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Bosonization and Dualities in Lower Dimensional Quantum Field Theories

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Resumo

Investigações físicas de sistemas fortemente acoplados podem ser extremamente desafiadoras, essencialmente porque a teoria de perturbação falha em ser uma boa técnica em regimes onde as constantes de acoplamento são grandes. Este é um problema muito relevante, uma vez que existem objetos que apresentam grandes interações em regimes específicos, por exemplo, a física da Cromodinâmica Quântica, os Líquidos Hall Quânticos, Líquidos Spin Quânticos e outros. O objetivo do trabalho atual é investigar e compreender as dualidades da teoria quântica de campos $2D$ e $3D$ abeliana. Essas relações de dualidade não perturbativas podem fornecer ferramentas para estudar tais objetos de interesse. Alcançamos este estudo argumentando a veracidade dessas dualidades em duas abordagens: a abordagem macroscópica e a abordagem dos fios quânticos. Na abordagem macroscópica, comparamos as fases macroscópicas das teorias, as simetrias globais e o padrão de quebra de simetria espontânea das teorias. Na abordagem dos fios quânticos, comparamos as teorias mapeando com precisão uma teoria na outra, entendendo as teorias como sendo composições de vários fios quânticos $2D$. Esta última abordagem também nos permite usar técnicas de bosonização $2D$ e teoria de campos conformes, cujas são técnicas bem compreendidas.
Palavras-chave: Teoria Quântica de Campos. Bosonização. Dualidades. Fios Quânticos.

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Abstract

Physical investigations of strongly coupled systems can be extremely challenging, essentially because perturbation theory fails to be a good technique in regimes where the coupling constants are large. This is a very relevant problem, since there are objects which present large interactions in specific regimes, for example, the physics of Quantum Chromodynamics, the Quantum Hall Liquids, Quantum Spin Liquids and so on. The purpose of the actual work is to investigate and understand $2D$ and $3D$ abelian quantum field theory dualities. These non-perturbative duality relations may provide tools to study such objects of interest. We achieve this study by arguing the veracity of these dualities in two approaches: the macroscopic approach and the quantum wires approach. In the macroscopic approach, we compare the macroscopic phases of theories, the global symmetries and the spontaneous symmetry breaking pattern of the theories. In the quantum wires approach, we compare the theories by precisely mapping one theory into another, understanding the theories as being compositions of several $2D$ quantum wires. This last approach also allow us to use well-understood $2D$ bosonization and conformal field theory techniques.

Keywords: Quantum Field Theory. Bosonization. Dualities. Quantum Wires.

Lista de ilustrações

Sumário

Introduction

In the past decades, Quantum field theory (QFT) has proven its value as a suitable framework for theoretical physics by providing proper physical descriptions with accurate results. Some of its achievements come out of the descriptions of several complex phenomena that take place in different branches of physics, as in the Standard model of particles [1], condensed matter [2], statistical physics [3], cosmology [4] and string theory [5]. The QFT framework is constituted by a set of methods based on locality, symmetry principles and on the intrinsic quantum nature of fundamental particles. Also, QFT handles very well with nature at regimes where the interactions between physical constituents are sufficiently small.

However, even inside of the Standard Model (SM), which is entirely formulated in QFT language, there are strongly coupled phenomena which lie outside the perturbative framework. As an example, the Quantum Chromodynamics (QCD), theory which deals with the dynamics and interactions between quarks and gluons, is strongly coupled and exhibits confinement effects in low energies. As the quark interactions are strong, perturbative methods are not very useful. Beyond that, strongly coupled property appears in other important systems, as fractional quantum Hall liquids, quantum spin liquids and in the recent discovered fracton systems. It is, in principle, unpractical to use perturbation theory to describe the microscopic physics of these systems. Even so, physicists have struggled against this difficulty and developed new tools inside QFT to deal with it. Between these developments, we mention Large N expansion, holography and methods based on supersymmetry, gravity and string theory. Besides these, dualities and effective field theory (EFT) approach are extremely powerful methods to investigate strongly coupled systems . In this work, we will be focusing on these last ones.

Dualities between theories, at first, can be thought as accidents in which two different theories happen to describe the same underlying physics in a nontrivial way. This intimate relation between the theories opens the possibility to obtain quantitative results in one side of the duality using the other side, and vice-versa. In spite of that, it is natural to consider the possibility that these relations are not just accidents. If this turns out to be the case, dualities can produce a new systematic way of thinking about nature. The other method, effective field theory, is based on the fact that not every single microscopic degree of freedom plays a central rule in a low-energy description. Given that, EFT encodes the relevant degrees of freedom and symmetries in order to get a correct and computable low-energy theory. EFTs have shown their power not only in the context of strongly coupled system, describing, for example, the fractional quantum Hall effect, but also extracting physical results from nonrenormalizable quantum field theories [6].

This work is mostly of review character, intending to bring up important properties of $2D$ and $3D$ abelian dualities. We have organized the work as follows. The second chapter is devoted to the study of $2D$ bosonization. In the chapter three, we present some features of $3D$ quantum field theories. In chapter four, we have derived the master fermion/boson duality. In chapter five, we have deformed the master duality to the particle-vortex dualities and to the $3D$ massive Thirring model. In last chapter, we discuss the quantum wires approach, where these dualities are established in a more precise way. We conclude the work with final remarks.

1 $2D$ Bosonization

Due to the simple properties of low-dimensional systems, a natural theoretical interest to investigate their properties arises. In fact, $2D$ relativistic bosons and fermions are intimately related. In this chapter, we will investigate a relation between periodic scalar and Dirac theories. We will find the two-point functions of those two quantum theories and then do a match-up map between them. Using the bosonization dictionary constructed, we will see how massive Thirring and Sine-Gordon models are connected and the relation between their solutions using the canonical quantization .

1.1 $2D$ Fermion/Boson Duality

1.1.1 Fermionic Theory

Let us consider the $2D$ dimensional massless free Dirac action,

$$S_{Dirac} = \int d^2 x i\bar{\psi}\partial\!\!\!/\psi, \tag{1.1}$$

where $\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$, $\bar{\psi} = \psi^\dagger\gamma^0$ and γ^μ are the Dirac matrices, which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_{2x2}$. We can use $\gamma^0 = \sigma^1$, $\gamma^1 = i\sigma^2$ as a realization of the Clifford algebra, with σ^i being Pauli matrices. Since these γ's are real, we can choose the spinors to be real. Another useful property of γ matrices in this representation is that $(\gamma^\mu)^T = \gamma^0\gamma^\mu\gamma^0$. This can be easily checked performing the matricial products.

We may split the Dirac fermion in decoupled Weyl fermions as $\psi^T = (\psi_+, \psi_-)$, since the theory is massless. In this way, we have

$$\begin{aligned} S_{Dirac} &= \int d^2 x \psi^T \gamma^0 \left(i\gamma^0\partial_0 + i\gamma^1\partial_1\right)\psi \\ &= \int d^2 x \left(i\psi_+\partial_-\psi_+ + i\psi_-\partial_+\psi_-\right), \quad \partial_\pm \equiv \partial_t \pm \partial_x. \end{aligned} \tag{1.2}$$

The equations of motions for Dirac fermions in $2D$ dimensions can be easily checked to be

$$\partial\!\!\!/\psi = 0 \quad \Longleftrightarrow \quad \partial_\pm \psi_\mp = 0. \tag{1.3}$$

In addition, there exist two independent conserved currents in the system, which arises from internal structure of the theory. One of these is related to $U(1)$ and the other to an axial global transformations. Under $U(1)$ infinitesimal transformations, $\psi \to e^{-i\epsilon}\psi$, the Lagrangian does not change at all. The form of this conserved current is

$$J^\mu_{charge} = \bar{\psi}\gamma^\mu\psi. \tag{1.4}$$

The last relation was obtained by the use of the Noether's current [7],

$$\epsilon J^{\mu} = \frac{\partial \mathcal{L}}{\partial \partial_{\mu}\phi}\delta\phi - V^{\mu}, \quad \partial_{\mu}V^{\mu} = \delta\mathcal{L}. \tag{1.5}$$

In fact, we can check the current conservation using the equation of motion of ψ. Deriving J^{μ}_{charge} leads to

$$\partial_{\mu}J^{\mu}_{charge} = (\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi + \bar{\psi}(\partial\!\!\!/\psi). \tag{1.6}$$

From the algebra, $\gamma^0\gamma^0 = \mathbb{I}_{2x2}$, resulting

$$\begin{aligned}\partial_{\mu}J^{\mu}_{charge} &= \partial_{\mu}\psi^T\gamma^0\gamma^{\mu}\gamma^0\gamma^0\psi + \bar{\psi}(\partial\!\!\!/\psi)\\ &= \partial_{\mu}\psi^T(\gamma^{\mu})^T\gamma^0\psi + \bar{\psi}(\partial\!\!\!/\psi)\\ &= (\partial\!\!\!/\psi)^T\gamma^0\psi + \bar{\psi}(\partial\!\!\!/\psi) = 0.\end{aligned} \tag{1.7}$$

To show that the current is conserved, we have used the identities: $\gamma^0\gamma^0 = \mathbb{I}_{2x2}$, $(AB)^T = B^TA^T$, $\gamma^0\gamma^{\mu}\gamma^0 = (\gamma^{\mu})^T$ and $\partial\!\!\!/\psi = 0$. Under axial transformations $\psi \to e^{i\epsilon\gamma^3}\psi$, with $\gamma^3 = \gamma^0\gamma^1$, the conserved current of Dirac theory is

$$J^{\mu}_{axial} = \bar{\psi}\gamma^{\mu}\gamma^3\psi, \tag{1.8}$$

which could be checked to be conserved using the equation of motion and properties of γ matrices. By the end, these two currents will help us, giving more credibility in our future bosonization map.

Ir order to compute two-point functions for the chiral fermions ψ_- and ψ_+, let us quantize this field theory using the canonical quantization procedure. To do so, we will expand our fields ψ_- and ψ_+ in terms of a sum in momentum of creation $(b^{\dagger}_{\pm}, c^{\dagger}_{\pm})$ and destruction $(b_{\pm}, c_{\pm})$ operators, where $c_{\pm}|0\rangle = b_{\pm}|0\rangle = 0$. Also, we will demand that our quantum fields satisfies the classic equations. Looking to the classical equations in terms of Weyl fermions, we see that the ψ_- field moves to the right (positive momentum p), while ψ_+ moves in the opposite direction (negative momentum p). In this spirit, we will show that we can expand ψ_- as

$$\psi_-(x,t) = \int_0^{\infty}\frac{dp}{2\pi}\left(b_-(p)e^{ip(t-x)} + c^{\dagger}_-(p)e^{-ip(t-x)}\right)e^{-\frac{p\epsilon}{2}}. \tag{1.9}$$

To do so, let us consider the Fourier expansion of ψ_-,

$$\psi_-(x,t) = \int_{-\infty}^{\infty}\frac{dp_0dp_1}{2\pi}\tilde{\psi}_-(p_0,p_1)e^{-ip\cdot x}, \quad p\cdot x = p_0t - p_1x. \tag{1.10}$$

Demanding $\partial_+\psi_- = (\partial_t + \partial_x)\psi_- = 0$ implies the constrain

$$(-i)\int_{-\infty}^{\infty}\frac{dp_0dp_1}{2\pi}\tilde{\psi}_-(p_0,p_1)(p_0-p_1)e^{-ip\cdot x} = 0 \Longrightarrow \tilde{\psi}_-(p_0,p_1) = a(p_0,p_1)\delta(p_1-p_0). \tag{1.11}$$

for an arbitrary well-behaved momentum function $a(p_0, p_1)$. Now, rewriting ψ_- in terms of the previous constrain gives

$$\psi_-(x,t) = \int_{-\infty}^{\infty} \frac{dp_0 dp_1}{2\pi} a(p_0, p_1)\delta(p_1 - p_0)e^{-ip\cdot x}. \tag{1.12}$$

The integral over p_0 results

$$\psi_-(x,t) = \int_{-\infty}^{0} \frac{dp_1}{2\pi} a(p_1, p_1)e^{-ip_1(t-x)} + \int_{0}^{\infty} \frac{dp_1}{2\pi} a(p_1, p_1)e^{-ip_1(t-x)}. \tag{1.13}$$

Now, changing $p_1 \mapsto -p$ in the first integral and $p_1 \mapsto p$ in the second, we have

$$\psi_-(x,t) = \int_{0}^{\infty} \frac{dp}{2\pi} \underbrace{a(-p,-p)}_{\equiv\, b_-(p)} e^{ip(t-x)} + \int_{0}^{\infty} \frac{dp}{2\pi} \underbrace{a(p,p)}_{\equiv\, c_-^\dagger(p)} e^{-ip(t-x)}. \tag{1.14}$$

With the previous result and introducing a UV regulator $e^{-\frac{p\epsilon}{2}}$, we finally get

$$\psi_-(x,t) = \int_{0}^{\infty} \frac{dp}{2\pi} \left(b_-(p)e^{ip(t-x)} + c_-(p)^\dagger e^{-ip(t-x)}\right) e^{-\frac{p\epsilon}{2}}. \tag{1.15}$$

The Fourier expansion of ψ_+ which satisfies the classical equation $\partial_-\psi_+ = 0$ runs analogously, resulting

$$\psi_+(x_+) = \int_{-\infty}^{0} \frac{dp}{2\pi} \left(b_+(p)e^{ipx_+} + c_+(p)^\dagger e^{-ipx_+}\right) e^{-\frac{|p|\epsilon}{2}}. \tag{1.16}$$

In this construction, we impose the canonical anti-commutation rules between the creation and destruction operators, $\{b, b\} = \{c, c\} = \{b_\pm, c_\mp\} = 0$ and $\{b_\pm(p), b_\pm^\dagger(q)\} = \{c_\pm(p), c_\pm^\dagger(q)\} = 2\pi\delta(p - q)$.

We can now compute the two-point functions of fields, $\langle 0| \psi_\pm(x_+)\psi_\pm^\dagger(x'_+) |0\rangle$. Let us start with ψ_+. Discarding terms with destruction operators on the right and creation operators on the left, we have

$$\begin{aligned} G_+(x_+, x'_+) \equiv \langle 0| \psi_+(x_+)\psi_+^\dagger(x'_+) |0\rangle &= \int_{-\infty}^{0} \frac{dpdq}{4\pi^2} \langle 0| b_+(p)b_+^\dagger(q) |0\rangle\, e^{ipx_+ - iqx'_+} e^{-\frac{(|q|+|p|)\epsilon}{2}} \\ &= \int_{-\infty}^{0} \frac{dp}{2\pi} e^{ip(x_+ - x'_+)} e^{-|p|\epsilon} \\ &= -\frac{i}{2\pi} \frac{1}{(x_+ - x'_+) - i\epsilon}. \end{aligned} \tag{1.17}$$

Notice also that $G_+(x_+, x'_+) = \langle 0| \psi_+^\dagger(x_+)\psi_+(x'_+) |0\rangle$, with the analogous being valid to ψ_- field. Using the fact that we are dealing with a free theory, the temporal evolution of states or operators are trivial; hence, we can work in the Schrodinger picture, were correlation functions are putted in equal times. The equal-time correlation function for ψ_+ is

$$G_+(x,x') = -\frac{i}{2\pi}\frac{1}{(x-x')-i\epsilon}. \tag{1.18}$$

Analogously for ψ_-,

$$G_-(x,x') = \frac{i}{2\pi}\frac{1}{(x-x')+i\epsilon}. \tag{1.19}$$

To sum up, the equal-time propagators for the free Dirac theory are

$$G_\pm(x,x') = \mp\frac{i}{2\pi}\left(\frac{1}{(x-x')\mp i\epsilon}\right). \tag{1.20}$$

Now that we have the two-point correlations for Weyl fermions, we can carry on and proceed to the bosonic side of the story.

1.1.2 Bosonic Theory

Let us now consider a compact real scalar field $\phi(x,t)$ governed by the action

$$S_{boson} \equiv \int d^2x \mathcal{L}_{boson} = \int d^2x \frac{\lambda^2}{2}(\partial\phi)^2, \quad \phi \sim \phi + 2\pi, \tag{1.21}$$

where we have defined $d^2x \equiv dxdt$. This theory might appear to be simple, because it is massless and free. However, its dimension and non-trivial boundary conditions give to it a rich structure, which we will explore to establish a boson-fermion duality. The equation of motion of ϕ can be obtained by varying the action in ϕ. This leads to

$$\partial^2\phi = 0. \tag{1.22}$$

There are two independent solutions to this differential equation, which are left and right-moving fields, $\phi_\pm(x^\pm)$ (excluding the zero mode), with $x^\pm = t \pm x$. The combined solution is

$$\phi(x,t) = \phi_+(x^+) + \phi_-(x^-). \tag{1.23}$$

This scalar theory possesses two different conserved quantities itself, which we will evidence below. The first one is related to continuous shifts of the field, $\phi' = \phi + \alpha$. Accoding to Noether's theorem, this theory possesses a conserved current J^μ associated to field shifts. Under an infinitesimal shift as above, $\mathcal{L}_{boson}$ is not affected, and the conserved current is

$$J^\mu_{shift} = \lambda^2\partial^\mu\phi. \tag{1.24}$$

At the same time, due to the spacetime dimensionality, there is a second conserved current, J^{μ}_{wind}, which is

$$J^{\mu}_{wind} = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_{\nu}\phi, \tag{1.25}$$

where $\epsilon_{\mu\nu}$ is the totally antisymmetric symbol. Notice that, due to the properties of Minkowski spacetime $\mathbb{R} \times \mathbb{R}$, partial derivatives commute and the current is conserved, independently of dynamics.

This bosonic theory can be described in terms of a dual field. If we define the dual field as $\sim \epsilon_{\mu\nu}\partial^{\nu}\phi$, with $\partial\phi$ being interpreted as primary field, the usual Bianchi identity $\sim \partial\epsilon\partial\phi = 0$ would impose a null value for the wind current (1.25). To handle this problem, remember that the ϕ field is periodic. Then $\phi \sim \phi + 2\pi$ implies

$$\frac{1}{2\pi}\oint dx^{\mu}\partial_{\mu}\phi \in \mathbb{Z}. \tag{1.26}$$

In this way, let us consider

$$S'[\phi,\tilde{\phi}] = \int d^2x\left[\frac{\lambda^2}{2}(\partial\phi)^2 - \frac{1}{2\pi}\epsilon_{\mu\nu}\partial^{\mu}\phi\partial^{\nu}\tilde{\phi}\right], \tag{1.27}$$

with $\tilde{\phi} \sim \tilde{\phi} + 2\pi$. Let us now see how the duality is established. Consider the equation of motion of $\tilde{\phi}$,

$$\frac{1}{2\pi}\partial^{\nu}\epsilon_{\mu\nu}\partial^{\mu}\phi = 0. \tag{1.28}$$

Putting this equation in S', we recover (1.21). On the other hand, the equation of motion for ϕ,

$$\partial_{\mu}\phi = \frac{1}{2\pi\lambda^2}\epsilon_{\mu\nu}\partial^{\nu}\tilde{\phi}, \tag{1.29}$$

gives a non-local relation between ϕ and $\tilde{\phi}$. Due to this non-locality, we would expect that the two-point functions of ϕ and $\tilde{\phi}$ would not match in a directly way.

Now, using (1.23) in (1.29), we can obtain a simple relation between the dual and chiral fields,

$$\tilde{\phi} = 2\pi\lambda^2(\phi_- - \phi_+). \tag{1.30}$$

Using this relation and $\phi = \phi_+ + \phi_-$, we can express $\phi_\pm$ in terms of ϕ and $\tilde{\phi}$,

$$\phi_{\mp} = \frac{1}{2}\left[\phi(x,t) \pm \frac{1}{2\pi\lambda^2}\tilde{\phi}(x,t)\right]. \tag{1.31}$$

This relation will be useful when we calculate the two-point correlations in the quantized theory.

To quantize, we will adopt the canonical procedure, imposing that the creation and destruction operators $a(k)$ and $a^{\dagger}(k')$ have the commutation rule

$$\left[a(k), a^{\dagger}(k')\right] = 2\pi\delta(k-k'), \quad \left[a(k), a(k')\right] = \left[a^{\dagger}(k), a^{\dagger}(k')\right] = 0. \tag{1.32}$$

As usual, we set $a(p)\left|0\right\rangle = 0$. The field Fourier decomposition compatible with the equation of motion can be achieved in the very similar way to the previous section. For the same reason stated in the previous section, we will work in the Schrodinger picture, where the field decomposition is

$$\phi(x) = \frac{1}{\lambda}\int \frac{dp}{2\pi}\sqrt{\frac{1}{2|p|}}\left[a(p)e^{-ipx} + a^{\dagger}(p)e^{ipx}\right]e^{-\frac{|p|\epsilon}{2}}. \tag{1.33}$$

Above, ϵ is small and acts like a inverse-cutoff parameter to exclude divergences. Notice that the λ appears in the ϕ expression as an imprint of the periodicity of the field. As the conjugated field π is defined as $\pi \equiv \frac{\partial \mathcal{L}_{boson}}{\partial\partial_0\phi}$, we have formally $\pi = \lambda^2\partial_0\phi$, and then

$$\pi = -i\lambda\int \frac{dp}{2\pi}\sqrt{\frac{|p|}{2}}\left[a(p)e^{-ipx} - a^{\dagger}(p)e^{ipx}\right]e^{-\frac{|p|\epsilon}{2}}. \tag{1.34}$$

Now we can compute explicitly the equal-time commutator between ϕ and π,

$$[\phi(x),\pi(y)] = i\int \frac{dpdq}{4\pi^2}\sqrt{\frac{|q|}{4|p|}}\left([a(p),a^{\dagger}(q)]e^{-i(px-qy)} - [a^{\dagger}(p),a(q)]e^{+i(px-qy)}\right)e^{\frac{-(|p|+|q|)\epsilon}{2}}.$$

Using the commutation rule between a and $a^{\dagger}$ and integrating in q,

$$\begin{aligned}
[\phi(x),\pi(y)] &= i\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\frac{1}{2}\left(e^{-ip(x-y)} + e^{ip(x-y)}\right)e^{-|p|\epsilon} \\
&= \frac{i}{\pi}\int_0^{\infty} cos(p(x-y))e^{-\frac{p\epsilon}{2}} \\
&= \frac{i}{\pi}\frac{\epsilon}{(x-y)^2+\epsilon^2}.
\end{aligned} \tag{1.35}$$

The limit of the commutator above as $\epsilon \to 0$ is $i\delta(x-y)$.

Intending to do the match between bosonic and fermionic correlations, let us give a step ahead and evaluate the bosonic ones for the chiral fields. First, we need the relation between $\phi_{\pm}$, ϕ and π, because we already have expressions for the last ones. Inserting (1.29) for $\mu = 0$, which is

$$\partial_0\phi = -\frac{1}{2\pi\lambda^2}\partial_x\tilde{\phi} = \frac{\pi(x,t)}{\lambda^2}, \tag{1.36}$$

in (1.31), we get

$$\phi_{\pm} = \frac{1}{2}\left[\phi(x,t) \pm \frac{1}{\lambda^2}\int_{-\infty}^{x} dy\pi(y,t)\right]. \tag{1.37}$$

The previous expression allows us to compute explicitly commutators between ϕ_- and ϕ_+. Directly,

$$[\phi_{\pm}(x),\phi_{\pm}(x')] = \pm\frac{1}{4\lambda^2}\left(\int_{-\infty}^{x'} dy'\underbrace{[\phi(x),\pi(y')]}_{=i\delta(x-y')} + \int_{-\infty}^{x} dy\underbrace{[\pi(y),\phi(x')]}_{=-i\delta(y-x')}\right). \tag{1.38}$$

Notice that when $x > x'$, we always have $x > y'$ in the first integral, and in the second integral we might have both $x' > y$ and $x' < y$ as y runs. So, in this case, only the second integral remains, which gives $-i$. However, when $x < x'$, we always have $y < x'$ in the second integral, and might have $x > y'$ and $x < y'$ as y' runs. In the second case, then, the integral results in i. Finally, we can express these dependencies using the signal function $\text{sign}(x - x')$, which is 1 when $x > x'$ and -1 when $x < x'$. Finally, we can write

$$[\phi_\pm(x), \phi_\pm(x')] = \mp\frac{i}{4\lambda^2}\text{sign}(x - x'). \tag{1.39}$$

In a similar route, one can compute $[\phi_+(x), \phi_-(x')]$ and check that

$$[\phi_+(x), \phi_-(x')] = -\frac{i}{4\lambda^2}. \tag{1.40}$$

We are ready to derive the two-point functions of $\phi_\pm$, $G_\pm(x, x')$, which we usually define as

$$G_\pm(x, x') = \langle 0| \phi_\pm(x)\phi_\pm(x') |0\rangle - \langle 0| \phi_\pm^2(0) |0\rangle . \tag{1.41}$$

Observe that we have removed the infinite constant term in order to avoid short-distance divergences. Using (1.37) and the expressions for ϕ and π, we reach

$$\begin{aligned} \phi_\pm &= \frac{1}{2\pi\lambda}\int \frac{dp}{2|p|}\left\{a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\right. \\ &\mp \left. i|p| \lim_{b\to-\infty}\int_b^x dy\left[a(p)e^{-ipy} - a^\dagger(p)e^{ipy}\right]\right\} e^{-|p|\epsilon}. \end{aligned} \tag{1.42}$$

Performing the integral in y, and using that $a(-p) = a^\dagger(p)$, we get

$$\phi_\pm = \frac{1}{2\pi\lambda}\int \frac{dp}{2|p|}\left(1 \mp \frac{|p|}{p}\right)\left(a^\dagger(p)e^{ipx} + a(p)e^{-ipx}\right) e^{-|p|\epsilon}. \tag{1.43}$$

Now, we are finally ready to obtain the correlations (1.41), and it is straightforward to compute the integrals using a computer software. The results are

$$G_\pm(x, x') = \frac{1}{4\pi\lambda^2}\ln\left(\frac{\epsilon}{\epsilon \pm i(x - x')}\right). \tag{1.44}$$

We see that the Lagrangian parameter λ appears in the two-point functions. This indicates that this parameter is a physical one, as a consequence of the periodicity of the field. The bare propagators form tells us that $\phi_\pm$ are not primary, as their propagators do not have a power law, but a logarithmic law. It is, therefore, natural to consider operators as $\partial\phi_\pm$ or $\exp\{\sim \phi_\pm\}$.

Comparing these correlations with the fermionic ones in (1.20), we see that they have different functional structures. In the spirit to match the correlations, let us consider the following operators

$$V_\pm(x) =: e^{i\phi_\pm(x)} :, \tag{1.45}$$

where $:\bullet:$ serves as the symbol of normal ordering[1] operation, which transports creation operators to the left and destruction operators to the right side of the expressions. The V operators, which are called *vertex* operators, are nice ones, because they already inherit the periodicity of the fields. Let us now compute correlations of vertex operators.

When two quantities F and G have a c-number commutator, the BCH theorem says that

$$e^F e^G = e^{F+G+\frac{1}{2}[F,G]} = e^G e^F e^{[F,G]}. \tag{1.46}$$

If F and G have the forms $F = f_1 a + f_2 a^\dagger$ and $G = g_1 a + g_2 a^\dagger$, it s easy to show that

$$\begin{aligned} :e^F::e^G: &= e^{f_2 a^\dagger} e^{f_1 a} e^{g_2 a^\dagger} e^{g_1 a} \\ &= e^{f_2 a^\dagger} e^{g_2 a^\dagger} e^{f_1 a} e^{g_1 a} e^{f_1 g_2 [a, a^\dagger]} \\ &= :e^{F+G}: e^{f_1 g_2 [a,a^\dagger]}, \end{aligned} \tag{1.47}$$

with $\exp\left\{f_1 g_2 [a, a^\dagger]\right\}$ being a c-number; notice that $f_1 g_2 = \langle 0| FG |0\rangle$. We can use the previous expression to compute $\langle 0| V_\pm V_\pm^\dagger |0\rangle$, since the fields in the exponetials are just integrals (sums) of a and $a^\dagger$. Considering this, we finally have

$$\begin{aligned} S_\pm(x,x') &\equiv \langle 0| V_\pm(x) V_\pm^\dagger(x') |0\rangle \\ &= \langle 0| : e^{i\phi_\pm(x) - i\phi_\pm(x')} : |0\rangle \, e^{(-i)i\langle 0|\phi_\pm(x)\phi_\pm(x')|0\rangle} \\ &= e^{G_\pm(x,x')}, \end{aligned} \tag{1.48}$$

because only the constant expansion term coming from the first exponential does not have null contribution in the vacuum expectation value. Notice that we have computed correlations of "uncharged" operators. We only did this in order to achieve correlation functions which are invariant under field shifts.

In this way, we have

$$S_\pm(x,x') = \left(\frac{\epsilon}{\epsilon \pm i(x-x')}\right)^{\frac{1}{4\pi\lambda^2}}. \tag{1.49}$$

Observe that these correlations are very similar to the ones in the expression (1.20).

1.1.3 The Bosonization Map

Let's start to establish this duality map by placing the two sets of equal-time correlations together in a clever way,

$$G_-(x,x') = \frac{i}{2\pi}\left(\frac{1}{(x-x')+i\epsilon}\right) \quad \text{and} \quad S_-(x,x') = \left(\frac{\epsilon}{\epsilon - i(x-x')}\right)^{\frac{1}{4\pi\lambda^2}}, \tag{1.50}$$

$$G_+(x,x') = -\frac{i}{2\pi}\left(\frac{1}{(x-x')-i\epsilon}\right) \quad \text{and} \quad S_+(x,x') = \left(\frac{\epsilon}{\epsilon + i(x-x')}\right)^{\frac{1}{4\pi\lambda^2}}. \tag{1.51}$$

[1] Normal ordering chooses a convenient order to put operators in a string, and by its own definition, for any function $f(a,a^\dagger)$ of a and $a^\dagger$, $\langle 0| : f(a,a^\dagger) : |0\rangle = 0$.

We see that if we choose $\lambda^2 = 1/(4\pi)$ and put the i's in the right place, we get

$$G_-(x,x') = \frac{1}{2\pi}\left(\frac{i}{(x-x')+i\epsilon}\right) \quad \text{and} \quad S_-(x,x') = \epsilon\left(\frac{i}{(x-x')+i\epsilon}\right), \tag{1.52}$$

$$G_+(x,x') = -\frac{1}{2\pi}\left(\frac{i}{(x-x')-i\epsilon}\right) \quad \text{and} \quad S_+(x,x') = -\epsilon\left(\frac{i}{(x-x')-i\epsilon}\right) \tag{1.53}$$

This shows us that there is a dual map between these two theories, which can be schematically written as

$$\psi_\pm(x) \leftrightarrow \sqrt{\frac{1}{2\pi\epsilon}} e^{\mp i\phi_\pm}. \tag{1.54}$$

Indeed, using this dual map,

$$\begin{aligned}\langle 0|\,\psi_-(x)\psi_-^\dagger(x')\,|0\rangle &\leftrightarrow \frac{1}{2\pi\epsilon}\langle 0|\,e^{i\phi_-(x)}e^{-i\phi_-(x')}\,|0\rangle \\ &\leftrightarrow \frac{1}{2\pi\epsilon}\epsilon\left(\frac{i}{(x-x')+i\epsilon}\right) \\ &\leftrightarrow \frac{1}{2\pi}\left(\frac{i}{(x-x')+i\epsilon}\right).\end{aligned} \tag{1.55}$$

As this identity happens to be valid, the others also do with this duality. The concept important to highlight here is that the map between the operators is valid under vacuum expectation values, and it is not a truly operatorial equality.

We can also obtain the map between composite operators, as $\bar{\psi}\psi$ and $i\bar{\psi}\gamma^3\psi$, which will be useful when studying the next two-dimensional duality. Using our bosonization map and remembering that $\phi = \phi_- + \phi_+$, these fermionic operators are

$$\bar{\psi}(x)\psi(x) = \psi_-^\dagger(x)\psi_+(x) + \psi_+^\dagger(x)\psi_-(x) \leftrightarrow \frac{1}{2\pi\epsilon}\left(e^{-i\phi_-(x)}e^{-i\phi_+(x)} + e^{i\phi_+(x)}e^{i\phi_-(x)}\right) \tag{1.56}$$

Using (1.40) and (1.46), the above expression became

$$\begin{aligned}\bar{\psi}(x)\psi(x) &\leftrightarrow \frac{1}{2\pi\epsilon}\left(e^{-i\phi(x)-\frac{1}{2}[\phi_-(x),\phi_+(x)]} + e^{i\phi(x)-\frac{1}{2}[\phi_+(x),\phi_-(x)]}\right) \\ &\leftrightarrow \frac{1}{2\pi\epsilon}\left(e^{-i\phi(x)} + e^{i\phi(x)}\right)e^{i\pi} \\ &\leftrightarrow -\frac{1}{\pi\epsilon}cos(\phi(x)).\end{aligned} \tag{1.57}$$

With a very similar computation, it is not hard to show that

$$i\bar{\psi}\gamma^3\psi \leftrightarrow -\frac{1}{\pi\epsilon}sin(\phi(x)). \tag{1.58}$$

Let us study the map between currents in bosonic and fermionic theories. Beginning with $j^\mu_{charge} = \bar{\psi}\gamma^\mu\psi$, and making a point splitting (to avoid divergences):

$$\begin{aligned}
: j^0_{charge}(x) : &= : \bar{\psi}(x)\gamma^0\psi(x) : \\
&= : \psi^\dagger_-(x)\psi_-(x) + \psi^\dagger_+(x)\psi_+(x) : \\
&\leftrightarrow : \lim_{x'\to x} \frac{1}{2\pi\epsilon}\left(e^{-i\phi_-(x)}e^{i\phi_-(x')} + e^{i\phi_+(x)}e^{-i\phi_+(x')}\right) : \\
&= : \lim_{x'\to x} \frac{1}{2\pi\epsilon}\left(e^{-i(\phi_-(x)-\phi_-(x'))}e^{S_-(x,x')} + e^{i(\phi_+(x)-\phi_+(x'))}e^{S_+(x,x')}\right) : \\
&= : \lim_{x'\to x} \frac{i}{2\pi}\left[1 - i(x-x')\frac{\partial\phi_-(x)}{\partial x} + \mathcal{O}\left((x-x')^2\right)\right]\left(\frac{1}{x-x'}\right) : \\
&- : \lim_{x'\to x} \frac{i}{2\pi}\left[1 + i(x-x')\frac{\partial\phi_+(x)}{\partial x} + \mathcal{O}\left((x-x')^2\right)\right]\left(\frac{1}{x-x'}\right) : . \quad (1.59)
\end{aligned}$$

As normal ordering operator eliminates constant terms, we finally have

$$: j^0_{charge} :\leftrightarrow \frac{1}{2\pi}\partial_x\phi. \quad (1.60)$$

Analogously,

$$\begin{aligned}
: j^1_{charge}(x) : &= : -\psi^\dagger_+(x)\psi_+(x) + \psi^\dagger_-(x)\psi_-(x) : \\
&\leftrightarrow \frac{1}{2\pi}\left(\frac{\partial\phi_-(x)}{\partial x} - \frac{\partial\phi_+(x)}{\partial x}\right).
\end{aligned}$$

Using the equation (1.37) and that $\pi(x) = \lambda^2\partial_0\phi$, we obtain $\partial_x(\phi_- - \phi_+) = -\partial_0\phi$. Finally, we have

$$: j^1_{charge} :\leftrightarrow -\frac{1}{2\pi}\partial_0\phi. \quad (1.61)$$

Briefly, we can write

$$: j^\mu_{charge} :\leftrightarrow \frac{\epsilon^{\mu\nu}}{2\pi}\partial_\nu\phi = j^\mu_{wind} \quad (1.62)$$

In a similar way, we could show that

$$: j^0_{chiral} : = : -\psi^\dagger_+(x)\psi_+(x) + \psi^\dagger_-(x)\psi_-(x) :\leftrightarrow -\frac{1}{2\pi}\partial_0\phi, \quad (1.63)$$

$$: j^1_{chiral} : = : \psi^\dagger_+(x)\psi_+(x) + \psi^\dagger_-(x)\psi_-(x) :\leftrightarrow -\frac{1}{2\pi}\partial^x\phi. \quad (1.64)$$

The above relations can be resumed as

$$: j^\mu_{chiral} :\leftrightarrow -\partial^\mu\phi = -\frac{1}{\lambda^2}j^\mu_{shift}. \quad (1.65)$$

We have finally concluded the discussion of bosonization duality map between the massless-free compact bosonic and fermionic theory. To achieve this, we have matched the two-point correlations and currents of these theories. Such duality relations are sufficient to explore all the properties of both theory using each side we want. This possibility isn't scarcely surprising owing to the fact that there isn't, in this case, a proper rotation group to differentiate these types of particles.

1.2 Bosonization of $2D$ Massive Thirring Model

Now that we have developed our bosonization language, let us push it forward. We already now that the duality between the free compact scalar and free Dirac theories happens precisely when $\lambda^2 = (4\pi)^{-1}$. We could now explore what type of dual fermionic theory we would obtain if we choose other parameters. Let us consider that the bosonic Lagrangian parameter is

$$\lambda^2 = \frac{1}{4\pi} + \frac{g}{2\pi^2}. \tag{1.66}$$

In our bosonic theory, this would only add a kinetic term to the Lagrangian. Fortunately, we have a nice way to construct such operator using the wind current. Note that

$$j^{\mu}_{wind} j_{wind\mu} = \frac{1}{4\pi^2} \underbrace{\epsilon^{\mu\nu}\epsilon_{\mu\rho}}_{=-\eta^{\nu}{}_{\rho}} \partial_\nu\phi\partial^\rho\phi = -\frac{1}{4\pi^2}(\partial\phi)^2. \tag{1.67}$$

We are able to see what this correspond in the fermionic side by cause of the duality expressed in (1.62), which ensures that

$$-\frac{1}{4\pi^2}(\partial\phi)^2 \leftrightarrow j^{\mu}_{charge} j_{charge\mu} = (\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi). \tag{1.68}$$

This operator is an interaction in the fermionic side, and it is called *Thirring* interaction. This bosonization example teach us something very intriguing: a compact scalar theory can be dual to an interacting fermionic theory.

We could also introduce a non-linear term in the bosonic side, as $cos(\phi)$, and see how it is translated in the other side. So, if we consider the Sine-Gordon Lagrangian

$$\mathcal{L}_{Sine-Gordon} = \frac{\lambda^2}{2}(\partial\phi)^2 + \frac{m}{\epsilon\pi}cos(\phi), \tag{1.69}$$

and recall the expression (1.57), we obtain the massive Thirring model,

$$\mathcal{L}_{Thirring} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - g(\bar{\psi}\gamma^\mu\psi)^2. \tag{1.70}$$

Notice that this shows us that the massive Thirring and Sine-Gordon theories are dual under bosonization map. As this map accepts large positive values of g, one can study non-perturbative regimes of both theories [8].

1.3 Solitons and Fermions

In this section, we will utilize our bosonization duality to study how Dirac fermions, which interact via Thirring term, are described in the Sine-Gordon theory. Let us start with a smart guess in the bosonic side. We want to consider a simple bosonic classical solution which is stable and that interpolates two minima in the potential. Recall that the Sine-Gordon theory is defined by the Lagrangian

$$\mathcal{L}_{Sine-Gordon} = \frac{\lambda^2}{2}(\partial\phi)^2 + \frac{m}{\epsilon\pi}cos(\phi) \equiv \mathcal{T} - \mathcal{V}. \quad (1.71)$$

From the definitions above one can see that the minima of the potential $\mathcal{V} \sim -cos\phi$ lie at $\phi = 2n\pi, n \in \mathbb{Z}$.

From the Sine-Gordon Lagrangian, one can compute the bosonic energy $E(t)$ of any solution. As the Hamiltonian is defined by

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial\partial_0\phi}\partial_0\phi - \mathcal{L}, \quad (1.72)$$

using that $cos(2a) = cos^2(a) - sin^2(a)$ and $cos^2(a) + sin^2(a) = 1$, one gets

$$E(t) = \int_{-\infty}^{\infty} dx\ \mathcal{H} = \int_{-\infty}^{\infty} dx \left(\frac{\lambda^2}{2}(\dot{\phi}^2 + \phi'^2) + \frac{2m}{\epsilon\pi}sin^2\left(\frac{\phi}{2}\right)\right) \quad (1.73)$$

Static solutions do not have variation in time, hence, time derivatives can be dropped off. In this case, the energy is

$$\begin{aligned} E &= \int_{-\infty}^{\infty} dx \left(\frac{\lambda^2}{2}\phi'^2 + \frac{2m}{\epsilon\pi}sin^2\left(\frac{\phi}{2}\right)\right) \\ &= \int_{-\infty}^{\infty} dx \left[\frac{\lambda^2}{2}\left(\phi' \pm \sqrt{\frac{4m}{\lambda^2\pi\epsilon}}sin\left(\frac{\phi}{2}\right)\right)^2 \mp \sqrt{\frac{4m\lambda^2}{\pi\epsilon}}\phi' sin\left(\frac{\phi}{2}\right)\right]. \end{aligned} \quad (1.74)$$

It is notable that the first integrand term is positive definite and that the second term is a total derivative. The second term, then, can be seen as a lower-bound to the energy, i.e.,

$$E \geq \mp 4\sqrt{\frac{m\lambda^2}{\pi\epsilon}}cos\left(\frac{\phi}{2}\right)\Bigg|_{-\infty}^{\infty}. \quad (1.75)$$

As the bosonic field is compact and we want a static solution, we can choose a configuration that interpolates the two potential minima as $\phi(x \to -\infty) = 2\pi$ to $\phi(x \to \infty) = 0$. In this case, the field has no variation in time, i.e., it is a stable solution. Also, it s non-singular everywhere, and possesses a positive energy, which is

$$E = 8\sqrt{\frac{m\lambda^2}{\pi\epsilon}}. \quad (1.76)$$

The above properties allow us to recognize this solution as a soliton: a well-behaved energy and stable configuration with no singularities. We know that the bosonization duality between Sine-Gordon and Thirring theories happens when $\lambda^2 = (4\pi)^{-1}$ with $g = 0$; with the duality, the energy is,

$$E_{Soliton} = \frac{4}{\pi}\sqrt{\frac{m}{\epsilon}}. \quad (1.77)$$

If we choose $\epsilon = 16(m\pi^2)^{-1}$, we obtain $E_{Soliton} = m$, and this low-energy solution corresponds in the fermionic side to an object with the fermion mass.

To explore which fermionic solution is related to this interpolating bosonic solution, recall that the $j_{wind} \sim j_{charge}$ under the bosonization map. The wind conserved quantity is

$$\begin{aligned} Q_{wind} &= \int_{-\infty}^{\infty} dx j^0_{wind} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\phi' \\ &= \frac{1}{2\pi}\phi\Big|_{-\infty}^{\infty} = -1. \end{aligned} \tag{1.78}$$

The bosonization map ensures, therefore, that the fermionic solution possesses $U(1)$ charge equal to -1. Thus, the free fermionic solution is an object with mass m and charge -1, which can be identified as fermions with negative charge. This is valuable lesson about fermion-boson dualities: free fermions, which are in this context created by local quantum operators, are mapped into solitons, the last being non-local solutions. This impressive feature was not expected, and we were able to easily catch it due to the bosonization duality.

2 $3D$ Quantum Field Theory

In the last chapter, we have discussed the $2D$ bosonization procedure, which can be applied to quantum field theories. Before we discuss the generalization of this $2D$ procedure, we will review in the actual chapter some basic aspects $3D$ quantum field theories, such as the charge confinement, flux attachment, the infrared behavior of coupled fermions and the Wilson-Fisher fixed point of the complex ϕ^4 theory.

2.1 Basics of Electrodynamics

2.1.1 Maxwell Theory

Let us start considering the $3D$ Maxwell theory coupled to a conserved current j,

$$S_{Maxwell}[f,j] = \int d^3x \left(-\frac{1}{4e^2}f^2 - j\cdot a\right), \quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu = -f_{\nu\mu}. \tag{2.1}$$

This theory is constructed in such way that it is invariant under $U(1)$ gauge transformations. The electromagnetic three-potential is defined as $a^\mu = (\phi, a_x, a_y)$ and $a_\mu = (\phi, -a_x, -a_y)$. The electric $\vec{E}$ and magnetic B fields are $E_i \equiv -\partial_i\phi - \partial_0 a^i = \partial_0 a_i - \partial_i a_0 = f_{0i} = f^{i0}$ and $B = \varepsilon^{ij}\partial_i a_j = \partial_1 a_2 - \partial_2 a_1 = f_{12}$. We can obtain the equation of motion for a^μ imposing that $\delta S_{Maxwell} = 0$ under a^ν variation. This process gives us the classical equations

$$\frac{1}{e^2}\partial_\mu f^{\mu\nu} = j^\nu. \tag{2.2}$$

Performing standard dimensional analysis, we see that $[a] = 1$ implies $[f^2] = 4$, $[j] = 2$ and $[e^2] = 1$. The last identity indicates that $3D$ charged particles are strongly coupled at the IR regime [6]. We can argue in favor of this statement in a simple way: the IR regime can be described as $E \ll e^2$, and we can think about it as $e^2 \to \infty$. As the electric coupling goes to infinity, it is natural to think that these interactions becomes huge in this regime.

To see precisely that $3D$ charged particles are strongly coupled in low energies, let us compute the electric potential ϕ associated with a particle with charge Q lying in the origin. The current associated with this configuration is $j^0 = Q\delta^2(r)$. Putting $\nu = 0$ in (2.2) and choosing the Coulomb gauge ($\nabla \cdot \vec{a} = 0$) leads to

$$\frac{1}{e^2}\partial_i f^{i0} = \frac{1}{e^2}\partial_i E_i = \frac{1}{e^2}\left(-\nabla^2\phi - \partial_0 \underbrace{\nabla\cdot\vec{a}}_{=0}\right) = j^0 = Q\delta^2(r). \tag{2.3}$$

We can reorganize the last result as

$$\nabla^2\phi(r) = -e^2 Q\delta^2(r) \equiv -J(r). \tag{2.4}$$

The general solution for the previous differential equation is

$$\phi(r) = -\frac{e^2 Q}{2\pi} ln(r) + c', \quad c' \text{ constant.} \tag{2.5}$$

Supposing that there exists a particle with charge $-Q$ at the position r, its potential energy $V(r)$ is

$$V(r) = -Q\phi(r) = \frac{e^2 Q}{2\pi} ln(r) + c'', \quad c'' \text{ constant,} \tag{2.6}$$

indicating that the interaction energy between $3D$ electric charged particles grows logarithmically with the distance. This is an confinement effect in low energies, and highlights the claim that charged matter is strongly coupled at large distances in $3D$ electromagnetism. There is another $3D$ electromagnetism property, which is related to the number of degrees of freedom of Maxwell theory. This subtlety will be addressed in the next subsection.

2.1.2 Dual Photon Theory

The structure of $3D$ Maxwell theory indicates that there is no intrinsic magnetic monopoles. To verify this, let us consider the Maxwell equations (2.2) again. The theory was constructed such that these equations generate the inhomogeneous Maxwell equations, and the Bianchi identity $\epsilon^{\mu\nu\rho}\partial_\mu f_{\nu\rho} = 0$ generates the homogeneous ones.

For $\nu = 0$, (2.2) is the electric Gauss's law $\nabla \cdot \vec{E} = e^2 j^0$, which indicates that j^0 generates electric flux. For $\nu = i$, (2.2) becomes the Ampère-Maxwell's law $\partial_j f^{ji} - \partial_0 E_i = e^2 j^i$, which says that electric current densities and time-variations of E can produce magnetic fields (because B is related to spatial components of f). On the other hand, the Bianchi identity gives rise to the Faraday equation $\partial_0 B - \nabla \times \vec{E} = 0$, stating that time-variations of magnetic fields produce electric fields. There is no magnetic Gauss's law contained in the Bianchi identity due to the dimensionality of this Maxwell theory. In sum, electric fields can be produced either by charge densities or time-variations of B. Magnetic fields can be produced either by currents or time-variations of $\vec{E}$.

In order to explore more about magnetic monopole configurations, suppose that we are interested in the quantum field theory associated with current-free $3D$ Maxwell theory, which we define through the path integral

$$Z_{Maxwell}[0] = \int \mathcal{D}a \, \exp\left(iS_{Maxwell}[f, j=0]\right). \tag{2.7}$$

The path integral above must be carried carefully due to the $U(1)$ gauge invariance. Notice that the free Maxwell action depends only on f only. Thus, in principle, we can change the functional measure from a to f if we, somehow, do not forget about the Bianchi identity which restricts the f's we can sum up. A convenient way to implement this is to consider a scalar Lagrange multiplier σ in the Maxwell Lagrangian, whose form is $\sim \sigma\epsilon^{\mu\nu\rho}\partial_\mu f_{\nu\rho}$. The Maxwell generating functional with a f measure is

$$Z_{Maxwell} = \int \mathcal{D}f\mathcal{D}\sigma \exp\Big\{ i\int d^3x \left(-\frac{1}{4e^2} f^2 + \frac{1}{4\pi}\sigma\epsilon^{\mu\nu\rho}\partial_\mu f_{\nu\rho} \right) \Big\}. \tag{2.8}$$

Intending to get a nice quantum theory, f must satisfy the Dirac quantization condition,

$$\frac{1}{2}\int dS f_{12} \in 2\pi\mathbb{Z}. \tag{2.9}$$

In this case, it is enough that σ be compact in the interval $[0, 2\pi)$. Now, varying the above action with Lagrange multiplier in respect to $f^{\mu\nu}$ leads to the equation

$$\begin{aligned} -\frac{1}{2e^2}f^{\mu\nu} + \frac{1}{4\pi}\epsilon^{\rho\mu\nu}\partial_\rho\sigma = 0 \quad &\Longleftrightarrow \quad f^{\mu\nu} = \frac{e^2}{2\pi}\epsilon^{\rho\mu\nu}\partial_\rho\sigma \\ &\Longleftrightarrow \quad \partial^\mu\sigma = \frac{\pi}{e^2}\epsilon^{\mu\alpha\beta}f_{\alpha\beta}. \end{aligned} \tag{2.10}$$

As $Z_{Maxwell}$ is quadratic in f, the path integral over f is straightforward, resulting in

$$Z_{Maxwell} = \int \mathcal{D}[\sigma]\exp\left(i\int d^3x \frac{e^2}{8\pi^2}(\partial\sigma)^2\right). \tag{2.11}$$

The last identity says that $3D$ Maxwell photons can be described by a compact scalar degree of freedom: the photon in $3D$ has a single degree of freedom, related to the only polarization state that it can possesses. The compact field σ is called *dual photon*, for clear reasons. Notice that this theory has a global continuous $U(1)_?$ symmetry (the nature of the "?" will be discussed below). In the dual description, this symmetry has the form $\sigma \to \sigma + \alpha,\ \alpha$ constant. Utilizing Noether's theorem, the associated conserved current is

$$j^\mu_? = \frac{e^2}{4\pi^2}\partial^\mu\sigma. \tag{2.12}$$

Due to the equation (2.10), we can rewrite this current in terms of f field,

$$j^\mu_? = \frac{1}{4\pi}\epsilon^{\mu\alpha\beta}f_{\alpha\beta}. \tag{2.13}$$

In the photon description, it is easy to see that $j_?$ is conserved by the Bianchi identity, independently of equations of motions for f. Remember that the relation (2.10) was achieved by Bianchi identity; hence, it is not surprising that the $j_?$ conservation arises from Bianchi identity. We can also say that $j_?$ is off-shell-conserved (from the point of view of Maxwell theory), and this is the reason why we will call it *topological* current j^μ_{top}, corresponding to the symmetry $U(1)_{top}$.

2.1.3 Magnetic Monopole Operators

The j^μ_{top} conservation achieved in the previous subsection implies that

$$Q_{top} = \int d^2x\ j^0_{top} = \int \frac{1}{2\pi}d^2x\ f_{12} = \frac{1}{2\pi}\int d^2x\ B \tag{2.14}$$

is conserved in time. According to our previous discussions, this is precisely the flux of B through spatial regions. Therefore, purely free $3D$ Maxwell theory encodes time-conservation of magnetic flux. Another way to see that is to show that any local operators in Maxwell theory (which essentially is powers of f) commutes with the charge operator Q_{top}.

We will now explore some captivating and non-trivial properties of $U(1)_{top}$ symmetry in this context. Suppose that exists an operator $\mathcal{M}_{\tilde{q}}^{\dagger}(x)$ that creates $\tilde{q}$ magnetic monopoles at the position x in spacetime. In this situation, there are $\tilde{q}$ unit-fluxes crossing the sphere S^2 around x^μ; mathematically, $\int_{S^2} d^2x\ f_{12} = 2\pi\tilde{q}$. In order to visualize how this process occurs, we will consider a $\mathbb{R}^3$ Minkowski spacetime which is free from magnetic monopoles at time $t_- \equiv t - \epsilon$. Then, a magnetic monopole $\tilde{q}$ configuration is created at $x^\mu = (t, x, y)$ by the insertion of $\mathcal{M}_q^{\dagger}(x^\mu)$. The creation of the magnetic monopole essentially removes a point from space, allowing the identification $\mathbb{R}^3 \mapsto \mathbb{R}_{time} \times S^2$.

Also, this insertion results in a magnetic flux through spacetime regions with $t_+ = t + \epsilon$. Defining $S_\pm$ as a constant-time sphere at time $t_\pm$ and using the equation $\int_S d^2x \frac{1}{2\pi} f_{12} = \tilde{q}$, we can compute the magnetic flux through the surface $S \equiv S_+ \cup S_-$,

$$\tilde{q} = \int_S d^2x\ \frac{1}{2\pi} f_{12} = \overbrace{\int_{S_+} d^2x\ \frac{1}{2\pi} f_{12}}^{Q_{top}(t_+)} - \overbrace{\int_{S_-} d^2x\ \frac{1}{2\pi} f_{12}}^{Q_{top}(t_-)}. \tag{2.15}$$

That is,

$$Q_{top}(t_+) - \cancelto{0}{Q_{top}(t_-)} = \tilde{q}. \tag{2.16}$$

This result tells us that the monopole operator $\mathcal{M}_q^{\dagger}$ removes a point from space and gives specific non-trivial boundary conditions (related to the magnetic charge $\tilde{q}$) for the gauge fields on the surface around it, in order to attribute the correct magnetic flux value.

Let us now discuss a little more about the hidden global symmetry of $3D$ Maxwell theory $U(1)_{top} : \sigma \mapsto \sigma + \alpha$ and its charge Q_{top}. As mentioned before, the σ field must be compact to ensure Dirac quantization. Also, we know that the object $\mathcal{M}_{\tilde{q}}^{\dagger}(x)$ creates a magnetic monopoles $\tilde{q}$ at x and that Q_{top} measures the magnetic charge. This means that $\mathcal{M}_1^{\dagger}(x)$ possesses unit charge under $U(1)_{top}$. The previous statement can be written as

$$\mathcal{M}_{\tilde{q}}(x) \sim \int \mathcal{D}\sigma e^{i\tilde{q}\sigma(x)}, \quad \tilde{q} \in \mathbb{Z}. \tag{2.17}$$

Clearly, a magnetic monopole operator of this form is certainly invariant under the field transformation $\sigma \mapsto \sigma + 2\pi$.

2.2 Chern-Simons Theory

2.2.1 The Action

As usual, the Maxwell term $\sim f^2$, which is constructed from f, naturally is gauge invariant. In even dimensions, this is the simplest quadratic term which depends only on f. In odd dimensions, however, there are other kinds of terms which can be added to the

Lagrangians in order to get a gauge-invariant theory. In our three-dimensional case, this term is [9]

$$S_{Chern-Simons} \equiv S_{CS} = \frac{\kappa}{4\pi}\int d^3x \; \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho. \tag{2.18}$$

This is the so-called *Chern-Simons* theory.

At first, the CS action appears to be not gauge-invariant, because it depends explicitly on a_μ. However, implementing the gauge transformation $a_\mu \mapsto a_\mu + \partial_\mu \lambda$ over S_{CS} leads to

$$\begin{aligned} S'_{CS} &\sim \int d^3x \; \epsilon^{\mu\nu\rho} \left(a_\mu + \partial_\mu\lambda\right) \partial_\nu \left(a_\rho + \partial_\rho \lambda\right) \\ &\sim S_{CS} + \int d^3x \; \left(\epsilon^{\mu\nu\rho} a_\mu \partial_\nu \partial_\rho \lambda + \epsilon^{\mu\nu\rho}\partial_\mu\lambda\partial_\nu a_\rho + \epsilon^{\mu\nu\rho}\partial_\mu\lambda\partial_\nu\partial_\rho\lambda\right). \end{aligned} \tag{2.19}$$

The first and third integrands in r.h.s. are zero, due the antisymmetry property of ϵ and commutativity of partial derivatives in trivial topological spacetimes. An integral by parts in the middle integrand in r.h.s. will give us a surface term and a term which is the product of an symmetric and a antisymmetric object; this will permit us to rewrite the transformed action as

$$S'_{CS} \sim S_{CS} + \int d^3x \; \partial_\mu \left(\epsilon^{\mu\nu\rho}\lambda\partial_\nu a_\rho\right). \tag{2.20}$$

The integral above is performed in the entire spacetime, and the surface term can be discarded only if the field a_μ have trivial conditions in the boundaries of spacetime. If this is the case, then S_{CS} action is, indeed, invariant under gauge transformations. For this reason, we will call this theory $U(1)_\kappa$ gauge theory, where κ represents the level of the Chern-Simons.

To finish this subsection, notice that the CS theory also has the topological current j^μ_{top} due to the $U(1)_{top}$ symmetry. This current can be minimally coupled to an external field A_μ as

$$A_\mu j^\mu_{top} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho. \tag{2.21}$$

As j^μ_{top} is conserved by Bianchi identity, A^μ can be a gauge field. This kind of coupling is called *BF* term [10]. At the quantum level, the normalization of the BF coupling guarantees that it is a gauge invariant term.

2.2.2 Flux-Attachment

The equations of motion associated with CS theory are $f_{\mu\nu} = 0$. This means that the free CS theory is dynamically trivial as a classical field theory. If we consider instead the physics of a Chern-Simons theory coupled to a conserved current j^μ, whose action is

$$S_{CS}[j] = \int d^3x \; \left(\frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho - a_\mu j^\mu\right), \tag{2.22}$$

the equations of motion for a_μ are

$$\frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}f_{\nu\rho}=j^\mu \iff f_{\mu\nu}=\frac{2\pi}{\kappa}\epsilon_{\mu\nu\rho}j^\rho. \tag{2.23}$$

The r.h.s. of the above equation tells us that the CS coupling attach fields fluxes to matter currents. This mechanism is called *flux-attachment.* Inserting $(\mu,\nu)=(1,2)$ and $(\mu,\nu)=(0,i)$ in (2.23) give us

$$f_{12} = B=\frac{2\pi}{\kappa}\rho, \tag{2.24}$$

$$f_{0i} = E_i=\frac{2\pi}{\kappa}\epsilon_{ij}j^j. \tag{2.25}$$

The equations above tell us that there is a magnetic field B attached to every electric-charged particle (the opposite is also true) and that these magnetic fields follows each charged particle. In other words, charged particles coupled to a Chern-Simons gauge field acquire a magnetic field locally tied to them. The attached B-field arises from the continuity equation for j.

A remarkable property of the CS theory is that its term is metric-independent, different from Maxwell kinetic term $f^2\equiv f^{\mu\alpha}\eta_{\alpha\beta}f^{\beta\nu}$. A natural conclusion is that Chern-Simons properties are independent on the local geometry of the manifold in that the physical system is inserted in. Instead, its property depend on the topology of the manifold. This remark serves as a motivation to categorize this term as a *topological* one.

Let us discuss the effects of exchanging non-relativistic charged particles in the presence of a Chern-Simons field. CS theory tells us that particles acquire magnetic flux attached to them when coupled to a Chern-Simons field. Thus, when charged particles are interchanged (via translation after rotation, as shown in the figure 1), the total wavefunction wavefunction must suffer a phase shift due Aharonov-Bohm effect.

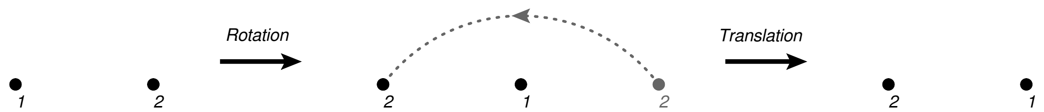


Figura 1 – The process of interchanging two particles is essentially a translation after a rotation.

This phase shifts is the one present in the appendix A, resulting in

$$\Delta\varphi=\frac{q\Phi_R}{2}. \tag{2.26}$$

The factor $\frac{1}{2}$ is due to the fact that one particle does not go completely around the magnetic-flux (particle). Instead, it encompasses half of flux, justifying the factor.

As an example, suppose a system with two charged particles with charge $q=1$ that are coupled via a CS field. By using the equation (2.24), the magnetic field that one particles experiments is

$$B(x)=\frac{2\pi}{\kappa}\delta^2(x-y), \tag{2.27}$$

with y being the position of the other particle.

Then, by using their statistics, when the particles are interchanged, the system wavefunction acquires a phase ± 1, depending on the nature of the particles (bosons or fermions, respectively). On the other hand, as they are coupled via a CS field, the wavefunction also acquires a phase $\Delta\varphi$ due to the Aharonov-Bohm effect. The total phase shift $e^{i\Delta\phi}$ is

$$e^{i\Delta\phi} = \pm e^{i\Delta\varphi} = \pm e^{i\frac{1}{2}\Phi_R}. \tag{2.28}$$

The phase shift in this case is determined, then, by

$$\Phi_R = \int_R dS\ B = \frac{2\pi}{\kappa} \implies e^{i\Delta\phi} = \pm e^{i\frac{\pi}{\kappa}}. \tag{2.29}$$

Choosing $\kappa = \pm 1$, we see that the phase acquired in this process is $e^{i\Delta\phi} = \mp 1$. Therefore, we can read the effect of our $\kappa = \pm 1$ CS field on bosons (fermions) as changing their statistics to a fermionic (bosonic) one. Therefore, flux attachment mechanism provided by CS physics is an essential key to bosonize fermionic theories.

It is remarkable that different values of κ leads to different statistics transmutations. For a generic κ, it is evident that the effective statistics of particles coupled to CS fields do not need to be an integer one. Indeed, they can have fractional statistics, where a generic phase is obtained when two particles are exchanged. To these particles with fractional statistics is given the name *anyons* [11].

2.2.3 Quantization of Compact Chern-Simons Theory

We have seen that the Chern-Simons theory is invariant under gauge transformations if the total derivatives in the action could be thrown away, i.e., if the boundary conditions of the fields are trivial; otherwise, the Chern-Simons theory is not gauge-invariant (for generic values for κ). In order to investigate the possible gauge invariant values of κ, let us consider the Chern-Simons theory in the compact Euclidean spacetime $\mathcal{M} \equiv S^1 \times S^2$. Considering that the time coordinate x^0 possesses periodicity β, a large gauge transformation, which winds around the time circle should be valid. Let us then consider the transformation

$$a_\mu \mapsto a_\mu + \partial_\mu \lambda, \quad \lambda = \frac{2\pi\hbar x^0}{e\beta} n, \quad n \in \mathbb{Z}. \tag{2.30}$$

Clearly, the time component of gauge field is simply shifted,

$$a_0' = a_0 + \partial_0 \frac{2\pi\hbar x^0}{e\beta} n = a_0 + \frac{2\pi\hbar}{e\beta} n, \tag{2.31}$$

which essentially means that the time component is also periodic, as both configurations must be identified. The gauge group itself is essentially compact in a compact manifold.

We are interested in particular configurations which contain a magnetic monopole. The previous statement means that

$$\frac{1}{2\pi}\int\limits_{S^2} da = \frac{\hbar}{e}. \tag{2.32}$$

A generic variation of the action is

$$\begin{aligned}\delta S_{CS} &= \frac{\kappa}{4\pi}\int d^3x\epsilon^{\mu\nu\rho}\left(a_\mu+\delta a_\mu\right)\partial_\nu\left(a_\rho+\delta a_\rho\right) - S_{CS}\\ &= \frac{\kappa}{4\pi}\int d^3x\epsilon^{\mu\nu\rho}\delta a_\mu f_{\nu\rho}.\end{aligned} \tag{2.33}$$

Now, remembering that the a_0 transforms as (2.31) and imposing the flux quantization condition, it is straightforward to obtain the CS action transformation, which is

$$S'_{CS} = S_{CS} + \frac{2\pi\kappa\hbar^2}{e^2}. \tag{2.34}$$

As a quantum theory, the vacuum generating function $Z = e^{iS/\hbar}$ transforms like

$$Z'_{CS} = e^{2\pi i\frac{\kappa\hbar}{e^2}} Z_{CS}. \tag{2.35}$$

In order to preserve the gauge invariance of the quantum theory, it is necessary to impose

$$\frac{\kappa\hbar}{e^2} \in \mathbb{Z}. \tag{2.36}$$

If the matter fields have charge ± 1 ($e = \pm 1$), we see that, in natural units ($\hbar = 1$), $\kappa \in \mathbb{Z}$ is a necessary condition to obtain a Chern-Simons quantum theory which is invariant under large gauge transformation. As large gauge transformations appears in compact gauge groups, and these are valid gauge transformations, always that we have a compact $U(1)_\kappa$ gauge theory, we will demand $\kappa \in \mathbb{Z}$.

2.2.4 Maxwell-Chern-Simons Theory

Let us now investigate the behavior of a Chern-Simons theory in the presence of a Maxwell kinetic term. The action of interest is

$$S_{MCS} = \int d^3x\left(-\frac{1}{4e^2}f^2 + \frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho\right). \tag{2.37}$$

The equations of motion are obtained by varying this action with respect to a_μ fields, which turn out to be

$$\partial_\mu f^{\mu\nu} + \frac{\kappa e^2}{4\pi}\epsilon^{\nu\rho\beta}f_{\rho\beta} = 0. \tag{2.38}$$

Writing the f field in terms of the dual field $\tilde{f}$,

$$\tilde{f}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho}f^{\nu\rho} \iff f^{\mu\nu} = \epsilon^{\mu\nu\alpha}\tilde{f}_\alpha, \tag{2.39}$$

the equations of motion become

$$\epsilon^{\mu\nu\alpha}\partial_\mu \tilde{f}_\alpha + \frac{\kappa e^2}{2\pi}\tilde{f}^\nu = 0 \iff \tilde{f}^\nu = \frac{2\pi}{\kappa e^2}\epsilon^{\nu\mu\alpha}\partial_\mu \tilde{f}_\alpha. \tag{2.40}$$

Putting the $\tilde{f}$ in the l.h.s. above, in order to obtain a equation purely in terms of $\tilde{f}$, results

$$\left[\partial_\mu\partial^\mu + \left(\frac{\kappa e^2}{2\pi}\right)^2\right]\tilde{f}^\nu = 0. \tag{2.41}$$

This is the Klein-Gordon equation for $\tilde{f}^\nu$, whose mass is identified as $m = \dfrac{\kappa e^2}{2\pi}$.

The previous analysis indicates that when we consider both Chern-Simons and Maxwell terms, the photons are massive. To show this precisely, we need to look to the Green's function related to the MCS action. To do so, let us consider the MCS action with a appropriated gauge fixing term,

$$S_{MCS+\text{gauge fixing}} = \int d^3x\left(-\frac{1}{4e^2}f^2 + \frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho - \frac{1}{2\xi e^2}(\partial_\mu a^\mu)^2\right) \equiv \int d^3x\mathcal{L}_{gf}. \tag{2.42}$$

The Lagrangian $\mathcal{L}_{gf}$ can be rewritten as

$$\mathcal{L}_{gf} = \frac{1}{2}a_\mu\left\{\frac{1}{e^2}\left[\eta^{\rho\mu}\partial^\alpha\partial_\alpha - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\rho\right] + \frac{\kappa}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu\right\}a_\rho. \tag{2.43}$$

This Lagrangian leads to the equations of motions below:

$$\left\{\frac{1}{e^2}\left[\eta^{\rho\mu}\partial^\alpha\partial_\alpha - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\rho\right] + \frac{\kappa}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu\right\}a_\rho \equiv \mathcal{O}^{\rho\mu}a_\rho = 0. \tag{2.44}$$

The Green's function $\Delta_{\mu\nu}(x,x')$ associated with this equation satisfies

$$\mathcal{O}^{\rho\mu}\Delta_{\rho\nu}(x,x') = -\eta^\mu_{\ \nu}\delta^4(x-x'). \tag{2.45}$$

Expressing $\Delta_{\rho\nu}(x,x')$ and $\delta^4(x-x')$ in the space of momenta p leads to a tensor equation, which is

$$\left\{\frac{1}{e^2}\left[\eta^{\rho\mu}p^2 - \left(1-\frac{1}{\xi}\right)p^\mu p^\rho\right] + \frac{i\kappa}{2\pi}\epsilon^{\mu\alpha\rho}p_\alpha\right\}\Delta_{\rho\nu}(p) = \eta^\mu_{\ \nu}. \tag{2.46}$$

From this point, it is straightforward to obtain the solution of this tensor equation. Assuming that this equation supports the following structure

$$\Delta_{\rho\nu}(p) = ap^2\eta_{\rho\nu} + bp_\rho p_\nu + c\epsilon_{\rho\nu\sigma}p^\sigma, \tag{2.47}$$

one obtain a solution for (a,b,c), which gives

$$\Delta_{\rho\nu}(p) = e^2\left[\frac{p^2\eta_{\rho\nu} - p_\rho p_\nu + im\epsilon_{\rho\nu\sigma}p^\sigma}{p^2(p^2-m^2)} + \xi\frac{p_\rho p_\nu}{(p^2)^2}\right], \quad m \equiv \frac{\kappa e^2}{2\pi}. \tag{2.48}$$

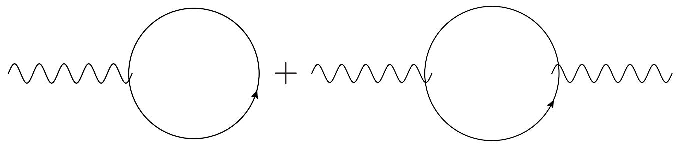
The poles of the Green's function unveil the physical spectrum of propagating modes. Above, the poles are encountered at $p^2 = m^2$. Therefore, we see that the MCS theory has massive degrees of freedom. A remarkable fact about this theory is that it is gauge invariant. This allows us to give mass to gauge fields in a gauge-invariant way without Higgs mechanism.

2.2.5 Infrared Collective Behavior of Coupled Fermions

It is a well-known fact that the low energy description of massive fermions in the presence of a background gauge field coincides to be a Chern-Simons theory [8]. For this reason, we will demonstrate how this CS effective field theory can be achieved from fermions. The quantum theory of charge $+1$ Dirac fermions minimally coupled to a background $U(1)$ gauge field is defined by the vacuum generating function

$$Z[A;m] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{i\int d^3x \bar{\psi}(i\not{\mathcal{D}}_A - m)\psi}, \quad \mathcal{D}_A \equiv \partial - iA. \tag{2.49}$$

If we are interested in the Infrared (IR) description of this system, turns out that the effective action is essentialy [12]

$S_{eff} = constant\ +$  . (2.50)

Precisely, it is

$$S_{eff} = -i\ln(Z[A;m]) = -i\ln\det(i\not{\mathcal{D}}_A - m) = -i\,\mathrm{tr}\ln(i\not{\partial} + \not{A} - m). \tag{2.51}$$

In the IR regime, where $E \ll m$, the effective action can be expanded in powers of A and p^2. Performing a Taylor expansion of logarithm and using that the trace operation is linear leads to

$$S_{eff} = -i\,\mathrm{tr}\ln(i\not{\partial} - m) - i\,\mathrm{tr}\left(\frac{1}{i\not{\partial} - m}\not{A}\right) - \frac{i}{2}\,\mathrm{tr}\left(\frac{1}{i\not{\partial} - m}\not{A}\frac{1}{i\not{\partial} - m}\not{A}\right) + \cdots. \tag{2.52}$$

The first term of this expansion is a constant and the second one is not gauge invariant and must be vanishing. In fact, the second term is a tadpole, is identically null, as it contains an odd number of photon external legs. The Feynman rules for this theory permit us to obtain the one-loop integral $\Gamma^{\mu\nu}(p;m)$ for the third term, which is

$$\begin{aligned} \Gamma^{\mu\nu}(p;m) &= \mathrm{tr}\int \frac{d^3k}{(2\pi)^3}\gamma^\mu \frac{\not{k} + m}{k^2 - m^2 + i\epsilon}\gamma^\nu \frac{\not{k} - \not{p} + m}{(k-p)^2 - m^2 + i\epsilon} \\ &= \int \frac{d^3k}{(2\pi)^3}\left(\frac{2m^2\eta^{\mu\nu} + 2im\epsilon^{\mu\rho\nu}p_\rho + 4k^\mu k^\nu - 2k^\mu p^\nu - 2k^\nu p^\mu - 2\eta^{\mu\nu}k^\alpha(k-p)_\alpha}{(k^2 - m^2 + i\epsilon)((k-p)^2 - m^2 + i\epsilon)}\right). \end{aligned} \tag{2.53}$$

The uncommon term ($\sim \epsilon^{\mu\rho\nu}$) in the previous expansion leads to a CS term in the effective action. Explicitly

$$S_{eff}[A;m] = const. - \frac{i}{2}\int \frac{d^3p d^3k}{(2\pi)^6}\frac{2mi\epsilon^{\mu\rho\nu}A_\mu(-p)p_\rho A_\nu(p)}{(k^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} + \cdots. \tag{2.54}$$

In coordinate space, this action takes the form

$$S_{eff}[A;m] = const. - \frac{i}{2}\int d^3x\ \epsilon^{\mu\rho\nu}A_\mu\partial_\rho A_\nu \int \frac{d^3k}{(2\pi)^3}\frac{2m}{(k^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} + \cdots. \tag{2.55}$$

Using that

$$\int \frac{d^3k}{(2\pi)^3} \frac{2m}{(k^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} = \frac{i}{4\pi} \mathrm{sign}(m) \tag{2.56}$$

gives us the final result

$$S_{eff}[A; m] = const. + \frac{1}{4\pi} \frac{\mathrm{sign}(m)}{2} \int d^3x \; \epsilon^{\mu\rho\nu} A_\mu \partial_\rho A_\nu + \cdots . \tag{2.57}$$

Hence, taking the IR limit and keeping only non-trivial terms, we have

$$S_{eff}[A; m] = \frac{1}{4\pi} \frac{\mathrm{sign}(m)}{2} \int d^3x \; \epsilon^{\mu\rho\nu} A_\mu \partial_\rho A_\nu . \tag{2.58}$$

The final result is that the low-energy effective theory of interacting fermions is a $U(1)_{1/2}$ Chern-Simons theory.

Notice that, according to the subsection 2.2.3, the effective action for interacting fermions results in a wrong CS level. Therefore, it is not a gauge invariant theory. The previous result is very weird: low-energy quantum descriptions of interacting fermions does not appear to be consistent. Obviously, this is not desirable, and the solution to this problem problem rises from the knowledge of its origin. The problem have emerged from the UV ill-defined integral loops which are unrelated to the CS term in the above description. These ill-defined quantities must be regularized in order to get a meaningful physical theory.

This can be done using the Pauli-Villars regularization, which inserts, in this case, a fermionic field with a huge mass M_{UV}; this procedure basically replaces the logarithm argument in (2.51) by [8]

$$\frac{\det(i\not{D}_A - m)}{\det(i\not{D}_A - M_{UV})}, \tag{2.59}$$

The resultant effective action in the large M_{UV} limit is

$$S_{eff} = \frac{1}{4\pi} \frac{(\mathrm{sign}(m) - 1)}{2} \int d^3x \; \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho . \tag{2.60}$$

This effective theory has good CS levels: $\kappa = 0$ for $m > 0$ and $\kappa = -1$ for $m < 0$, which are for sure integer values. In both case, the quantum theory is gauge invariant, as desirable, and the low energy quantum descriptions of interacting fermions are physically right. In short, the regularization procedure above inserted in the theory an $U(1)_{-1/2}$ Chern-Simons term.

2.3 Wilson-Fisher Fixed Point of the Complex Scalar ϕ^4 Theory

In this section, we want to uncover the existence of a possible nontrivial fixed point for the complex ϕ^4 theory in $D = 3$. The existence of this fixed point is a key ingredient for the establishment of the dualities to be discussed in the next chapters.

With this objective in mind, consider the $3D$ complex scalar field theory with quartic interactions defined by

$$S_{\phi^4} \equiv \int d^3x \left(|\partial\phi|^2 - M^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4 \right). \tag{2.61}$$

By dimensional analysis, we have $[\phi] = \frac{1}{2}$, $[M^2] = 2$ and $[\lambda] = 1$. The fact that $[\lambda] > 0$ indicates that the ϕ^4 operator is relevant, and consequently the theory is strong coupled in the infrared regime. In this way, we can not make use of perturbative methods to investigate the existence of low-energies fixed point for this theory in $D = 3$.

To handle this problem, we will perform the dimensional regularization of this theory in $D = 4 - \epsilon$ and then we will extrapolate the expansion by making $\epsilon = 1$. This is interesting because the theory is perturbative in $D = 4$, just because $[\lambda] = 0$. Although this procedure is somewhat tricky, it provides strong evidences to the existence of a fixed point.

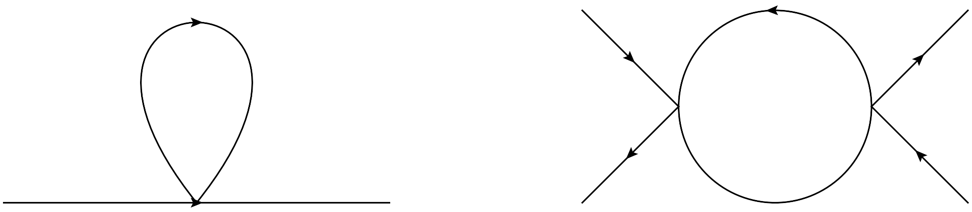
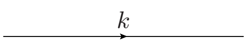
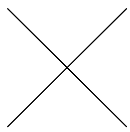


Figura 2 – 1-loop corrections for the 2-point and 4-point functions. In $D = 4$, the superficial degree of divergence of these diagrams is 2 and 0, respectively.

Using the Feynman rules of this theory,



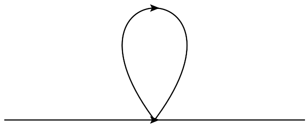
$$= \frac{i}{k^2 - M^2} \quad \text{and}$$



$$= -i\lambda,$$

we can compute the relevant diagrams depicted in 2.

Let us start by performing dimensional regularization on these diagrams. This consists in setting $D = 4 - \epsilon$ and rewriting the coupling constant λ in terms of a dimensionless coupling $\tilde{\lambda}$ as $\lambda = \tilde{\lambda}\mu^{4-D}$. Implementing this and the Feynman rules, the 1-loop correction to the 2-point function is

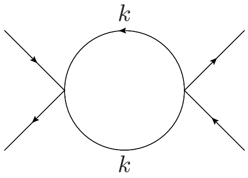


$$\equiv i\Sigma_2(p^2) = 1(-i\tilde{\lambda}\mu^{4-D}) \int \frac{d^Dk}{(2\pi)^D} \frac{i}{k^2 - M^2}.$$

The integral results in

$$\begin{aligned} i\Sigma_2(p^2) &= -\frac{i\tilde{\lambda}\mu^{4-D}}{(4\pi)^{D/2}} \left(\frac{1}{M^2}\right)^{1-\frac{D}{2}} \Gamma\left(1 - \frac{D}{2}\right) \\ &= \frac{i\tilde{\lambda}M^2}{8\pi^2}\frac{1}{\epsilon} + \mathcal{O}(\epsilon) + finite. \end{aligned} \tag{2.62}$$

Analagously for the 4-point function,



$$+\,\text{perm.} \equiv i\Gamma_4(0) = 3(-i\tilde{\lambda}\mu^{4-D})^2 \int \frac{d^Dk}{(2\pi)^D}\frac{i}{k^2-M^2}\frac{i}{k^2-M^2}.$$

The factor of 3 in the numerator comes from the possible scattering channels, which for $p = 0$ give the same contribution. The integration in k results in

$$\begin{aligned} i\Gamma_4(p^2) &= 3\frac{i\tilde{\lambda}^2\mu^{8-2D}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(\frac{4-D}{2}\right)\int_0^1 dx\ (M^2)^{\frac{D}{2}-2} \\ &= 3\frac{i\tilde{\lambda}^2}{8\pi^2}\frac{1}{\epsilon} + \mathcal{O}(\epsilon) + finite. \end{aligned} \tag{2.63}$$

Summing up, the divergent part of the relevant $1PI$ diagrams for small ϵ are

$$\text{div}\left[\Sigma_2(p^2)\right] = \frac{\tilde{\lambda}M^2}{8\pi^2}\frac{1}{\epsilon}, \tag{2.64}$$

$$\text{div}\left[\Gamma_4(p^2)\right] = \frac{3\tilde{\lambda}^2}{8\pi^2}\frac{1}{\epsilon}. \tag{2.65}$$

Let us now renormalize the ϕ^4 theory by introducing counterterms in the Lagrangian as

$$\phi = \sqrt{Z_\phi}\phi_R, \tag{2.66}$$

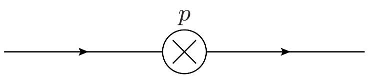
$$M^2 = Z_M M_R^2, \tag{2.67}$$

$$\lambda = Z_\lambda \lambda_R. \tag{2.68}$$

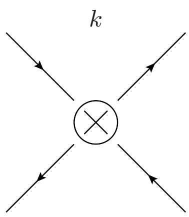
Above, all the Z quantities have the form $Z = 1 + \delta$. In this fashion, the renormalized Lagrangian takes the form

$$\begin{aligned} \mathcal{L}_{(R,\phi^4)} &= Z_\phi|\partial\phi_R|^2 - Z_M Z_\phi M_R^2|\phi_R|^2 - \frac{\lambda_R}{4}Z_\lambda Z_\phi^2|\phi_R|^4 \\ &= (1+\delta_\phi)|\partial\phi_R|^2 - (1+\delta_M)(1+\delta_\phi)M_R^2|\phi_R|^2 - \frac{\lambda_R}{4}(1+\delta_\lambda)(1+\delta_\phi)^2|\phi_R|^4 \\ &= \mathcal{L}_{\phi^4}(\phi_R,\partial\phi_R) + \delta_\phi|\partial\phi_R|^2 - (\delta_\phi+\delta_M)M_R^2|\phi_R|^2 - (\delta_\lambda+2\delta_\phi)\frac{\lambda_R}{4}|\phi_R|^4 \end{aligned} \tag{2.69}$$

It is direct to extract the Feynman rules for this new interactions from the renormalized Lagrangian:



$$= i\delta_\phi\left(p^2 - M_R^2\right) - i\delta_M M_R^2, \tag{2.70}$$



$$= -i\left(\delta_\lambda + 2\delta_\phi\right)\lambda_R. \tag{2.71}$$

Up to this order, the divergent part of the quantum corrections do not involve the external momentum p. This means that we do not need the δ_ϕ part to this order of perturbation, and we can already set $\delta_\phi = 0$.

Now, in order to cancel the divergent part of the relevant $1PI$ renormalized diagrams, we have to require

$$\frac{\tilde{\lambda}_R M_R^2}{8\pi^2}\frac{1}{\epsilon} - \delta_M M_R^2 = 0, \tag{2.72}$$

$$\frac{3\tilde{\lambda}^2}{8\pi^2}\frac{1}{\epsilon} - \delta_\lambda \lambda_R = 0. \tag{2.73}$$

Notice that we have already extracted the corrections for the bosonic propagator coming from Z_ϕ and Z_M. The solutions of the above equations are

$$\delta_M = \frac{\tilde{\lambda}_R}{8\pi^2}\frac{1}{\epsilon}, \tag{2.74}$$

$$\delta_\lambda = \frac{3\tilde{\lambda}_R}{8\pi^2}\frac{1}{\epsilon}. \tag{2.75}$$

The beta functions can be obtained using the fact that the bare parameters $M^2 = Z_M M_R^2$ and $\lambda = \mu^\epsilon Z_\lambda \tilde{\lambda}_R$ are μ-independent. This means that

$$0 = \mu\frac{d}{d\mu}\lambda = \mu\frac{d}{d\mu}\left[\mu^\epsilon Z_\lambda \tilde{\lambda}_R\right] \tag{2.76}$$

and

$$0 = \mu\frac{d}{d\mu}M^2 = \mu\frac{d}{d\mu}\left[Z_M M_R^2\right]. \tag{2.77}$$

The above equations can be rewritten as

$$0 = \mu^\epsilon \tilde{\lambda}_R Z_\lambda \left(\epsilon + \frac{\mu}{\tilde{\lambda}_R}\frac{d}{d\mu}\tilde{\lambda}_R + \frac{\mu}{Z_\lambda}\frac{d}{d\mu}\delta_\lambda\right), \tag{2.78}$$

$$0 = M_R^2 Z_M \left(\frac{1}{M_R^2}\mu\frac{d}{d\mu}M_R^2 + \frac{1}{Z_M}\mu\frac{d}{d\mu}\delta_M\right), \tag{2.79}$$

which imply the following solutions

$$\beta_{\tilde{\lambda}_R} \equiv \mu\frac{d}{d\mu}\tilde{\lambda}_R = \tilde{\lambda}_R\left(-\epsilon + \frac{3\tilde{\lambda}_R}{8\pi^2}\right) + \mathcal{O}\left(\tilde{\lambda}_R^3\right), \tag{2.80}$$

$$\gamma_M \equiv \frac{1}{M_R^2}\mu\frac{d}{d\mu}M_R^2 = \frac{\tilde{\lambda}_R}{8\pi^2}M_R^2 + \mathcal{O}\left(\tilde{\lambda}_R^3\right). \tag{2.81}$$

The fixed points of the theory are the pairs of values $(M_R^*, \tilde{\lambda}_R^*)$ for which the beta functions vanish. They can be immediately read from the above expressions,

$$M_R^* = 0 \tag{2.82}$$

$$\tilde{\lambda}_R^* = \frac{8\pi^2\epsilon}{3}. \tag{2.83}$$

Therefore, we have unveiled a nontrivial fixed point by renormalizating the complex ϕ^4 theory via dimensional regularization. We have expanded the spacetime dimension as $D = 4 - \epsilon$. In this way, as we take $\epsilon \to 1$, we reach the fixed point $\left(M_R^* = 0, \tilde{\lambda}_R^* = \frac{8\pi^2}{3}\right)$ for the $(2+1)D$ complex ϕ^4 theory. This theory, which is also known as XY model, is important in the context of the bosonic particle-vortex duality, appearing as one of the protagonists. The existence of this fixed point is a central aspect to support the existence of such duality [1, 12].

3 $3D$ Bosonization

Due to the existence $3D$ rotational differentiation of fermions and bosons, one cannot bosonize fermions with a similar approach to the $2D$'s one. Nevertheless, physicists have figured out a manner to do it using a different and elegant mechanism, which emerges due the special properties of $3D$ (and odd-dimensional) systems. The main objective of this chapter is to bosonize $3D$ fermionic systems using the flux-attachment mechanism.

3.1 $3D$ Master Duality

3.1.1 Bosonic Theory

To start, let us consider the theory of a complex scalar field ϕ coupled to a dynamical $U(1)_1$ gauge field with a BF term, whose action is

$$S_{boson} \equiv \int d^3x \left(-\frac{1}{4e^2} f^2 + \frac{1}{2\pi} + \frac{ada}{4\pi} + \frac{Ada}{2\pi} + \mathcal{L}_{\phi^4} \right), \tag{3.1}$$

with $ada \equiv \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$ and $\mathcal{L}_{\phi^4}$ defined as (2.61). This theory contains only bosonic degrees of freedom and three parameters. By dimensional analysis, we see that $[\lambda] = [M] = 1$, which means that the mass and coupling term are relevant in low energies.

In the previous chapter, we got the intuition that bosons coupled to an $U(1)_1$ gauge field behaves like fermions, and this is the motivation to consider such an action. This theory naturally possesses the global $U(1)_{top}$ symmetry discussed before,

$$j^\mu_{top} = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} f_{\nu\rho}, \tag{3.2}$$

whose local charged operators are the monopole operators $\mathcal{M}_{\tilde{q}}(x)$. In a $3D$ spacetime, charged particles are strongly coupled in IR regime. Let us explore the phases of this theory in both these regimes.

Case $M^2 \gg 0$

At this regime, the theory has a single vacuum, and the very massive field ϕ can be integrated out to achieve a low-energy description. In other words, the scalar field decouple from the gauge fields, and the effective theory at low energies has the form

$$\begin{aligned} \mathcal{L}'_{boson} &= -\frac{1}{4e^2} f^2 + \frac{1}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \frac{1}{2\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho \\ &= -\frac{1}{4e^2} f^2 + \frac{1}{4\pi} \epsilon^{\mu\nu\rho} (a_\mu + A_\mu) \partial_\nu (a_\rho + A_\rho) - \frac{1}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \end{aligned} \tag{3.3}$$

This combination gives to the photon a_μ a gauge invariant mass, which implies that this kind of excitation does not occur at low energies. It turns out then that the theory in this phase is also trivial in relation to a_μ and ϕ. Explicitly, if we also integrate out a_μ, we obtain the low-energy partition function

$$Z_{boson}[A] = \exp\left(-\frac{i}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho\right), \tag{3.4}$$

which is a $U(1)_{-1}$ Chern-Simons theory.

Case $M^2 \ll 0$

In this phase, the field is very massive field ϕ has a non-vanishing VEV, and the gauge particles acquires a mass by Higgs mechanism. At sufficiently low energies scales, the theory is essentially empty, because nothing is excitable. In this situation, the vacuum generating function is numerically trivial, namely $Z_{boson}[A] = 1$.

3.1.2 Fermionic Theory

In the subsection 2.2.5, we learned that the low-energy physics of interacting fermions is described by a Chern-Simons theory, which are sign(m)-dependent. Recalling, the result of that subsection is

$$S_{eff} = \frac{1}{4\pi}\frac{(\text{sign}(m)-1)}{2}\int d^3x\ \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho. \tag{3.5}$$

A fermionic theory which contains the previous action as a low-energy effective action is

$$S_{fermion} = \int d^3x\ \left[i\bar{\psi}\not{D}_A\psi - m\bar{\psi}\psi - \frac{1}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho\right]. \tag{3.6}$$

The insertion of the $U(1)_{-1/2}$ CS term is effectively the result of the Pauli-Villars regularization, as we have shown in the subsection 2.2.5. Now, the phases analysis is standard, which can be done by deforming the mass parameter, just as in the bosonic side.

Case $m \gg 0$

In this case, the fermionic theory is described effectively by (3.5) with sign$(m) = 1$, which turns out to be a trivial quantum field theory, as $S_{eff} = 0$ implies $Z_{fermion}[A] = 1$. By comparing this to the bosonic side, this result matches the bosonic behavior at $M^2 \ll 0$ regime.

Case $m \ll 0$

The last phase is described by a theory equivalent to (3.5) with sign$(m) = -1$. In this scenario, the vacuum generating function is

$$Z_{fermion}[A] = \exp\left(-\frac{i}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho\right), \tag{3.7}$$

which matches the bosonic side in $M^2 \gg 0$ regime. In the next chapter, we will make more explicit the reason for the form of this mass identification.

3.1.3 The IR Bosonization Map

The whole duality discussed so far in this section can be summarized as

$$i\bar{\psi}\not{D}_A\psi - \frac{1}{2}\frac{AdA}{4\pi} \quad \Longleftrightarrow \quad |\mathcal{D}_a\phi|^2 - \frac{\lambda_*}{4}|\phi|^4 + \frac{Ada}{2\pi} + \frac{ada}{4\pi}. \tag{3.8}$$

As the gauge kinetic terms become irrelevant in IR regime, they are not expressed in the duality relation. To justify the IR property of this duality, we can develop a line of thought as follows. We can insert in the duality expression (3.8) IR relevant terms, as kinetic, mass and interaction terms. This process recovers the Lagrangians (3.1) and (3.6). Afterwards, we can take them to the IR regime by throwing the bosonic and fermionic masses to infinity and integrate out non-excitable excitations. The analysis walks as in the previous subsections, comparing phases and symmetries in both sides, justifying the duality relation itself.

Notice that the fermionic theory in the duality is clearly conformal invariant as it does not depend on dimensional parameters. On the other hand, the bosonic theory is not conformal invariant, since it has one parameter λ. For this reason, in the precise duality, one should put the bosonic theory at the conjectured IR Wilson-Fischer fixed point, with $\lambda = \lambda_*$.

Let us now explore a possible consequence of this IR duality. If we including mass terms in both sides of the duality, keeping in mind that one needs to take the large mass limit, the relation (3.8) becomes [13]

$$i\bar{\psi}\not{D}_A\psi - m\bar{\psi}\psi - \frac{1}{2}\frac{AdA}{4\pi} \quad \Longleftrightarrow \quad |\mathcal{D}_a\phi|^2 - M^2|\phi|^2 - \frac{\lambda_*}{4}|\phi|^4 + \frac{Ada}{2\pi} + \frac{ada}{4\pi}. \tag{3.9}$$

To finish this chapter, we will express the duality relation in (3.8) in the path integral language. Defining

$$Z_{fermion}[A] \equiv \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(iS_{fermion}[\psi, A]\right), \tag{3.10}$$

$$Z_{scalar+flux}[A] \equiv \int \mathcal{D}\phi^*\mathcal{D}\phi\mathcal{D}a \exp i\left(S_{scalar}[\phi, a] + S_{CS}[a] + S_{BF}[a, A]\right), \tag{3.11}$$

with

$$S_{fermion}[\psi, A] \equiv \int d^3x\, i\bar{\psi}\not{D}_A\psi, \qquad S_{scalar}[\phi, a] \equiv \int d^3x \left(|\mathcal{D}_a\phi|^2 - \frac{\lambda_*}{4}|\phi|^4\right) \tag{3.12}$$

$$S_{CS}[a] \equiv \int d^3x\, \frac{ada}{4\pi}, \qquad S_{BF}[a, A] \equiv \int d^3x\, \frac{adA}{2\pi}, \tag{3.13}$$

it is immediate to write (3.8) as

$$Z_{fermion}[A] \exp\left(-\frac{i}{2}S_{CS}[A]\right) = Z_{scalar+flux}[A]. \tag{3.14}$$

In the same sense of the massive extension present in (3.9), the above path integral notation for the duality (3.8) allows us to easily obtain other dualities, as we will do in the next chapter. Clearly, these extension do not need to be correct. When a extension is done, it is necessary to check its limits and predictions in order to get a physical confidence about it. Also, in the next chapter, when mandatory, we will do such a physical and genuine consistency checks.

4 Applications of $3D$ Bosonization

In the previous chapter, we made a simple derivation of the master duality. In order to provide more evidences about its correctness, we will discuss the relation between the master duality and the *particle-vortex* dualities. These dualities are well-known and tested in several contexts, justifying our actual interest [14, 15]. Also in this chapter, we will bosonize the $3D$ Massive Thirring theory using the master duality, based on [16].

4.1 Bosonic Particle-Vortex Duality

In order to further test the master duality, we will relate it to the particle-vortex dualities via physical deformations. We start with the master duality in (3.14),

$$Z_{fermion}[A]\exp\left(-\frac{i}{2}S_{CS}[A]\right) \Longleftrightarrow Z_{scalar+flux}[A]. \tag{4.1}$$

To indicate that the master duality is related to the bosonic particle-vortex duality, it is convenient to express (4.1) in different form.

To do so, we will deform both sides of the above duality by promoting A to a dynamical field b (meaning that it is integrated in the partition function) and couple it to a background gauge field A through the BF term $-\frac{bdA}{2\pi}$. These deformations give us

$$Z_{fermion}[A]\exp\left(-\frac{i}{2}S_{CS}[A]\right) \mapsto \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}b\exp i\left(S_{fermion}[\psi,b]-\frac{1}{2}S_{CS}[b]-S_{BF}[b,A]\right) \tag{4.2}$$

in the fermionic side and

$$Z_{scalar+flux}[A] \mapsto \int \mathcal{D}\phi^*\mathcal{D}\phi\mathcal{D}a\mathcal{D}b\exp i\left(S_{scalar}[\phi,a]+S_{CS}[a]+S_{BF}[a,b]-S_{BF}[b,A]\right) \tag{4.3}$$

in the bosonic one.

The equation of motion for b in the bosonic side is

$$\delta_b\left(\frac{bda}{2\pi}-\frac{bdA}{2\pi}\right)=0 \implies a=A. \tag{4.4}$$

Hence, integrating out b in the bosonic side and using the identity $S_{BF}=[A,b]=S_{BF}[b,A]$, we have

$$Z_{scalar+flux}[A] \mapsto Z_{scalar}[A]\exp\left(iS_{CS}[A]\right). \tag{4.5}$$

We can express the fermionic side of (4.2) in a more compact form as

$$Z_{fermion}[A]\exp\left(-\frac{i}{2}S_{CS[A]}\right) \mapsto Z_{fermion+flux}[A] \equiv \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}b\exp\left(iS_{fermion+flux}[\psi,b]\right) \tag{4.6}$$

where

$$S_{fermion+flux}[\psi, b] \equiv S_{fermion}[\psi, b] - \frac{1}{2}S_{CS}[b] - S_{BF}[b, A]. \tag{4.7}$$

Therefore, we have deformed the duality (4.1) to

$$Z_{fermion+flux}[A] \Longleftrightarrow Z_{scalar}[A] \exp\left(iS_{CS}[A]\right), \tag{4.8}$$

which can also be expressed as

$$Z_{fermion+flux}[A] \exp\left(-iS_{CS}[A]\right) \Longleftrightarrow Z_{scalar}[A]. \tag{4.9}$$

Now that we have put the duality in a convenient form, we will relate it to the bosonic particle-vortex duality. Deforming (4.9) by promoting A to a dynamical gauge field b and coupling it to a new background field A through the BF term $\frac{bdA}{2\pi}$, the fermionic side transforms as

$$\begin{aligned} Z_{fermion+flux}[A] \exp\left(-iS_{CS}[A]\right) \quad \mapsto \quad & \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}a\mathcal{D}b \exp i \left(S_{fermion}[\psi, a]\right. \\ & \left. -\frac{1}{2}S_{CS}[a] - S_{BF}[a, b] + S_{BF}[b, A] - S_{CS}[b]\right), \end{aligned} \tag{4.10}$$

while the bosonic side transforms like

$$Z_{scalar}[A] \mapsto Z_{scalar\ QED} \equiv \int \mathcal{D}\phi^*\mathcal{D}\phi\mathcal{D}b \exp i \left(S_{scalar}[\phi, b] + S_{BF}[b, A]\right). \tag{4.11}$$

In the fermionic side, the equation of motion for b is

$$\delta_b \left(-\frac{bda}{2\pi} + \frac{bdA}{2\pi} - \frac{bdb}{4\pi}\right) = 0 \implies b = A - a. \tag{4.12}$$

Integrating out the gauge field b in the fermionic side leads us to

$$\begin{aligned} Z_{fermion+flux}[A] \exp\left(-iS_{CS}[A]\right) \quad \mapsto \quad & \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}a \exp i \left(S_{fermion}[\psi, a]\right. \\ & \left. +\frac{1}{2}S_{CS}[a] - S_{BF}[a, A] + S_{CS}[A]\right). \end{aligned} \tag{4.13}$$

Under time-reversal transformation $\mathcal{T}$, the gauge fields transform like $(a_0, a_i) \to (a_0, -a_i)$ in the Dirac representation for the Clifford algebra, meaning that both CS and BF terms are odd under $\mathcal{T}$ [9]. In counterpart, $S_{fermion}$ and S_{scalar} are $\mathcal{T}$-invariant. Hence, the time-reversed version of $Z_{fermion+flux}[-A]$ defined in (4.7) is

$$Z'_{fermion+flux}[-A] \equiv \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}a \exp i \left(S_{fermion}[\psi, a] + \frac{1}{2}S_{CS}[a] - S_{BF}[a, A]\right). \tag{4.14}$$

Using (4.14) in (4.13), we have

$$Z_{fermion+flux}[A] \exp\left(-iS_{CS}[A]\right) \mapsto Z'_{fermion+flux}[-A] \exp\left(iS_{CS}[A]\right). \tag{4.15}$$

Until now, we have the following relation

$$Z'_{fermion+flux}[-A] \exp\left(iS_{CS}[A]\right) \Longleftrightarrow Z_{scalar\ QED}[A]. \tag{4.16}$$

Comparing the previous relation with the time-reversed version of the master duality (with $A \mapsto -A$),

$$Z'_{fermion+flux}[-A] \exp\left(iS_{CS}[A]\right) \Longleftrightarrow Z_{scalar}[-A], \tag{4.17}$$

which remarkably leads to

$$Z_{scalar\ QED}[A] \Longleftrightarrow Z_{scalar}[-A], \tag{4.18}$$

which is the bosonic particle-vortex duality. We can express it in a more explicit way as

$$|\mathcal{D}_a\tilde{\phi}|^2 - \frac{\tilde{\lambda}}{4}|\tilde{\phi}|^4 + \frac{adA}{2\pi} \Longleftrightarrow |\mathcal{D}_{-A}\phi|^2 - \frac{\lambda}{4}|\phi|^4. \tag{4.19}$$

The bosonic particle-vortex duality above relates bosons coupled to a dynamical gauge field with free dual bosons. The "particle-vortex" nature of this duality will be clarified in the next chapter.

4.2 Fermionic Particle-Vortex Duality

Similarly to what we did in the previous section, we want to derive a duality relation between fermionic theories starting from the master duality. To do so, consider again the master duality in (3.14),

$$Z_{fermion}[A] \exp\left(-\frac{i}{2}S_{CS}[A]\right) \Longleftrightarrow Z_{scalar+flux}[A], \tag{4.20}$$

Passing the CS term in the fermionic side to the bosonic one results in

$$Z_{fermion}[A] \Longleftrightarrow Z_{scalar+flux}[A] \exp\left(\frac{i}{2}S_{CS}[A]\right). \tag{4.21}$$

We are not so much concerned with the gauge symmetry breaking at the present time, as this is only an intermediate step. Now, we promote the background field A to a dynamical field b and introduce a new background field A with a BF term $\frac{bdA}{2\pi}$. This process leads to

$$Z_{fermion}[A] \mapsto Z_{QED}[A] \equiv \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}b \exp i\left(S_{fermion}[\psi, b] + S_{BF}[b, A]\right) \tag{4.22}$$

in the fermionic side and

$$\begin{aligned} Z_{scalar+flux}[A] \quad \mapsto \quad & \int \mathcal{D}\phi^*\mathcal{D}\phi\mathcal{D}a\mathcal{D}b \exp i\left(S_{scalar}[\phi, a]\right. \\ & \left. + S_{BF}[a, b] + S_{CS}[a] + \frac{1}{2}S_{CS}[b] + S_{BF}[b, A]\right), \end{aligned} \tag{4.23}$$

in the bosonic side.

Integrating out the field b in the bosonic side using its equation of motion $b = -2a - 2A$, and scaling A as $C \mapsto \frac{A}{2}$ gives

$$\begin{aligned} Z_{scalar+flux}[A] &\mapsto \int \mathcal{D}\phi^* \mathcal{D}\phi \mathcal{D}a \exp i \left(S_{scalar}[\phi, a] - S_{CS}[a] \right. \\ &\qquad \left. -S_{BF}[a, A] - \frac{1}{2} S_{CS}[A]\right) \\ &\equiv Z'_{boson}. \end{aligned} \tag{4.24}$$

From the previous steps, at first, we can conclude the

$$Z_{QED}[A] \Longleftrightarrow Z'_{boson} \tag{4.25}$$

is valid. To finish the demonstration, let us get back to the equation (4.20). Performing a deformation in both sides of $+\frac{S_{CS}[A]}{2}$ reads

$$\begin{aligned} Z_{fermion}[A] &\Longleftrightarrow Z_{scalar+flux}[A] \exp\left(\frac{i}{2} S_{CS}[A]\right) \\ &\Longleftrightarrow \int \mathcal{D}\phi^* \mathcal{D}\phi \mathcal{D}a \exp \left(S_{scalar}[\phi, a] + S_{CS}[a] \right. \\ &\qquad \left. +S_{BF}[a, A] + \frac{1}{2} S_{CS}[A]\right). \end{aligned} \tag{4.26}$$

Under time-reversal transformation $\mathcal{T}$, both BF and CS terms are odd. In counterpart, $S_{fermion}$ and S_{scalar} are $\mathcal{T}$-invariant. Hence, the time-reversed version of the above duality is

$$\begin{aligned} Z_{fermion}[A] &\Longleftrightarrow \int \mathcal{D}\phi^* \mathcal{D}\phi \mathcal{D}a \exp \left(S_{scalar}[\phi, a] - S_{CS}[a] \right. \\ &\qquad \left. -S_{BF}[a, A] - \frac{1}{2} S_{CS}[A]\right). \\ &\Longleftrightarrow Z'_{boson}. \end{aligned} \tag{4.27}$$

From this result, we can conclude

$$Z_{QED}[A] \Longleftrightarrow Z_{fermion}[A]. \tag{4.28}$$

This is the so-called fermionic particle-vortex duality, which can also be expressed as

$$i\bar{\psi} \not{D}_a \psi + \frac{adA}{2\pi} \Longleftrightarrow i\bar{\psi} \not{D}_A \psi. \tag{4.29}$$

This duality is important because it is directly related to the physical content of $\nu = \frac{1}{2}$ Fractional Quantum Hall phase [15]. In the next chapter, we will further discuss properties of both particle-vortex dualities.

We have seen explicit cases where nontrivial dualities were achieved by manipulating the master duality (3.14). These results suggest that it is possible to obtain a infinitely large number of equivalences between quantum field theories by proceeding in a similar way as we have done above. The result of such a process is called **web of dualities**. It is worth to stress the power of this assertion: the theories in the web of dualities are somehow connected. This is quite remarkable in that one can find intimately relations between distinct theories and learn a new way of thinking about physical descriptions when dual descriptions are possible. There are several papers which present different faces of this web of dualities and also its generalizations (see [12, 13] and references therein). In the next section, we will provide a bosonized version for the $3D$ massive Thirring model by using the master duality.

4.3 Bosonization of $3D$ Massive Thirring Model

The fermion/boson duality achieved in the previous section allows us to investigate what would happen if we add in one side of the duality a certain field operator which are not physically present in the original duality scheme. Intuitively, it is obvious that the duality would break down over such process if we do not do something similar in the other side of the duality. In the sense, we will deform the bosonization duality (3.8) in order to get a known bosonization of $3D$ Thirring model. This IR duality can be view in the relation

$$\begin{aligned} i\bar{\psi}\not{D}_A\psi - m\bar{\psi}\psi - \frac{g}{2}\left(\bar{\psi}\gamma^\mu\psi\right)^2 \quad &\Longleftrightarrow \quad |\mathcal{D}_a\phi|^2 - M^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4 + \frac{ada}{4\pi} - \\ &- \quad \frac{cdc}{2\pi} - \frac{g}{16\pi^2}f^2_{a+c} + \frac{(a+c)dA}{2\pi}, \end{aligned} \tag{4.30}$$

together with some extra requirements about the compactness of gauge fields. This duality is interesting since it is valid for arbitrary values of Thirring coupling and finite, yet large, fermionic masses. In order to get a feeling about how this duality was achieved, we will first present the large mass limit behavior of massive Thirring model, and together with the duality encountered in (3.8), we will do consistency checks of the full duality above [16, 17].

4.3.1 Massive Thirring Model and MCS Theory

The $3D$ massive Thirring model is described by the action

$$S_{Thirring} = \int d^3x \left[i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2 \right]. \tag{4.31}$$

In principle, the Thirring model is very different from the Dirac theory. The massive Thirring model owns two dimensional parameters, $[m] = -[g] = 1$. The previous analysis tells us that the Thirring interaction is irrelevant in low energies. In counterpart, the Dirac theory possesses a mass parameter and only relevant operators, implying that both theories are different in UV regime, yet similar in IR. To obtain the required fermion/boson duality in the large mass limit, consider the Thirring Lagrangian in the Euclidean space, which is

$$\mathcal{L}_{Thirring} = \bar{\psi}(i\partial\!\!\!/ + m)\psi - \frac{g}{2}(\bar{\psi}\gamma^{\mu}\psi)^2. \tag{4.32}$$

The partition function of such theory is defined as

$$Z_{Th} \equiv \int \mathcal{D}[\bar{\psi}]\mathcal{D}[\psi] \exp\left(-S_{Thirring}\right). \tag{4.33}$$

Expressing the quartic interactions $(\bar{\psi}\gamma^{\mu}\psi)^2$ using an auxiliary field a as $\sim a^2 + a\bar{\psi}\gamma\psi$ and performing the fermionic path-integral results in [17]

$$\begin{aligned} Z_{Th} &= \int D[a^{\mu}] \exp\left(-S_{eff}\right), \quad S_{eff} \equiv S_{SD} + \mathcal{O}\left(\frac{1}{m}\right), \\ S_{SD} &\equiv \frac{1}{2}\int d^3x \left(a_{\mu}a^{\mu} \mp \frac{ih^2}{4\pi}\epsilon^{\mu\rho\nu}a_{\mu}\partial_{\rho}a_{\nu}\right). \end{aligned} \tag{4.34}$$

The theory defined by S_{SD} is the so called *Self-Dual* (SD) theory, and appears in the context of topologically massive gauge theories. It is broadly known that the SD theory dual to the MCS theory [18]. This shortens our work, providing us a nice result, which can be lazy represented as

$$Z_{Th} \approx Z_{MCS} \quad \text{(large mass limit relation)}. \tag{4.35}$$

Precisely, this duality can be written as

$$i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{g}{2}(\bar{\psi}\gamma^{\mu}\psi)^2 \quad \overset{|m|\to\infty}{\Longleftrightarrow} \quad -\frac{g}{64\pi^2}f_b^2 - sign(m)\frac{bdb}{8\pi}. \tag{4.36}$$

This duality is important in the actual context because somehow it is related to the already know duality (see equation (3.8)) and the Thirring model. The exact relation relies in the fact that the computations were made considering energy regimes E that respect $E \sim \frac{1}{g} \ll |m|$, which allowed the expansion in powers of $\frac{1}{m}$. In this regime, there is no single fermion excitations, because its mass is huge. As the Lagrangian (4.32) contains Dirac fermions interacting via Thirring coupling, this theory can be thought as describing the bound-state sector of the Thirring model. Therefore, one can expect that this duality may be inside of a larger Thirring duality, and this bound-state section should be encountered in the larger duality in some region of parameter space.

4.3.2 The Duality Map

To motivate the desired duality, let us consider again the master duality (3.8). To start, we will deform both sides of this duality with the term $+\frac{1}{2g}(A-B)^2 + \frac{1}{2}\frac{AdA}{4\pi}$. This process results in

$$i\bar{\psi}D\!\!\!\!/_A\psi + \frac{1}{2g}(A-B)^2 \quad \Longleftrightarrow \quad |\mathcal{D}_a\phi|^2 - \frac{\lambda}{4}|\phi|^4 + \frac{Ada}{2\pi} + \frac{ada}{4\pi} + \frac{1}{2g}(A-B)^2 + \frac{1}{2}\frac{AdA}{4\pi} \tag{4.37}$$

with B^μ is a new background gauge field and a is a compact field, $\int da \in 2\pi\mathbb{Z}$. We will now promote the A^μ to a dynamical field and perform the changes of variables $A^\mu \to eb^\mu$ and $B^\mu \to A^\mu$. It is important to highlight that, by promoting a certain gauge field to a dynamical one means that we also integrate over this field to obtain the quantum partition function associated with the system.

The result of this is

$$\begin{aligned} i\bar{\psi}(\not{\partial} - ie\not{b})\psi + \frac{1}{2g}(eb-A)^2 \iff & |\mathcal{D}_a\phi|^2 - \frac{\lambda}{4}|\phi|^4 + e\frac{bda}{2\pi} + \frac{ada}{4\pi} \\ + & \frac{1}{2g}(eb-A)^2 + \frac{e^2}{2}\frac{bdb}{4\pi}. \end{aligned} \tag{4.38}$$

In the bosonic side, the Lagrangian that describes the physics of the gauge field b^μ is

$$\mathcal{L}_b \equiv e^2\frac{bdb}{8\pi} + e\frac{bda}{2\pi} + \frac{1}{2g}(eb-A)^2. \tag{4.39}$$

There is a Lagragean $\mathcal{L}_{int}$ which interpolates between $\mathcal{L}_b$ and other useful Lagrangian $\mathcal{L}'_b$. This interpolating Lagrangian is

$$\begin{aligned} \mathcal{L}_{int} &= -\frac{cdc}{8\pi} + e\frac{cdb}{4\pi} + \frac{1}{2g}(eb-A)^2 + e\frac{bda}{2\pi} \\ &= -\frac{cdc}{8\pi} + \frac{1}{2g}(eb-A)^2 + \frac{eb(dc+2da)}{4\pi}, \end{aligned} \tag{4.40}$$

with c^μ being a emergent gauge field. In fact, the equation of motion associated to c^μ is $c^\mu = eb^\mu$. Plugging this in $\mathcal{L}_{int}$ gives us

$$\begin{aligned} \mathcal{L}_{int}[c=eb] &= -e^2\frac{bdb}{8\pi} + e^2\frac{bdb}{4\pi} + \frac{1}{2g}(eb-A)^2 + e\frac{bda}{2\pi} \\ &= e^2\frac{bdb}{8\pi} + \frac{1}{2g}(eb-A)^2 + e\frac{bda}{2\pi} = \mathcal{L}_b, \end{aligned} \tag{4.41}$$

which guarantees that the two Lagrangian are equivalent in this sense. Now, integrating out the b field in $\mathcal{L}_{int}$ using its equation of motion, which is $eb = A - \frac{g}{4\pi}(dc+2da)$ results in a effective theory for c, a and A. This effective theory is

$$\begin{aligned} \mathcal{L}'_{int} &= -\frac{cdc}{8\pi} + \frac{1}{2g}\frac{g^2}{16\pi^2}(dc+2da)^2 + \frac{A(dc+2da)}{4\pi} - \frac{g}{4\pi}\frac{(dc+2da)^2}{4\pi} \\ &= -\frac{cdc}{8\pi} - \frac{g}{64\pi^2}f^2_{c+2a} + \frac{(c+2da)dA}{4\pi}. \end{aligned} \tag{4.42}$$

The last step was performed using that $(da)^2 = \frac{1}{2}f_a^2$, $(f_a)_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. To finish, let us make a scaling of the gauge field c as $c \mapsto 2c$. This results in

$$\mathcal{L}'_b = -\frac{cdc}{2\pi} - \frac{g}{16\pi^2}f^2_{a+c} + \frac{(c+a)dA}{2\pi}. \tag{4.43}$$

Hence, the full Lagrangian of the bosonic side after integrating out the b field is

$$\mathcal{L}_{boson} = |\mathcal{D}_a\phi|^2 - \frac{\lambda}{4}|\phi|^4 + \frac{ada}{4\pi} - \frac{cdc}{2\pi} - \frac{g}{16\pi^2}f^2_{a+c} + \frac{(a+c)dA}{2\pi}. \tag{4.44}$$

Let us now analyze what happens if one integrates out the gauge field b in the fermionic side of the duality (4.38). As the equation of motion for b is $eb^\mu = -g\bar{\psi}\gamma^\mu\psi + A^\mu$, this integration results the following fermionic Lagrangian

$$\begin{aligned}
\mathcal{L}_{fermion} &= i\bar{\psi}\partial\!\!\!/\psi + eb^\mu\bar{\psi}\gamma_\mu\psi + \frac{1}{2g}(eb^\mu - A^\mu)^2 \\
&= i\bar{\psi}\partial\!\!\!/\psi + (A^\mu - g\bar{\psi}\gamma^\mu\psi)\bar{\psi}\gamma^\mu\psi + \frac{1}{2g}\left(-g\bar{\psi}\gamma^\mu\psi\right)^2 \\
&= i\bar{\psi}(\partial\!\!\!/ - iA\!\!\!/)\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2 \\
&= i\bar{\psi}D\!\!\!\!/_A\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2,
\end{aligned} \tag{4.45}$$

which is precisely the massless Thirring model Lagrangian. This finishes the derivation of the bosonization of $3D$ massless Thirring model, which is contained in [16]. Therefore, we can write the following duality relation

$$i\bar{\psi}D\!\!\!\!/_A\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2 \Longleftrightarrow |\mathcal{D}_a\phi|^2 - \frac{\lambda}{4}|\phi|^4 + \frac{ada}{4\pi} - \frac{cdc}{2\pi} - \frac{g}{16\pi^2}f_{a+c}^2 + \frac{(a+c)dA}{2\pi}, \tag{4.46}$$

with $\int_{S^2} da \equiv \int_{S^2} d^2x f_{12} = 2\pi\mathbb{Z}$, ensuring that electric charge of the scalar under the gauge field a is quantized.

4.3.3 Consistency Checks and Predictions

In this subsection, we will check the validity of the Thirring duality in different energy regimes. A logical way to do so is to insert mass operators in both sides of the duality and perform physical consistency checks in different energy regimes. So, let us consider again the middle-step duality (4.38) and assume that it is allowed to insert mass operators in it. Therefore, the middle-step duality allows us to identify, under $a \mapsto a - c$, the fermionic and bosonic Lagrangians as being

$$\begin{aligned}
\mathcal{L}_{fermion} &= i\bar{\psi}(\partial\!\!\!/ - ieb\!\!\!/)\psi - m\bar{\psi}\psi + \frac{1}{2g}(eb - A)^2, \\
\mathcal{L}_{boson} &= |\mathcal{D}_{(a-c)}\phi|^2 - \frac{\lambda}{4}|\phi|^4 + \frac{(a-c)d(a-c)}{4\pi} + e^2\frac{bdb}{8\pi} \\
&+ e\frac{bd(a-c)}{2\pi} + \frac{1}{2g}(eb - A)^2.
\end{aligned} \tag{4.47}$$

We already know that the Thirring interaction is irrelevant in low energies, $E \ll |m|$. In this regime, one can integrate out the fermions in the same spirit of the subsection 2.2.5, up to order $\mathcal{O}(1/m^2)$. This process results in [16]

$$\mathcal{L}'_{fermion} = \frac{e^2\text{sign}(m)}{8\pi}bdb - \frac{e^2}{48\pi|m|}f_b^2 + \frac{1}{2g}(eb - A)^2 + \mathcal{O}(1/m^2). \tag{4.48}$$

The previous Lagrangian is related to

$$\mathcal{L}''_{fermion} = -\frac{\text{sign}(m)}{8\pi}cdc + \frac{cdA}{4\pi} - \left(\frac{g}{64\pi^2} + \frac{1}{48\pi|m|}\right)f_c^2 \tag{4.49}$$

by the interpolation

$$c^{\mu} = \text{sign}(m)eb^{\mu} - \frac{e^2}{6|m|}\epsilon^{\mu\nu\sigma}f^{b}_{\nu\sigma} + \mathcal{O}(1/m^2) \tag{4.50}$$

and integration over b.

On the bosonic side, we will perform analogous computations in each one of the bosonic phases. In the $M^2 \ll 0$ Higgs phase, the gauge field $(a - c)$ acquires a mass associated with the scalar vacuum expectation value, $\langle 0|\,\phi\,|0\rangle = \sqrt{-2M^2/\lambda}$. The Lagrangian which describes this phase, where matter modes have been integrated out, is [16]

$$\mathcal{L}'_{boson} = -\frac{ada}{8\pi} - \left(-\frac{\lambda}{64\pi^2 M^2} + \frac{g}{64\pi^2}\right) f_a^2 + \frac{adA}{4\pi}. \tag{4.51}$$

Looking at (4.49) and (4.51), we see that

$$M^2 = -\frac{3}{4\pi}|m|\lambda, \tag{4.52}$$

is a sufficient condition to ensure this fist side of the duality at low energies. Moreover, the Higgs phase ($M^2 \ll 0$) is dual to the $m > 0$ fermionic phase.

In the $M^2 \gg 0$ symmetric phase, the effective field theory is given by the Lagrangian

$$\mathcal{L}''_{boson} = \frac{ada}{8\pi} - \left(\frac{1}{96\pi|M|} + \frac{g}{64\pi^2}\right) f_a^2 + \frac{adA}{4\pi}, \tag{4.53}$$

which is dual to (4.49) under the conditions

$$M^2 = \frac{3}{4\pi}|m|\lambda, \quad \frac{\lambda}{|M|} = \frac{2\pi}{3}, \tag{4.54}$$

with $m < 0$, i.e., the bosonic symmetric phase is dual to the $m < 0$ fermionic phase.

We conclude, therefore, that the two theories are dual in the $E << |m|$ regime at the fixed point $\lambda/|m| = 2\pi/3$, with the mass identification

$$M^2 = -\frac{3}{4\pi}m\lambda. \tag{4.55}$$

This serves as a consistency check due the following arguments. First, the two Lagrangian are dual in this regime. Second, in the master duality expression, one can express the bosonic quartic interaction as

$$-\frac{\lambda}{4}|\phi|^4 \sim -\sigma|\phi|^2 + \frac{1}{\lambda}\sigma^2, \tag{4.56}$$

with σ being a Lagrange multiplier. Operatorially, $|\phi|^2$ is related to $\bar{\psi}\psi$, which suggests, by looking at the equation of motion for σ, that $\bar{\psi}\psi \sim -\sigma$. Hence, if one inserts a mass term $-m\bar{\psi}\psi$ at the fermionic side, one should expect to get a bosonic term like $M\sigma$. That is to say that $M^2 \sim -\lambda m$.

Taking advantage of the previous discussion, it is straightforward to obtain information about the regime where the Thirring interaction becomes relevant and the fermionic mass m is large, regime which can be mathematically stated as $E \sim \frac{1}{g} \ll |m|$. Taking the described regime in the duality expression leads to the duality

$$i\bar{\psi}\not{D}_A\psi - m\bar{\psi}\psi - \frac{g}{2}\left(\bar{\psi}\gamma^\mu\psi\right)^2 \overset{|m|\to\infty}{\Longleftrightarrow} -\text{sign}(m)\frac{ada}{8\pi} - \frac{g}{64\pi^2}f_a^2 + \frac{adA}{4\pi}, \tag{4.57}$$

which can be directly compared with the bound-state-sector duality present in (4.36).

To finish the discussion, let us now to stress the duality (4.46) at the strict UV energy regime. As $[g^{-1}] = [\lambda] = 1$, wee see that only the Thirring interaction plays a relevant role in UV regime. Another way to express the previous statement is: the g Thirring coupling assumes large values in high energies, while λ became small. Therefore, only kinetic terms and terms follow by the g coupling will be interesting at high energies, indicating that the bosonic side possesses a non-trivial fixed point. Hence, the analysis enables us to write down the UV duality

$$i\bar{\psi}\not{\partial}\psi - \frac{g_*}{2}(\bar{\psi}\gamma^\mu\psi)^2 \Longleftrightarrow |\partial\phi|^2 - \frac{g_*}{16\pi^2}f_{a+c}^2, \tag{4.58}$$

with g being taken at the fixed point. Therefore, if the duality is credible and correct, the $3D$ Thirring model must possesses a UV fixed point, where it become a conformal field theory (CFT). This property of Thirring model has been tested in several recent papers [19, 20, 21, 22, 23]. This is a honest prediction of the duality present at [16], which indicates the existence of a powerful tool to explore the strongly coupled sector of the Thirring model, as the bosonic side of the duality is a free theory in the UV regime.

5 Quantum Wires

So far, we have tested the $3D$ master duality, as discussed in the chapter 4, by indicating that it is connected with particle-vortex dualities. In the current chapter, we will approach the dualities from a different perspective. We discretize one of the spatial dimensions of the systems and, in this way, we express a $3D$ theory in terms of a set of $2D$ interacting theories. This procedure can be thought as a kind of dimensional deconstruction, which is known as the *quantum wires* formalism. The advantage of this approach is that it allows us to make use of powerful methods of $2D$ theories, as the bosonization and conformal field theory tools. This chapter is mostly based on the paper [24].

5.1 Bosonic Particle-Vortex Duality Revisited

5.1.1 Macroscopic Construction

Before starting to study the bosonic particle-vortex duality from the quantum wires perspective, let us consider its macroscopic version, derived in the previous chapter. The duality relation of interest is (4.19),

$$|\mathcal{D}_{-A}\phi|^2 - \frac{\lambda}{4}|\phi|^4 \Longleftrightarrow |\mathcal{D}_a\tilde{\phi}|^2 - \frac{\tilde{\lambda}}{4}|\tilde{\phi}|^4 + \frac{adA}{2\pi}. \tag{5.1}$$

In order to analyze the phases of both sides in low energies, we turn off the external field A and deform the duality, including mass terms in both sides, as well as rising up a Maxwell term for the dynamical gauge field a. With this, the duality reads

$$|\partial\phi|^2 - M^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4 \Longleftrightarrow -\frac{1}{4e^2}f_{\mu\nu}f^{\mu\nu} + |\mathcal{D}_a\tilde{\phi}|^2 - \tilde{M}^2|\tilde{\phi}|^2 - \frac{\tilde{\lambda}}{4}|\tilde{\phi}|^4. \tag{5.2}$$

Notice that the left hand side is the XY model, discussed in chapter 2. As this model has an infrared nontrivial fixed point, and it is supposed to be dual to the theory in the right hand side (known as Abelian Higgs model), the later model should also possesses a nontrivial infrared fixed point.

Let us start by analyzing the phases of the XY model, based on the mass parameter M^2 and the spontaneous breaking pattern of its global $U(1)$ symmetry. When $M^2 > 0$, we see that the theory is theory is gapped, and the $U(1)$ symmetry is unbroken. The lowest excitation of this phase is the $\phi - particles$, with mass M^2 and unit charge under $U(1)$. On the other hand, when $M^2 < 0$, the scalar field acquires a nontrivial vacuum expectation value $\langle\phi\rangle \equiv v \sim -M^2/\lambda$. As the vacuum breaks spontaneously the $U(1)$ global symmetry, the low-lying excitations are the Goldstone bosons. Hence, this phase is gapless.

Now, let us find the phases of the Abelian Higgs model, whose global symmetry is the $U(1)_{top}$ discussed in chapter 2. This theory also has the $U(1)_{gauge}$ symmetry. In the phase $\tilde{M}^2 > 0$, the $\tilde{\phi}$ excitations are massive and decouple from the gauge field a. The $U(1)_{gauge}$ local symmetry is unbroken, leaving the photons massless, which can be viewed as the Goldstone bosons coming from the spontaneous symmetry breaking of $U(1)_{top}$. On the other hand, in the phase $\tilde{M}^2 < 0$, the field $\tilde{\phi}$ acquires a vacuum expectation value $\tilde{v} \neq 0$, and the Higgs mechanism gives the photon a mass. Hence, this phase is gapped with unbroken $U(1)_{top}$.

Comparing the phases on both sides of the duality, we see that there is a phase matching with $M^2 \sim -\tilde{M}^2$. In the gapless phase, the XY Goldstone bosons correspond to the massless photons in the Abelian Higgs model. In contrast, the $\tilde{\phi}$-vortices, which have finite mass, correspond to the ϕ-particles in the gapped phase. This is the essence of the macroscopic argument, which sustains the duality map between the XY and Abelian Higgs models. In the next sections, we will present a more precise discussion of this relation in terms of the quantum wires construction.

5.1.2 Quantum Wires Construction

In order to give a more precise sense of the duality between the XY and the Abelian Higgs model, we need a theory which essentially contains the physical properties of these both models. Hence, we need a theory which exhibits a phase transition between gapped and gapless phases in the infrared regime.

In this fashion, let us consider the quantum wires Euclidean action $S_{original}$ of $2D$ bosonic fields $\Phi_y \sim \exp(i\phi_y)$ defined by

$$S_{original} = \int d\tau \; dx \sum_y \left(\frac{i}{\pi} \partial_x \theta_y \partial_\tau \phi_y + \mathcal{L}_{LL} + \mathcal{L}_{hop} + \mathcal{L}_{phase-slip} \right), \tag{5.3}$$

where

$$\begin{aligned} \mathcal{L}_{LL} &= \frac{v}{2\pi} \left(\partial_x \phi_y\right)^2 + \frac{u}{2\pi} \left(\partial_x \theta_y\right)^2, && (5.4) \\ \mathcal{L}_{hop} &= -g_1 cos(\phi_{y+1} - \phi_y), && (5.5) \\ \mathcal{L}_{phaseslip} &= -g_2 cos(2\theta_y), && (5.6) \end{aligned}$$

and the index y labels the different wires of the system.

To properly interpret the physics of the quantum wires system, let us discuss its quantization. By the definition of the Euclidean action, we can see that the $\phi_y(x)$ and $\sim \partial_x \theta_y(x)$ form a canonical pair. Indeed, by defining the momentum field as

$$\Pi_y \equiv i \frac{\partial \mathcal{L}}{\partial(\partial_\tau \phi_y)} = -\frac{1}{\pi} \partial_x \theta_y, \tag{5.7}$$

the canonical commutation relation is

$$\Big[\phi_{y'}(x'), \Pi_y(x)\Big] = i\delta_{yy'}\delta(x - x') \quad \Rightarrow \quad \Big[\partial_x \theta_y(x), \phi_{y'}(x')\Big] = i\pi\delta_{yy'}\delta(x - x'). \tag{5.8}$$

This implies

$$\left[\theta_y(x), \phi_{y'}(x')\right] = \frac{i\pi}{2}\delta_{yy'}\text{sign}(x-x') \tag{5.9}$$

and

$$\left[\theta_y(x), \partial_{x'}\phi_{y'}(x')\right] = -i\pi\delta_{yy'}\delta(x-x'), \tag{5.10}$$

As we have defined, the bosonic field $\Phi_y^\dagger \sim \exp(-i\phi_y)$ is the operator which creates bosonic particles. Let us, then, compute the commutation rule between $\rho_y(x) \equiv \frac{1}{\pi}\partial_x\theta_y$ and $\Phi_y^\dagger(x)$:

$$\left[\rho_y(x), \Phi_{y'}^\dagger(x')\right] = \frac{1}{\pi}\left[\partial_x\theta_y(x), e^{-i\phi_{y'}(x')}\right]. \tag{5.11}$$

Using the identity

$$[A, e^B] = \int_0^1 ds\ e^{(1-s)B}[A,B]e^{sB}, \tag{5.12}$$

we have

$$\begin{aligned}
\left[\rho_y(x), \Phi_{y'}^\dagger(x')\right] &= \frac{1}{\pi}\int_0^1 ds\ e^{-i(1-s)\phi_{y'}(x')}\left[\partial_x\theta_y(x), -i\phi_{y'}(x')\right]e^{-is\phi_{y'}(x')} \\
&= \frac{1}{\pi}\int_0^1 ds\ e^{-i(1-s)\phi_{y'}(x')}(-i)i\pi\delta_{yy'}\delta(x-x')e^{-is\phi_{y'}(x')} \\
&= \delta_{yy'}\delta(x-x')e^{-i\phi_{y'}(x')}\int_0^1 ds \\
&= \delta_{yy'}\delta(x-x')\Phi_{y'}^\dagger(x').
\end{aligned} \tag{5.13}$$

Considering $\rho_y(x)$ as the density operator in wire y and at position x, and also considering $|q\rangle$ as a state with q bosons, we have

$$\begin{aligned}
\rho_y(x)\left(\Phi_{y'}^\dagger(x')\,|q\rangle\right) &= \left[\delta_{yy'}\delta(x-x')\Phi_{y'}^\dagger(x') + \Phi_{y'}^\dagger(x')\rho_y(x)\right]|q\rangle \\
&= \left[\delta_{yy'}\delta(x-x') + q_y(x)\right]\left(\Phi_{y'}^\dagger(x')\,|q\rangle\right).
\end{aligned} \tag{5.14}$$

Defining $|q+1\rangle \equiv \Phi_{y'}^\dagger(x')\,|q\rangle$, we then have

$$\rho_y(x)\,|q+1\rangle = \left[q_y(x) + \delta_{yy'}\delta(x-x')\right]|q+1\rangle. \tag{5.15}$$

The above identity shows that ρ_y is a good wire density operator, because it measures the unit increment in the charge number in wire y due to the application of the operator $\Phi_{y'}^\dagger(x')$.

Another important object of the theory is the phase slip operator, defined as

$$\Xi_y(x) = e^{2i\theta_y(x)}. \tag{5.16}$$

Let us find the transformation of $\Phi^\dagger_{y'}(x') \sim e^{-i\phi_{y'}(x')}$ under the action of $\Xi_y(x)$ using the BCH formula,

$$\begin{aligned} e^{-i\phi_{y'}(x')} \mapsto \Xi_y(x)e^{-i\phi_{y'}(x')}\Xi^\dagger_y(x) &= e^{2i\theta_y(x)}e^{-i\phi_{y'}(x')}e^{-2i\theta_y(x)} \\ &= e^{-i\phi_{y'}(x')}e^{2i\theta_y(x)}e^{[2i\theta_y(x),-i\phi_{y'}(x')]}e^{-2i\theta_y(x)} \\ &= e^{-i\phi_{y'}(x')}e^{2i\theta_y(x)}e^{i\pi\ \delta_{yy'}\mathrm{sign}(x-x')}e^{-2i\theta_y(x)} \\ &= e^{-i\left[\phi_{y'}(x')-\pi\delta_{yy'}\mathrm{sign}(x-x')\right]}. \end{aligned} \tag{5.17}$$

This means that $\Xi_y(x)$ operator translates $\phi_{y'}(x')$ by $-\pi\delta_{yy'}\mathrm{sign}(x-x')$. With these appropriate definitions and properties, we can interprete the theory: the v term is a spatial kinetic term for ϕ in the x direction, while the u term is a density-density interwire interaction. From the above results, $g_1 cos(\phi_{y+1} - \phi_y)$ can be thought as a kinetic term in the y direction and $g_2 cos(2\theta_y)$ is a Mott-insulator term which is able to open a gap in the system whenever it is relevant [24]. The relevance of these operators is ultimately determined by the parameters u and v, as we shall see below.

To properly interpret the effect of $\Xi_y(x)$ in the system, consider the figure below.

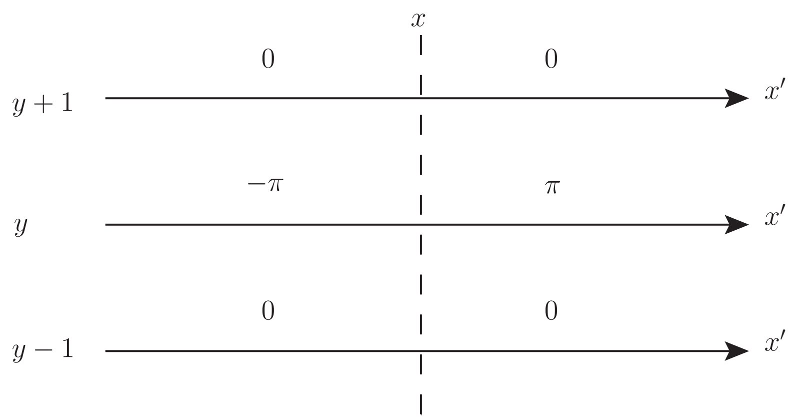


Figura 3 – Phase-shift generated by $\Xi_y(x)$. We have omitted the other wires in this diagram for simplicity.

The numbers 0 and $\pm\pi$ above each wire represent the phase shift generated by $\Xi_y(x)$ on $\phi_{y'}(x')$ depending on the position x'.

Let us define the phase slip $P(y)$ as the phase difference between the regions $x' > x$ and $x' < x$ at the wire y',

$$P(y) \equiv \phi_y(x' > x) - \phi_y(x' < x). \tag{5.18}$$

Looking to the figure 3, we see that the operation of $\Xi_y(x)$ at the wire y produces a phase slip $P(y) = 2\pi$. For $y' \neq y$, $P(y') = 0$.

It is important to highlight that this phase slip is a global property of each wire. In this sense, it is illustrative to redraw the figure 3, indicating the phase slip of each wire, as shown in 4.

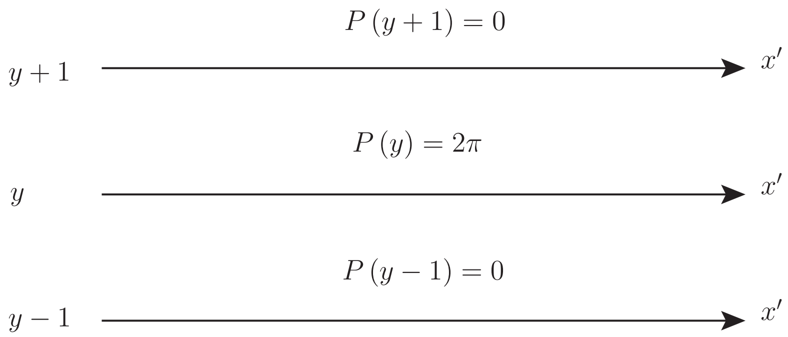


Figura 4 – Phase slip of the wires generated by $\Xi_y(x)$.
.

The quantity above each wire represents the total phase shift at that respective wire. Therefore, we can properly interpret the effect of $\Xi_y(x)$ at each wire: it produces a phase slip of 2π at the wire y, leaving the remaining ones unaffected.

The complete categorization of the effect of $\Xi_y(x)$ on the system can be done by defining

$$\Delta_y P \equiv P\left(\phi_y, y\right) - P\left(\phi_{y-1}, y\right). \tag{5.19}$$

Essentially, $\Delta_y P$ measures the phase slip difference of consecutive wires generated by $\Xi_y(x)$. The definition implies that the only nontrivial ΔP are

$$\Delta_y P = 2\pi, \tag{5.20}$$

$$\Delta_{y+1} P = -2\pi. \tag{5.21}$$

Notice that $\Delta_y P$ takes into account the net phase slip in both spatial directions. In this way, a phase slip of 2π can be naturally interpreted as signaling the presence of vortex configurations localized at the dual wires. This is shown in the figure below. Notice that we

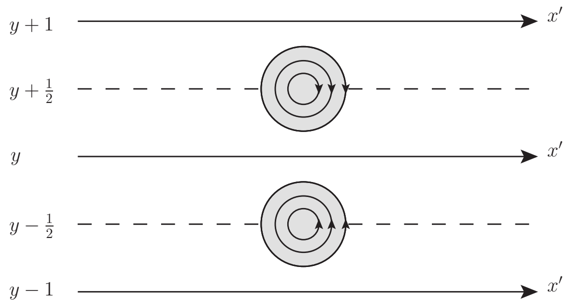


Figura 5 – Vortex-antivortex pair creation by $\Xi_y(x)$.
.

have denoted the $\Delta_y P = 2\pi$ as a clockwise vortex, and $\Delta_y P = -2\pi$ as a counterclockwise vortex.

To conclude this subsection, it is interesting to call attention to the effect of $\Xi_y(x)$ in a state where already exists a vortex configuration. As this operator creates a vortex-antivortex pair, its effect (with appropriated argument sign, depending on the winding of the vortex) essentially translates the vortex. This process is represented below.

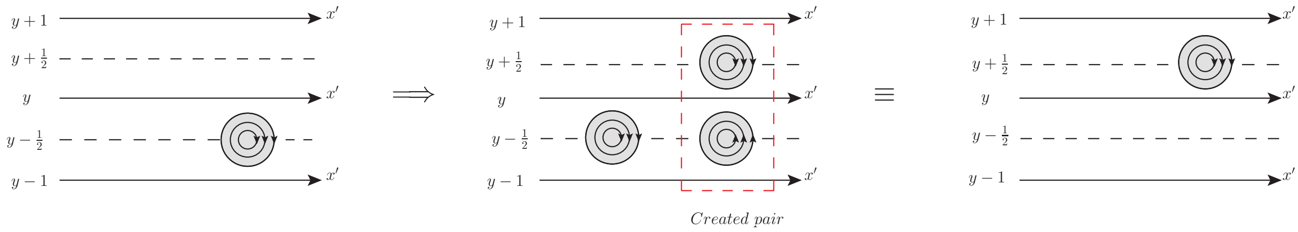


Figura 6 – $\Xi_y(x)$ as a vortex translation operator in the y direction.

5.1.3 Renormalization of Parameters and Wilson-Fisher Fixed Point

Before dualizing the theory, let us analyse the renormalization flow of the coupling constantes g_1 and g_2. Starting from (5.3), we can define $v \equiv v_0 K$ and $u \equiv \frac{v_0}{K}$, factorizate the global parameter v_0 and absorb the parameter K in the fields ϕ and θ by making the rescaling

$$\phi \mapsto \frac{1}{\sqrt{K}}\phi, \tag{5.22}$$

$$\theta \mapsto \sqrt{K}\theta. \tag{5.23}$$

This rescaling changes the cosine operators as

$$-g_1 cos\left(\phi_{y+1}-\phi_y\right) \mapsto -g_1 cos\frac{1}{\sqrt{K}}\left(\phi_{y+1}-\phi_y\right), \tag{5.24}$$

$$-g_2 cos\left(2\theta_y\right) \mapsto -g_2 cos\sqrt{K}\left(\theta_y\right). \tag{5.25}$$

We can predict the quantum dimension of the coupling constants g_1 and g_2 using the identity

$$cos(A) =: cos(A) : \exp\left(-\frac{1}{2} < AA >\right). \tag{5.26}$$

In order to do so, we need the 2-point functions of ϕ and θ, which essentially are (compare the structure with the one in (1.44), up to a finite renormalization)

$$\langle\phi\phi\rangle = -\frac{1}{4}ln\left(\mu^2 a^2\right), \tag{5.27}$$

$$\langle\theta\theta\rangle = -\frac{1}{4}ln\left(\mu^2 a^2\right), \tag{5.28}$$

where a is the lattice spacing and μ is the energy scale. Using the above equations, we see that the quantum operators can be written as

$$-g_1 : cos\left(\phi_{y+1} - \phi_y\right) : (\mu a)^{\frac{1}{2K}}, \tag{5.29}$$
$$-g_2 : cos\left(2\theta_y\right) : (\mu a)^{K}. \tag{5.30}$$

Hence, considering the quantum corrections, the scale dependency of g_1 and g_2 on μ is

$$\mu^2 \bar{g}_1(\mu) \equiv g_1(\mu a)^{\frac{1}{2K}}, \tag{5.31}$$
$$\mu^2 \bar{g}_2(\mu) \equiv g_2(\mu a)^{K}. \tag{5.32}$$

We have defined in the above equations the couplings $\bar{g}_1$ and $\bar{g}_2$ in a such way that the their classical dimension are zero. These equations can be written as

$$\bar{g}_1(\mu) = g_1 a^{\frac{1}{2v}} \mu^{-\left(2-\frac{1}{2K}\right)}, \tag{5.33}$$
$$\bar{g}_2(\mu) = g_2 a^{\frac{1}{u}} \mu^{-(2-K)}, \tag{5.34}$$

which imply

$$\mu \frac{d}{d\mu}\bar{g}_1(\mu) = -\left(2 - \frac{1}{2K}\right)\bar{g}_1(\mu), \tag{5.35}$$
$$\mu \frac{d}{d\mu}\bar{g}_2(\mu) = -\left(2 - K\right)\bar{g}_2(\mu). \tag{5.36}$$

The reparametrization of u and v in terms of K and v_0 implies that

$$K^2 = \frac{v}{u}. \tag{5.37}$$

Using this relation, we finally have

$$\mu \frac{d}{d\mu}\bar{g}_1(\mu) = -\left(2 - \frac{1}{2}\sqrt{\frac{u}{v}}\right)\bar{g}_1(\mu), \tag{5.38}$$
$$\mu \frac{d}{d\mu}\bar{g}_2(\mu) = -\left(2 - \sqrt{\frac{v}{u}}\right)\bar{g}_2(\mu). \tag{5.39}$$

Therefore, we have obtained the running of the coupling constants g_1 and g_2.

Let us interpret this running structure. When $u \gg v$, there is a strong repulsion between particles, since the u term is the coupling associated to the density-density interactions. In this case, g_1 decreases while g_2 grows, resulting in a Mott insulator gapped phase. The other case, $u \ll g$, g_2 decreases while g_1 grows, resulting in a superfluid gapless phase.

To conclude, we have seen that the renormalization group of this theory tells us that this theory describe the phase transition between a gapped phase and a gapless one. This phase transition phenomenology is extremely similar to the one involved the bosonic particle-vortex duality, as we have discussed in the previous subsection. Also from the beta functions, we can conclude that the running of g_1 and g_2 are the same over the separatrix $u = 2v$. Hence, starting from a nontrivial point over this separatrix with $\text{sign}(\bar{g}_1) = \text{sign}(\bar{g}_2) = -1$, $\bar{g}_1$ and $\bar{g}_2$ flow to 0, which is a parameter point where the theory is scale-invariant. This analysis essentially indicates that this theory has a nontrivial infrared Wilson-Fisher fixed point, which is the critical point of the phase transition between the superfluid and Mott insulator phases.

5.1.4 The Dual Mapping

We will now rewrite the original quantum wires model into a dual model, by expressing (θ, ϕ) in terms of dual fields $(\tilde{\theta}, \tilde{\phi})$. The explicit map which will be used is

$$\phi_y = 2\sum_{y_1} \Delta^{-1}_{y,y_1} \tilde{\theta}_{y_1+1/2} \Longleftrightarrow \phi = 2\Delta^{-1}\tilde{\theta}, \tag{5.40}$$

$$\theta_y = -\frac{1}{2}\sum_{y_2} \Delta_{y,y_2} \tilde{\phi}_{y_2-1/2} \Longleftrightarrow \theta = -\frac{1}{2}\Delta\tilde{\phi}, \tag{5.41}$$

where $\theta, \tilde{\phi}, \tilde{\theta}$ are also column "vectors". The Δ is a matrix whose components are $\Delta_{y,y'} = \left(\delta_{y+1,y'} - \delta_{y,y'}\right)$, and ϕ is a column "vector" whose components are ϕ_y. It is possible to check that the Δ matrix possesses the following properties

$$\Delta^{-1}_{y,y'} = -\frac{1}{2}\text{sign}\left(y' - y + \frac{1}{2}\right), \tag{5.42}$$

$$\Delta\Delta^T = \Delta^T\Delta, \tag{5.43}$$

where "sign" is the usual signal function.

Before applying this map to the original theory, let us investigate the nature of the operator $\tilde{\Phi} \sim e^{i\tilde{\phi}}$, based on the interpretations given to the original model. Looking to map (5.41) related to the field $\tilde{\phi}$,

$$\tilde{\phi}_{y+\frac{1}{2}}(x) = \sum_{y'} \text{sign}\left(y' - y - \frac{1}{2}\right)\theta_{y'}(x), \tag{5.44}$$

we see that $\tilde{\Phi}_{y+\frac{1}{2}} \sim \exp\left(i\tilde{\phi}_{y+\frac{1}{2}}\right)$ is the infinite product of terms which are similar to the Ξ operators (as θ operators commutes). The difference between the pieces of $\tilde{\Phi}$ and Ξ is that the exponential argument pre-factor in the first one is 1, while it is 2 in the second one. The sign of each exponential product term depends on the wire considered, due to the piece "$\text{sign}(y' - y)$". Below, we present a schematic figure for this transport.

Recalling that Ξ transports vortices between dual wires, we can conclude that the effect of $\tilde{\Phi}_{y+\frac{1}{2}}$ is transport "half-vortex" from the the wire $y = \infty$ and a "half-antivortex" from $y = -\infty$ to the wire $y + ...$ The "half" term is to indicate that there is a factor of 2 missing in the $\tilde{\Phi}$ exponential argument, in comparison to the one in Ξ. Explicitly, the phase slips created by $\tilde{\Phi}_{y+\frac{1}{2}}$ in the dual lattice are

$$P = \begin{cases} \pi, & \text{if } y' > y, \\ -\pi, & \text{if } y' \leq y. \end{cases} \tag{5.45}$$

Hence, $\tilde{\Phi}_y$ creates a vortex ($\Delta_{y+1}P = 2\pi$) at the dual wire $y+\frac{1}{2}$, as represented below.

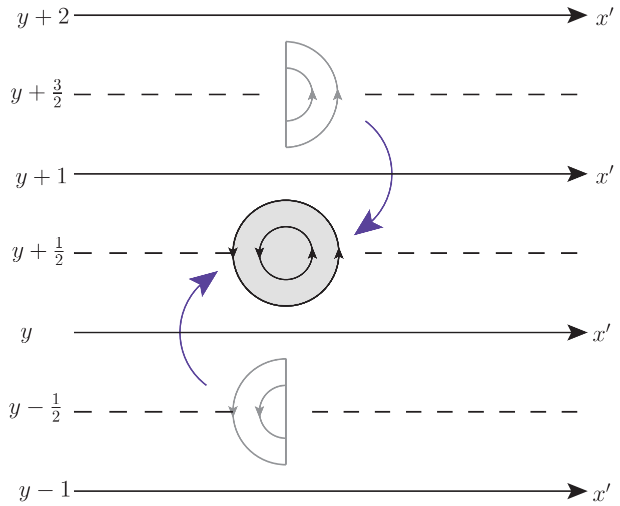


Figura 7 – The $\tilde{\Phi}_{y+\frac{1}{2}}$ operator combines two half-vortices at the dual wire $y+\frac{1}{2}$.

Now, let us dualize the original bosonic theory using the dual map defined in the beginning of this section. Under this map, the first term of the Lagrangian (5.3) becomes

$$\begin{aligned} \frac{i}{\pi}\sum_y \partial_x\theta_y\partial_\tau\phi_y &= \frac{i}{\pi}\sum_y \partial_x\left(-\frac{1}{2}\sum_{y_2}\Delta_{y,y_2}\tilde{\phi}_{y_2-1/2}\right)\partial_\tau\left(2\sum_{y_1}\Delta^{-1}_{y,y_1}\tilde{\theta}_{y_1+1/2}\right) \\ &= -\frac{i}{\pi}\sum_{y,y_1,y_2}\Delta_{y,y_2}\Delta^{-1}_{y,y_1}\partial_x\tilde{\phi}_{y_2-1/2}\partial_\tau\tilde{\theta}_{y_1+1/2}. \end{aligned} \tag{5.46}$$

Using the property

$$\sum_y \Delta_{y,y_2}\Delta^{-1}_{y,y_1} = -\delta_{y_1,y_2-1}, \tag{5.47}$$

we get

$$\begin{aligned} \frac{i}{\pi}\sum_y \partial_x\theta_y\partial_\tau\phi_y &= \frac{i}{\pi}\sum_{y_1,y_2}\delta_{y_1,y_2-1}\partial_x\tilde{\phi}_{y_2-1/2}\partial_\tau\tilde{\theta}_{y_1+1/2} \\ &= \frac{i}{\pi}\sum_{y_1}\partial_x\tilde{\phi}_{y_1+1/2}\partial_\tau\tilde{\theta}_{y_1+1/2}. \end{aligned} \tag{5.48}$$

Under the integral sign $\int_{\tau,x}$, we can exchange the derivatives ∂_x and ∂_τ by discarding surface terms, resulting in

$$\frac{i}{\pi}\sum_y \partial_x\theta_y\partial_\tau\phi_y = \frac{i}{\pi}\sum_y \partial_x\tilde{\theta}_{y+1/2}\partial_\tau\tilde{\phi}_{y+1/2}. \tag{5.49}$$

As the above term maintain its functional form under the map and the other terms can not produce terms like $\partial_\tau\tilde{\phi}$, we can conclude that the dual fields have the same commutation rules as the original ones,

$$\left[\partial_x\tilde{\theta}_{y+1/2}(x), \tilde{\phi}_{y'+1/2}(x')\right] = i\pi\delta_{y,y'}\delta(x-x'). \tag{5.50}$$

In order to obtain the desired particle-vortex duality, we will also include in the original theory the short-range interaction term

$$\frac{\tilde{u}}{8\pi}\left(\partial_x \Delta\phi\right)^2. \tag{5.51}$$

The mapping transforms this term as

$$\frac{\tilde{u}}{8\pi}\left(\partial_x \Delta\phi\right)^2 = \frac{\tilde{u}}{2\pi}\left(\partial_x \tilde{\theta}\right). \tag{5.52}$$

The remaining terms can be straightly expressed using $\tilde{\phi}$ and $\tilde{\theta}$. The action as a functional new variables becomes

$$\begin{aligned} \tilde{S}_{dual} &= \int d\tau\ dx \sum_{\tilde{y}} \left(\frac{i}{\pi}\partial_x \tilde{\theta}_{\tilde{y}} \partial_\tau \tilde{\phi}_{\tilde{y}} + \frac{v}{2\pi}\left(2\Delta^{-1}\partial_x \tilde{\theta}_{\tilde{y}}\right)^2 + \frac{u}{8\pi}\left(\partial_x \Delta \tilde{\phi}_{\tilde{y}}\right) + \right. \\ &+ \left. \frac{\tilde{u}}{2\pi}\left(\partial_x \tilde{\theta}_{\tilde{y}}\right) - g_1 \cos\left(2\Delta\tilde{\theta}_{\tilde{y}}\right) - g_2 \cos\left(\Delta\tilde{\phi}_{\tilde{y}}\right)\right), \quad \tilde{y} \equiv y + \frac{1}{2}. \end{aligned} \tag{5.53}$$

Notice that the $\sim v(2\Delta^{-1}\partial_x\tilde{\theta})^2$ term is a highly nonlocal density-density interaction in dual theory. This essentially means that the dual bosons may interact at large distances. Even so, can express this long-range interaction in a local way using an auxiliary field $\tilde{a}_0$ as

$$\sum_{\tilde{y}} \frac{v}{2\pi}(2\Delta^{-1}\partial_x\tilde{\theta}_{\tilde{y}})^2 = \sum_{\tilde{y}} \left(-\frac{i}{\pi}\partial_x \tilde{\theta}_{\tilde{y}} \tilde{a}_{0,\tilde{y}} + \frac{(\Delta \tilde{a}_{0,\tilde{y}})^2}{8\pi v} \right), \quad (\Delta a_0)^2 \sim (\Delta a_0)^T (\Delta a_0). \tag{5.54}$$

The equation of motion associated to of this auxiliary field is $\tilde{a}_0 = 4iv(\Delta^T\Delta)^{-1}\partial\tilde{\theta}$. Observe that, by plugging this equation of motion in the r.h.s. of (5.54), we the l.h.s.,

$$\begin{aligned} +\frac{4v}{\pi}(\partial\tilde{\theta})^T\left(\Delta^T\Delta\right)^{-1}\partial\tilde{\theta} - \frac{2v}{\pi}(\partial\tilde{\theta})^T\left(\Delta^T\Delta\right)^{-1}\partial\tilde{\theta} &= +\frac{v}{2\pi}(2\partial\tilde{\theta})^T(\Delta^{-1})^T\Delta^{-1}(2\partial\tilde{\theta}) \\ &= \sum_{\tilde{y}} \frac{v}{2\pi}\left(2\Delta^{-1}\partial_x\tilde{\theta}_{\tilde{y}}\right)^2, \end{aligned} \tag{5.55}$$

as desired.

For convenience, we will similarly replace the u term as

$$\begin{aligned} \sum_{\tilde{y}} \frac{u}{8\pi}\left(\partial_x\Delta\tilde{\phi}_{\tilde{y}}\right) &= \sum_{\tilde{y}} \frac{u}{8\pi}\left[(\partial_x\tilde{\phi}_{\tilde{y}} - \tilde{a}_{1,\tilde{y}})^2 + (\Delta\tilde{a}_{1,\tilde{y}})^2\right] \\ &+ \sum_{\tilde{y},\tilde{y}'} \frac{u}{8\pi} V_{\tilde{y},\tilde{y}'}\partial_x\left(\Delta\tilde{\phi}_{\tilde{y}}\right)\partial_x\left(\Delta\tilde{\phi}_{\tilde{y}'}\right), \end{aligned} \tag{5.56}$$

where $\tilde{a}_{1,\tilde{y}}$ is another auxiliary field and $V \equiv \Delta^T\left(1+\Delta^T\Delta\right)^{-1}\Delta$. It is possible to see that $V_{\tilde{y},\tilde{y}'}$ decays exponentially with $|\tilde{y}-\tilde{y}'|$.

Therefore, the effect of the map is change the original wire theory to dual theory on the dual wire. In order to interpret the dual theory as a theory of dual bosons minimally coupled to a gauge field $\tilde{a}$, we have to restore the explicit gauge invariance, as the second spatial gauge field component a_2 is missing. This lack can be reversed by saying that we are working with the gauge choice $\tilde{a}_2 = 0$. Making explicit the gauge invariance of the dual model, the Lagrangian density of each wire is [24]

$$\tilde{\mathcal{L}}_{dual\ wire} = \frac{i}{\pi}\partial_x\tilde{\theta}_{\tilde{y}}\partial_\tau\tilde{\phi}_{\tilde{y}} + \tilde{\mathcal{L}}_{wire} + \tilde{\mathcal{L}}_{gauge} + \tilde{\mathcal{L}}_{int} + \tilde{\mathcal{L}}_{hop+phase-slip}. \tag{5.57}$$

Here, summation over $\tilde{y}$ is implicit and we have defined

$$\tilde{\mathcal{L}}_{wire} \equiv \frac{u}{8\pi}\left(\partial_x\tilde{\phi}_{\tilde{y}} - \tilde{a}_{1,\tilde{y}}\right)^2 + \frac{\tilde{u}}{2\pi}\left(\partial_x\tilde{\theta}_{\tilde{y}}\right)^2 - \frac{i}{\pi}\partial_x\tilde{\theta}_{\tilde{y}}\tilde{a}_{0,\tilde{y}}, \tag{5.58}$$

$$\tilde{\mathcal{L}}_{gauge} \equiv \frac{1}{8\pi v}\left(\Delta\tilde{a}_{0,\tilde{y}} - \partial_\tau\tilde{a}_{2,\tilde{y}}\right)^2 + \frac{u}{8\pi}\left(\Delta\tilde{a}_{1,\tilde{y}} - \partial_x\tilde{a}_{2,\tilde{y}}\right)^2, \tag{5.59}$$

$$\tilde{\mathcal{L}}_{int} \equiv \frac{u}{8\pi}\sum_{\tilde{y}'} V_{\tilde{y},\tilde{y}'}\partial_x\left(\Delta\tilde{\phi}_{\tilde{y}} - \tilde{a}_{2,\tilde{y}}\right)\partial_x\left(\Delta\tilde{\phi}_{\tilde{y}'} - \tilde{a}_{2,\tilde{y}'}\right), \tag{5.60}$$

$$\tilde{\mathcal{L}}_{hop+phase-slip} \equiv -g_1\cos\left(2\tilde{\theta}_{\tilde{y}}\right) - g_2\cos\left(\Delta\tilde{\phi}_{\tilde{y}} - \tilde{a}_{2,\tilde{y}}\right). \tag{5.61}$$

Hence, the dual theory essentially describe dual vortices with long-range interactions minimally coupled to gauge fields.

5.1.5 Comparison

To compare the bosonic particle-vortex duality in the macroscopic approach with the one in the quantum wires one, consider the following schematic diagram, which presents the properties of the both dualities.

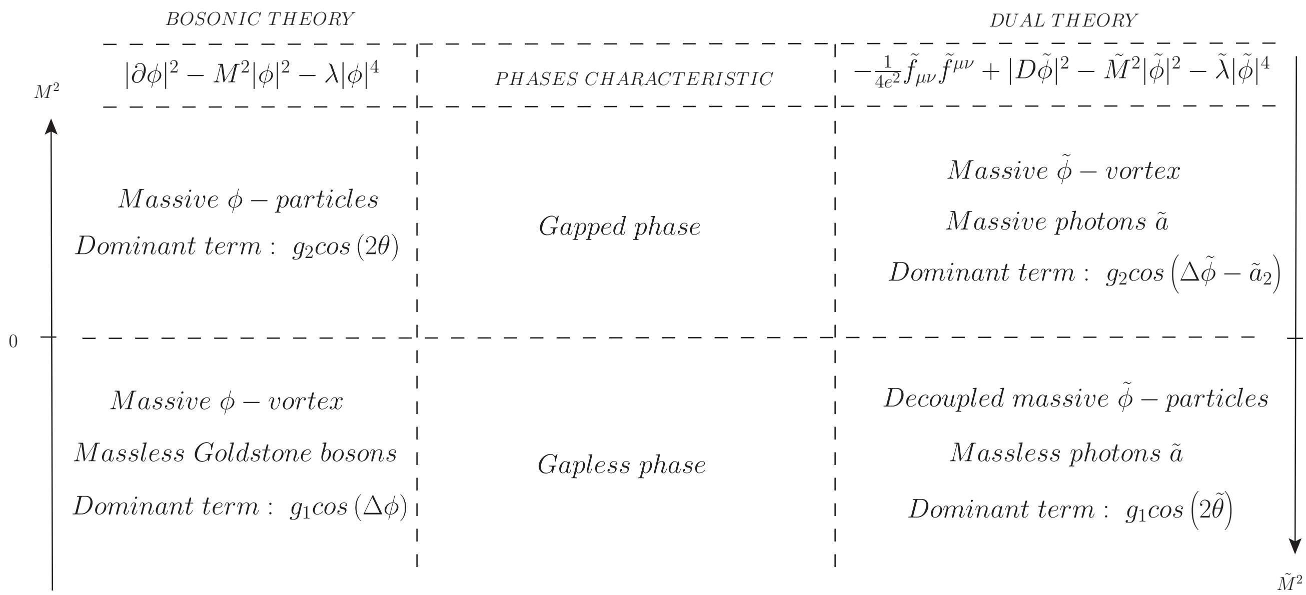


Figura 8 – Comparison between phase diagrams.

Notice that we have putted the wire terms in the diagrams based on the dominance of g_1 or g_2 in the original and in the dual Lagrangian (5.57), to facilitate the role identification of each term in the quantum wires construction. The qualitative relation between the mass parameters M^2 and $\tilde{M}^2$ and the wires coupling constants g_1 and g_2 is

$$M^2 \sim (g_2 - g_1) \sim -\tilde{M}^2. \tag{5.62}$$

The time to utterly compare these dualities has arrived.

From the macroscopic perspective, the XY vacuum gap exists because its only excitations are the massive $\phi - particles$, while the Abelian Higgs gap is generated by the Higgs mechanism, which gives a mass to the photon particles. On the other hand, the XY gapless phase occurs due to the appearance of the massless Goldstone bosons, whilst the Abelian Higgs gapless phase is a consequence of the presence of the massless photons.

Through the microscopic perspective, the gap is opened when g_2 is relevant, where the Φ mass gap happens due to the $g_2 cos\,(2\theta)$ term in the original theory. Whereas, the dual theory is gapped because of the photon mass, caused by the dominance of the term $g_2 cos\left(\Delta\tilde{\phi} - a_2\right)$. Meanwhile, the gapless phase arises when g_1 is relevant, as the original model is ruled by the superfluid term $g_1 cos\,(\Delta\phi)$, while the dual model is also gapless as a consequence of the term $g_1 cos\left(2\tilde{\theta}\right)$, permitting the photons to be massless.

In this way, we have conclude the duality connection between two bosonic theory, where bosonic particles with short-range interactions are mapped onto vortices with long-range interactions. It is remarkable to notice that these two particle-vortex dualities seem to be compatible, arguing in favor of nature redundancies.

5.2 Fermionic Particle-Vortex Duality Revisited

To give a brief motivation for the quantum wires construction of the fermionic particle-vortex duality, let us consider the macroscopic version of such a duality,

$$i\bar{\psi}\not{D}_a\psi + \frac{adA}{2\pi} \Longleftrightarrow i\bar{\psi}\not{D}_A\psi. \tag{5.63}$$

The right hand side represents the relativistic Landau problem, while the left hand side describes electrons coupled to a dynamical gauge field a and a background gauge field A. Historically, this duality was proposed by the physicist Dam Thahn Son in order to handle the $\nu = \frac{1}{2}$ quantum hall phase [15].

Let us derive the fermionic version of the particle-vortex duality in the quantum wires approach. We will do similar process to the ones done in the previous section. The starting point is to discretize the right hand side of (5.63) in terms of quantum wires. We will do it by first discretizing the y direction into wires, and then separating each of these wires in two wires, where the chirality of the fermions alternates with the label y. This discretization leads to the Euclidean action describing a system of a single chiral fermions ψ_j living in wires labeled by the integers j. This action is defined as

$$S = \int d\tau dx \sum_j \left[\psi_j^\dagger \partial_\tau \psi_j + \mathcal{L}_{chiral} + \mathcal{L}_{tunnel}\right], \tag{5.64}$$

$$\mathcal{L}_{chiral} = v(-1)^j \psi_j^\dagger \left(-i\partial_x\right)\psi_j, \tag{5.65}$$

$$\mathcal{L}_{tunnel} = g_j\left(i\psi_j^\dagger\psi_{j+1} - i\psi_{j+1}^\dagger\psi_j\right). \tag{5.66}$$

Notice that, by definition, the chirality is alternating along the wires due the factor $(-1)^j$. Also by definition, $g_j = g_1$ for odd j and $g_j = g_2$ for even j. We can interpret the theory as: the term $\mathcal{L}_{chiral}$ is a kinetic term for the chiral fermion in the wire j, while the $\mathcal{L}_{tunnel}$ allows the chiral fermions to tunneling to neighbor wires.

Let us now bosonize the theory based on the discussion in the chapter 1. We will write the chiral field as $\psi_j \sim e^{i\phi_j}$, where ϕ_j is a chiral bosonic field satisfying (compare with (1.39), (1.40) and (1.54))

$$\left[\phi_j(x), \phi_{j'}(x')\right] = i\pi(-1)^j \delta_{jj'} \text{sign}(x - x') + i\pi \left(1 - \delta_{jj'}\right) \text{sign}(j - j'). \tag{5.67}$$

These commutation rules between the bosonic chiral fields implies the correct anticommutation rules for the chiral fermions.

Transforming (5.64) with this bosonization map results in

$$\begin{aligned} S &= \int d\tau dx \sum_j \left[(-1)^j \frac{i}{4\pi} \partial_x \phi_j \partial_\tau \phi_j + \mathcal{L}_{chiral} + \mathcal{L}_{tunnel}\right], & (5.68) \\ \mathcal{L}_{chiral} &= \frac{v}{4\pi} \left(\partial_x \phi_j\right)^2, & (5.69) \\ \mathcal{L}_{tunnel} &= -g_j cos\left(\phi_j - \phi_{j+1}\right). & (5.70) \end{aligned}$$

Similarly to the procedure made in the previous section, we will dualize this theory with the map

$$\tilde{\phi}_j = \sum_{j' \neq j} (-1)^{j'} \text{sign}\left(j - j'\right) \phi_{j'}. \tag{5.71}$$

This transformation changes the commutator rule by a -1 factor:

$$\left[\tilde{\phi}_j(x), \tilde{\phi}_{j'}(x')\right] = -\left[\phi_j(x), \phi_{j'}(x')\right] \tag{5.72}$$

The relation above physically means that the dual bosons in each wire have opposite chirality to the original ones, and that the berry phase term in the Lagrangian will also gain a -1 factor.

From the map itself, we can verify that

$$\tilde{\phi}_{j+1} - \tilde{\phi}_j = (-1)^{j+1} \left(\phi_{j+1} - \phi_j\right). \tag{5.73}$$

This identity ensures that the tunnel term is invariant under this map.

Now, we need to deal with the chiral term, which has the form $\partial_x \phi_j$. Likewise in the previous section, this term will introduce non-local interactions between the dual variables, which reflects long-range interactions between the excitations of the dual theory. The introduction of gauge fields $\tilde{a}$ is sufficient to restore the locality.

In the same spirit of the bosonic case, introducing the gauge fields, as auxiliary fields, into the dual theory to restore the locality of interactions, the dual theory (in the gauge $\tilde{a}_2 = 0$) becomes

$$
\begin{aligned}
\tilde{S}_{dual} &= \int d\tau dx \sum_j \left[(-1)^j \frac{-i}{4\pi}\partial_x\tilde{\phi}_j\partial_\tau\tilde{\phi}_j + \mathcal{L}_{QED_3}\right], && (5.74)\\
\mathcal{L}_{QED_3} &= \mathcal{L}_0 + \mathcal{L}_{CS} + \mathcal{L}_{MW} + \mathcal{L}_{tunnel}[\tilde{\phi}], && (5.75)\\
\mathcal{L}_0 &= (-1)^j \frac{i}{2\pi}\partial_x\tilde{\phi}_j\tilde{a}_{0,j} + \frac{u}{4\pi}\left(\partial_x\tilde{\phi}_j - \tilde{a}_{1,j}\right)^2, && (5.76)\\
\mathcal{L}_{CS} &= (-1)^j \frac{i}{8\pi}\left(\tilde{a}_{1,j+1} + \tilde{a}_{1,j}\right), && (5.77)\\
\mathcal{L}_{MW} &= \frac{1}{16\pi}\left[\frac{1}{v}\left(\Delta\tilde{a}_{0,j}\right)^2 + v\left(\Delta\tilde{a}_{1,j}\right)^2\right]. && (5.78)
\end{aligned}
$$

Using a bosonization map with the form $\tilde{\psi}_j \sim e^{i\tilde{\phi}_j}$, one gets a theory of dual fermions coupled to gauge fields. This argumentation essentially proofs the duality discussed at chapter 4 between free fermions and QED, which is the fermionic particle-vortex duality.

Final Remarks and Perspectives

In this work, we have studied several properties of $2D$ and $3D$ abelian dualities between quantum field theories, which are important in their own and also play a fundamental role in the description of topological phases of matter. In a brief way, we have studied the $2D$ duality relation between bosons and fermions, the generalization of bosonization for the $3D$ case, the macroscopic and quantum wires construction of the particle-vortex dualities. We have also highlighted some reasons why the master duality is reliable, as well as the web of dualities that follows from it.

Almost all discussions about quantum field theory in the $3D$ web of dualities are given and justified in a more heuristic form, relying essentially on comparisons of phases according to the global symmetries and the corresponding spontaneous symmetry breaking pattern. The quantum wires approach provides a new perspective to this problem, in the sense that it allows more explicit derivation of the dualities discussed along the work, capturing precisely the underlying microscopic physics. We do expect that a deeper understanding of these dualities can allow us to explore several generalizations of them, improving the way we understand nature. This deepening process is currently under progress.

To close the work, the methods discussed here unravel possible paths to the investigation of more general dualities and their applications. In special, we would like to pursue the following directions: i) quantum wires approach in non-abelian dualities; ii) inspection of holography-like dualities; iii) applications of quantum wires and conformal field theory techniques to topological phases of matter. Finally, in a more fundamental level, we do expect to understand why theoretical physics allows this duality "redundancies" in the space of theories.

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Apêndices

APÊNDICE A – Aharonov-Bohm Effect

When a charged particle moves near to a magnetic field, its wavefunction suffers a phase shift according to its path. This is a quantum effect called *Aharonov-Bohm effect.* To see this explicitly, consider a massive charged particle with position $\vec{x}$ encircling a magnetic field described by potential $A^\mu = (A^0, \vec{A})$ confined on a "small"region in spacetime. We can derive the Aharonov-Bohm phase, in a very intuitive way, using the path integral approach. Along the dynamics, the particle's wavefunction $\Psi(x')$ is transported to $\Psi(Q)$ by the two-point function $K(Q, x')$ as $\Psi(Q) = \int d^2x'\ K(Q, x')\Psi(x')$. The action that describes this physics is

$$S = \int dt \left(\frac{m}{2} \dot{\vec{x}}^2 + q\dot{\vec{x}} \cdot \vec{A} \right) \equiv S_0 + \int dtq\dot{\vec{x}} \cdot \vec{A} = S_0 + \int qd\vec{x} \cdot \vec{A}. \tag{A.1}$$

Above, $K(x', x)$ is given in the path integral approach by [25]

$$K \sim \sum_{\text{all paths}} \exp\{iS\}. \tag{A.2}$$

In the actual setup, this sum made over all paths that passes through slits 1 and 2. The important quantity to measure is difference of phases

$$e^{i\Delta\varphi} \equiv \exp\left(i \int_P^Q qd\vec{x} \cdot \vec{A}|_{\text{slit 1}} - i \int_P^Q qd\vec{x} \cdot \vec{A}|_{\text{slit 2}} \right) = \exp\left(i \oint_C qd\vec{x} \cdot \vec{A} \right), \tag{A.3}$$

C being the path $P \to P$, first passing through slit 1 and then slit 2, which encircles the magnetic flux, as shown in the figure 9. If we consider a diagram of the setup, is clear that this phases difference can be made using the indicated closed path.

Using the Stokes theorem, we obtain

$$\Delta\varphi = q \oint_C d\vec{x} \cdot \vec{A} = q \int_R \underbrace{\nabla \times \vec{A}}_{=B} \cdot d\vec{S} = q\Phi_R, \tag{A.4}$$

where Φ_R is the B-flux across the region R enclosed by the closed line $C \equiv (P \to Q) \bigcup (Q \to P)$. The phase shift $\Delta\varphi = q\Phi_R$ derived before is acquired by the particle's wavefunction when encircling the magnetic region, without entering in it.

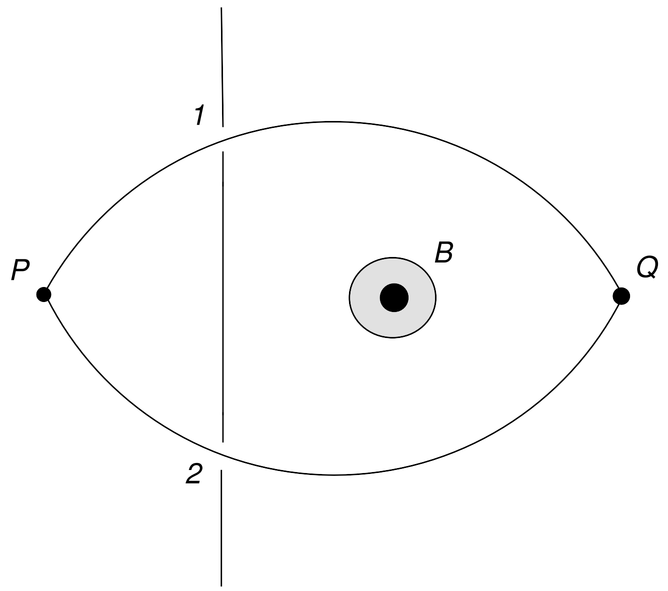

Figura 9 – Charged particle trajectory $P \to Q$ "through"slit 1 and $Q \to P$ "through"slit 2 around a magnetic field B confined in a small region of space. The difference of phases can be realized using the closed path above.