta^{d/2} \left[(2-d)\left(-\frac{d}{2}\Delta^{-1}\Gamma\left(\frac{2-d}{2}\right)\right)\right.$$
$$\left. + (dm^2 - (2-d)q^2x(1-x))\left(\Delta^{-2}\Gamma\left(\frac{4-d}{2}\right)\right)\right].$$

Taking now the limits of $2-d \to 0$, $m^2 \to 0$ and expanding

$$\Gamma\left(\frac{2-d}{2}\right) \approx \frac{2}{2-d} + \gamma + \mathcal{O}(2-d),$$

where $\gamma = 0,577...$ is the Euler-Mascheroni constant, we find that after several simplifications

$$\begin{aligned} T(q^2) &= \frac{2di}{4\pi}\int_0^1 dx \\ &= \frac{i}{\pi}, \end{aligned}$$

such that

$$\mathcal{M}^{\mu\nu}(q) = \frac{i}{\pi}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right).$$

Recalling (2.13) and (2.15), we conclude that the Chiral anomaly in $d = 1+1$ is

$$q_\nu \mathcal{M}_5^{\mu\nu}(q) = \frac{1}{\pi}\epsilon^{\mu\nu} q_\nu. \tag{2.21}$$

To relate this result with the violation of the current's conservation law, we take the functional derivative of the effective action

$$\langle j_5^\mu(p)\rangle = \frac{\delta}{\delta A_\mu(p)}\Gamma\bigg|_{A=0} = \mathcal{M}_5^{\mu\nu}(p) V_\nu(p)$$

and use (2.21) to find that upon an inverse Fourier transformation we have

$$\begin{aligned} \partial_\mu \langle j_5^\mu(x)\rangle &= \frac{1}{\pi}\epsilon^{\mu\nu}\partial_\mu V_\nu \\ &= \frac{1}{2\pi}\epsilon^{\mu\nu} F_{\mu\nu}. \end{aligned} \tag{2.22}$$

Lastly, note that both $(1+1)d$ and $(3+1)d$ chiral anomalies are total derivatives of the gauge fields. For $1+1$ dimensions this is obvious, since $\epsilon^{\mu\nu}F_{\mu\nu} \sim \partial_\mu\left(\epsilon^{\mu\nu}V_\nu\right)$. On the other hand, from (2.2) it is straightforward to show that the chiral anomaly can be rewritten as

$$\partial_\mu \langle j_5^\mu \rangle = \partial_\mu\left(\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\sigma\rho} A_\nu \partial_\sigma A_\rho\right). \tag{2.23}$$

Therefore, it is tempting to think that we still have a conserved current, since a redefinition of j_5^μ could take into account the anomalous term. However, this would result in an gauge dependent observable, which in turn prohibits us from exploiting this procedure. Nevertheless, with the recent discovery of *non-invertible symmetry*, it was observed by [29, 30] that we are able to persist in the previous argumentation. For more details see Appendix D.

2.2 't Hooft Anomaly

Suppose that we want to analise a generic QFT and this theory has a global group symmetry G. If we try to gauge this symmetry we will find a partition function dependent on the gauge fields $A(x) \in G$

$$Z[A] = \int \mathcal{D}f e^{-S},$$

where $\mathcal{D}f$ stands for the integration over all dynamical fields. So, at this stage $A(x)$ is just a set of classical background gauge fields, which upon a gauge transformation may induce an extra unwanted phase in $Z[A]$

$$Z[A^\lambda] = \exp\left(i \int_{\mathcal{M}} \alpha(\lambda, A)\right) Z[A], \tag{2.24}$$

where $\alpha(\lambda, A)$ is some local function of the gauge fields $A(x)$ and the gauge parameters $\lambda(x)$. If this phase cannot be removed even after adjusting the partition function with local counter-terms, we say that the symmetry in question has a 't Hooft anomaly.

Notice, however, that 't Hooft anomalies are different from gauge anomalies [5], in the sense that if we do not gauge the symmetry G, the theory continues to make sense. 't Hooft anomalies are to be thought as an obstruction to gauging and not as something that brings inconsistency to the theory. On the contrary, 't Hooft anomalies can say a lot about the behavior of the theory, as we will see.

The chiral anomaly is an example of a mixed 't Hooft anomaly, since the theory only becomes inconsistent when we gauge both vector and axial symmetries. The phase (2.24) for the chiral anomaly is the Jacobian of the transformation (2.3). For $(1+1)d$, as discussed in the previous section, we have

$$\alpha(\phi, V) = \phi \ \frac{1}{\pi} \epsilon^{\mu\nu} \partial_\mu V_\nu.$$

Another feature of 't Hooft anomalies is the *anomaly inflow* mechanism. Consider for a moment that our manifold $\mathcal{M}$, in which our field theory is defined, is the boundary of a one dimension higher manifold $\mathcal{V}$, i.e. $\partial\mathcal{V} = \mathcal{M}$. If in this new setup the following equality holds

$$\exp\left(i \int_{\mathcal{V}} \omega(A^\lambda) - i \int_{\mathcal{V}} \omega(A)\right) = \exp\left(-i \int_{\mathcal{M}} \alpha(\lambda, A)\right), \tag{2.25}$$

where $\omega(A)$ defines a theory [3] on the bulk $\mathcal{V}$ and is a local function of background gauge fields. Then, we can redefine the partition function as

$$\mathcal{Z}[A] \equiv \exp\left(i \int_{\mathcal{V}} \omega(A)\right) Z[A],$$

[3] In the condensed matter literature, $\omega(A)$ is referred to as an *invertible field theory.*

such that, under a gauge transformation we would find

$$\mathcal{Z}[A] \to \mathcal{Z}[A^\lambda] = \exp\left(i\int_{\mathcal{V}} \omega(A^\lambda)\right)\left(\exp\left(i\int_{\mathcal{M}} \alpha(\lambda, A)\right)\right) Z[A]$$
$$= \exp\left(i\int_{\mathcal{V}} \omega(A)\right) Z[A] = \mathcal{Z}[A],$$

making once again the theory gauge invariant and, therefore, consistent.

As an example of this procedure, let us consider a massive free fermion in $(2+1)d$, coupled to a background electromagnetic field

$$S = \int d^3x \bar{\psi}\left(i\not{D} + m\right)\psi. \tag{2.26}$$

By integrating out these fermions, the low-energy physics is dominated by the following term [7, 31]

$$\Gamma = \frac{1}{8\pi}\frac{m}{|m|}\int_{\mathcal{V}} d^3x \epsilon^{\mu\nu\sigma} A_\mu \partial_\nu A_\sigma + ..., \tag{2.27}$$

where m is the fermion mass. Notice, however, that differently from the classical theory (2.26), the effective action Γ does not enjoy a parity symmetry in the limit of $m^2 \to 0$ and it is not gauge invariant, since (2.27) is a Chern-Simons theory [4] with level $k = 1/2$, showing, therefore, a mixed 't Hooft anomaly between parity and $U(1)$ symmetry.

We can still restore one of those two symmetry by adding the counter-term

$$S \supset \frac{\epsilon}{8\pi}\int_{\mathcal{V}} d^3x \epsilon^{\mu\nu\sigma} A_\mu \partial_\nu A_\sigma$$

in the action. We can choose $\epsilon = \pm 1$ to either eliminate the Chern-Simons term in (2.27) or to make the level $k = 1$. Depending on the choice, we restore parity symmetry or gauge invariance, but not both of them simultaneously.

In what follows we will take $\epsilon + \text{sing}(m) = 2$, such that, under a gauge transformation Γ changes as

$$\delta_\theta \Gamma = \frac{1}{2\pi}\int_{\mathcal{V}} d^3x \epsilon^{\mu\nu\sigma} \partial_\mu \theta \partial_\nu A_\sigma$$
$$= -\frac{1}{2\pi}\int_{\mathcal{M}} dS_\mu \epsilon^{\mu\nu\sigma} \partial_\nu A_\sigma \theta, \tag{2.28}$$

where we have made an integration by parts and used the fact that $\partial\mathcal{V} = \mathcal{M}$. We then see that (2.28) can be used to cancel the chiral anomaly in $(1+1)d$. Note that, both theories are inconsistent. If we insist in gauging the chiral symmetry for the $(1+1)d$ theory, thus we have a gauge anomaly. On the other hand, if $\mathcal{V}$ has a boundary, then (2.27) is by nature gauge dependent. However, when considered together, they give rise to a healthy theory.

4 See section (4.2) for a introduction to the topic or follow the references [32, 16].

Moving on in our discussion, we can always identify an 't Hooft anomaly, provided that our theory enjoys from a global symmetry lifted by a projective representation, that acts on the Hilbert space by a set of unitary operators U_g. If some pair of these unitary operators satisfies the composition rule

$$U_{g_1}U_{g_2} = e^{2\pi i\alpha}U_{g_1\times g_2}, \tag{2.29}$$

where $g_1, g_2 \in G$, and we cannot eliminate the phase in (2.29) by a redefinition of U_{g_i}, then G is represented projectively.

As a consequence, once G is gauged, any state $|\psi\rangle$ transforming under U_g must be projected out form our Hilbert space. But if (2.29) holds, then we can always produce a gauge dependent state as follows

$$U_{g_i}|\psi\rangle = |\psi\rangle \ \forall\, g_i \in G \Rightarrow U_{g_1}U_{g_2}|\psi\rangle = e^{2\pi i\alpha(x)}U_{g_1\times g_2}|\psi\rangle \tag{2.30}$$

$$|\psi\rangle = e^{2\pi i\alpha(x)}|\psi\rangle\,. \tag{2.31}$$

This should be viewed as an obstruction to gauging and we must infer that projective representations produce 't Hooft anomalies.

2.2.1 Matching condition

The most interesting fact about 't Hooft anomalies is that they are rigid against the RG flow [5], that is, the anomaly $\mathcal{A}(x)$ is the same in the IR as well as in the UV. This is known as the anomaly matching condition. The original 't Hooft's argument can be found in [22]. Here we provide a more modern approach.

Given that the IR and UV physics are respectively related with large and short distances, we can view the RG flow as a scale transformation acting on the metric

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = e^{\sigma}g_{\mu\nu}(x), \tag{2.32}$$

such that, under an infinitesimal transformation the effective action changes as

$$\delta_\sigma\Gamma = \sigma\int g_{\mu\nu}\langle T^{\mu\nu}\rangle\ d^dx, \tag{2.33}$$

where $T^{\mu\nu}$ is the energy-momentum tensor. One can justify (2.33) by arguing that $T^{\mu\nu}$ is the only tensor at our disposal that contracts with $\delta_\sigma g_{\mu\nu} = \sigma g_{\mu\nu}$.

If we now invoke the Wess-Zumino consistency condition [6], use the fact that $\delta_{\sigma\times\lambda} = 0$ and define $\delta_\lambda\Gamma \equiv G(\lambda)$ as the anomaly, it can be shown that

$$\delta_\sigma\delta_\lambda\Gamma - \delta_\lambda\delta_\sigma\Gamma = \delta_\sigma G(\lambda) - \delta_\lambda\langle T^{\mu}{}_{\mu}\rangle = 0. \tag{2.34}$$

Based on (2.34) we are then led to conclude that $G(\lambda)$ is scale invariant, once $\langle T^{\mu}{}_{\mu}\rangle$ is by construction gauge invariant. In conclusion, the anomaly in the UV must be the same as in the IR. In other words, *a 't Hooft anomaly is rigid against the RG flow.*

[5] It is possible to relate this robustness with the fact that $\alpha(\lambda, A)$ is in a discrete cohomology group. More details on that can be found in [33].

[6] See the appendix B for more details.

2.2.2 Implication of 't Hooft anomaly in the IR spectrum

Given a generic QFT, there are only four possible phases that the theory can access in the IR. They are divided into three categories:

- Gapless: In this phase there is no gap above the ground state, the theory becomes scale invariant at long distances and is described by some CFT.
- Gapped: In this phase there is a gap between the ground state and the first excited state.

    ◦ Non-Degenerate: The ground state is unique and nothing can happen in the IR. This is a *trivially gapped* phase.

    ◦ Degenerate: The ground state is not unique and topological correlations functions can be found. The IR theory is described by some TQFT.

- Higgs: In this phase the symmetry is spontaneously broken and the spectrum of particles has Goldstone bosons.

Notice, however, that in a trivially gapped phase there is no dynamics in the IR and correlation functions reduce to contact terms. So in the IR, $Z[A]$ can be replaced by an effective partition function dependent only on the background gauge fields A

$$Z_{eff}[A] \sim \exp\left(i\int_{\mathcal{M}} f(A)\right),$$

which can be eliminated by a redefinition of the partition function. So, the conclusion is that if a theory has an 't Hooft anomaly, the IR spectrum cannot be trivially gapped, once 't Hooft anomalies are due to non-trivial phases.

2.2.3 A Particle on a circle

It will be instructive for us to study the system of a particle on a ring, which has the following lagrangian

$$L = \frac{1}{2}\dot{q}^2 - V(q) + \frac{\theta}{2\pi}\dot{q}, \tag{2.35}$$

with $q \simeq q + 2\pi$ and $V(q)$ being 2π periodic function. Despite the simplicity of the classical model, where the difficulty rest in the potential, the quantum version of the theory will present itself with a much richer structure, providing very useful insights of 't Hooft anomalies and some parallels with Yang Mills theory.

Classically, the θ-term plays no role in the theory. But once the particle is embedded in S^1, we can have configurations where the winding number is non-zero

$$\frac{1}{2\pi}\int_{S^1} \dot{q}dt = \mathbb{Z},$$

and this fact must be accounted when summing over all configurations. Consequently, the partition function $Z[\theta]$ comes with a factor of $e^{i\theta n}$ and the theory is invariant under $\theta \to \theta + 2\pi$.

This fact can also be derived in the canonical formalism. Once we find the conjugate momentum $\pi_q = \dot{q} + \frac{\theta}{2\pi}$, it is easily checked that

$$H_\theta = \frac{1}{2}\left(\pi_q - \frac{\theta}{2\pi}\right)^2 + V(q),$$

and that the similarity transformation

$$e^{iq} H_\theta e^{-iq} = H_{\theta+2\pi} \tag{2.36}$$

furnishes the desired identification.

If we take $V(q) = 0$, the Schrödinger equation can be solved explicitly. The wave function and the spectrum are

$$\begin{aligned} \psi_n(q) &= \frac{1}{\sqrt{2\pi}} e^{iqn}, \\ E_n &= \frac{1}{2}\left(n - \frac{\theta}{2\pi}\right)^2. \end{aligned} \tag{2.37}$$

Notice, that the spectrum is invariant under $\theta \to \theta + 2\pi$, provide that we shift all states as $|n\rangle \to |n+1\rangle$. This phenomenon is known as *level crossing* or *spectral flow.* As we will see later, this system has an 't Hooft anomaly at $\theta = \pi$, indicating, therefore, the non-triviality of the ground state. At the present stage, we can already foresee that $|0\rangle$ and $|1\rangle$ become degenerated at $\theta = \pi$.

Let us now analise the symmetries of the theory. Since the particle is free, we have a spatial translation symmetry. Classically this symmetry acts like $q \to q + \alpha$. In the quantum theory is implemented through the operator T_α, which is given by the map

$$T_\alpha : |n\rangle \to e^{i\alpha n} |n\rangle \,. \tag{2.38}$$

This translation symmetry forms a representation of $SO(2)$.

For $\theta = 0$ and $\theta = \pi$ we also have a parity symmetry, that is, $q \to -q$. For $\theta = 0$ this is obvious. But the fact that θ is 2π periodic, also allows us to also perform a parity transformation at $\theta = \pi$. The action of the parity operator P at $\theta = 0$ is simply

$$P : |n\rangle \to |-n\rangle \,.$$

Thus, when we combine these two transformations we find

$$PT_\alpha P = T_{-\alpha}.$$

So, for $\theta = 0$ the translation and parity combine into $O(2) \simeq \mathbb{Z}_2 \ltimes SO(2)$.

For $\theta = \pi$ we find a more intricate structure. Now the ground state is two-fold degenerated. Translation still respect (2.38), but parity now acts like

$$P : |n\rangle \to |-n+1\rangle \,,$$

where we have identified $\theta = \pi$ with $\theta = -\pi$ to undo the parity transformation.

As a result, at $\theta = \pi$ we find

$$PT_\alpha P = e^{i\alpha} T_{-\alpha}, \tag{2.39}$$

which indicates that the trivial $O(2)$ representation has evolved to a projective representation. Thus, we can infer that a particle on a ring exhibits an 't Hooft anomaly at $\theta = \pi$.

Another way to derive the 't Hooft anomaly is by trying to gauge the translation symmetry. We can accomplish this with the introduction of a gauge field A_0 and modifying the theory as

$$S_{\theta,k}[A_0] = \int dt \left(\frac{1}{2}(\dot{q} - A_0)^2 + \frac{\theta}{2\pi}(\dot{q} - A_0) + kA_0 \right), \; k \in \mathbb{Z}. \tag{2.40}$$

Apart from the last term, the action is invariant under $\dot{q} \to \dot{q} + \dot{\alpha}(t)$ and $A_0 \to A_0 + \dot{\alpha}(t)$. Here the kA_0 term is needed to guarantee the 2π periodicity of θ, which can be fulfilled provided we also shift k as $k \to k-1$. However, as already mentioned, this is not a gauge invariant term and by consequence the action becomes ill-defined. But, once we are only interested on a well-defined quantum theory, we just need to ensure that the partition function is gauge invariant, and given that k is a integer this will be the case[7].

Now let us look at the parity symmetry. For $\theta = \pi$ the parity maps (θ, k) into $(-\theta, -k)$. To get back at $\theta = \pi$ we must perform the 2π shift and by consequence $-k$ goes to $-k+1$. In light of this, the result is

$$(\pi, k) \sim (-\pi, -k+1).$$

But there is no integer solution for $k = -k+1$. Consequently, the theory is not parity invariant and, therefore, we must conclude that we have a mixed 't Hooft anomaly between translation and parity symmetry.

If a generic potential is added to (2.35), then we lost translation symmetry and the mixed 't Hooft anomaly vanishes away. Nonetheless, an interesting possibility arises when $SO(2)$ is only partially broken. As an example, we could add a potential with the following Fourier expansion

$$V(q) = \sum_{k\in\mathbb{Z}} c_{2k} \cos(2kq). \tag{2.41}$$

[7] This follows from the fact that after a gauge transformation, which has a non-zero winding number n, the partition function is proportional to $Z \sim e^{2\pi i n k} \underset{k\in\mathbb{Z}}{\Rightarrow} 1$.

In this set up, the subgroup $q \to q + \pi$ is preserved and the degeneracy at $\theta = \pi$ remains. Depending on the complexity of (2.41) the system can become unsolvable. Nevertheless, the 't Hooft anomaly overcomes this difficult and allow us to infer something non-trivial about the spectrum, that is, the ground state is not trivially gapped.

3 Line operators

To fully comprehend a theory, one should determine all possible correlation functions. Besides the well-known local operators defined from the fields, there is another class of operators that are gauge invariant and non-local. These are the line operators, e.g. Wilson and 't Hooft loops. They play a crucial role in classifying different phases of the theory [15]. To achieve a complete understanding of the subject, it will be necessary to categorize the various representations of the gauge group. This in turn will require an in-depth exploration of representation theory.

3.1 Wilson Loops

Before we begin to address the topic of line operators, it is worthy to review some aspects of non-abelian gauge theories. Suppose our matter fields ψ have a global symmetry determined by some Lie group G, with a general transformation of the form

$$U(\theta) = e^{ig\theta^a T^a} \in G,$$

where θ^a is a set of real numbers and T^a are the generators of G in some representation $R(G)$. For the fundamental representation we have

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{\delta_{ab}}{2}.$$

Given that G represents a symmetry, two different observers must agree on the values of the matter fields,However, if we promote the global parameters θ^a to local ones $\theta^a(x)$ we lose the ability to compare the fields experienced by these observers, once $\theta^a(x)$ introduce an ambiguity. As a consequence, the derivative $\partial_\mu \psi(x)$ becomes ill-defined. To account for this ambiguity, we promote all ordinary derivatives to their covariant form and consider a set of gauge fields $A_\mu(x) \equiv A_\mu^a(x) T^a$, such that

$$\partial_\mu \to D_\mu \equiv \partial_\mu - igA_\mu(x).$$

In this approach, the quantum excitations of $A_\mu(x)$ will give rise to photons or gluons, depending if $G \cong U(1)$ or $G \cong SU(3)$, respectively. The gauge transformation of A_μ can be determined by imposing that

$$\begin{aligned} D_\mu \left(U(x)\psi(x)\right) &= U(x) D_\mu \psi(x) \\ \Rightarrow A_\mu(x) &= U^\dagger(x) \left(A_\mu(x) + \frac{i}{g}\partial_\mu\right) U(x), \end{aligned}$$

or, alternatively, $D_\mu \to U^\dagger(x) D_\mu U(x)$.

After the covariant derivative is found, we can construct the field strength as

$$\begin{aligned} F_{\mu\nu} &\equiv \frac{i}{g}[D_\mu, D_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \end{aligned}$$

A short calculation shows that its gauge transformation is

$$F_{\mu\nu} \to U^\dagger(x) F_{\mu\nu} U(x). \tag{3.1}$$

Moreover, we can give dynamics to the gauge fields by considering the following action

$$S_{YM} = -\frac{1}{2}\int \mathrm{Tr}\left(F \wedge \star F\right), \tag{3.2}$$

since the cyclic propriety of the trace makes (3.2) gauge invariant. This is the Yang-Mills action and its equation of motion is

$$D_\mu F^{\mu\nu} = 0. \tag{3.3}$$

In addition to that, we also have Bianchi identities

$$D_\mu\left(\epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho}\right) = 0. \tag{3.4}$$

Besides (3.2), we are also free to consider the θ-term

$$iS_\theta = i\theta\frac{1}{2}\int_M \mathrm{Tr}\left(\frac{F}{2\pi} \wedge \frac{F}{2\pi}\right) = i\theta\nu, \tag{3.5}$$

with $\nu \in \mathbb{Z}$ being the *instanton number*. However, a brief analysis shows that (3.5) is in fact a total derivative of the form

$$F \wedge F = d\left(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A\right),$$

and because of that (3.5) will not contribute in the classical theory. Nonetheless, since we can have gauge configurations with $\nu \neq 0$, the θ-term might affect the quantum theory, in the same way as the particle on a ring does. As a result, the partition function becomes invariant under the shift $\theta \to \theta + 2\pi$.

So far, our focus has been primarily on local aspects. However, it is time to address the construction of Wilson loops and the role these non-local gauge invariant operators play in the theory.

The parallel transport of ψ_i on our manifold is given by the equation

$$\frac{dx^\mu(\tau)}{d\tau} D_\mu \psi_i(x) = 0 \Rightarrow \frac{d}{d\tau}\psi_i(x) = ig\frac{dx^\mu(\tau)}{d\tau}(A_\mu\psi)_i(x). \tag{3.6}$$

A formal solution can be obtained by introducing an operator $W[x_i, x_f; C]$ that rotates $\psi_i(x)$ to its new configuration

$$\psi_i(x_f) = W_{ij}[x_f, x_i; C]\psi_j(x_i), \tag{3.7}$$

where C is the path traced out by $x^\mu(\tau)$. Given the initial condition $W[x_i, x_i] = 1$, we can plug (3.7) in (3.6) to conclude that

$$W[x_f, x_i; C] = 1 + ig \int_{x_i}^{x_f} d\tau \frac{dx^\mu(\tau)}{d\tau} A_\mu(x) W[x_f, x_i; C]. \tag{3.8}$$

After iterating (3.8) we arrive at

$$W[x_f, x_i; C] = \mathcal{P} \exp\left(ig \int_{x_i}^{x_f} A_\mu dx^\mu \right), \tag{3.9}$$

where $\mathcal{P}$ denotes the path ordering product. It means that when expanding the exponential, we should care about the order we multiply the matrices A_μ and place those corresponding to early positions to the left. The operator (3.9) is a Wilson line and can be used to construct gauge invariant objects such as $\bar{\psi}(x_f) W[C] \psi(x_i)$.

On the other hand, by knowing the gauge transformation of the covariant derivative, we can use (3.6) to argue that the gauge transformation of $W[x_f, x_i; C]$ is

$$W[x_f, x_i; C] \to U^\dagger(x_f) W[x_f, x_i; C] U(x_i). \tag{3.10}$$

Otherwise, the cancellations will not happen and the equation will become gauge-dependent. Once (3.10) is discovered, we can engineer a gauge-invariant operator by taking the trace of (3.9) and setting $x_i = x_f$. By doing so, we arrive at

$$W[C] = \mathrm{Tr}\left(\mathcal{P} \exp\left(ig \oint_C A \right) \right), \tag{3.11}$$

thereby completing the construction of Wilson loops. If our spacetime manifold $\mathcal{M}$ has a non-trivial holonomy [34], then $W[C]$ will measure it. Notice that, this is a topological type of question and, therefore, Wilson loops must be related to the non-perturbative sector of the theory. As we will see, they can verify if the theory confines or not.

3.1.1 Confinement criteria

In this part of the text, we will see how the vacuum expectation value of Wilson loops can provide some insights on the mechanism of confinement. In the case of QCD, confinement can be stated as the absence of isolated quarks in nature. One can rephrase this by saying that *all physical states are color singlets*. It is widely believed that Yang-Mills theory, for a generic group G, also confines. The foundation of this belief is rooted in numerical results provided by lattice gauge theory [15]. But a formal proof is still an open problem. In fact this one of the 7 millennium problems.

Let us consider a pure gauge theory, not necessarily Yang-Mills. We will choose C to be a rectangle, as shown in Figure 3, use the $A_0 = 0$ gauge and define

$$w^i{}_j(t) \equiv \left[\mathcal{P} \exp\left(ig \int_0^R A_i(t, x) dx^i \right) \right]^i{}_j, \tag{3.12}$$

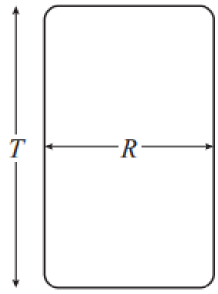

Figura 3 – Rectangular Wilson loop, with $T >> R$ [31].

where i, j are internal indices from the gauge group.

As a consequence of the $A_0 = 0$ gauge, the path C has only spatial contributions to the expectation value $\langle W[C]\rangle$. On the other hand, the path ordering product splits the Wilson loop into two pieces

$$\begin{aligned}\langle W[C]\rangle &= \langle \left[\mathcal{P}\exp\left(ig\int_0^R A_i(T,x)dx^i\right)\right]^i_{\ j} \left[\mathcal{P}\exp\left(ig\int_R^0 A_i(0,x)dx^i\right)\right]^j_{\ i}\rangle \\ &= \langle w^i_{\ j}(T)(w^\dagger)^j_{\ i}(0)\rangle\,. \end{aligned} \tag{3.13}$$

If we recall the time-evolution in the Heisenberg picture $w^i_{\ j}(T) = e^{iHT}w^i_{\ j}(0)e^{-iHT}$ and insert a completeness relation in energy states, we find

$$\begin{aligned}\langle W[C]\rangle &= \sum_n \langle 0|e^{iHT}w^i_{\ j}(0)e^{-iHT}|n\rangle\,\langle n|(w^\dagger)^j_{\ i}(0)|0\rangle \\ &= \sum_n e^{iE_nT}|\,\langle 0|w^i_{\ j}(0)|n\rangle\,|^2. \end{aligned} \tag{3.14}$$

After a Wick rotation and taking the limit of $T \to \infty$, we see that only the ground state will contribute to (3.14). Thus, we can conclude that

$$\langle W[C]\rangle = e^{-E_0(R)T}, \tag{3.15}$$

where $E_0(R)$ measures the energy separation of two probe particles, which are created at $t = 0$ and annihilated at $t = T$.

Once convinced of (3.15), we can analyze the case where the energy separation grows linearly with the distance $E(r) = \sigma r$, that is, the theory confines. In this scenario, the Wilson loop respect an area law

$$\langle W[C]\rangle \sim e^{-\sigma\mathcal{A}[C]},$$

where $\mathcal{A}[C] = TR$ is the area of the loop. Therefore, when the vacuum expectation value of the Wilson loop respect an area law, we can conclude that the theory is in the confining phase.

If now we consider a spacelike loop, that lives in a fixed point in time, then $W[C]$ has the interpretation of an operator that acts on the Hilbert space and creates an electric flux along C. To see this, let us consider the Abelian case for a moment, such that the Wilson loop is simply given by

$$W[C] = \exp\left(ie \oint_C A\right).$$

From the canonical formalism, we known that

$$[A_i(x), E_j(y)] = i\delta_{ij}\delta^{(3)}(x-y). \tag{3.16}$$

So, by using the BCH theorem, we find

$$\begin{aligned}\exp\left(-ie\oint A\right) E_i(y) \exp\left(ie\oint A\right) &= E_i(y) - ie\oint_C [A_\mu(x(\tau)), E_i(y)]dx^\mu \\ &= E_i(y) + e\oint_C \delta^{(3)}(x(\tau)-y)\frac{dx^i(\tau)}{d\tau}d\tau. \end{aligned}\tag{3.17}$$

As already mentioned, (3.17) leads to the conclusion that Wilson loop creates an electric flux along C. Consequently, the expectation value $\langle W[C]\rangle$ is the amplitude of the electric flux to be annihilated back to the vacuum. In this point of view, there are three possibilities scenarios:

- First, the flux tube can dissipate away as it expands. In this case the energy separation goes to zero. One example of such phenomenon is the *QED* vacuum, where the energy separation is the Coulomb potential, i.e. $E(r) \sim \mathcal{O}(1/r)$.
- In the second case, the flux tube is quickly absorbed by the vacuum, breaking itself in smaller strings due to screening effects, and in this case the absorption is proportional to the perimeter of the loop. We find $\langle W[C]\rangle \sim e^{-\mu L[C]}$, where μ is the decay rate of the these strings. This is what happens in an electron plasma.
- Finally, the flux tube can be locally stable, differently from what happens in the previous case. In this scenario, the expectation value of the Wilson loop is measuring the probability for the flux tube to instantaneously disappear, this must be proportional to the area swept out. So, we find $\langle W[C]\rangle \sim e^{-\sigma\mathcal{A}[C]}$, that is, an area law.

3.1.2 *QED* calculation

As a pedagogical example, we will perform the calculation of $\langle W[C]\rangle$ for a pure $U(1)$ gauge theory, in Euclidean space-time, that is

$$S_E = \frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}. \tag{3.18}$$

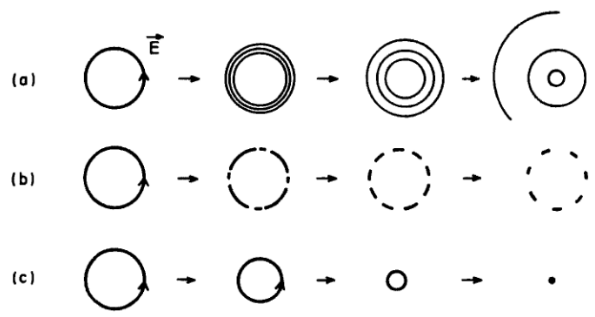

Figura 4 – The three different scenarios, in order of discussion, for an electric flux tube to be annihilated [15].

So, our goal is to compute the following functional integral

$$\langle W[C]\rangle = \int \mathcal{D}A \exp\left(ie\oint_C A\right) e^{-S_E}. \tag{3.19}$$

For that we identify the integral over C as an external current term

$$\begin{aligned}\oint_C A_\mu dx^\mu &= \int d\tau A_\mu \frac{dx^\mu}{d\tau} \\ &= \int d\tau A_\mu \frac{dx^\mu}{d\tau}\delta^{(3)}(x-y)d^3y \\ &= \int A_\mu J^\mu d^4y, \quad J^\mu \equiv \frac{dx^\mu(\tau)}{d\tau}\delta^{(3)}(x(\tau)-y)\end{aligned} \tag{3.20}$$

and consider the Fourier transformation of (3.19)

$$\begin{aligned}S_E = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(-k)(k^2 g^{\mu\nu} - k^\mu k^\nu)A_\nu(k) \\ + A_\mu(k)J^\mu(-k) + A_\mu(-k)J^\mu(k).\end{aligned}$$

The next step would be to use the Green function to shift the field, in such a way that $J_\mu(k)$ and $A_\mu(k)$ decouple. But this involves inverting the matrix

$$P^{\mu\nu} = g^{\mu\nu} - k^\mu k^\nu/k^2, \;\; P^{\mu\sigma}P_{\sigma\nu} = \delta^\mu_\nu,$$

which is singular. Note that, $P^{\mu\nu}$ is a projection matrix. However, if we decompose $A_\mu(k)$ into its transverse and longitudinal components, then the component parallel to k^μ will not contribute to the path integral, once $P^{\mu\nu}k_\nu = 0$ and $k_\mu J^\mu(k) = 0$ [1]. As a consequence, we can restrict ourselves to work only with the subspace orthogonal to k_μ. But in this subspace $P^{\mu\nu}$ acts like the identity, which is invertible.

Once this detail is understood, we decouple J^μ from A^μ by the redefinition

$$\bar{A}_\mu(k) = A_\mu + \frac{P_{\mu\nu}}{k^2}J^\nu(k),$$

[1] This follows from the conservation law $\partial_\mu J^\mu = 0$.

finding so

$$\langle W[C]\rangle = Z_0 \exp\left(-\frac{e^2}{2}\int \frac{d^4k}{(2\pi)^4} J_\mu(-k)\frac{P^{\mu\nu}}{k^2}J_\nu(k)\right),$$

where Z_0 is a number that can be eliminated after the normalization. If we come back to the coordinate space and substitute (3.20), then we find out that

$$\langle W[C]\rangle = \exp\left(-\frac{e^2}{2}\oint_C dx^\mu \oint_C dy_\mu \frac{1}{4\pi^2|x-y|^2}\right).$$

But, once the integrand depends only on the difference $|x-y|$, we arrive at a perimeter law

$$\langle W[C]\rangle = \exp\left(-\mu(a,C)\oint_C dy\right) = e^{-\mu(a,C)L[C]},$$

where $\mu(a,C) \equiv \dfrac{e^2}{8\pi^2}\oint_C \dfrac{du}{|u|^2}$, with $u = x - y$, is a number that depends on path C and the cutoff a, which removes the divergence as $x \to y$. Therefore, we can conclude that *QED* does not confine.

3.2 ’t Hooft Loops

In the previous sections, we saw how a gauge theory responds to the insertion of electric probes. This response was characterized by Wilson lines. If we now want to understand the effect of magnetic monopole insertions, then we must construct a line operator capable of creating magnetic fluxes, such that it behaves in the same fashion as their electric partners.

For that, it is convenient to introduce the dual field $\widetilde{A}_\mu$, defined by

$$\frac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho} \equiv \widetilde{F}_{\mu\nu} \equiv \partial_\mu \widetilde{A}_\nu - \partial_\nu \widetilde{A}_\mu, \tag{3.21}$$

which in language of differential forms can be simply stated as $\star dA \equiv d\widetilde{A}$, where $\star$ is the Hodge dual [34]. Note that, (3.21) is valid only locally. In fact, we might have to define several $\widetilde{A}$, on different patches, in order to cover the whole manifold. This is similar to what we do when discussing the Dirac monopole, where we define a north and a south vector potential on the S^2 sphere surrounding the monopole.

Once (3.21) is given, a ’t Hooft loop is defined as

$$T[C'] = \exp\left(i\mathit{\Phi}\oint_{C'} \widetilde{A}\right), \tag{3.22}$$

where $\mathit{\Phi}$ is the magnetic flux that $T[C']$ creates along C'. We can rewrite (3.22) in terms of the electric field. For that we set C' to be a spacelike loop, use the Stokes’ theorem and apply (3.21), as follows

$$\begin{aligned}T[C'] &= \exp\left(i\varPhi\oint_{C'}\widetilde{A}\right) = \exp\left(i\varPhi\int_{\Sigma}d\widetilde{A}\right)\\ &= \exp\left(i\varPhi\int_{\Sigma}\star dA\right) = \exp\left(\frac{i\varPhi}{2}\int_{\Sigma}\epsilon_{ijk}E_i dy^j\wedge dy^k\right),\end{aligned} \tag{3.23}$$

with $\partial\Sigma = C'$.

From the BCH theorem and the canonical quantization (3.16), it follows that

$$\begin{aligned}T[C']\left(\oint_C A\right)T^{-1}[C'] &= \oint_C A + \epsilon_{ijk}\frac{i\varPhi}{2}\int_{\Sigma}\oint_C [E_i(x'),A_l(x)]dx^l\wedge dy^j\wedge dy^k\\ &= \oint_C A + \epsilon_{ijk}\frac{\varPhi}{2}\int_{\Sigma}\oint_C \delta^{(3)}(x-x')dx^i\wedge dy^j\wedge dy^k\\ &= \oint_C A + \varPhi L(\Sigma,C),\end{aligned} \tag{3.24}$$

where $L(\Sigma,C)$ is the intersection number and it counts how many times C pierces Σ and, by consequence, measures if C is linked with C'. Its definition is given by

$$L(\Sigma,C)\equiv\oint_C\left(\epsilon_{ijk}\int_{\Sigma}\delta^{(3)}(x-x')dy^j\wedge dy^k\right)\wedge dx^i. \tag{3.25}$$

Note that, $\oint_C A$ is measuring the magnetic flux through the surface with boundary C. Thus, the addition of the $\varPhi$ term must be interpreted as the creation of a magnetic flux. So, the operator (3.22) is in fact what we would expect from a spacelike 't Hooft loop. It also follows from (3.24) that the Wilson loop and 't Hooft loop have a non-trivial commutation relation

$$W[C]T[C'] = e^{ie\varPhi L(\Sigma,C)}T[C']W[C]. \tag{3.26}$$

Until now, we have restricted ourselves to loops contained in $3d$ manifolds, where the intersection number has a topological meaning. This implies that we cannot undo the link between two lines by a continuous deformation. But, if we want to consider correlation functions between $W[C]$ and $T[C']$, then we must set the extra phase on (3.26) to unity. To understand this, let us consider two closed loops C and C', initially intertwined. In order to calculate the correlation function between these two lines, we must consider them on the whole $4d$ spacetime. By doing that, the intersection number becomes ill-defined, once we can use the fourth dimension to undo the link, making the correlation function, a physical observable, ill-defined.

We can get around this by setting

$$e^{ie\Phi}\equiv 1 \Rightarrow \Phi = \frac{2\pi}{e}n,\ \ n\in\mathbb{Z}. \tag{3.27}$$

The condition (3.27) is nothing other than the Dirac quantization [35]. Such a condition introduces a fundamental *quantum flux*, that we will define as the charge of a single monopole $m\equiv 2\pi/e$.

3.3 Line operators for non-Abelian gauge theories

When discussing 't Hooft loops, we have restricted ourselves to the Abelian case, for the sake of simplicity. Now we will study the Dirac quantization in the non-Abelian context. If a monopole is inserted at the origin, from analogy with the electric monopole, we expect the magnetic field to be

$$F_{ij} = \epsilon_{ijk}\frac{x^k}{4\pi r^3}Q(x) \sim B^k = \frac{x^k}{4\pi r^3}Q(x), \tag{3.28}$$

where $Q(x) = Q^a(x)T^a$. The factor of $Q(x)$ can be interpreted as some kind of non-Abelian magnetic charge. However, different from the Abelian case, this charge needs to be a function of spacetime, once F_{ij} has a non-trivial gauge transformation

$$F_{ij} \to U^\dagger F_{ij} U \Rightarrow Q \to U^\dagger Q U. \tag{3.29}$$

Considering (3.3) with $E_i = 0$ and B_i given by (3.28), we find

$$\begin{aligned} \epsilon^{ijk}D_iB_j = 0 \Rightarrow \epsilon^{ijk}Q\partial_i\left(\frac{x_j}{r^3}\right) + \epsilon_{ijk}\frac{x^j}{r^3}D_iQ &= \\ \epsilon^{ijk}Q\underbrace{\left(\frac{\delta_{ij}}{r^3} - 3\frac{x_ix_j}{r^5}\right)}_{=0} + \epsilon^{ijk}\frac{x_j}{r^3}D_iQ &= \\ \epsilon^{ijk}x_jD_iQ &= 0. \end{aligned} \tag{3.30}$$

Additionally, (3.4) gives

$$\begin{aligned} D_0B_k = 0 \Rightarrow \frac{x_k}{r^3}D_0Q &= \\ D_0Q &= 0 \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} D_iB^i = 0 \Rightarrow Q\partial_i\left(\frac{x^i}{r^3}\right) + x_iD^iQ &= \\ Q\underbrace{\left(\frac{\delta^i{}_i}{r^3} - 3\frac{x_ix^i}{r^5}\right)}_{=0} + x^iD_iQ &= \\ x^iD_iQ &= 0. \end{aligned} \tag{3.32}$$

From (3.30), (3.31) and (3.32) we can conclude that $D_\mu Q = 0$ for all spacetime, that is, Q is covariantly constant[2]. More than that, we can take $Q(x)$ to be constant, provided that we choose the potentials as

$$A_\pm = \frac{Q}{4\pi}(\pm 1 - \cos\theta)d\phi. \tag{3.33}$$

[2] By covariantly constant, we mean that the covariant derivative gives zero.

Here the A_+ denotes the northern hemisphere, and A_- the southern hemisphere, such that both fields produce the magnetic field (3.28) and avoid their respective singularities [3]. By setting (3.33), we find that the covariant derivative of Q reduces itself to an ordinary derivative

$$D_\mu Q = \partial_\mu Q - i \underbrace{[A_\mu, Q]}_{\sim [Q,Q]=0} = 0.$$

We are also free to choose the basis where $Q(x)$ is diagonal. If we define the *Cartan sub-algebra*[4] $\mathfrak{h} \subset \mathfrak{g}$ as the maximal sub-set of mutually commuting generators, then $Q(x)$ can be written as the linear combination

$$Q(x) = m^i(x) H^i, \text{ with } H^i \in \mathfrak{h} \subset \mathfrak{g}. \tag{3.34}$$

Let us now analyze the Wilson loop. Since Q is in the Cartan sub-algebra, the path ordering can be ignored. On the other hand, setting C to be the circuit at the overlapping region between northern and southern hemispheres, i.e. $\theta = \pi/2$, the following condition on $W[C]$ must hold true

$$\mathrm{Tr}\left(\exp\left(i \oint_C A_+\right)\right) = \mathrm{Tr}\left(\exp\left(i \oint_C A_-\right)\right).$$

This in turn leads to

$$\mathrm{Tr}\left(\exp\left(i \int_0^{2\pi} \frac{Q}{4\pi} d\phi\right)\right) = \mathrm{Tr}\left(\exp\left(-i \int_0^{2\pi} \frac{Q}{4\pi} d\phi\right)\right) \Rightarrow \mathrm{Tr}\left(e^{iQ}\right) = 1. \tag{3.35}$$

Recalling (3.34) and denoting μ^i as the eigenvalues of H^i in a specific representation $R(G)$, the condition (3.35) can be stated as

$$\vec{m} \cdot \vec{\mu} = 2\pi \mathbb{Z}, \tag{3.36}$$

which is the non-Abelian generalization of the Dirac quantization. From Appendix A [5], we known that (3.36) can be respected if m^i is in the dual weight lattice

$$\frac{\vec{m}}{2\pi} \in \Lambda_w^*(G)/W,$$

where we are already eliminating the gauge redundancy introduced by the Weyl group. If $\widetilde{G}$ is the covering group, that is, the first homotopy group is trivial $\pi_1(\widetilde{G}) = 0$, then the dual weight lattice $\Lambda_w^*(G)$ exactly matches the co-root lattice $\Lambda_r^\vee(G)$. In conclusion, an allowed magnetic charge must come from tensorial products of the adjoint representation, once m^i is a co-root. On the other hand, for gauge groups of the form $G = \widetilde{G}/\Gamma_G$, in

[3] To make the singularities evident we must use $d\phi = \hat{\phi}/r \sin\theta$ to rewrite (3.33) as a vector.

[4] In some literatures, the Cartan sub-algebra is also referred as the *maximal torus* of G.

[5] A detailed discussion concerning the formalism of weights and roots can be found in Appendix A. The results obtained therein will be extensively utilized throughout the following discussion.

which $\pi_1(G) \neq 0$, the dual weight lattice allows the existence of more magnetic charges than those in the adjoint representation, at the cost that the weight lattice lose some of the allowed electric charges.

Let us now focus on a generic line operator which is charged both electrically and magnetically, these are said to be *dyonic lines*. In general, the electric and magnetic charges of these dyonic lines are in their respective weights lattice, modulo the Weyl group,

$$\left(\vec{e}, \frac{\vec{m}}{2\pi}\right) \in \Lambda_w^e(G)/W \times \Lambda_w^m(G)/W, \tag{3.37}$$

such that (3.36) is generalized to [14]

$$\vec{m} \cdot \vec{e}\,' - \vec{m}' \cdot \vec{e} \in 2\pi\mathbb{Z}. \tag{3.38}$$

This condition ensures that two dyonic lines with different charges will commute and, as a consequence, have a well-defined correlation function. However, the classification provided in (3.37) can be further reduced, once these charges form an equivalence class. If $\vec{r}_e \in \Lambda_r^e(G)$ and $\vec{r}_m \in \Lambda_r^m(G)$, then it follows that the weights $(\vec{e}, \vec{m})$ and $(\vec{e}+\vec{r}_e, \vec{m}+\vec{r}_m)$ are in the same equivalence class. Given that roots are always present in the weight lattice, regardless the exact choice of G, their addition cannot affect the quantization (3.38), since $\vec{m} \cdot \vec{r}_e$ and $\vec{e} \cdot \vec{r}_m$ are always multiples of 2π.

After we eliminate this redundancy, by modding out those roots, we are left with

$$\left(\vec{e}, \frac{\vec{m}}{2\pi}\right) \in Z(G) \times Z(G), \tag{3.39}$$

where $Z(G)$ is the center of the group G and we used that $\Lambda_w(G)/\Lambda_r(G) \cong Z(G)$, which is derived in [36]. To better illustrate the previous discussion about weights and roots, we now turn to the specific example of gauge theories based on $\mathfrak{su}(2)$, namely $SU(2)$ and $SU(2)/\mathbb{Z}_2$.

3.3.1 $SU(2)$ vs $SU(2)/\mathbb{Z}_2$

We begin by examining the $SU(2)$ theory, which has $\sigma_z \in \mathfrak{h}$ and the algebra given by

$$[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm. \tag{3.40}$$

Firstly, note that the Cartan sub-algebra is one-dimensional and, therefore, the weights and roots are simply numbers. The weights are determined by the eigenvalues of σ_z, i.e., $\mu = \pm 1$. Conversely, for the roots we compare (3.40) with (A.6) to infer that $\alpha = \pm 2$. A generic line is then characterized by two integers

$$\left(e, \frac{m}{2\pi}\right) \in \mathbb{Z} \times 2\mathbb{Z}, \tag{3.41}$$

such that (3.38) is respected. In summary, differently from 't Hooft lines which can only be found in the adjoint representation, a Wilson can be taken in any representation. It is worth noting that all magnetic charges for a $SU(2)$ gauge theory are defined as $m = 0$ mod 2. This is the first signal that 't Hooft lines are uncharged by the gauge group center. The implications of this will be explored in Chapter 5. As a sanity check, we can verify that the coset

$$\Lambda_w(SU(2))/\Lambda_r(SU(2)) \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$$

matches exactly with the center of $SU(2)$.

Alternatively, for $SU(2)/\mathbb{Z}_2$, which is isomorphic to $SO(3)$, we have $J_z \in \mathfrak{h}$ and the algebra

$$[J_z, J_\pm] = \pm J_\pm. \tag{3.42}$$

Given that the gauge group is centerless, the electric charges are now defined as $e = 0$ mod 2. Conversely, from (3.42) it follows that $\alpha = \pm 1$. As a result, we find

$$\left(e, \frac{m}{2\pi}\right) \in 2\mathbb{Z} \times \mathbb{Z}. \tag{3.43}$$

This suggests that Wilson lines in the fundamental representation are prohibited, resulting in a smaller electric weight lattice compared to the $SU(2)$ theory. In contrast, 't Hooft lines can now be constructed in any representation, hence the magnetic weight lattice is enlarged. This exchange of roles guarantees that the physical content in the spectrum of lines is conserved.

A third spectrum of lines can be lifted by making use of the Witten effect [37]. The Witten effect states that after a shift in the θ parameter, recall (3.5), the electric and magnetic charges are mapped into

$$\left(e, \frac{m}{2\pi}\right) \underset{\theta+2\pi}{\rightarrow} \left(e + \frac{m}{2\pi}, \frac{m}{2\pi}\right). \tag{3.44}$$

For $SU(2)$ this identification has no consequences, since the spectrum is invariant under (3.44). However, when considering a $SO(3)$ gauge theory, the spectrum we found in (3.43) is mapped into

$$\left(e, \frac{m}{2\pi}\right)_{\theta+2\pi} \in \mathbb{Z} \times \mathbb{Z}, \tag{3.45}$$

where the condition $e + m/2\pi \in 2\mathbb{Z}$ returns the original electric charge. As a result, we should conclude that a $SU(2)/\mathbb{Z}_2$ gauge theory is only invariant under a 4π shift in the θ parameter.

The discussion given above can be easily generalized to all $SU(N)$ theories. It is worth noting that for $PSU(N) \cong SU(N)/\mathbb{Z}_N$ theories, an interesting idea arises. Provided that the theory is only invariant under $\theta \to \theta + 2\pi N$, there will be some values of N

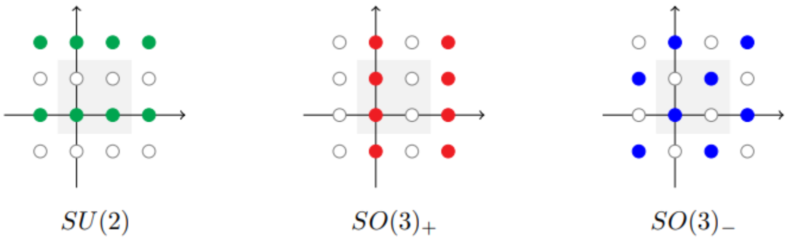

Figura 5 – The horizontal axis stands for the allowed electric charges and the vertical axis for allowed magnetic charges [14].

that we will not be able to undo a time-reversal transformation through a shift in the θ parameter. Therefore, given a generic $\mathbb{Z}_N$ global symmetry within a $SU(N)$ gauge theory, this observation must imply that upon gauging this hypothetical symmetry time-reversal is lost. We are then led to conclude that there is an 't Hooft anomaly in the theory. But before proving that is indeed the case, we must delve into the topic of *higher-forms symmetries*.

4 Higher form Symmetries

The recent discover of *higher forms symmetries* [1] demonstrated that there are other objects, beyond point-like fields, that are charged by a new type of symmetry. These are extended operators, e.g. Wilson loop and surface defects, which have q-dimensional probe excitations. As we will see, all results of ordinary symmetries ($q = 0$) can be generalized straightway to those *q-form* symmetries. They can break spontaneously, be gauged or have anomalies. In each of these scenarios, a lot of insights about the theory can be obtained.

4.1 The general framework

We begin our study of higher form symmetries by considering a $U(1)$ gauge theory embedded in $\mathcal{M} = S^1 \times \mathcal{M}^3$, where time is a periodic variable $t \in [0, \tau)$ [1]. Besides the ordinary gauge transformations

$$a_\mu \to a'_\mu = a_\mu + \partial_\mu \Lambda, \tag{4.1}$$

we are also allowed to choose a gauge parameter Λ which is not single-valued, a feature that is only present on a manifold with non-trivial topology, such as S^1. For example, if we take $\Lambda = (2\pi n/\tau)t$, then the gauge field transforms as

$$a_0 \to a'_0 = a_0 + \frac{2\pi}{\tau} n, \;\; n \in \mathbb{Z}. \tag{4.2}$$

These type of gauge transformations are said to be *large gauge transformation.*

Consider now that we shift our gauge field by a closed one-form $\lambda \in [0, 2\pi/\tau]$, as follows

$$a_\mu \to a'_\mu = a_\mu + \lambda_\mu. \tag{4.3}$$

If our action only depends on the field strength $f = da$, then (4.3) must represent a symmetry of our theory, since λ is closed, i.e. $d\lambda = 0$. Also, note that (4.3) cannot be undone by a gauge transformation, once λ is not exact, i.e. $\lambda = d\Lambda$, and it is outside the range of allowed large gauge transformations. However, someone could argue that the gauge field a_μ is not a physical observable and, by consequence, (4.3) does not represent a symmetry. But this only holds true for local operators. The charged objects here are Wilson loops

$$W[C] \to W[C]' = \exp\left(ie \oint_C \lambda \right) W[C], \tag{4.4}$$

[1] As Witten argues in [16], we might not be able to engineer this set up in the real world. Nevertheless, this theoretical study can provide us with some information, since the theory still must make sense regardless on the manifold it is embedded in.

which in turn lead us to conclusion that non-local observables can have symmetries by their own. We now proceed on the precise description of these new type of symmetries.

Suppose we have a continuous symmetry with global parameter λ and a current j^μ. If we consider the local version of this symmetry, the action must change as

$$\delta_\lambda S = \int_{\mathcal{M}^d} d^d x j^\mu \partial_\mu \lambda = \int_{\mathcal{M}^D} \star j \wedge d\lambda. \tag{4.5}$$

Normally, we would invoke the principle of least action and conclude that the current must be conserved [2]

$$\partial_\mu j^\mu = d \star j = 0. \tag{4.7}$$

Notice, however, that by writing these results in the language of differential forms we are allowed to consider cases beyond those in which λ is a function of spacetime, that is, a *0-form*. If λ is in fact a *q-form*, then we must have $d\lambda = 0$ to (4.5) represent a true symmetry. The parameter λ is said to be a *flat connection* that parameterize the *q-form symmetry.*

For ordinary symmetries, we have point-like charged excitations, and to measure their charges we must enclose them by a Σ^{d-1} closed manifold, as usual from the Gauss law. In the case of continuous *q-form* symmetries, we have *q-dimensional* excitations, e.g. line operators, and they are measured by the operator

$$Q(\Sigma^{d-q-1}) = \oint_{\Sigma^{d-q-1}} \star j. \tag{4.8}$$

As an example of (4.8), let us consider $q = 0$ and that Σ^{d-1} is the whole space. In this set up we find

$$\begin{aligned} Q(\Sigma^{d-1}) &= \frac{1}{(d-1)!} \int_{\Sigma^{d-1}} j_\mu \epsilon^\mu{}_{\nu_1 \dots \nu_{d-1}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{d-1}} \\ &= \int_{\Sigma^{d-1}} j_0 \Bigg(\underbrace{\epsilon^0{}_{i_1 \dots i_{d-1}} \frac{dx^{i_1} \wedge \dots \wedge dx^{i_{d-1}}}{(d-1)!}}_{=dV} \Bigg) = \int_{\Sigma^{d-1}} j_0 dV, \end{aligned} \tag{4.9}$$

which is our usual definition of charge.

---

[2] To verify (4.7) we must use $\star j = \frac{1}{(d-1)!} \epsilon^\mu{}_{\mu_1 \dots \mu_{d-1}} j_\mu dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{d-1}}$ and $dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{d-1}} = \epsilon^{\nu \mu_1 \dots \mu_{d-1}} dV$, such that

$$d \star j = \frac{1}{(d-1)!} \epsilon^\mu{}_{\mu_1 \dots \mu_{d-1}} \partial_\nu j_\mu dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{d-1}} = \underbrace{\frac{\epsilon^\mu{}_{\mu_1 \dots \mu_{d-1}} \epsilon^{\nu \mu_1 \dots \mu_{d-1}}}{(d-1)!}}_{=g^{\mu\nu}} \partial_\nu j^\mu dV. \tag{4.6}$$

Thus, implying $d \star j = \partial_\mu j^\mu dV$.

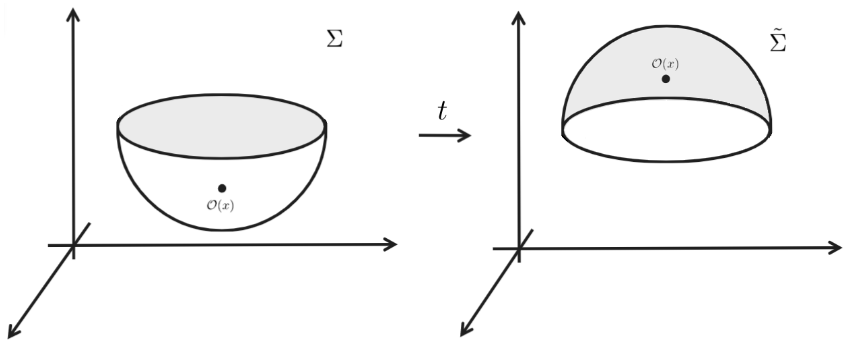

Figura 6 – On the left: A closed surface Σ composed of a southern hemisphere and a disk. On the right: Another closed surface $\tilde{\Sigma}$ composed of a northern hemisphere and a disk. These two manifolds are homeomorphic to each other, such that their union gives an sphere, with a charged insertion $\mathcal{O}(x)$ inside it.

We can also construct a symmetry transformation by exponentiating the charge (4.8)

$$U_\alpha(\Sigma^{d-q-1}) = e^{i\alpha Q(\Sigma^{d-q-1})}. \tag{4.10}$$

Note that, Σ^{d-q-1} can extend itself along time [3]. If this is the case, then U_α is said to be a topological defect. Otherwise, U_α is an operator that acts on the Hilbert space. These symmetry transformations satisfy an usual group law multiplication

$$U_{g_1}(\Sigma^{d-q-1})U_{g_2}(\Sigma^{d-q-1}) = U_{g_1 g_2}(\Sigma^{d-q-1}), \tag{4.11}$$

where $g_1, g_2 \in G$. When possible we will prefer to work with U_α, instead of Q, since those operators are defined both in discrete and continuous symmetries.

So far, we have been writing that both Q and U_α depend on manifold the Σ^{d-q-1}. However, this dependence is only topological, and not geometric. This means that we can consider a continuous deformation $\tilde{\Sigma}^{d-q-1}$ of our original manifold and still have the same operator. To see why, let us evaluate the difference

$$Q(\tilde{\Sigma}^{d-q-1}) - Q(\Sigma^{d-q-1}) = \int_{\tilde{\Sigma}^{d-q-1}} \star j - \int_{\Sigma^{d-q-1}} \star j.$$

We note that $\Sigma^{d-q-1} \cup \tilde{\Sigma}^{d-q-1}$ is the boundary of another closed manifold V^d. Take the Figure 6 as an example, where we deform a three dimensional surface Σ into $\tilde{\Sigma}$, such that the topology of the system never changes, that is, the charged insertion $\mathcal{O}(x)$ never cross the surface while deforming it. So, using the Stokes' theorem we find

$$Q(\tilde{\Sigma}^{(d-1)}) - Q(\Sigma^{(d-1)}) = \oint_{\Sigma \cup \tilde{\Sigma}} \star j = \int_{V^{(d)}} d \star j = 0, \tag{4.12}$$

which imply that both charges $Q(\Sigma^{d-1})$ and $Q(\tilde{\Sigma}^{d-1})$ coincide. This topological aspect implies that correlation functions are unaffected by the insertion of U_α, unless it cross a charged object.

[3] This feature is in agreement with the notion that space and time should be treated on equal footing in relativistic theories.

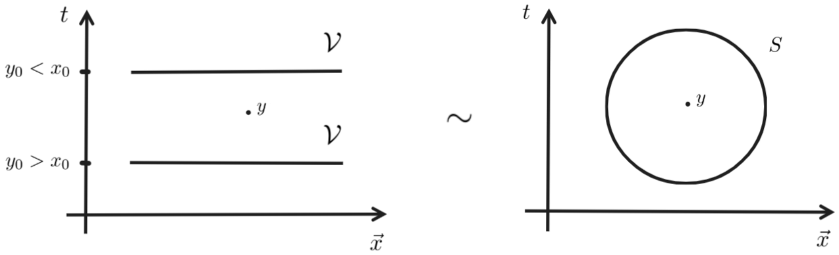

Figura 7 – Deformation of two insertions $Q(\mathcal{V})$ into one surrounding the charged operator $\mathcal{O}(y)$.

In order to see that, let us analyze an ordinary Ward identity

$$\partial_\mu \langle j^\mu(x)\mathcal{O}(y)\rangle = -i\delta^{(d)}(x-y)\langle\delta\mathcal{O}(y)\rangle . \tag{4.13}$$

First integrate both sides over a spacetime volume $\mathcal{V} \subset \mathcal{M}$ and invoke the divergent theorem

$$\int_{\mathcal{V}} d^d x \partial_\mu \langle j^\mu(x)\mathcal{O}(y)\rangle = \oint_{\partial\mathcal{V}} dS_\mu \langle j^\mu(x)\mathcal{O}(y)\rangle \tag{4.14}$$

$$= \langle T(Q(\mathcal{V})\mathcal{O}(y))\rangle = -i\left(\int_{\mathcal{Y}} \delta^{(d)}(x-y)d^d x\right)\langle\delta\mathcal{O}(y)\rangle , \tag{4.15}$$

such that, from (4.14) to (4.15) we have chosen $\partial\mathcal{V}$ to be a spatial volume [4] and used (4.9). Also note that in (4.15) we are explicitly denoting the time-ordering product $T(\cdot,\cdot)$, such that, in the limit of $x_0 \to y_0$ we are allowed to deform the two possible insertions of $Q(\mathcal{V})$, as shown in Figure 7, into a single one $Q(S)$ that surrounds the charged local operator $\mathcal{O}(y)$. Thus, we find

$$\langle T(Q(\mathcal{V}), \mathcal{O}(y))\rangle = \langle Q(S)\mathcal{O}(y)\rangle .$$

In this new set up, the integral on the right hand side of (4.15) has a interpretation of intersection number, since it is only non-zero when $y \in \mathcal{V}$. If we then define

$$L(\mathcal{V}, y) \equiv \int_{\mathcal{V}} \delta^{(d)}(x-y)d^d x = \begin{cases} 0, & \text{if } y \notin \mathcal{V} \\ 1, & \text{if } y \in \mathcal{V} \end{cases} , \tag{4.16}$$

the Ward identity (4.13) can now be phrased as

$$\langle Q(S)\mathcal{O}(y)\rangle = -iL(\mathcal{V}, y)\langle\delta\mathcal{O}(y)\rangle ,$$

which then implies that

$$\langle U_\alpha(\mathcal{V})\mathcal{O}(y)\rangle = e^{-i\alpha L(\mathcal{V},y)}\langle\mathcal{O}(y)\rangle , \tag{4.17}$$

[4] The fact that $\partial\mathcal{V}$ is a spatial volume implies that $dS_\mu = \frac{1}{(d-1)!}\epsilon_{\mu\nu_1\ldots\nu_{d-1}}dx^{\nu_1}\wedge\ldots\wedge dx^{\nu_{d-1}}$ will reduce itself to $dS_0 = dx^1 \wedge \ldots dx^{d-1} = dV$.

where we have used the fact that we are dealing with an internal symmetry and, by consequence, $\delta\mathcal{O}(y) = \mathcal{O}(y)$. In conclusion, (4.17) is stating the fact that U_α only manifests itself on the correlation function when crosses a charged object, providing, therefore, a topological meaning to symmetries and their conservation laws.

For a generic *q-form* symmetry we have

$$\langle U_\alpha(\Sigma^{d-q-1})\Gamma(\mathcal{P}^q)\rangle = e^{-i\alpha L(\Sigma,\mathcal{P})}\langle\Gamma(\mathcal{P}^q)\rangle\,, \tag{4.18}$$

with $\Gamma(\mathcal{P}^q)$ being a *q-dimensional* excitation, e.g. a Wilson line, and $L(\Sigma,\mathcal{P})$ is the intersection number for a generic pair of manifolds Σ^{d-q-1} and $\mathcal{P}^q$.

It is interesting to note that all higher form symmetries are Abelian, even if they were constructed from a non-Abelian structure. The idea is that in a spacetime with trivial topology we can always deform a certain ordering, in which the insertions were made, into any other configuration. This is not the case of ordinary symmetries, once operators inserted at different times have a well-defined ordering.

We close our construction of higher form symmetries by explicit characterizing the *q-form* parameter λ. If the manifold $\widehat{\lambda}^{d-q}$ has no boundary, then a generic closed *q-form* λ can be written as

$$\lambda_{\mu^1\dots\mu^q}(x) = \frac{1}{(d-q)!}\int_{\widehat{\lambda}^{d-q}}\delta^{(d)}(x-y)\epsilon_{\mu_1\dots\mu_q\nu_{q+1}\dots\nu_d}dy^{\nu_{q+1}}\wedge\dots\wedge dy^{\nu_d}. \tag{4.19}$$

We prove this statement by taking the exterior derivative of λ and using the Stokes theorem to integrate over the boundary of $\widehat{\lambda}$, which is a null region.

The expression (4.19) establishes a duality between the form λ and the manifold $\widehat{\lambda}$. This relation is known as *Poincaré duality.* Conversely, if we compose (4.19) with any *(d-q)-form* ω, we find

$$\int_{\mathcal{M}^d}\omega\wedge\lambda = \int_{\widehat{\lambda}^{d-q}}\omega, \tag{4.20}$$

with $\widehat{\lambda}^{d-q}\subset\mathcal{M}^d$. Let us take $q=1$ as an example. In this case λ is a *1-form* and ω is a *(d-1)-form*, such that, (4.20) gives

$$\begin{aligned}\int_{\mathcal{M}^d}\omega\wedge\lambda &= \int_{\mathcal{M}^d}\int_{\widehat{\lambda}^{d-1}}\frac{\omega_{\mu_1\dots\mu_{d-1}}(x)}{(d-1)!}\frac{\delta^{(d)}(x-y)}{(d-1)!}\epsilon_{\mu\nu_1\dots\nu_{d-1}}\times\\ &\qquad\times dx^{\mu_1}\wedge\dots\wedge dx^{\mu_{d-1}}\wedge dx^{\mu}\wedge dy^{\nu_1}\wedge\dots\wedge dy^{\nu_{d-1}}\\ &= \int_{\mathcal{M}^d}\int_{\widehat{\lambda}^{d-1}}\frac{\omega_{\mu_1\dots\mu_{d-1}}(x)}{(d-1)!}\frac{\delta^{(d)}(x-y)}{(d-1)!}\left(\epsilon_{\mu\nu_1\dots\nu_{d-1}}\epsilon^{\mu^1\dots\mu^{d-1}\mu}d^dx\right)dy^{\nu_1}\wedge\dots\wedge dy^{\nu_{d-1}}.\end{aligned}$$

If we now use $\epsilon^{\mu^1\dots\mu^{d-1}\mu}\epsilon_{\mu\nu_1\dots\nu_{d-1}}dy^{\nu_1}\wedge\dots\wedge dy^{\nu_{d-1}} = (d-1)!dy^{\mu_1}\wedge\dots\wedge dy^{\mu_{d-1}}$, we conclude that

$$\begin{aligned}\int_{\mathcal{M}^d}\omega\wedge\lambda &= \int_{\widehat{\lambda}^{d-1}}\left(\int_{\mathcal{M}^d}\frac{\omega_{\mu_1\dots\mu_{d-1}}(x)}{(d-1)!}\delta^{(d)}(x-y)d^dx\right)dy^{\mu_1}\wedge\dots\wedge dy^{\mu_{d-1}}\\ &= \int_{\widehat{\lambda}^{d-1}}\frac{\omega_{\mu_1\dots\mu_{d-1}}(y)}{(d-1)!}dy^{\mu_1}\wedge\dots\wedge dy^{\mu_{d-1}} = \int_{\widehat{\lambda}^{d-1}}\omega.\end{aligned}$$

In this sense, the parameter λ works like an extended delta function, which takes objects embedded in $\mathcal{M}^d$ and evaluate then on the manifold $\widehat{\lambda}^{d-q} \subset \mathcal{M}^d$.

Lastly, take $\lambda = A \wedge B$ to be a *d-form*, which is Poincaré dual to the intersection $\widehat{A} \cap \widehat{B}$, and set $\omega = 1$. The Poincaré duality (4.20) then gives

$$\int_{\mathcal{M}^d} A \wedge B = \int_{\widehat{A}\cap\widehat{B}} 1, \tag{4.21}$$

that is, the wedge product $A \wedge B$ counts how many times $\widehat{A}$ intersects $\widehat{B}$, since we will have to add 1 in the right side of (4.21) whenever $\widehat{A}$ touches $\widehat{B}$, thus, providing us with a precise definition of the intersection number $L(\Sigma, \mathcal{P})$ between two arbitrary closed manifolds Σ and $\mathcal{P}$. More details on Poincaré duality can be found in [38].

4.2 Chern-Simons Theory

The simplest example of a higher form symmetry is provided by the Chern-Simons theory, which has the action

$$S_{CS} = \int_{\mathcal{M}^{(3)}} \frac{k}{4\pi} \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma = \int_{\mathcal{M}^{(3)}} \frac{k}{4\pi} a \wedge da, \tag{4.22}$$

with a_μ a dynamical $U(1)$ gauge field. Its applications encompass a variety of topics inside condensed matter physics, since it is capable of describing the quantum Hall effect [32, 16].

It is interesting to note that (4.22) does not depend on the metric of our spacetime manifold and, therefore, can only be affected by the topology of $\mathcal{M}^{(3)}$. One can straightway verify that the equation of motion reads

$$k \epsilon_{\mu\nu\sigma} f^{\mu\nu} = 0, \tag{4.23}$$

implying that no local degrees of freedom can propagate in the Chern-Simons theory. In contrast, we can access the global information of the theory by the condition

$$\oint_{S^{(2)}} \frac{f}{2\pi} \in \mathbb{Z}, \tag{4.24}$$

which states that the flux due to a magnetic monopole is quantized. This is a feature that all compact gauge groups have and that we will use throughout this work.

As a consequence of (4.24), we must have $k \in \mathbb{Z}$. This follows from the fact that under a large gauge transformation the action transforms as

$$\delta_\lambda S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}^{(3)}} d\lambda \wedge da = k \underbrace{\left(\int_{S^{(1)}} \dot{\lambda} \right)}_{\in 2\pi\mathbb{Z}} \underbrace{\left(\frac{1}{4\pi} \int_{S^{(2)}} \epsilon_{ij} f_{ij} \right)}_{\in \mathbb{Z}}, \tag{4.25}$$

where we have chosen $\mathcal{M}^{(3)} = S^{(1)} \times S^{(2)}$. If k is a integer, then $i\delta_\lambda S_{CS}$ is a multiple of $2\pi i$ and the partition function is gauge invariant.

With this introduction in mind, we now develop on the $\mathbb{Z}_k$ *1-form* symmetry present in the Chern-Simons theory. In order to lift the symmetry, we must study the correlation between different Wilson lines

$$W_n[C] = \exp\left(in \oint_C a\right),$$

where n labels the possible charges of $W_n[C]$.

Given the canonical quantization rule

$$[a_1(x), a_2(y)] = \frac{2\pi i}{k}\delta^{(2)}(x-y), \tag{4.26}$$

which one can determine by calculating the momentum $\pi_1 = \frac{k}{2\pi}a_2$, we find that

$$\begin{aligned}\langle W_m[C']W_n[C]\rangle &= \exp\left((in)(im)\oint_C\oint_{C'}[a_1(x), a_2(y)]dx^1 \wedge dy^2\right)\langle W_n[C]\rangle \\ &= \exp\left(-2\pi i\frac{nm}{k}\oint_C\oint_{C'}\delta^{(2)}(x-y)dx^1 \wedge dy^2\right)\langle W_n[C]\rangle \\ &= \exp\left(-2\pi i\frac{nm}{k}L(C,C')\right)\langle W_n[C]\rangle, \end{aligned} \tag{4.27}$$

where we used the Poincaré duality to identify the intersection number $L(C, C')$.

Comparing (4.27) with (4.18), we see that $W_n[C]$ is simultaneously the charged object and the symmetry operator, since it induces a transformation law on itself. It also follows that $W_0[C]$ and $W_k[C]$ provide the same holonomy around C, once $W_k[C]$ behaves as the identity line in (4.27), leading, therefore, to a $\mathbb{Z}_k$ symmetry with k linear independent elements.

However, we should point out that the holonomy n and the statistics ν are two different quantum numbers. This prohibits us from directly identifying W_0 with W_k. By considering $n = m$ in (4.27), we verify that after a particle wraps another of the same type we get a phase, which represents half of the phase we would find by the particle statistic, that is,

$$e^{i\pi\nu} = e^{i\pi\frac{n^2}{k}} \Rightarrow \nu = \frac{n^2}{k} \mod 2 \tag{4.28}$$

If k is even, then (4.28) is invariant under $n \to n + k$, implying that the spectrum of lines ranges in $n = 0, ..., k-1$ and W_k is a transparent line describing the worldline of a boson, i.e. the identity line. For k odd, we find $n \to n + 2k$ and the spectrum becomes $n = 0, ..., 2k-1$. This forbids us from identifying $W_k \sim W_0$, since now we have the worldline of a fermion.

Conversely, if we interpret (4.27) as the composition rule of two symmetry operators a projective representation can be found, which then signals a 't Hooft anomaly. As a consequence, the theory spectrum cannot be trivially gapped. In our present case, we have a topological quantum field theory.

4.3 QED in (3+1)d

As our next example of a higher form symmetry, we will study free Maxwell theory in $(3+1)d$. We know from the equation of motion $d\star f = 0$ and the Bianchi identity $df = 0$ that both respect a conservation law as (4.7), with $\star j$ being a *2-form*. In the present case, we have an electric current $j_e = \star f$ and a magnetic current $j_m = f/2\pi$, thus implying that

$$U_{(e)}(\Sigma^{(2)}) = \exp\left(i\alpha \oint_{\Sigma^{(2)}} \star f\right), \tag{4.29}$$

$$U_{(m)}(\Sigma^{(2)}) = \exp\left(i\beta \oint_{\Sigma^{(2)}} \frac{f}{2\pi}\right), \tag{4.30}$$

are two valid topological *1-form* symmetry operators.

We begin our discussion by examining $U_{(e)}$, which has Wilson loops as charged objects. To see this, we must first use Poincaré duality to write the charge as

$$Q_e(\mathcal{M}^{(3)}) = \oint_{\Sigma^{(2)}} \star f = \oint_{\mathcal{M}^{(3)}} \star f \wedge \widehat{\Sigma}, \tag{4.31}$$

where $\widehat{\Sigma}$ is Poincaré dual to $\Sigma^{(2)}$. We then choose $\mathcal{M}^{(3)}$ to be a space volume at fixed time and use the canonical quantization (3.16) to obtain

$$\begin{aligned} U_{(e)}(\mathcal{M}^{(3)}) \exp\left(i\oint_C a\right) U^{\dagger}_{(e)}(\mathcal{M}^{(3)}) &= W[C] \exp\left(-\frac{\alpha}{e^2}\oint_C \oint_{\mathcal{M}^{(3)}} [\star f, a] \wedge \widehat{\Sigma}\right) \\ &= W[C] \exp\left(i\alpha \oint_C \oint_{\mathcal{M}^{(3)}} \delta^{(3)}(x-y)\left(\widehat{\Sigma}_k(x) dy^k\right) d^3x\right) \\ &= W[C] \exp\left(i\alpha \oint_C \widehat{\Sigma}\right), \end{aligned} \tag{4.32}$$

which is precisely what we have found in (4.4). If we apply (4.20) and (4.21)

$$\oint_C \widehat{\Sigma} = \oint_{\mathcal{M}^{(3)}} \widehat{\Sigma} \wedge \widehat{C} = \oint_{\Sigma\cap C} 1,$$

it is possible to identify the intersection number $L(\Sigma, C)$.

Similarly, the magnetic symmetry (4.30) acts on the 't Hooft lines as

$$U_{(m)}(\Sigma^{(2)}) T[C] U^{\dagger}_{(m)}(\Sigma^{(2)}) = e^{i\beta L(\Sigma,C)} T[C]. \tag{4.33}$$

In order to prove (4.33) we must express the action in terms of $\widetilde{f}$ and calculate $\Pi_{\widetilde{a}}$. Using $\star f = d\widetilde{a}$ and $f = -\star(\star f) = -\star d\widetilde{a}$ [5] we know that the action becomes

$$S = -\frac{1}{2}\int f \wedge \star f \to S = \frac{1}{2}\int \widetilde{f} \wedge \star \widetilde{f} = \frac{1}{4}\int \widetilde{f}_{\mu\nu}\widetilde{f}^{\mu\nu}.$$

[5] There is a signal ambiguity here, depending whether we choose between Euclidean or Minkowski signatures. We have chosen to work with the Minkowski spacetime.

A short calculation then shows $(\Pi_{\widetilde{a}})^{\mu} = \frac{\partial L}{\partial \dot{a}_{\mu}} = \widetilde{f}_{0\mu} \to (\Pi_{\widetilde{a}})^{j} = -B^{j}$, implying that the canonical quantization for $\widetilde{a}$ is given by

$$[\widetilde{a}_i(x), B_j(y)] = i\delta^{(3)}(x-y)\delta_{ij}. \tag{4.34}$$

Once found (4.34), we follow the same steps in (4.32) to derive (4.33). Note that, if we were considering $D \neq 4$, 't Hooft operators would become $D-3$ dimensional objects, since $\widetilde{a}$ is now a *(D-3)-form*. This implies that the magnetic symmetry is in fact a *(D-3)-form* symmetry [6].

Next, consider that we add charged matter to our theory. As consequence, we lose the electric *1-form* symmetry, once j_e is not conserved anymore, implying that the operator $U_{(e)}(\Sigma^{(2)})$ is no longer topological. The physical interpretation in this scenario is that screening effects will suppress the charge of $W[C]$, once we add dynamical matter to our theory, making, therefore, the effective charge of the line dependent on the distance and not only on the topology.

However, we can get around this by assuming that

$$\int_{V^{(3)}} \star J = q_e \in \mathbb{Z}, \tag{4.35}$$

where J is the matter electric current. The consequence of (4.35) is that the continuous $U(1)$ electric symmetry breaks down to a discrete $\mathbb{Z}_{q_e}$ *1-form* symmetry. This follows from the fact that (4.12) now gives

$$\begin{aligned} Q_e(\tilde{\Sigma}^{(2)}) - Q_e(\Sigma^{(2)}) &= \int_{\tilde{\Sigma}^{(2)}} \star j_e - \int_{\Sigma^{(2)}} \star j_e \\ &= \oint_{\tilde{\Sigma}\cup\Sigma} \star j_e \\ &= \int_{V^{(3)}} d\star j_e. \end{aligned}$$

Considering the equation of motion $d \star f = \star J$, we then find

$$\exp\left(i\alpha Q_e(\tilde{\Sigma}^{(2)}) - i\alpha Q_e(\Sigma^{(2)})\right) = \exp\left(i\alpha \int_{V^{(3)}} \star J\right) = e^{i\alpha q_e}$$

which still implies in a topological operator, provided that $\alpha = 2\pi n/q_e$, with $n = 0, ..., q_e - 1$.

Let us now study the gauging of these two *1-form* symmetries. First we gauge the magnetic symmetry by adding in the action the following coupling

$$\frac{1}{2\pi}\int j_m \wedge B = \frac{1}{2\pi}\int f \wedge B,$$

6 For $D = 2$ this result in a *(-1)-form* symmetry, some comments on this can be found in [33].

where B is a *2-form* background gauge field, transforming as $B \to B + d\Lambda$. Consequently, the equation of motion becomes

$$d \star f = d\left(\frac{B}{2\pi}\right), \tag{4.36}$$

which breaks the conservation law of j_e. We could try to define $\star f - B/2\pi$ as our new electric current, but this would not be a gauge-invariant quantity. Therefore, the magnetic *1-form* symmetry cannot be gauged or, otherwise, we will lose the electric *1-form* symmetry. As a consequence, we must conclude that these two symmetries have a mixed 't Hooft anomaly.

Despite the presence of this 't Hooft anomaly, we can still gauge the subgroup $\mathbb{Z}_k \subset U(1)$ in the magnetic symmetry. This can be accomplished by the action

$$S = \int -\frac{1}{2e^2} f \wedge \star f + \frac{1}{2\pi} B \wedge f + \frac{k}{2\pi} c \wedge dB,$$

where c is a Lagrange multiplier which sets $d(kB) = 0$. If we now follow the consequence of (4.36), it is possible verify that $U_{(e)}$ regains its topological feature, since $\oint B \in 2\pi\mathbb{Z}$. Alternatively, we could also gauge the $\mathbb{Z}_k \subset U(1)$ electric symmetry.

A fact that we will not explore, but is worth commenting, is that the photon can be understood as Goldstone boson arising from the sponteanoulsy breaking of the electric *1-form* symmetry. More details on that can be found in [9, 38].

5 't Hooft Anomaly in $SU(N)$ Yang-Mills Theory

The Yang-Mills theory in the presence of the θ-term is described by the action

$$S_{YM}[a] = \int_{\mathcal{M}^{(3+1)}} \mathrm{Tr}\left(-\frac{1}{g^2} f \wedge \star f + \frac{\theta}{8\pi^2} f \wedge f\right), \tag{5.1}$$

where $f = da - ia \wedge a$ and a is a gauge field valued in the algebra $\mathfrak{g}$. It is interesting to note that at $\theta = 0, \pi$ the theory enjoys from a time-reversal symmetry, since θ is a 2π periodic parameter. We will return to this point when deriving the 't Hooft anomaly. Additionally, as we already know, we also have Wilson lines given by

$$W_R[C] = \mathrm{Tr}_R\left(\mathcal{P}\exp\left(ig\oint_C A\right)\right), \tag{5.2}$$

where we have made explicit the dependence on the representation $R(G)$.

One might think that the electric *1-form* symmetry is not present in the non-Abelian theory, since the gauge fields are charged by themselves. This implies that no conservation law can be constructed, since any charge will be suppressed by screening effects. But, there is a caveat in this argument. It is true that the dynamical nature of the gauge field will interfere in any measurement that we try to perform. See Figure 8 where the net charge inside the surface is affected by excitations of a located at the boundary. However, if we mod out from the measured charge the gauge field charge, then we will find a "conservation law", since the gauge field always contributes with the same amount for the total charge.

This heuristic conservation law arises from the observation that the set of all line operators form an equivalence class. More specifically, two lines are said to be in the same equivalence class if they can be connected by some local operator $\mathcal{O}(x)$, since they

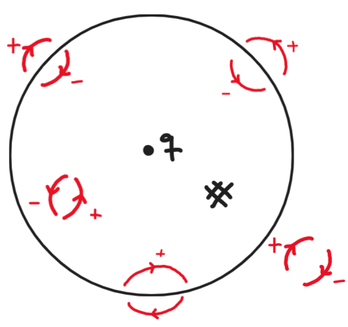

Figura 8 – A charge q surrounded by creation and annihilation processes coming from the gauge field a.

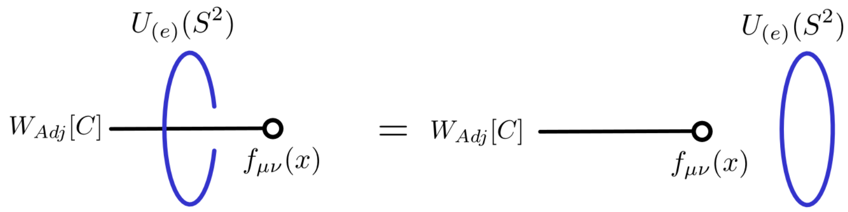

Figura 9 – An Adjoint Wilson line ending on the field strength.

will essentially have the same charge. See figure (9), where a Wilson line in the adjoint representation is allowed to end on the field strength operator. As a result, the adjoint Wilson line has the same charge as the trivial line. Mathematically, this is captured by the spectrum of allowed charges (3.39).

To lift the symmetry we assume the existence of some local transformation $\widehat{U}(x) \in G$ which has a non-trivial holonomy

$$\widehat{U}(x+\tau) = \gamma \widehat{U}(x), \tag{5.3}$$

where $x \sim x + \tau$ and $\gamma \in Z(G)$. Invoking (3.1), we see that all local operators are uncharged by $\widehat{U}(x)$, since $\mathrm{Tr}(f'_{\mu\nu}(x)) = \mathrm{Tr}(f_{\mu\nu}(x))$. On the other hand, using (3.10) we find

$$\begin{aligned} W[x+\tau, x; C] \to W'[x+\tau, x; C] &= \widehat{U}^\dagger(x+\tau) W[x+\tau, x; C] \widehat{U}(x) \\ &= \gamma \widehat{U}^\dagger(x) W[x+\tau, x; C] \widehat{U}(x) \\ \Rightarrow \mathrm{Tr}(W'[C]) &= \gamma \mathrm{Tr}(W[C]), \end{aligned} \tag{5.4}$$

that is, Wilson lines are charged by $\gamma \in Z(G)$. In conclusion, we have found an electric *1-form* symmetry valued in the center $Z(G)$ of G. Note that, despite this construction was based on a non-Abelian structure the symmetry itself is Abelian.

If we define $\hat{t} \in \mathfrak{g}$ as the generator that returns the identity, i.e. $\exp\left(2\pi i \hat{t}\,\right) = I$, then we have $\widehat{U} \equiv \exp\left(2\pi i \frac{\hat{t}}{\tau} x\right)$ and the gauge field transforming as

$$a_\mu(x) \to a'_\mu(x) = \widehat{U}(x) a_\mu(x) \widehat{U}^\dagger(x) + \frac{2\pi}{\tau} \hat{t}$$

under the *electric center symmetry* (5.4). For $G \cong SU(N)$ we find $Z(SU(N)) \cong \mathbb{Z}_N$ and

$$\hat{t}_{N^2-1} = \begin{pmatrix} \frac{1}{N} & & & \\ & \frac{1}{N} & & \\ & & \ddots & \\ & & & \frac{1}{N} - 1 \end{pmatrix}, \tag{5.5}$$

such that, $\gamma = e^{2\pi i k/N}$ with $k = 0, ..., N-1$.

Let us now consider that the fields ψ^i and ψ_i are respectively in the fundamental and anti-fundamental representations of $SU(N)$, and that their gauge transformations are

$$(\psi')^i = (U)^i{}_j \psi^j, \;\; (\psi')_i = (U^*)_i{}^j \psi_j. \tag{5.6}$$

It can be shown that the product $\psi^i\psi_j$ splits itself into a scalar and an adjoint part, that is,

$$\psi^i\psi_j = \underbrace{\left[\frac{1}{N}\psi^2\delta^i{}_j\right]}_{Scalar} + \underbrace{\left(\psi^i\psi_j - \frac{1}{N}\psi^2\delta^i{}_j\right)}_{Adjoint}.$$

Denoting the adjoint part as $\psi^{\widehat{ij}} \equiv \psi^i\psi_j - \frac{1}{N}\psi^2\delta^i{}_j$, we find that its gauge transformation $U^{\widehat{ij}}_{\widehat{pq}}$ will be given by

$$\begin{aligned}
(\psi')^{\widehat{ij}} &= U^{\widehat{ij}}_{\widehat{pq}}\psi^{\widehat{pq}} = (\psi')^i(\psi')_j - \frac{1}{N}(\psi')^2\delta^i{}_j \\
&= (U)^i{}_p(U^*)^q{}_j\psi^p\psi_q - \frac{1}{N}\psi^2\delta^i{}_j \\
&= (U)^i{}_p(U^*)^q{}_j\left\{\left(\psi^p\psi_q - \frac{1}{N}\psi^2\delta^p{}_q\right) + \left[\frac{1}{N}\psi^2\delta^p{}_q\right]\right\} \\
&\qquad\qquad - \frac{1}{N}\delta^q{}_p\delta^i{}_j\left\{\left(\psi^p\psi_q - \frac{1}{N}\psi^2\delta^p{}_q\right) + \left[\frac{1}{N}\psi^2\delta^p{}_q\right]\right\} \\
&= \left((U)^i{}_p(U^*)^q{}_j - \frac{1}{N}\delta^q{}_p\delta^i{}_j\right)\left(\psi^p\psi_q - \frac{1}{N}\psi^2\delta^p{}_q\right) \\
&\qquad\qquad + \underbrace{\left[(U)^i{}_p(U^*)^q{}_j\delta^p{}_q - \frac{1}{N}\delta^q{}_p\delta^i{}_j\delta^p{}_q\right]}_{=0}\frac{1}{N}\psi^2 \\
\Rightarrow (\psi')^{\widehat{ij}} &= \left((U)^i{}_p(U^*)^q{}_j - \frac{1}{N}\delta^q{}_p\delta^i{}_j\right)\psi^{\widehat{pq}},
\end{aligned} \tag{5.7}$$

where we have used the unitary condition $(U)^i{}_p(U^*)^p{}_j = \delta^i{}_j$. Once (5.7) is determined, we can infer that

$$U^{\widehat{ij}}_{\widehat{pq}} = (U)^i{}_p(U^*)^q{}_j - \frac{1}{N}\delta^q{}_p\delta^i{}_j. \tag{5.8}$$

From (5.8) we see that the set of all matrices which act on the adjoint representation can be rewritten in terms of fundamental and anti-fundamental indices, and once Wilson lines are included in this set, we must have

$$(W_A)^{\widehat{ij}}_{\widehat{pq}}[C] = (W_f)^i{}_p[C](W_f^*)^q{}_j[C] - \frac{1}{N}\delta^q{}_p\delta^i{}_j. \tag{5.9}$$

Given (5.9), we can construct a Wilson loop in the adjoint representation by setting $i = p$ and $q = j$, thus finding

$$\mathrm{Tr}_{Adj}\left(W[C]\right) = \left|\mathrm{Tr}_f\left(\exp\left(i\oint a\right)\right)\right|^2 - 1. \tag{5.10}$$

However, since the electric center symmetry changes Wilson lines by a phase, it means that $W_A[C]$ is uncharged by (5.4). As a consequence, we can always gauge any subgroup $\Gamma \subset \mathbb{Z}_N$ from the electric center symmetry and still have the adjoint representation in the spectrum of lines, once $W_A[C]$ will always be a gauge invariant operator. This is related to the fact that roots will always be present in $\Lambda_w^e(G)$, regardless of G.

Finally, if we consider a generic gauge group G which is not the covering group, for example the $SU(N)/\Gamma$ that we have mentioned previously, then $\Lambda_w^m(G)$ admits weights beyond those roots which are naturally dual to $\vec{e} \in \Lambda_w^e(G)$. Now the equivalence class

$$\vec{m} \in \Lambda_w^m(G)/\Lambda_r^m(G) \cong Z(G), \tag{5.11}$$

that characterizes all magnetic charges, will not collapse itself into just one element. In this set up, we have magnetic charges which are not associated with the adjoint representation. Consequently, we are able to measure something non-trivial when modding out charges coming from screening effects. In other words, given a 't Hooft loop that does not belongs to the adjoint representation, it is possible to lift a *magnetic center symmetry*, in the same fashion as the electric one.

Having set the stage with this introduction, we explicit derive the 't Hooft anomaly discussed in the article *Theta, Time Reversal and Temperature* [2]. The main result is that pure $SU(N)$ gauge theory has a mixed 't Hooft anomaly between the electric $\mathbb{Z}_N$ *1-form* symmetry and time reversal, at $\theta = \pi$. The argument is similar to what we have done when discussing the particle on a circle in section 2.2.3. Here we try to gauge $\mathbb{Z}_N$ symmetry, by introducing a set of gauge fields, at the price that we lose the time reversal symmetry. Thus identifying an 't Hooft anomaly.

For that we introduce a pair of $U(1)$ gauge fields (A, B) and add to the action

$$S \supset \frac{1}{2\pi} \int z \wedge (dA - NB)\,, \tag{5.12}$$

where z is a *2-form* Lagrange multiplier which sets $dA = NB$. As a result, B is constrained to be a $\mathbb{Z}_N$ gauge field. The gauge transformations are given by

$$B \to B + d\lambda, \;\; A \to A + N\lambda, \tag{5.13}$$

and they leave the condition $dA = NB$ unchanged as well as the surface operator

$$W_B = \exp\left(i \oint_\Sigma B\right), \tag{5.14}$$

which satisfies $(W_B)^N = 1$.

Next, we recall that for a global electric symmetry we shift the gauge field by a closed *1-form*, e.g. λ, such that the field strength remains invariant. Alternatively, when we consider the gauged version of the same symmetry a modification in f is required, since

λ is no longer closed. We can overcome this gauge dependence by replacing the $SU(N)$ field strength with

$$f \to f - BI. \tag{5.15}$$

However, there is an inconsistency in this construction. Locally, the field strength can be expressed as $f - \frac{I}{N}dA$, where we used $B = dA/N$. This indicates that the auxiliary gauge field A has acquired its own dynamic throughout the gauging process, since it now has a kinetic term in the action. As a result, we have an extra $U(1)$ degree of freedom in our $SU(N)/\mathbb{Z}_N$ gauge theory. In order to eliminate this $U(1)$ field, we extend the original $SU(N)$ gauge field a to an $U(N) \cong [U(1) \times SU(N)]/\mathbb{Z}_N$ gauge field

$$\mathcal{A} \equiv a + \frac{I}{N}A, \tag{5.16}$$

and use the gauge freedom provided by λ to prevent the A field from propagating.

In this construction, the kinetic term of a $SU(N)/\mathbb{Z}_N$ theory is given by

$$S \supset -\frac{1}{2g^2}\int \mathrm{Tr}\left[(\mathcal{F} - BI) \wedge \star(\mathcal{F} - BI)\right], \tag{5.17}$$

where $\mathcal{F}$ is the $U(N)$ field strength, and the θ-term becomes

$$S \supset \frac{\theta}{8\pi^2}\int \mathrm{Tr}\left[(\mathcal{F} - BI) \wedge (\mathcal{F} - BI)\right]. \tag{5.18}$$

Note that, the $SU(N)/\mathbb{Z}_N$ gauge field is the whole triplet $(\mathcal{A}, A, B)$. It is the gauge transformations of A and B that ensures we will constrain the $U(N)$ gauge field $\mathcal{A}$ to the desired $SU(N)/\mathbb{Z}_N$ theory.

As a sanity check, we verify that the spectrum of electric lines has reduced itself to allow only products of the adjoint representations. Given that, the $U(N)$ gauge field transforms by a gauge transformation as

$$\mathcal{A} \to \mathcal{A}' = \mathcal{A} + I\lambda \tag{5.19}$$

Wilson lines in the fundamental representation are now attached to a surface operator

$$W_f[C, \Sigma] = \mathrm{Tr}\left(\mathcal{P}\exp\left(i\oint_C \mathcal{A}\right)\right)\exp\left(i\int_\Sigma B\right), \;\; C = \partial\Sigma. \tag{5.20}$$

Otherwise, we would not find a gauge invariant operator. Using (5.10) we see that

$$W_A[C] = \left|\mathrm{Tr}\left(\mathcal{P}\exp\left(i\oint_C \mathcal{A}\right)\right)\right| - 1, \tag{5.21}$$

namely, the adjoint representation is the only one which continue to be a pure electric line, as we would expect from the $SU(N)/\mathbb{Z}_N$ gauge theory. Consequently, the magnetic weight lattice becomes larger and a magnetic center symmetry should appear in our theory. This emergent magnetic symmetry is lifted by the magnetic *1-form* symmetry of the B-field. Naively, we might think that B has also an electric symmetry. However, the electric *2-form symmetry* is broken by (5.17) and (5.18), since those terms are not invariant under the shift $B \to B + \Xi$.

With these comments out of the way, we continue our derivation of the 't Hooft anomaly. Provided that we consistently gauged the $\mathbb{Z}_N$ *1-form* electric symmetry, we now investigate the θ-term (5.18), which under the shift $\theta \to \theta + 2\pi$ becomes

$$\begin{aligned}\delta_\theta S &= \frac{1}{4\pi}\int \mathrm{Tr}\left(\mathcal{F}\wedge\mathcal{F}\right) - \frac{1}{2\pi}\int \mathrm{Tr}\left(\mathcal{F}\right)\wedge B + \frac{N}{4\pi}\int B\wedge B \\ &= \frac{1}{4\pi}\int \mathrm{Tr}\left(\mathcal{F}\wedge\mathcal{F}\right) - \frac{N}{4\pi}\int B\wedge B,\end{aligned} \tag{5.22}$$

where we have used $\mathrm{Tr}\left(\mathcal{F}\right) = dA = NB$.

Note that the first term in (5.22) can be rewritten as the second Chern number [34] of the $U(N)$ fiber bundle

$$-2\pi c_2 = \frac{1}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(\mathcal{F}\wedge\mathcal{F}\right) - \mathrm{Tr}\left(\mathcal{F}\right)\wedge \mathrm{Tr}\left(\mathcal{F}\right), \tag{5.23}$$

which is always an integer, implying that

$$\delta_\theta S = -2\pi c_2 + \frac{N(N-1)}{4\pi}\int B\wedge B. \tag{5.24}$$

Since $c_2 \in \mathbb{Z}$, the partition function is unaffected by the first term in (5.24). On the other hand, if we add to the action the counter-term

$$S \supset -\frac{pN}{4\pi}\int B\wedge B, \tag{5.25}$$

then the second term in (5.24) can absorbed under the shift

$$p \to p + N - 1. \tag{5.26}$$

However, we must be careful when adding (5.25). If we exploit the fact that $\oint d\lambda = 2\pi\mathbb{Z}$ and $\oint B = \frac{2\pi}{N}\mathbb{Z}$, we are able to derive a constrain on the possible values of p. Under (5.13) the counter-term (5.25) transforms as

$$\delta_\lambda S = pN\int \frac{d\lambda}{2\pi}\wedge B + \pi pN\int \frac{d\lambda}{2\pi}\wedge\frac{d\lambda}{2\pi} = (2\pi p + \pi pN)\mathbb{Z}. \tag{5.27}$$

The first term in (5.27) can be dropped out provided that $p \in \mathbb{Z}$, since only then the partition function $Z \sim e^{i\delta_\lambda S}$ will become invariant. In contrast, the second term is only consistent if we set

$$\frac{pN}{2} \in \mathbb{Z}. \tag{5.28}$$

This means that for N even p can be any integer, and for N odd p must be even. Note that in both case we are allowed to identify $p \sim p + 2N$, since (5.25) only changes by a integer under this shift.

With all of these elements at hand, we are now in position to examine the fate of time reversal symmetry, which acts on θ and p as

$$T : (\theta, p) \to (-\theta, -p), \tag{5.29}$$

after we have gauged the $\mathbb{Z}_N$ center symmetry. In order to better elucidate our discussion we divide it into odd and even N:

- N even: For $\theta = \pi$ it is expected that we find a symmetry under (5.29), since we can use a 2π shift to undo the time reversal transformation. This can be accomplished if we also consider (5.26). We are then led to conclude that

$$(\pi, p) \underset{T}{\to} (-\pi, -p) \underset{\theta \to \theta + 2\pi}{\to} (\pi, -p + N - 1). \tag{5.30}$$

  However, as discussed previously, gauge invariance implies in (5.28), which cannot be satisfied, since $p = (N-1)/2 \notin \mathbb{Z}$. Therefore, we conclude that there is an 't Hooft anomaly for all $SU(N)$ gauge theories with N even and $\theta = \pi$.

- N odd: In this case, we find two possible solutions[1] at $\theta = \pi$

$$p = \frac{N-1}{2}, \; if \; N = 4k+1,$$
$$p = N + \frac{N-1}{2}, \; if \; N = 4k+3,$$

  implying that there is no 't Hooft anomaly for N odd. Nevertheless, we cannot simultaneously make both $\theta = 0$ and $\theta = \pi$ non-anomalous. For $\theta = 0$ time reversal symmetry acts as $p \to -p$, which only has the solution $p = 0$. And as the authors of [2] state *"This is almost as good as saying that there is an anomaly"*.

As expected, the presence of this 't Hooft anomaly restricts the low-energy physics, by prohibiting the existence of trivially gapped phase in $SU(N)_{\theta=\pi}$ gauge theories. In fact, this 't Hooft anomaly and its constraints represent an important achievement towards solving the mass gap problem, since it finally provide us with information about the IR of Yang-Mills theory, making, therefore, this result one of the most remarkable in modern theoretical physics.

In addition to that, the 't Hooft anomaly also has implications on the phase diagram. Let us define T_D as the temperature of deconfinement transition and T_{CP} as the temperature where time reversal symmetry is restored. The presence of this 't Hooft anomaly

[1] The distinction between these different N's is needed once we must always find an even p. We should also point out that the second solution is only trivial under (5.30) upon the identification $p \sim p + 2N$.

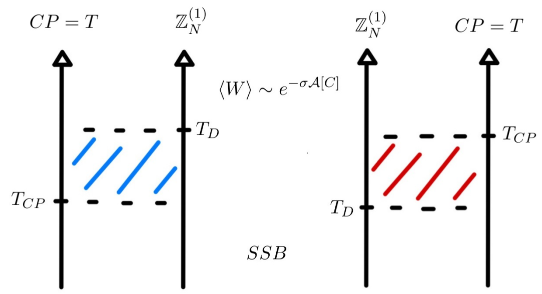

Figura 10 – The two possible orders in which phase transitions can happen. On the left we consider $T_D \geq T_{CP}$ and on the right $T_{CP} \geq T_D$.

implies that either $\mathbb{Z}_N$ *1-form* symmetry or $CP = T$ is broken. Assuming that we are at the confining phase [2] and that time reversal symmetry is spontaneously broken, we must conclude that

$$T_{CP} \geq T_D. \tag{5.31}$$

Otherwise, we would have two well-defined symmetries, which contradicts the anomaly. We illustrate this argument in Figure 10, where the red region is in accordance with our assumptions and the blue one is prohibited by the anomaly.

[2] This is equivalent to say that the center symmetry is unbroken [9, 38], once all $W_R[C]$ have an area law.

6 Perspectives and closing remarks

Throughout this work, we have explored the power of 't Hooft anomaly matching condition to infer non-trivial results across several theories, among them the $SU(N)$ Yang-Mills theory. For that the study of line operators and representation theory was needed to fully categorize the spectrum of lines, as well as, a complete understanding of *higher-forms symmetries*, since only then we were able to undercover the mixed anomaly between the *1-form* center symmetry and time-reversal. In fact, this 't Hooft anomaly and its constraints represent an important achievement towards solving the mass gap problem, since it finally provide us with information about the IR of Yang-Mills theory, making, therefore, this result one of the most remarkable in modern theoretical physics.

It is also interesting to note that with the discovery of higher-forms symmetries, the generalization to *non-invertible symmetries* becomes more natural, since those are primarily characterized by the presence of topological operators $U_i(M)$, which are not associated with conservation laws. As a result, they do not form a group structure and have non-trivial fusion rules

$$U_i(M) \times U_j(M) = \sum_k n_{ij}^k U_k(M), \tag{6.1}$$

This type of generalized symmetry are abundant in $1+1d$, appearing in many Rational Conformal Field Theories. The most prominent example is the Ising CFT model [39, 40], which has a duality defect D that fuses in a non-invertible away with itself. In general these non-invertible symmetries can led to constraints in the theory spectrum [18, 19, 20, 21], thereby providing a tool that is on an equal footing with 't Hooft anomalies. Additionally, as shown in the Appendix D, it is possible to rephrase anomalies in terms of non-invertible symmetries [29, 30, 41].

In conclusion, the field of *generalized symmetries*, which also encompasses *sub-system symmetries*, has proven itself as an effervescent line of research. The opening work of Gaiotto, Kasputin, Seiberg and Willett [1] has already led to a number of significant breakthroughs, and it is expected to continue to drive progress on many subjects, such as condensed matter [17, 9], quantum gravity [41, 42, 43, 44, 45] and particle physics [29, 30, 11, 46].

A Weights and roots lattices

Our main goal in this appendix is to prove that

$$2\frac{\alpha \cdot \mu}{\alpha^2} \in \mathbb{Z}, \tag{A.1}$$

where μ^i are weights and α^i are roots. The proof for that can be found in [47]. However, for completeness of this text, we will review such concepts.

For any group G, with algebra $\mathfrak{g}$, we define the *Cartan sub-algebra* as the maximal sub-set $\mathfrak{h} \subset \mathfrak{g}$ of mutually commuting generators. We refer to these generators as *Cartan generators* and denote then as H^i, with $i = 1, ..., dim(\mathfrak{h})$. The Cartan sub-algebra is unique for any group G. In summary, we have

$$H_i = H_i^\dagger, \;\; [H^i, H^j] = 0.$$

The dimension $dim(\mathfrak{h})$ defines the rank of G.

Since, all Cartan generators commute, they can be simultaneously diagonalized, that is,

$$H^i \left|\mu, R(G)\right\rangle = \mu^i \left|\mu, R(G)\right\rangle.$$

Note that, these eigenvectors are dependent on the representation $R(G)$. The eigenvalues $\vec{\mu} = (\mu_1, ..., \mu_{dim(\mathfrak{h})})$ are called *weights*. We define the scalar product of the vector $\vec{\mu}$ in an Euclidean fashion, that is, $\vec{\mu} \cdot \vec{\mu} = \mu_i \mu_i$.

Let us now examine the adjoint representation. From the algebra $\mathfrak{g}$, we known that

$$[T_a, T_b] = i f_{abc} T_c,$$

where f_{abc} are the structure constants of the group. It is possible to prove that the structure constants themselves furnish the adjoint representation of G. For that, we use the Jacobi identity

$$\begin{aligned}[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0 \\ \Rightarrow f_{ade} f_{bcd} + f_{bde} f_{cad} + f_{cde} f_{abd} = 0,\end{aligned}$$

but we can view the expression above as the commutation rule for the matrices $[T_a]_{bc} = -i f_{abc}$, that is,

$$\begin{aligned}(-i f_{aed})(-i f_{bdc}) - (-i f_{bed})(-i f_{cdc}) = i f_{abd}(-i f_{dec}) \\ [T_a, T_b] = i f_{abc} T_c.\end{aligned}$$

Since, our parametrization $[T_a]_{bc}$ acts on vectors of dimension $\dim(G)$, we are dealing with the adjoint representation. Therefore, the eigenvectors are in an one-to-one correspondence with the generators, that is,

$$|\mu, Adj\rangle \longleftrightarrow |T_a\rangle \,. \tag{A.2}$$

We can introduce a scalar product in this space as

$$\langle T_a|T_b\rangle = \lambda^{-1} tr(T_a^\dagger T_b),$$

where λ is a normalization constant. This normalization λ depends only on the *index* k of the representation, where the index is defined as $\mathrm{Tr}(T_aT_b) = k\delta_{ab}$. From (A.2) and knowing that these eigenvectors form a complete basis, we can prove that

$$\begin{aligned} T_a|T_b\rangle &= \sum_c |T_c\rangle \langle T_c|T_a|T_b\rangle = \sum_c |T_c\rangle (T_a)_{cb} \\ &= \sum_c -if_{acb}|T_c\rangle = \sum_c if_{abc}|T_c\rangle \\ &= |[T_a, T_b]\rangle \,. \end{aligned} \tag{A.3}$$

Once derived (A.3) we gain a lot of information about the adjoint representation. If we define *roots* as the weights of the adjoint representation, then it is easily verified that the Cartan generators have trivial roots, that is,

$$H_i|H_j\rangle = |[H_i, H_j]\rangle = 0. \tag{A.4}$$

On the other hand, if a state $|\alpha\rangle$ has a non-zero root, we must have

$$H_i|\alpha\rangle = \alpha_i|\alpha\rangle \,. \tag{A.5}$$

However, as we already argued, eigenvectors in the adjoint representation can mapped in the set of generators. In our case, we choose E_α to be the generator associated with $|\alpha\rangle$. Thus, from (A.3) and (A.5) we known

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{A.6}$$

In addition to that, after taking the dagger on both side, we find

$$[H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger \Rightarrow E_\alpha^\dagger = E_{-\alpha}.$$

Therefore, every root comes in pairs $(\alpha, -\alpha) \sim (E_\alpha, E_\alpha^\dagger)$ and this pair of roots form an $SU(2)$ sub-group in G.

To prove the previous statement, first note that the state $E_{\pm\alpha}|\mu\rangle$ has weight $\mu \pm \alpha$

$$\begin{aligned} H_i(E_{\pm\alpha}|\mu\rangle) &= [H_i, E_{\pm\alpha}]|\mu\rangle + E_{\pm\alpha}H_i|\mu\rangle \\ &= \pm\alpha_i E_{\pm\alpha}|\mu\rangle + \mu_i E_{\pm\alpha}|\mu\rangle \\ &= (\mu \pm \alpha)_i (E_{\pm\alpha}|\mu\rangle) \,. \end{aligned} \tag{A.7}$$

This look very much like the lowering and raising operators within $\mathfrak{su}(2)$.

If we now consider the state $E_\alpha |E_{-\alpha}\rangle$, from (A.7) we known that it has weight zero and, by consequence, must be some linear combination of the Cartan generators

$$E_\alpha |E_{-\alpha}\rangle = \beta_i |H_i\rangle .$$

In fact, we must have

$$\begin{aligned}\beta_i &= \langle H_i|E_\alpha|E_{-\alpha}\rangle = \lambda^{-1} tr\left(H_i[E_\alpha, E_{-\alpha}]\right) \\ &= \lambda^{-1} tr\left(E_{-\alpha}[H_i, E_\alpha]\right) = \lambda^{-1} tr\left(\alpha_i E_{-\alpha} E_\alpha\right) \\ &= \alpha_i,\end{aligned}$$

which implies in

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i. \tag{A.8}$$

In conclusion, once equipped with (A.6) and (A.8) we can prove that

$$J_\pm \equiv \frac{E_{\pm\alpha}}{|\alpha|}, \quad J_3 \equiv \frac{\alpha_i H_i}{|\alpha|^2}$$

exactly reproduce the $\mathfrak{su}(2)$ algebra. Moreover, by acting J_3 in $|\mu\rangle$ we find

$$J_3 |\mu\rangle = \frac{\vec{\alpha}\cdot\vec{\mu}}{\alpha^2} |\mu\rangle .$$

However, from our knowledge of $SU(2)$ we must have

$$2\frac{\vec{\alpha}\cdot\vec{\mu}}{\alpha^2} \in \mathbb{Z}, \tag{A.9}$$

since the eigenvalues of J_3 are integers or half-integers. Proving our claim from the beginning.

From now own we discuss how these weights and roots organize themselves into different lattices. The set of all weights and roots span two lattices

$$\mu \in \Lambda_w(G), \quad \alpha \in \Lambda_r(G).$$

However, since $\Lambda_r(G)$ is only associeted with the adjoint representation, we have

$$\Lambda_r(G) \subseteq \Lambda_w(G).$$

We can also introduce the *co-root*, defined as $\alpha^\vee \equiv 2\alpha/\alpha^2$, which span another lattice

$$\alpha^\vee \in \Lambda_r^\vee(G).$$

Therefore, (A.9) can be written as $\alpha_i^\vee \mu^i \in \mathbb{Z}$. In fact, it will be better to consider a dual weight lattice $\alpha \in \Lambda_w^*(G)$, with $\Lambda_r^\vee(G) \subseteq \Lambda_w^*(G)$, such that

$$\alpha \cdot (\_\_) : \Lambda_w(G) \to \mathbb{Z}$$

This ensures that no matter if we choose the covering group $\widetilde{G}$ or $G = \widetilde{G}/\Gamma_G$, with Γ_G contained in the center $Z(G)$, we will always have an integer. Once $\widetilde{G}$ and G may not necessarily share the same weight lattice.

The distinction between these two scenarios can be characterized by modding out the co-root lattice $\Lambda_r^\vee(G)$, which will always be dual to $\Lambda_w(G)$, from $\Lambda_w^*(G)$ and verifying if there is something non-trivial

$$\Lambda_w^*(G)/\Lambda_r^\vee(G) \cong \Gamma_G \cong \pi_1(G). \tag{A.10}$$

The prove for (A.10) can be found in [36].

Finally, a last lattice can be made. If we consider the covering group $\widetilde{G}$, than its weight lattice generates the *fundamental weights*, $w \in \Lambda_w(\widetilde{G})$, such that every other weight associeted with the group $G = \widetilde{G}/\Gamma$, can be written as a $\mathbb{Z}$-valued linear combination of all the fundamental weights. Therefore, we have

$$\Lambda_w(G) \subseteq \Lambda_w(\widetilde{G}).$$

The root lattice, on the other hand, does not know if we are in G or $\widetilde{G}$. This follows from the fact that the adjoint representation is completely characterized by the algebra $\mathfrak{g}$, not the global topology of G, i.e. $\Lambda_r(G) \cong \Lambda_r(\widetilde{G})$.

As an example, we will consider the $SU(3)$ group. The generators of $\mathfrak{su}(3)$ are the Gell-mann matrices, $T^a = \lambda^a/2$, with $a = 1, ..., 8$. The $SU(3)$ has rank 2. We can choose $H_1 = T_3$ and $H_2 = T_8$ to be in the Cartan sub-algebra, once these generators are already diagonal. We then find the weights

$$\vec{w}_{(1)} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{pmatrix}, \vec{w}_{(2)} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{pmatrix}, \vec{w}_{(3)} = \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{3} \end{pmatrix}. \tag{A.11}$$

The weights shown in (A.11) are the basis for the fundamental weight lattice. Conversely, the roots are given by

$$\vec{\alpha}_{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\alpha}_{(2)} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \vec{\alpha}_{(3)} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{A.12}$$

$$\vec{\alpha}_{(4)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \vec{\alpha}_{(5)} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}, \quad \vec{\alpha}_{(6)} = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \tag{A.13}$$

The weight lattice and root lattice are shown in figure (11).

Note that both lattices in figure (11) have a reflection symmetry through the hyperplane[1] orthogonal to the line that connects the pair of roots $\pm\alpha$. This is an effect of the

1 In the present case, this hyperplane is just a line.

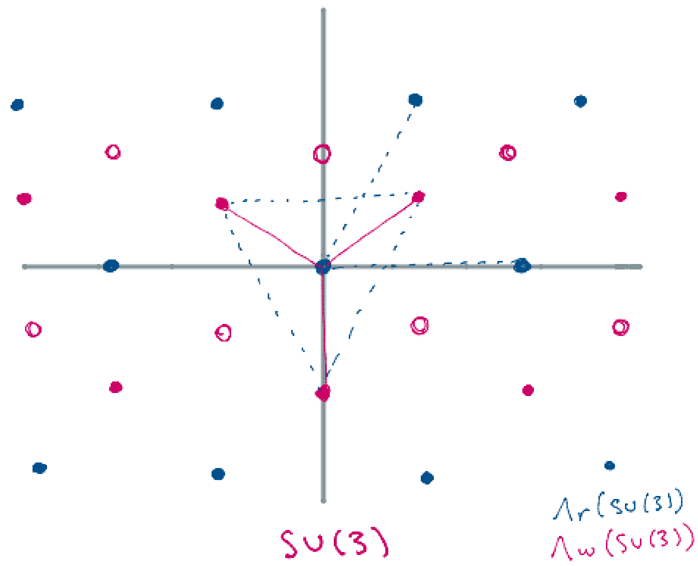

Figura 11 – Red dots are points in the fundamental weight lattice, blue points are in the root lattice and open red dots are given by the sum of two fundamental weights [36].

Weyl group. Once the roots comes in pairs, we can map one onto the another, with the transformation

$$\mathcal{W}_\alpha = \exp\left(i\pi\frac{E_\alpha + E_{-\alpha}}{|\alpha|}\right).$$

If $B \in \mathfrak{g}$, then the Weyl group acts as a similarity transformation

$$Weyl : B \to \mathcal{W}_\alpha B \mathcal{W}_\alpha^\dagger$$

and map weights into

$$\begin{aligned} Weyl : \mu_i \to \mu_i - 2\alpha_i\frac{\alpha_j\mu^j}{\alpha^2} &= \mu_i - \alpha_i^\vee(\alpha_j\mu^j) \\ &= \mu_i - \alpha_i(\alpha_j^\vee\mu^j). \end{aligned}$$

It is easily verified that, in fact, $Weyl : \alpha_i \to -\alpha_i$.

To conclude our discussion, note that weights are dependent on the representation of G. As an example, we can think in the angular momentum, where j identifies differents representations and m_j the projections in the z-axis. In fact, m_j are the weights of $SO(3)$ in a $2j+1$ dimensional representation. So the *highest weight*, which happens to be $m_j = j$, tell us in what representation we are in. More on that can be found in [47].

Thus, there must be a map between weights and representations. This map is given by

$$R(G) \leftrightarrow \Lambda_w(G)/W. \tag{A.14}$$

The expression (A.14) is telling us that, after we eliminate the redundance of the Weyl group, we find a one-to-one map between weights and representations.

B Wess-Zumino consistency condition

The Wess-Zumino consistency condition [48] states that the algebra of the gauge group, more specifically its fusion rules, has several implications that can be used to probe the theory. The discussion provided below is mainly based on [49]. More details can be found therein.

Let us consider a non-Abelian gauge theory and define the translation operator

$$\delta_\lambda(\_) \equiv \int D_\mu \lambda^a(x) \frac{\delta}{\delta A_\mu^a(x)} (\_) d^d x, \tag{B.1}$$

which implements a infinitesimal gauge transformation, i.e. $\delta_\lambda(A_\mu^a(x)) = D_\mu \lambda^a(x)$. Since (B.1) is defined entirely in terms of the algebra and furnishes a representation of the gauge group, it follows that

$$\delta_\lambda \delta_{\lambda'}(\_) - \delta_{\lambda'} \delta_\lambda(\_) = \delta_{\lambda \times \lambda'}(\_). \tag{B.2}$$

Consequently, the fusion rules (B.2) must hold true despite the functional they are acting in.

Take as example the effective action Γ. If the quantum theory has an anomaly associated with it, after a gauge variation we shall find

$$\delta_\lambda \Gamma \equiv G(\lambda) = \int d^d x \; \lambda^a(x) \mathcal{A}^a(x), \tag{B.3}$$

where $G(\lambda)$ denotes the anomaly. By applying (B.3) to (B.2) we are able to derive a consistency condition for the anomaly

$$\delta_\lambda G(\lambda') - \delta_{\lambda'} G(\lambda) = G(\lambda \times \lambda'). \tag{B.4}$$

As a consequence, it becomes clear that $G(\lambda)$ does not care about changes in the regularization scheme, since this would amount to shifting $G(\lambda)$ by a counter-term of the form $\delta_\lambda W$. However, (B.2) is automatically satisfied by $\delta_\lambda W$. We are then lead to conclude that anomalies cannot be eliminated by regularization process and, therefore, are physical quantities.

C Differential forms

The mathematical framework of differential forms provides a systematic language that naturally incorporates the orientation of manifolds we wish to integrate over. Some complementary information about the topic can be found in [5, 34, 49]. Here we only provide conventions we are using throughout the text and a summary of facts about the subject.

A generic *q-form* in a d dimensional manifold $\mathcal{M}$ is defined as

$$\omega \equiv \frac{1}{q!}\omega_{\mu_1\mu_2\ldots\mu_q}(x)dx^{\mu_1}\wedge dx^{\mu_2}\wedge \ldots \wedge dx^{\mu_q}, \tag{C.1}$$

where $\wedge$ denotes the anti-symmetric wedge product, that is $dx^{\mu}\wedge dx^{\nu} = -dx^{\nu}\wedge dx^{\mu}$. Consequently, we can verify that the highest form in a d dimensional manifold is the volume form

$$dV_{(d)} \equiv \frac{\epsilon_{\mu_1\mu_2\ldots\mu_d}}{d!}dx^{\mu_1}\wedge dx^{\mu_2}\wedge \ldots \wedge dx^{\mu_d}. \tag{C.2}$$

There is also a duality relation which maps a *q-form* into another *(d-q)-form* as follows

$$\begin{aligned}\star\omega &= \frac{\omega_{\mu_1\ldots\mu_q}}{q!}\left(\star dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_q}\right)\\ &= \frac{\omega_{\mu_1\ldots\mu_q}}{q!}\left(\frac{1}{(d-q)!}\epsilon^{\mu_1\ldots\mu_q}{}_{\mu_{q+1}\ldots\mu_d}dx^{\mu_{q+1}}\wedge dx^{\mu_d}\right),\end{aligned} \tag{C.3}$$

this operation is referred as the *hodge dual*. Depending on the signature of the metric we find that

$$\star\star = \begin{cases} (-1)^{q(d-q)}, & \text{Euclidean} \\ (-1)^{q(d-q)+1}, & \text{Minkowskian} \end{cases}. \tag{C.4}$$

Additionally, we can differentiate a form by introducing the *exterior derivative*, defined as

$$d\omega = \frac{1}{q!}\partial_{\mu}\omega_{\mu_1\mu_2\ldots\mu_q}(x)dx^{\mu}\wedge dx^{\mu_1}\wedge dx^{\mu_2}\wedge\ldots\wedge dx^{\mu_q}. \tag{C.5}$$

A form is said to be closed if $d\omega = 0$ and exact if $\omega = d\eta$. Note that, once the product of two partial derivatives is symmetry, all exact forms are closed implying, therefore, in $d^2 = 0$. Alternatively, the *Poincaré lemma* states that all closed forms are locally exact.

Finally, the Stokes theorem can be generalized to a generic *q-form* as follows

$$\oint_{\partial\Sigma}\omega = \int_{\Sigma}d\omega. \tag{C.6}$$

If ω is an *1-form*, then (C.6) reproduces the usual Stokes theorem. On the other hand, if we consider a *2-form*, then (C.6) furnishes the Gauss theorem.

D Non-invertible chiral symmetry

As we saw in chapter 2, the chiral anomaly is characterized by the violation of the conserved current j_5^μ at the quantum theory, namely,

$$d \star j_5 = \frac{1}{4\pi^2} F \wedge F. \tag{D.1}$$

Naively, we might try to resolve the anomaly by considering the modified current

$$\star \hat{j}_5 = \star j_5 - \frac{1}{4\pi^2} A \wedge dA, \tag{D.2}$$

since (D.2) is now a conserved quantity. As a consequence, we would find a gauge-dependent object, which in turn prohibits us from following this procedure.

However, there is a caveat in this argument. Let us insist in constructing a symmetry operator from (D.2)

$$\hat{U}_\alpha(M^{(3)}) \equiv \exp\left(i\alpha \oint_{M^{(3)}} \star j_5 - \frac{1}{4\pi^2} A \wedge dA \right), \tag{D.3}$$

where $\alpha \in [0, \pi)$ [1]. As expected, the second term in (D.3) is not gauge invariant, since it is a Chern-Simons theory with a non-integer level. Nevertheless, this observations suggests that we could construct a fractional level by reducing the angle to a rational number $\alpha = \pi/N$

$$\hat{U}_{1/N}(M^{(3)}) = \exp\left(i \oint_{M^{(3)}} \frac{\pi}{N} \star j_5 - \frac{1}{4\pi N} A \wedge dA \right). \tag{D.4}$$

In this case, the operator (D.4) can be made gauge invariant through the introduction of an auxiliary gauge field

$$\mathcal{D}_{1/N}(M^{(3)}) = \int \mathcal{D}a \exp\left(i \oint_{M^{(3)}} \frac{\pi}{N} \star j_5 + \frac{N}{4\pi} a \wedge da - \frac{1}{2\pi} a \wedge dA \right). \tag{D.5}$$

Integrating over the dynamical field in (D.5) locally sets $a = A/N$, which upon substitution returns (D.4).

In summary, the operator (D.5) is understood as the composition of an rational chiral rotation with a fractional quantum Hall state. This composition is simultaneously gauge invariant and topological. So, $\mathcal{D}_{1/N}$ generates a symmetry and, therefore, eliminates the chiral anomaly. The only observation here is that (D.5) is a non-invertible symmetry operator, since it fuses non-trivially with its dagger

$$\mathcal{D}^\dagger_{1/N}(M^{(3)}) = \int \mathcal{D}a' \exp\left(-i \oint_{M^{(3)}} \frac{\pi}{N} \star j_5 + \frac{N}{4\pi} a' \wedge da' - \frac{1}{2\pi} a' \wedge dA \right), \tag{D.6}$$

[1] Note that, for $\alpha = \pi$ a chiral transformation is reduced to a trivial phase, that is, $\psi \to e^{i\pi\gamma_5}\psi = -\psi$. Consequently, only the values of $\alpha < \pi$ represent true chiral transformations.

that is,

$$\mathcal{D}_{1/N}(M^{(3)}) \times \mathcal{D}^{\dagger}_{1/N}(M^{(3)}) = \int \mathcal{D}a\mathcal{D}a' \exp\left(i \oint_{M^{(3)}} \frac{N}{4\pi} a \wedge da - \frac{N}{4\pi} a' \wedge da' + \frac{1}{2\pi}(a - a') \wedge dA\right) \tag{D.7}$$

$$= \mathcal{C}(M^{(3)}) \neq 1. \tag{D.8}$$

Note that, while the chiral part of (D.5) is invertible, the Chern-Simons defect implies in the non-invertible aspect of this generalized symmetry. The operator $\mathcal{C}(M^{(3)})$ is said to be a condensation defect [50].

In addition to this discussion, we can also consider a generic rational angle $\alpha = \pi p/N$, in this case the $U(1)_N$ Chern-Simons theory is replaced by a minimal $\mathbb{Z}_N$ TQFT with Lagrangian $\mathcal{A}^{(N,p)}[dA/N]$

$$\mathcal{D}_{p/N}(M^{(3)}) \equiv \exp\left(i \oint_{M^{(3)}} \frac{\pi p}{N} \star j_5 + \mathcal{A}^{(N,p)}[dA/N]\right). \tag{D.9}$$

For more details on this abstraction see [29, 51].

It is worth mentioning that even though the anomaly has been eliminated, its physical implications persist. Conventionally, we would invoke the 't Hooft anomaly matching condition and derive a constraint on the low energy physics. However, once we have rephrased the anomaly in terms of $\mathcal{D}_{p/N}(M^{(3)})$, this implies that the non-invertible chiral symmetry is also rigid against the RG flow and, therefore, must be realized in the IR theory. An example of such phenomena is provided by the massless QCD Lagrangian, where the non-invertible chiral symmetry can be used to explain the coupling $\pi^0 F \wedge F$ in the low energy theory [29].

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