Universidade Estadual de Londrina

Bruno Oliveira de Araújo

Magnetic Dipole Moment of Neutrinos
Measurements and Perspectives

Londrina
2023

Bruno Oliveira de Araújo

Magnetic Dipole Moment of Neutrinos
Measurements and Perspectives

Dissertação apresentada ao Programa de Pós-graduação do Departamento de Física da Universidade Estadual de Londrina, como requisito parcial à obtenção do título de Mestre em Física.
Orientador: Prof. Dr. Pietro Chimenti

Londrina
2023

Ficha de identificação da obra elaborada pelo autor, através do Programa de Geração Automática do Sistema de Bibliotecas da UEL

A663m de Araújo, Bruno Oliveira .
Momento de dipolo magnético de neutrinos : medidas e perspectivas / Bruno Oliveira de Araújo. - Londrina, 2023.
112 f. : il.

Orientador: Pietro Chimenti.
Dissertação (Mestrado em Física) - Universidade Estadual de Londrina, Centro de Ciências Exatas, Programa de Pós-Graduação em Física, 2023.
Inclui bibliografia.

1. Física de neutrinos - Tese. 2. Além do Modelo Padrão - Tese. 3. Momento de dipolo magnético - Tese. 4. Neutrinos de baixas energias - Tese. I. Chimenti, Pietro . II. Universidade Estadual de Londrina. Centro de Ciências Exatas. Programa de Pós-Graduação em Física. III. Título.

CDU 53

Bruno Oliveira de Araújo

Magnetic Dipole Moment of Neutrinos

Measurements and Perspectives

Dissertação apresentada ao Programa de Pós-graduação do Departamento de Física da Universidade Estadual de Londrina, como requisito parcial à obtenção do título de Mestre em Física. Orientador: Prof. Dr. Pietro Chimenti

BANCA EXAMINADORA

____
Orientador: Prof. Dr. Pietro Chimenti
Universidade Estadual de Londrina - UEL

____
Christiane Frigerio Martins
Universidade Estadual de Londrina - UEL

____
Arturo Rodolfo Samana
Universidade Estadual de Santa Cruz - UESC

Londrina, _____ de ________________ de _______.

To whomever seeks knowledge.

Agradecimentos

Agradeço imensamente ao meu pai por todo o apoio e auxílio que me proporcionou ao longo dos diversos momentos da minha vida. Sua presença e orientação foram essenciais para meu crescimento.

Agradeço também à minha namorada, Nyara Tamarozi, por todo o apoio, compreensão e cuidado. Sua presença ao meu lado durante todo o período de desenvolvimento desta fase foi fundamental. Cada momento que compartilhamos foi um prazer e sempre será uma alegria dividir esse tempo com você.

Expresso minha gratidão ao meu primo, Mario Reinaldo da Silva, pela influência positiva que exerceu em minha vida, especialmente no que diz respeito aos estudos.

Gostaria de expressar minha sincera gratidão aos diversos amigos, tanto dentro quanto fora do ambiente acadêmico. Em particular, gostaria de agradecer à Lucas Queiroz Silveira, Luiz Felipe Demétrio e Marcelo Klug Pereira Pedrosa. Suas discussões sobre física, bem como os momentos de risadas, conversas e cervejas compartilhados ao longo dessa jornada, foram indispensáveis para a conclusão deste trabalho. Também gostaria de agradecer ao Matheus Rodrigues Silva e ao Gabriel Vinicius Silva Vieira de Souza pelas correções feitas neste trabalho, sua contribuição foi muito valiosa.

Não posso deixar de agradecer aos professores do departamento, que contribuíram significativamente para a minha formação como profissional e cidadão. Gostaria de expressar um agradecimento especial ao professor Dr. Pietro Chimenti por sua orientação neste trabalho. Além disso, sou grato por todos os bate-papos e pela amizade que construímos ao longo dos anos de colaboração. Também gostaria de estender meus agradecimentos à professora Dra. Christiane Frigerio Martins e ao professor Dr. Arturo Rodolfo Samana por aceitarem participar como membros da banca examinadora e por suas contribuições e correções neste trabalho. Seu envolvimento foi de grande importância e valor para o seu desenvolvimento.

Por fim, agradeço a CAPES pelo fornecimento de bolsas durante meu mestrado. Sem essa oportunidade, a conclusão desta etapa seria totalmente impossibilitada. Sou imensamente grato pela ajuda recebida. Também valorizo o papel das universidades públicas e de qualidade, que desempenham um papel fundamental na formação acadêmica e no avanço do conhecimento.

"No one's gonna take me alive
Time has come to make things right
You and I must fight for our rights
You and I must fight to survive"
Muse, Knights of Cydonia.

Araújo, B. O. **Magnetic Dipole Moment of Neutrinos: Measurements and Perspectives.** 2023. (112)fls. Dissertação de Mestrado (Programa de Pós-Graduação em Física).– Universidade Estadual de Londrina, Londrina, 2023.

Resumo

A descoberta do neutrino, predito por Wolfgang Pauli nos anos de 1930, desempenho papel de motor propulsor para o avanço da física de partículas e as mais variadas áreas correlacionadas. Quando Pauli propôs a existência dos neutrinos, foi levantado a possibilidade dessas partículas possuírem momento de dipolo magnético. A descoberta da oscilação de neutrinos, fenômeno que ocorre devido aos neutrinos serem partículas massivas, representou uma nova alavancada para o estudo dessas partículas e suas propriedades, trazendo também novas problemáticas sobre o assunto, uma primeira questão fundamental que surgiu é sobre a natureza do mecanismo de geração de massa dessas partículas, se esses são partículas de Dirac ou Majorana. Além disso, o valor de tais massas serem tão menores em comparação com os outros férmions do Modelo Padrão é de extrema curiosidade. Estimativas do valor do momento de dipolo magnético de neutrinos se diferente no caso deste ser uma partícula de Dirac ou Majorana. Neste trabalho, é apresentada uma revisão bibliográfica focado nas propriedades eletromagnéticas dos neutrinos, em especial no momento de dipolo magnético dos neutrinos, também é levado em consideração o uso de tal tema para o estudo de física além do Modelo Padrão.

A estrutura deste trabalho é organizada da seguinte forma: No capítulo 1, é apresentada uma breve visão geral das propriedades dos neutrinos, juntamente com algumas questões em aberto relacionadas ao tema. Já o capítulo 2 aborda os neutrinos de Dirac e Majorana, com foco nos mecanismos de geração de massa dessas partículas. O capítulo 3 concentra-se em apresentar a origem das propriedades eletromagnéticas dos neutrinos. No capítulo 4, é introduzido um Toy Model para o estudo do espalhamento entre neutrinos e núcleos por meio de interações eletromagnéticas, em que o estado final do núcleo encontra-se em um estado excitado. Neste capítulo, também são apresentados alguns resultados experimentais recentes das medições do momento de dipolo magnético, como o experimento XENONnT. O capítulo 5 concentra-se na apresentação do ^{6}Li como um possível detector para o estudo das interações de neutrinos em baixas energias. Além disso, são destacadas algumas das vantagens de utilizar esse detector. Por fim, são apresentadas as conclusões obtidas neste trabalho.
Palavras-chave: Física de Neutrinos. Além do Modelo Padrão. Momento de Dipolo Magnético. Neutrinos de Baixas Energias

Araújo,B. O. **Magnetic Dipole Moment of Neutrinos: Measurements and Perspectives.** 2023. (112)p. Master Deegre Thesis – Universidade Estadual de Londrina, Londrina, 2023.

Abstract

The discovery of the neutrino, predicted by Wolfgang Pauli in the 1930s, played a pivotal role in advancing particle physics and related fields. When Pauli proposed the existence of neutrinos, the possibility of these particles possessing a magnetic dipole moment was raised. The discovery of neutrino oscillation, a phenomenon that occurs due to the neutrinos' massiveness, opened new avenues for studying these particles and their properties, while also introducing new questions. One fundamental question that arose was the nature of the mechanism responsible for generating neutrino masses, whether they are Dirac or Majorana particles. Furthermore, the significantly smaller masses of neutrinos compared to other fermions in the Standard Model are of great interest. Estimates of the neutrino's magnetic dipole moment differ depending on whether it is a Dirac or Majorana particle. This work presents a literature review focused on the electromagnetic properties of neutrinos, particularly their magnetic dipole moment, while also considering the application of this topic in the study of physics beyond the Standard Model.

Keywords: Neutrino Physics, Beyond the Standard Model, Magnetic Dipole Moment, Low-Energy Neutrinos.

Lista de ilustrações

Lista de tabelas

Sumário

1 Introduction

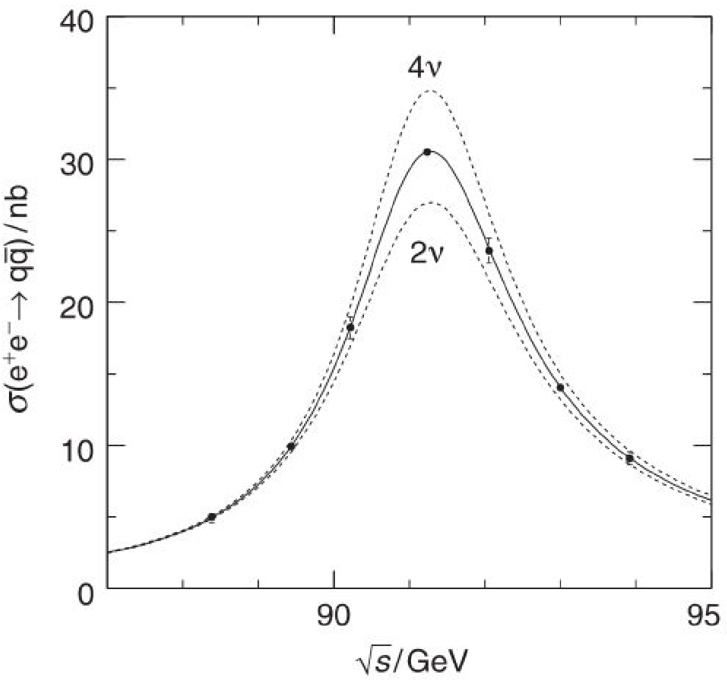
According to Pauli's definition, a neutrino is a particle with spin 1/2 that is both electrically neutral and extremely light. In Appendix A.3, we explore the essential labels required to characterize fundamental particles. In this work we will explore the eletromagnetic properties of neutrinos. The structure of this work is organized as follows: In Chapter 1, a brief overview of neutrino properties is presented, along with some open questions related to the topic. Chapter 2 discusses Dirac and Majorana neutrinos, focusing on the mechanisms responsible for their mass generation. Chapter 3 focuses on explaining the origin of the electromagnetic properties of neutrinos. In Chapter 4, a Toy Model is introduced to study the scattering between neutrinos and nuclei through electromagnetic interactions, where the final state of the nucleus is in an excited state. This chapter also presents some recent experimental results of measurements of the magnetic dipole moment, such as the XENONnT experiment. Chapter 5 focuses on the presentation of ^{6}Li as a potential detector for studying neutrino interactions at low energies. Additionally, some advantages of using this detector are highlighted. Finally, the conclusions drawn from this work are presented.This section will firstly present the fundamental knowledge about neutrinos, followed by listing significant unsolved problems concerning the properties of massive neutrinos.

1.1 Neutrinos: Fundamental Properties and Open Questions

1.1.1 Present Knowledge of Neutrinos

Three species of neutrinos, referred as flavors (ν_e, ν_μ, ν_τ), exist in accordance with the existence of three families of charged leptons (e, μ, τ), Q= +2/3 quarks (u, c, t), and Q= -1/3 quarks (d, s, b). The measurement of the decay width of the gauge boson Z^0 on the Large Electron-Positron Collider (LEP) experiment provides the most rigorous constraint on the number of neutrino species, indicating that $N_\nu = 2.984 \pm 0.008$.

Figura 1 – Cross section of the process $e^+ + e^- \longrightarrow q + \bar{q}$ near the resonance of the Z^0 boson.



Source: Modern Particle Physics [1].

This constraint is only applicable to the active neutrinos that participate in the standard weak interaction and have masses below $\frac{M_{Z^0}}{2} \approx 45.6$ GeV [2].

Owing to the precise measurements of electric charges of electrons, protons, and neutrons, the electric charge of an electron antineutrino in beta decay can be experimentally limited to $e_\nu < 3 \times 10^{-21}$e, in accordance with the fundamental law of electric charge conservation [3].

Angular momentum conservation, being a fundamental law, allows one to infer the spin of neutrinos as $1/2$ by considering processes such as $\pi^+ \longrightarrow \mu^+ + \nu_\mu$ and $\pi^+ \longrightarrow e^+ + \nu_e$. In both, the orbital-electron capture and decay mode $\pi^- \longrightarrow \mu^- + \bar{\nu}_\mu$, the helicity, denoted as λ, of the electron neutrino and muon antineutrino were found to be, respectively, left-handed ($\lambda = -1/2$) and right-handed ($\lambda = +1/2$) [4, 5].

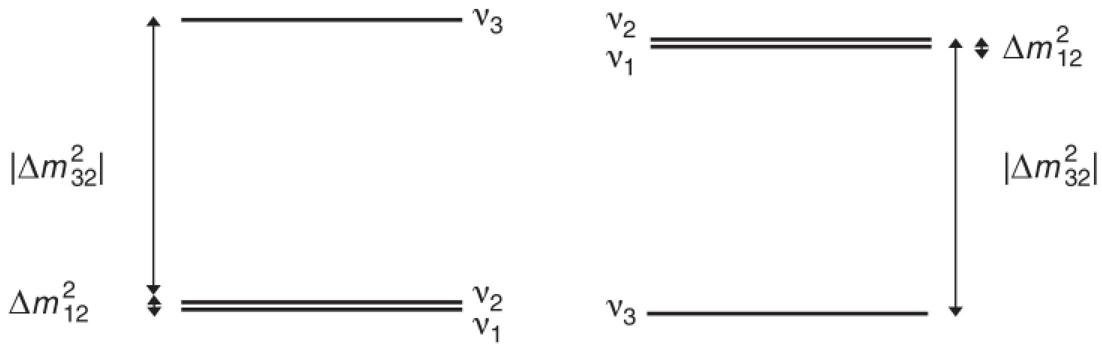
The helicity of the tau neutrino was determined by the ALEPH Collaboration through the measurement of the decay mode $\tau^- \longrightarrow l_\alpha + \bar{\nu}_\alpha + \nu\tau$, where $\alpha =$ e or μ. The measured value was found to be $\lambda = -0.496 \pm 0.006$, which is in agreement with the prediction of the Standard Model (SM)[6].

The detection of neutrino oscillations represents a significant advancement in particle physics as it suggests that neutrinos possess rest masses and that the SM is incomplete.

Current experimental data have revealed the values of two independent in difference of mass-squared of neutrinos: $\Delta m^2_{21} \equiv m^2_2 - m^2_1 \sim 7.7 \times 10^{-5}\text{eV}^2$ and $\Delta m^2_{32} \equiv$

$m_3^2 - m_2^2 = \pm 2.4 \times 10^{-3} \mathrm{eV}^2$. The absolute scale of neutrino masses cannot be determined from neutrino oscillations. As a consequence, two different mass hierarchies have been established: the normal hierarchy, in which $m_3 > m_2$, and the inverted hierarchy, where $m_2 > m_3$.

Figura 2 – Mass Hiearchies.



Source: Modern Particle Physics [1].

The neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$ and the neutrino mass eigenstates (ν_1, ν_2, ν_3) are not identical, resulting in the phenomenon of neutrino flavor mixing. To express the mass eigenstates in terms of the flavor eigenstates, we utilize the Pontecorvo-Maki-Nakagawa-Sakata (PMSN) matrix, which is a unitary matrix. We can express the PMNS matrix by utilizing three mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$ and one complex phase δ_{CP} (see Apendix E). The latest measurements of mixing angles, mass-squared differences, and the CP-violating phase δ_{CP} are presented in the following table

Tabela 1 – Oscillation parameters for three flavors, considering a global data fit.

Normal Hierarchy			Inverted Hierarchy		
	$bfp \pm 1\sigma$	3σrange		$bfp \pm 1\sigma$	3σrange
$\theta_{12}/°$	$33,44^{+0,78}_{-0,75}$	$31,27 \rightarrow 35,86$	$\theta_{12}/°$	$33,45^{+0,78}_{-0,75}$	$31,27 \rightarrow 35,87$
$\theta_{23}/°$	$49,00^{+1,10}_{-1,40}$	$39,60 \rightarrow 51,80$	$\theta_{23}/°$	$49,30^{+0,90}_{-1,10}$	$40,30 \rightarrow 51,80$
$\theta_{13}/°$	$08,57^{+0,13}_{-0,12}$	$08,20 \rightarrow 08,97$	$\theta_{13}/°$	$08,60^{+0,12}_{-0,12}$	$08,24 \rightarrow 08,96$
$\delta_{CP}/°$	$195,0^{+51,0}_{-25,0}$	$107,0 \rightarrow 403,0$	$\delta_{CP}/°$	$283,0^{+26,0}_{-30,0}$	$193,0 \rightarrow 352,0$
$\frac{\Delta m^2_{21}}{10^{-5}eV^2}$	$07,42^{+0,21}_{-0,20}$	$06,82 \rightarrow 08,04$	$\frac{\Delta m^2_{31}}{10^{-5}eV^2}$	$07,42^{+0,21}_{-0,20}$	$06,82 \rightarrow 08,04$
$\frac{\Delta m^2_{31}}{10^{-5}eV^2}$	$2,514^{+0,028}_{-0,027}$	$2,431 \rightarrow 2,598$	$\frac{\Delta m^2_{32}}{10^{-5}eV^2}$	$-2,498^{+0,028}_{-0,028}$	$-2,581 \rightarrow -2,414$

Source: The Fate Of Hints: Upadated Global Analysis Of Three-Flavor Neutrino Oscillations [7].

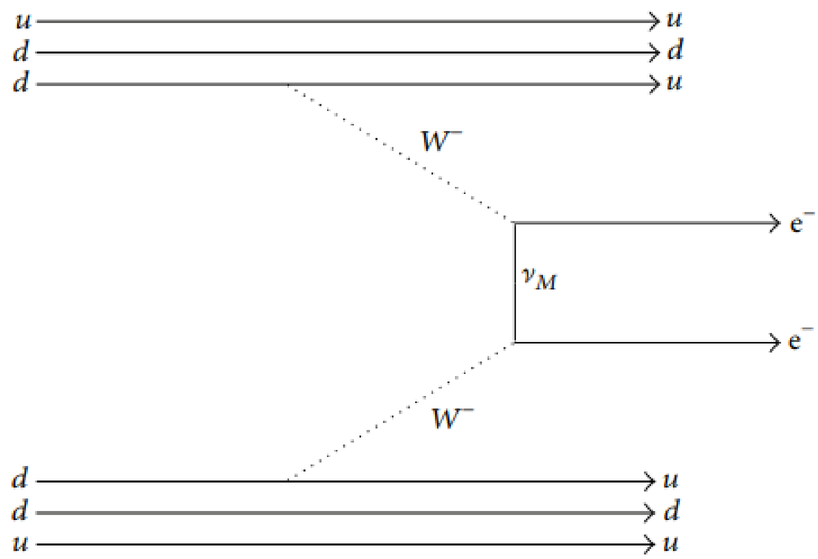
There is an equivalent quark mixing in the baryonic sector, which is described in terms of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. However, the values of θ_{12} and θ_{23} in the PMNS matrix are significantly greater than the largest quark mixing angle (Cabibbo angle $\vartheta_c \approx 8$). This suggests that there may be a fundamental distinction between the origins of lepton mixing and quark flavor mixing.

1.1.2 Open Questions on Neutrinos

There are numerous uncertainties regarding the intrinsic properties of neutrinos. Below, will provide a compilation of significant and urgent questions in the fields of neutrino physics, astronomy, and cosmology.

The question of whether massive neutrinos are Dirac or Majorana particles remains unresolved. If they are Dirac particles, then they can be distinguished from their antiparticles due to the conservation of lepton number. On the other hand, if they are Majorana particles, they are indistinguishable from their antiparticles and this can result in processes where lepton number conservation is violated. One possible approach to confirming the Majorana nature of the three known neutrinos is to detect the neutrinoless double-beta decay ($0\nu 2\beta$) (see Figure 3). Until now, there is no conclusive proof indicating the existence of $0\nu 2\beta$ decay.

Figura 3 – $0\nu 2\beta$ diagram.



Source: Neutrinoless Double Beta Decay: 2015 Review [8].

The second issue concerns the absolute mass scale of neutrinos, which can be investigated using three possible methods. One such method involves studying the effective mass value in tritium β-decay, which is defined by $\langle m \rangle_e$ and is currently being explored by the KATRIN collaboration. This value is calculated using the equation

$$\langle m \rangle_e \equiv \sqrt{m_1^2|U_{e1}|^2 + m_2^2|U_{e2}|^2 + m_3^2|U_{e3}|^2}.$$

The KATRIN collaboration has established an upper limit of $m_\nu < 0.9\text{eV}^2$[9]. In the case that massive neutrinos are Majorana particles, their masses may be constrained by detecting the $0\nu 2\beta$ decay. Additionally, valuable insights regarding the absolute neutrino mass scale can be obtained from cosmology and astrophysics.

Another interesting question is related to the detection of neutrinos cosmic background. Cosmological models suggest that the present-day abundance of relic neutrinos is approximately equal to that of relic photons [10]. Our understanding of the matter content and structure formation of the Universe has been greatly enhanced by the measurement of the Cosmic Microwave Background (CMB). However, detecting the Cosmic Neutrino Background (CνB) represents a significant obstacle, given that the energies of relic neutrinos cause them to interact too weakly with matter.Although various methods have been developed, the current technological limitations continue to restrict their sensitivities, which are several orders of magnitude insufficient for successfully detecting the CνB.

In addition to the previous question regarding cosmology and neutrinos, there is another issue to consider regarding the impact of neutrinos on the universe's evolution. The confirmation of neutrinos existence in the early universe has been indirect and based on the success of the Standard Model of Cosmology. The present observational data on the abundances of primordial light elements, the power spectra of CMB anisotropies, and the formation of large-scale structures all indicate that neutrinos play a vital role in the evolution of the Universe [11].

There is a possibility that the asymmetry in the baryon number observed in the Universe is linked to the presence of heavy Majorana neutrinos during the early stages of the Universe [12]. If this is the case, it means that the evolution of the Universe was influenced by these neutrinos. Testing the mechanisms of baryogenesis-via-leptogenesis is still a significant question in the field of particle physics and cosmology today [13].

2 Dirac and Majorana Neutrinos

The Dirac equation is satisfied by a massive spin-1/2 particles, in the case of massless particle, the Dirac equation reduce to the Weyl equation (see appendix A.2.5). The standard mass term for fermions can be written as:

$$\mathcal{L}_m = -m\overline{\psi}\psi. \tag{2.1}$$

where m represents the mass of the fermion and ψ is the fermionic field. By introducing the left and right projections defined as:

$$\psi_L = P_L\psi = \frac{1-\gamma^5}{2}\psi; \tag{2.2}$$

$$\psi_R = P_R\psi = \frac{1+\gamma^5}{2}\psi, \tag{2.3}$$

we can express the mass term as:

$$\mathcal{L}_m = -m\overline{\psi}\psi = -m\left(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L\right), \tag{2.4}$$

where we have used $P_LP_R = P_RP_L = 0$. However, it should be noted that this particular term does not preserve the $SU(2)$ gauge symmetry, and consequently, it violates renormalizability [14]. In the context of the Standard Model, fermion masses are generated through Yukawa couplings to the Higgs doublet. For example, we can consider the following term:

$$\mathcal{L}_{Y_e} = -\lambda_e(\overline{E_L}^i\phi^i)e_R + h.c.. \tag{2.5}$$

In this equation, E_L represents the electron lepton $SU(2)$-doublet:

$$E_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L . \tag{2.6}$$

The Higgs doublet is denoted by $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, and e_R represents the right-handed electron, which is a $SU(2)$-singlet. By implementing spontaneous symmetry breaking, as discussed in appendix (D.3), and considering the vacuum configuration $\phi^{(v)} = \begin{pmatrix} 0 \\ \frac{v+\chi(x)}{\sqrt{2}} \end{pmatrix}$, where v is the vacuum expectation value (vev), we obtain:

$$\mathcal{L}_{Y_e} = \frac{-\lambda_e}{\sqrt{2}}v\overline{e_L}e_R + h.c + \cdots \tag{2.7}$$

In the above expression, $\cdots$ denotes additional interaction terms involving the Higgs field. In this context, we find that $m_e = \frac{\lambda_e v}{\sqrt{2}}$, and we have successfully generated an electron mass term that preserves gauge invariance. This mechanism is employed in the Standard Model to generate masses for charged leptons while maintaining a global $U(1)$ symmetry, where the conserved charge corresponds to the lepton (or baryon) number (see appendix C.2). The Yukawa coupling constants for various channels were determined through the study of the decay of the Higgs boson into fermion-antifermion pairs at the Large Hadron Collider (LHC). These measurements are in good agreement with the predictions of the Standard Model [15].

The phenomenon of neutrino oscillation indicates that neutrinos are massive particles (see appendix E), where we observe that the three neutrino flavor states are associated with three massive states. However, introducing mass to neutrinos presents a challenge. By examining equation (2.4), we can see that it results in the coupling of left-handed (LH) and right-handed fermions. Nevertheless, due to parity violation in the weak interaction, the Standard Model does not include right-handed neutrinos. It is possible to introduce RH neutrinos in the Standard Model to generate neutrino masses through Yukawa couplings. However, the smallness of neutrino masses suggests that they may originate from an alternative mechanism [16].

In the framework of neutrino oscillation, the neutrino masses act as oscillation parameters, yet their mass smallness remains unexplained. To address this issue, Weinberg proposed a mechanism beyond the Standard Model that involves a Majorana mass term for neutrinos [17]. This mechanism is based on the effective Lagrangian approach and will be discussed subsequent to our exploration of Majorana neutrinos.

Majorana aimed to formulate the Dirac equation in a manner that exhibits complete symmetry between particles and antiparticles. In the bosonic sector, we encounter particles that exhibit the unique property of being their own antiparticles. One such example is the photon. Additionally, there exist bound states of quarks that possess this characteristic, such as the π^0 meson.

To facilitate our discussion, it is necessary to introduce the particle-antiparticle conjugation operator $\hat{C}$. Its action on a fermion field ψ is defined as follows:

$$\psi \xrightarrow{\hat{C}} \psi^c = \hat{C}\overline{\psi}^T, \tag{2.8}$$

in this expression, the matrix $\hat{C}$ satisfies the following properties:

$$\hat{C}^{-1}\gamma^\mu\hat{C} = -(\gamma^\mu)^T, \qquad \hat{C}^{-1}\gamma^5\hat{C} = (\gamma^5)^T, \qquad \hat{C}^\dagger = \hat{C}^{-1} = -\hat{C}^*. \tag{2.9}$$

In the Weyl representation of γ-matrices, given by:

$$\gamma^0 = \sigma^1 \otimes \sigma^0, \qquad \gamma^i = i\sigma^2 \otimes \sigma^i, \qquad \gamma^5 = -\sigma^3 \otimes \gamma^0,$$

where $\otimes$ represents the Kronecker product, σ^i $(i = 1,\ 2,\ 3.)$ denotes the 2×2 Pauli matrices, and $\sigma^0 = 1$ is the 2×2 identity matrix. We can choose $\hat{C} = i\gamma^2\gamma^0$. The anticommutation relation of the Dirac matrices implies that when $\hat{C}$ acts on a chiral field, it flips its chirality. For instance:

$$\begin{aligned} \psi_L \xrightarrow{\hat{C}} (\psi_L)^c &= \hat{C}\overline{(\psi_L)}^T = \hat{C}P_R^T\psi^* \\ &= \frac{\hat{C}}{2}\left(1+(\gamma^5)^T\right)\psi^*, \qquad \text{using } \hat{C}^{-1}\gamma^5\hat{C} = (\gamma^5)^T \\ \Rightarrow (\psi_L)^c &= P_R\psi^c = (\psi^c)_R. \end{aligned} \tag{2.10}$$

Similarly,

$$\psi_R \xrightarrow{\hat{C}} (\psi_R)^c = (\psi^c)_L. \tag{2.11}$$

Hence, the antiparticle of a left-handed fermion is a right-handed fermion.

In our discussion of the Dirac equation (see appendix A.2.4.3), we introduced a four-component spinor by combining two independent chiral spinors. The right-handed (RH) and left-handed (LH) components of the Dirac spinor were treated as independent. However, we can impose a constraint on this degree of freedom by recognizing that a RH antiparticle is equivalent to a LH particle. Therefore, we construct the following four-component spinor:

$$\psi = \psi_L + (\psi_L)^c = \psi_L + (\psi^c)_R. \tag{2.12}$$

In this case, we have a Majorana spinor field, which can be described using only one chiral field. In this picture we have a more economical case with comparasion with the Dirac field. It is straightforward to show that:

$$\psi = \psi_L + (\psi_L)^c \xrightarrow{\hat{C}} \psi^c = (\psi_L)^c + \psi_L, \tag{2.13}$$

where we have used $(\psi^c)^c = \psi$. This implies that particles associated with Majorana fields are genuinely neutral, meaning they are their own antiparticles[1]. The condition $\psi^c = \psi$ is called the Majorana condition. Majorana fermions have only two degrees of freedom and while they are essentially a two component spinor, it is often useful to write them in the four componente notation. In chiral representation, we consider $\psi_L = \begin{pmatrix}\phi\\0\end{pmatrix}$, therefore using the $\hat{C}$ operator we can write

$$\begin{aligned} \psi_R &= \hat{C}\overline{\psi_L}^T = i\gamma^2\psi_L^* \\ &= \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}\begin{pmatrix}\phi^*\\0\end{pmatrix} = \begin{pmatrix} 0 \\ -i\sigma^2\psi^* \end{pmatrix}. \end{aligned} \tag{2.14}$$

[1] A Majorana fermion is a neutral fermion, and according to the CPT theorem, the particle and its antiparticle have the same mass.

Therefore, a Majorana field can be written in the four-component form as:

$$\psi = \begin{pmatrix} \phi \\ -i\sigma^2\phi^* \end{pmatrix}. \tag{2.15}$$

Satisfying

$$\psi^c = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \phi^* \\ -i\sigma^2\phi \end{pmatrix} = \begin{pmatrix} \phi \\ -i\sigma^2\phi^* \end{pmatrix} = \psi. \tag{2.16}$$

Consider the Dirac equation (Equation $A.62$) for the Majorana spinor defined by Equation (2.15). It can be expressed as:

$$\begin{aligned} &\left(i\gamma^0\partial_0 + i\vec{\gamma}\cdot\vec{\nabla} - m\right)\begin{pmatrix} \phi \\ -i\sigma^2\phi^* \end{pmatrix} = 0 \\ &\begin{pmatrix} -m & i\partial_0 + i\vec{\sigma}\cdot\vec{\nabla} \\ i\partial_0 - i\vec{\sigma}\cdot\vec{\nabla} & -m \end{pmatrix}\begin{pmatrix} \phi \\ -i\sigma^2\phi^* \end{pmatrix} = 0, \end{aligned} \tag{2.17}$$

resulting in two equations

$$\begin{aligned} (\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\sigma^2\phi^* - m\phi = 0; \\ (\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\phi + m\sigma^2\phi^* = 0. \end{aligned} \tag{2.18}$$

These equations are equivalent, which can be seen by taking the complex conjugate of the first equation:

$$\begin{aligned} &(\partial_0 + \vec{\sigma}^*\cdot\vec{\nabla})(\sigma^2)^*\phi - m\phi^* = 0 \\ \Rightarrow &-\sigma^2(\partial_0 + \vec{\sigma}^*\cdot\vec{\nabla})\sigma^2\phi - m\sigma^2\phi^* = 0 \\ \Rightarrow &(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\phi + m\sigma^2\phi^* = 0. \end{aligned}$$

We have used the fact that $\sigma^2\vec{\sigma}^*\sigma^2 = -\vec{\sigma}$. Therefore, the two equations are equivalent, as we would expect since we are reducing the degrees of freedom in the theory.

The mass term of the Dirac Lagrangian (Equation (2.4)) can now be written as:

$$\begin{aligned} \mathcal{L}_m &= -m\left(\overline{\psi}_R\psi_L + h.c\right) \\ &= -m\left(\overline{(\psi_L)^c}\psi_L + h.c\right). \end{aligned} \tag{2.19}$$

Using the properties:

$$\begin{aligned} \overline{\psi^c} &= \overline{(\hat{C}\overline{\psi}^T)} = \left(\hat{C}\overline{\psi}^T\right)^\dagger\gamma^0 = \left[\hat{C}\left(\psi^\dagger\gamma^0\right)^T\right]^\dagger\hat{C}^\dagger\gamma^0 \\ &= \left(\psi^\dagger\gamma^0\right)^*\hat{C}^\dagger\gamma^0 = \psi^T\gamma^0\hat{C}^{-1}\gamma^0 = -\psi^T\hat{C}^{-1}, \end{aligned} \tag{2.20}$$

where we have used the first equation of (2.9), the Majorana mass term becomes:

$$\mathcal{L}_m = -m\left(-\psi_L^T\hat{C}^{-1}\psi_L + h.c\right), \tag{2.21}$$

and we define the Majorana Lagrangian as:

$$\mathcal{L}^M = \overline{\psi_L} i\gamma^\mu \partial_\mu \psi_L + \frac{m}{2}\left[\psi_L^T \hat{C}^{-1}\psi_L + h.c\right], \tag{2.22}$$

where the factor $\frac{1}{2}$ in the mass term is introduced for convenience since the term is quadratic in the field. This Lagrangian can be expressed in terms of the ϕ field using:

$$\psi_L = \begin{pmatrix}\phi\\0\end{pmatrix}, \qquad \psi_L^T = \begin{pmatrix}\phi^T & 0\end{pmatrix}, \qquad \overline{\psi_L} = \begin{pmatrix}0 & \phi^\dagger\end{pmatrix}, \qquad \hat{C}^{-1} = -i\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}.$$

Using this in Equation (2.22), we can express the Majorana Lagrangian as:

$$\mathcal{L}^M = i\left(\phi^\dagger \partial_0 \phi - \phi^\dagger \vec{\sigma}\cdot\vec{\nabla}\phi\right) - \frac{m}{2}\left(\phi^T i\sigma^2\phi + h.c\right), \tag{2.23}$$

It is explicitly not $U(1)$-invariant. Therefore, the Majorana Lagrangian breaks $U(1)$-charge conservation, such as lepton or baryon number, electric charge, etc. In the massless limit, the distinction between a Majorana or a Dirac particle vanishes, as both become Weyl particles. In this case, we recover $U(1)$ symmetry.

If Standard Model (SM) particles interact with heavy particles beyond the Standard Model, with masses much larger than the vacuum expectation value $v \approx$ 246GeV, the fields of the heavy particles in the electroweak region can be "integrated out"[2] and this new interaction induce a nonrenormalizable effective Lagrangian [17]. So the SM Lagrangian is written as:

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \frac{1}{\Lambda}Q_n^{(5)}, \tag{2.24}$$

where $Q^{(5)}$ is a set of operators with dimension $[m]^5$ written in terms of the SM fields. Imposing the SM gauge symmetry on $Q_n^{(5)}$ leaves only a single operator [18]:

$$Q_{\nu\nu} = \left(\tilde{\phi}^\dagger l_p\right)^T \hat{C}^{-1}\left(\tilde{\phi}^\dagger l_r\right), \tag{2.25}$$

where $\widetilde{\phi}_j = \epsilon_{jk}\phi^{*k}$ is the conjugate of the Higgs field, and $p,\ r = e,\ \mu,\ \tau$. Are the generation indexes, while l_p is the $SU(2)$-doublet. By using index notation and employing the unitary gauge to incorporate spontaneous symmetry breaking, we find:

$$\begin{aligned} Q_{\nu\nu} &= \epsilon_{jk}\epsilon_{mn}\phi^j\phi^m (l_p^k)^T \hat{C}^{-1} l_r^n = \frac{1}{2}\left(v + \chi(x)\right)^2 \nu_p^T \hat{C}^{-1}\nu_r \\ &= \frac{1}{2}v^2 \nu_p^T \hat{C}^{-1}\nu_r + v\chi(x)\nu_p^T \hat{C}^{-1}\nu_r + O\left(\chi^2(x)\right). \end{aligned} \tag{2.26}$$

The first part of the equation represents a Majorana mass term for neutrinos, indicating that physics beyond the Standard Model has the potential to generate such masses. This

[2] The term "integrated out"is employed in the context of the Feynman Path integral formalism of Quantum Field Theory, where the amplitude is computed as $A = \int D\phi_{\text{light}} D\phi_{\text{heavy}} \exp\left(iS\right) \prod_i V_i$, By integrating over all configurations of the ϕ_{heavy} field, an effective Quantum Field Theory is obtained.

phenomenon, in turn, can lead to neutrino flavor oscillation. The introduction of the parameter Λ in equation (2.24), which has mass dimension, acts as a suppression factor for the neutrino mass, due to the of new physics scale represented by Λ. Consequently, this provides a plausible explanation for the observed smallness of neutrino masses [17].

The Majorana mass term in the Lagrangian (2.22) applies for a single Majorana particle, we can extend the analysis to the case of multiple Majorana fermions. In this scenario, the Majorana mass term can be expressed as follows:

$$\mathcal{L}_m^M = \frac{1}{2}\left(\psi_L^T \hat{C}^{-1} m \psi_L + h.c\right), \tag{2.27}$$

where $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$ and m represents an $n \times n$ matrix that acts on the generation space, often referred to as the mass matrix. The m matrix is symmetric, since[3]

$$\begin{aligned} \psi^T M \psi &= \psi_i^T M_{ij} \psi_j \\ &= -\psi_j (M^T)_{ji} \psi_i^T = -\psi^T M^T \psi, \end{aligned} \tag{2.28}$$

in this case $M = \hat{C}^{-1} m \Rightarrow M^T = m^T (\hat{C}^T)^{-1} = -\hat{C}^{-1} m^T$, and in the last equation we use that $\hat{C}$ acts on spinor space and m acts on the generation space. Finally, we find

$$\begin{aligned} \psi^T \hat{C}^{-1} m \psi &= \psi^T \hat{C}^{-1} m^T \psi \\ &\Rightarrow m = m^T. \end{aligned} \tag{2.29}$$

Consider the case of n standard lepton generations, each consisting of a $SU(2)_L$-doublet of LH neutrino and charged lepton field denoted by $l_\alpha = \begin{pmatrix} \nu_\alpha \\ e_\alpha \end{pmatrix}_L$, and a $SU(2)_L$-single RH charged lepton field e_{α_R} $(\alpha = e,\ \mu,\ \tau, \cdots)$. In the Standard Model extended to include the mass generation mechanism for Majorana neutrinos, the terms of the Lagrangian relevant to neutrino oscillation include the charged-current weak interaction term and the mass term of charged leptons and neutrinos [16]:

$$\mathcal{L}_{w+m} = -\frac{g}{\sqrt{2}}\left(\overline{e'}_L \gamma^\mu \nu'_L\right) W_\mu - \overline{e}'_L m'_l e'_R + \frac{1}{2} \nu'^T_L \hat{C}^{-1} m' \nu'_L + h.c \tag{2.30}$$

The matrices $m'l$ and ml are generally non-diagonal in the weak-eigenstate basis. For n lepton generations, the mass matrix of charged leptons $m'l$ is a general complex $n \times n$

[3] In the second line, we apply the field quantization rule for fermionic fields. The procedure of field quantization is extensively discussed in many books on Quantum Field Theory, such as the reference [19]. This rule provides a framework for properly quantizing fermionic fields and is essential in the study of quantum phenomena in particle physics.

matrix, while the Majorana mass matrix m' is a complex symmetric $n \times n$ matrix. A square matrix A can be diagonalized by a bi-unitary transformation given by

$$A_{\text{diag}} = V_1^\dagger A V_2,$$

where A_{diag} is a diagonal $n \times n$ matrix with non-negative diagonal elements [20]. Similarly, a symmetric square matrix B can be diagonalized by a transformation using a single unitary matrix, resulting in

$$B_{\text{diag}} = U^T B U,$$

where all the diagonal elements of B_{diag} are non-negative. Therefore, we perform a basis transformation on the lepton fields as follows:

$$e'_L = V_L e_L, \qquad e'_R = V_R e_R, \qquad \nu'_L = U_L \nu_L.$$

In the new basis, the interaction term of the Lagrangian (2.30) takes the form:

$$\begin{aligned}
\mathcal{L}_w &= -\frac{g}{\sqrt{2}} \sum_{\alpha} \left(\overline{e'}_{L\alpha} \gamma^\mu \nu'_{L\alpha} \right) W_\mu + h.c \\
&= -\frac{g}{\sqrt{2}} \sum_{\alpha,\, \beta,\, i} \left(\bar{e}_{L_\beta} V^\dagger_{L_{\beta\alpha}} \gamma^\mu U_{L_{\alpha i}} \nu_{L_i} \right) W_\mu + h.c \\
&= -\frac{g}{\sqrt{2}} \sum_{\alpha,\, i} \left(\bar{e}_{L\alpha} \gamma^\mu U_{\alpha i} \nu_{L_i} \right) W_\mu + h.c.
\end{aligned}$$

A similar procedure can be applied to the Majorana neutrinos' mass term and the charged lepton mass term in the Lagrangian (2.30), resulting in the following expressions:

$$\mathcal{L}_{w+m} = -\frac{g}{\sqrt{2}} \sum_{\alpha,\, i} \left(\bar{e}_{L\alpha} \gamma^\mu U_{\alpha i} \nu_{L_i} \right) W_\mu - \sum_{\alpha} m_{l_\alpha} \bar{e}_{L\alpha} e_{R\alpha} + \frac{1}{2} \sum_{i} m_i \nu^T_{L_i} \hat{C}^{-1} \nu_{L_i} + h.c. \tag{2.31}$$

Here, m_{l_α} $(\alpha = e,\ \mu,\ \tau, \cdots)$ and m_i $(i = 1,\ 2, \cdots)$ are the diagonal elements of the mass matrices m_l and m respectively. These elements correspond to the masses of the charged leptons and the mass-eigenstate neutrinos. The PMNS matrix (see appendix E) is defined as

$$U \equiv V_L^\dagger U_L \tag{2.32}$$

The flavour-eigenstate neutrino fields can be expressed as

$$\nu_{L\alpha} \equiv \sum_{i=1}^{n} U_{\alpha i} \nu_{L_i}. \tag{2.33}$$

With these definitions, the charged-current interaction part of the Lagrangian (2.31) can be written as

$$\mathcal{L}_w = -\frac{g}{\sqrt{2}} \sum_{\alpha} \left(\bar{e}_{L\alpha} \gamma^\mu \nu_{L\alpha} \right) W_\mu + h.c, \tag{2.34}$$

where the flavour-eigenstate $\nu_e,\ \nu_\mu,\ \nu_\tau, \cdots$ correspond to the neutrinos emitted or absorbed together with the charged leptons $e,\ \mu,\ \tau, \cdots$ respectively. This description implies that massive neutrinos are generally mixed and gives rise to the phenomenon of neutrino flavour oscillations.

3 Neutrinos Eletromagnetic Interaction

Currently, one of the most active research fields in high-energy physics is the study of neutrino properties and interactions, which employs both theoretical and experimental methods. This research provides valuable insights into the mechanisms of the Standard Model of physics and serves as a powerful tool to explore physics beyond the Standard Model. In his famous 1930 letter addressed to "Dear Radioactive Ladies and Gentlemen", Pauli proposed the existence of neutrinos and speculated that they could possess a magnetic moment [21].

Although in the Standard Model neutrinos are electrically neutral and do not possess electric or magnetic dipole moments, they have a charge radius which is generated by radioactive corrections. We now know that neutrinos are massive, because many experiments observed neutrino oscillations (see appendix E). Hence, it is necessary to expand the Standard Model to accommodate the masses of neutrinos. Multiple extensions of the Standard Model exist, each with distinct predictions for the properties of neutrinos. One of the most significant properties among them is their fundamental Dirac or Majorana character, as discussed in Chapter 2.

Through quantum loop effects, neutrinos obtain electromagnetic properties in numerous extensions of the Standard Model. As a result, neutrinos can directly interact with electromagnetic fields and charged particles through electromagnetic interactions. Consequently, investigating the electromagnetic interactions of neutrinos, both theoretically and experimentally, is a powerful method to explore the fundamental theory beyond the Standard Model. Additionally, these interactions can produce significant effects, particularly in astrophysical settings, where neutrinos travel long distances in vacuum and matter and encounter magnetic fields [21].

The electromagnetic properties of neutrinos are crucial in discerning between Dirac and Majorana neutrinos. For instance, these properties can assist in differentiating between the two types since Dirac neutrinos can possess both diagonal and off-diagonal magnetic and electric dipole moments, whereas Majorana neutrinos are restricted to off-diagonal moments [22, 23].

3.1 Eletromagnetic Current Decomposition

This approach identifies form factors that are generally known but often neglected in literature. The omission of these factors was previously attributed to the assumption that discrete symmetries were always exact. However, the discovery of their violation and the recognition that time-reversal violation is a fundamental component of baryogenesis during the big bang necessitates the inclusion of all terms and rigorous experimental testing.

It has been discovered that the electromagnetic current can possess four form factors. The physical significance of these form factors in the non-relativistic limit is also examined.

To obtain the current's general expression, we will consider the expectation value given by the equation:

$$\langle p_f | j^\mu(x) | p_i \rangle = e^{-i(p_i - p_f)\cdot x} \langle p_f | j^\mu(0) | p_i \rangle, \tag{3.1}$$

where we have used a translation transformation

$$e^{iy\hat{\mathcal{P}}} j^\mu(x) e^{-iy\hat{\mathcal{P}}} = j^\mu(x+y),$$

with $\hat{\mathcal{P}}$ the four-momentum operator [21]. We can write

$$\langle p_f | j^\mu(0) | p_i \rangle = \overline{u}(\vec{p}_f) \mathcal{O}^\mu(l,\ q) u(\vec{p}_i), \tag{3.2}$$

where $u(\vec{p})$ is the Dirac spinors in momentum space. The l and q variables are the kinematic quantities we have, i.e., $l^\alpha \equiv p_i^\alpha + p_f^\alpha$ and $q^\alpha \equiv p_i^\alpha - p_f^\alpha$ and $\mathcal{O}^\mu$ being an operator whose matrix element between the spinors is a Lorentz vector[1]. Our focus is to derive the explicit form of $\mathcal{O}^\mu$.

The current j^μ must be Hermitian, i.e., $j_\mu^\dagger = j_\mu$. This amounts to $\langle p_f | j^\mu(x) | p_i \rangle = \langle p_i | j^\mu(x) | p_f \rangle^*$ or

$$\overline{u}(\vec{p}_f) \mathcal{O}_\mu(l,\ q) u(\vec{p}_i) = \left[\left(\overline{u}(\vec{p}_i) \mathcal{O}_\mu(l, -q) u(\vec{p}_f) \right)^\dagger \right]^T,$$

where we have taken the transpose operation twice on the right-hand side. If we now take the dagger and transpose operation, we get:

$$\overline{u}(\vec{p}_f) \mathcal{O}_\mu(l,\ q) u(\vec{p}_i) = \overline{u}(\vec{p}_f) \gamma^0 \mathcal{O}_\mu^\dagger(l, -q) \gamma^0 u(\vec{p}_i) \tag{3.3}$$

where we have used the fact that there is no effect of transposing a number, and $q^\alpha \equiv p_i^\alpha - p_f^\alpha$. Therefore, the $\mathcal{O}_\mu$ operator must satisfy

$$\mathcal{O}_\mu^\dagger(l,\ q) = \gamma^0 \mathcal{O}_\mu(l, -q) \gamma^0. \tag{3.4}$$

[1] To be more precise, the matrix element $\overline{\psi}(p_f) \mathcal{O}^\mu \psi(p_i)$ transforms like a four-vector.

The condition of gauge invariance can be expressed as follows:

$$\int d^4x J_\mu(x) A'^\mu(x) = \int d^4x j_\mu(x) A^\mu(x) - \int d^4x j_\mu(x) \partial^\mu \chi(x), \tag{3.5}$$

where the transformation of the photon field is given by

$$A'^\mu(x) \longrightarrow A^\mu(x) - \partial^\mu \chi(x). \tag{3.6}$$

The second integrand in the right-hand side of equation 3.5 can be solved by partial integration as

$$\int d^4x j_\mu(x) \partial^\mu \chi(x) = \int d^4x \partial^\mu (j_\mu(x)\chi(x)) - \int d^4x \partial^\mu j_\mu(x) \chi(x), \tag{3.7}$$

where the first term in equation 3.7 is a surface term that vanishes. Therefore, the gauge invariance condition implies

$$\int d^4x \partial^\mu j_\mu(x) \chi(x) = 0, \tag{3.8}$$

and since $\chi(x)$ is arbitrary, it follows that

$$\partial^\mu j_\mu = 0. \tag{3.9}$$

By utilizing equation 3.1, it can be inferred that $\partial_\mu j^\mu = 0$ leads to the conclusion that

$$q_\mu \overline{u}(\vec{p}_f) O^\mu(l,\ q) u(\vec{p}_i) = 0. \tag{3.10}$$

It is worth noting that without the requirement of gauge invariance we just can derive the weak form-factor decomposition [2].

In a theory with a massive spin-1 field, there are three vector polarizations $\epsilon^\mu(p)$, and a common basis is given by:

$$\begin{aligned} \epsilon_1^\mu &= (0,\ 1,\ 0,\ 0); \\ \epsilon_2^\mu &= (0,\ 0,\ 1,\ 0); \\ \epsilon_L^\mu &= (\frac{p_z}{m},\ 0,\ 0,\ \frac{E}{m}). \end{aligned}$$

In the massless limit, the longitudinal polarization becomes $\epsilon_L^\mu \propto q^\mu = (E, 0, 0, E)$. The gauge transformation (3.6) can be expressed in momentum space as:

$$\epsilon^\mu(p) \longrightarrow \epsilon^\mu(p) + ip^\mu. \tag{3.11}$$

In quantum field theory, we calculate matrix elements with the field A_μ. These matrix elements depend on the vector polarization and must have the form:

$$\mathcal{M} = \epsilon^\mu M_\mu,$$

[2] To be more precise, in an electromagnetic theory, current conservation is equivalent to the demand for gauge invariance. This requirement, in turn, necessitates that the photon has zero mass [19].

where M_μ transforms as a four-vector, ensuring that $\mathcal{M}$ is Lorentz invariant. In the case of a massless field, there are only two vector polarizations, as the longitudinal polarization becomes the momentum of the photon. Therefore, demanding Lorentz invariance and imposing gauge invariance implies:

$$\mathcal{M} \longrightarrow \left(\epsilon_1^{'\mu}(q) + \epsilon_2^{'\mu}(q)\right) M'_\mu + q^\mu M'_\mu = \left(\epsilon_1^{'\mu}(q) + \epsilon_2^{'\mu}(q)\right) M'_\mu. \tag{3.12}$$

This is only possible if $q^\mu M'_\mu = 0$, which is known as the Ward identity.

To construct the 4×4 matrix $\mathcal{O}^\mu(l,\ q)$ we have access to $\{l^\mu,\ q^\mu\}$, the matrices in the set $\mathcal{S} = \{1, \gamma^\mu,\ \sigma^{\mu\nu},\ \gamma^\mu\gamma^5,\ \gamma^5\}$[3], the metric tensor $g^{\mu\nu}$, and the Levi-Civita anti-symmetric tensor $\epsilon^{\mu\nu\sigma\rho}$. We define the first set such that q^μ and l^μ carry the Lorentz-index, giving us

$$\mathcal{O}_1 = \{q^\mu,\ l^\mu,\ q^\mu\gamma^5,\ l^\mu\gamma^5\}.$$

We can expand this set to include

$$\mathcal{S}_1 = \{q^\mu \not{q},\ q^\mu \not{l},\ q^\mu\gamma_5\not{q},\ q^\mu\gamma_5\not{l},\ q^\mu\sigma^{\alpha\beta}q_\alpha l_\beta, \text{ and } \quad q^\mu \longleftrightarrow l^\mu\},$$

but these terms either vanish or are proportional to those in $\mathcal{O}_1$. For instance,

$$\begin{aligned}
\overline{u}(\vec{p}_f) q^\mu \gamma_5 \not{q} u(\vec{p}_i) &= \overline{u}(\vec{p}_f) q^\mu \gamma_5 (\not{p}_i - \not{p}_f) u(\vec{p}_i) \\
&= \overline{u}(\vec{p}_f) q^\mu \gamma_5 (\not{p}_i u(\vec{p}_i) - \not{p}_f u(\vec{p}_i)) \\
&= 2m\overline{u}(\vec{p}_f) q^\mu \gamma_5 u(\vec{p}_i),
\end{aligned} \tag{3.13}$$

where we have used the Dirac equation $(A.62)$.

The next possible set of candidates is characterized by demanding Lorentz-index be carried by one of the matrix in $\mathcal{S}$

$$\mathcal{O}_2 = \{\gamma^\mu,\ \gamma_5\gamma^\mu,\ \sigma^{\mu\nu}q_\nu,\ \sigma^{\mu\nu}l_\nu\}.$$

It is worth noting that a term

$$\gamma_5\sigma^{\mu\nu}q_\nu = \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\nu$$

will be added to the next set. In the third set the Lorentz-index μ is carried by the Levi-Civita symbol

$$\mathcal{O}_3 = \{\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\nu,\ \epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}l_\nu,\ \epsilon^{\mu\nu\alpha\beta}\gamma_\nu q_\alpha l_\beta,\ \epsilon^{\mu\nu\alpha\beta}\gamma_\nu\gamma_5 q_\alpha l_\beta,\ \epsilon^{\mu\nu\alpha\beta}q_\alpha l_\beta\sigma_{\nu\rho}q^\rho,\ \epsilon^{\mu\nu\alpha\beta}q_\alpha l_\beta\sigma_{\nu\rho}l^\rho\}.$$

The Gordon-like identity (B.50) indicates that we can replace $\gamma^5 l^\mu$ with $\sigma^{\mu\nu}\gamma^5 q_\nu$. However, based on equation (B.10), $\sigma^{\mu\nu}\gamma^5 q_\nu = \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\nu$, which represents the first element in $\mathcal{O}3$. Moreover, using equation (B.35), we can eliminate $\sigma^{\mu\nu}l\nu$ in favor of q^μ,

[3] This set of matrices are linearly independent (see appendix B.1).

while equation (B.33) allows us to replace l^μ with γ^μ and $\sigma^{\mu\nu}q^\nu$. Therefore we will use the following sets:

$$\mathcal{O}'_1 = \{q^\mu,\ q^\mu\gamma^5\}; \tag{3.14}$$

$$\mathcal{O}'_2 = \{\gamma^\mu,\ \gamma^5\gamma^\mu,\ \sigma^{\mu\nu}q_\nu\}; \tag{3.15}$$

$$\mathcal{O}'_3 = \{\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\nu\}. \tag{3.16}$$

Note that the elements in $\mathcal{O}3$ are linear combinations of the elements in $\mathcal{O}'1$ and $\mathcal{O}'2$, as well as the element in $\mathcal{O}'3$. Hence, we can represent the set $\mathcal{O}_3$ with just one element, as we will see next.

The second term in $\mathcal{O}_3$ can be writen as follows:

$$\begin{aligned}\overline{u}(\vec{p}_f)[\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}l_\nu]u(\vec{p}_i) &\overset{\text{B.10}}{=} -2\overline{u}(\vec{p}_f)(i\gamma^5\sigma^{\mu\nu}l_\nu)u(\vec{p}_i)\\ &\overset{\text{B.34}}{=} \overline{u}(\vec{p}_f)\left(\frac{\gamma^5 q^\mu - \gamma^\mu\gamma^5}{m}\right)u(\vec{p}_i),\end{aligned}$$

which is a linear combination of elements in $\mathcal{O}'_1$ and $\mathcal{O}'_2$. The third term in $\mathcal{O}'_3$ can be expressed as

$$\begin{aligned}\overline{u}(\vec{p}_f)[\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}l_\nu]u(\vec{p}_i) \overset{\text{B.52}}{=} \overline{u}(\vec{p}_f)\Big[&i(\not{l}\, q^{\mu\gamma^5} - \not{q}\, l^\mu\gamma^5) + iq^2\gamma^5\gamma^\mu\\ &- 2im(l^\mu\gamma^5 + q^\mu\gamma^5)\Big]u(\vec{p}_i),\end{aligned}$$

this is again elements $\in \mathcal{O}'_1$, $\mathcal{O}'_1$ and $\mathcal{O}'_3$.

The fourth element can be writen as

$$\begin{aligned}\overline{u}(\vec{p}_f)[\epsilon^{\mu\nu\alpha\beta}\gamma_\nu\gamma_5 q_\alpha l_\beta]u(\vec{p}_i) \overset{\text{B.51}}{=} \overline{u}(\vec{p}_f)\Big[&i(q^\mu\not{l} - l^\mu\not{q}) - i(q^2-4m^2)\gamma^\mu\\ &+ 2im(l^\mu + q^\mu)\Big]u(\vec{p}_i),\end{aligned}$$

that is a linear combination of $\mathcal{O}'_1$, $\mathcal{O}'_2$. The fifth element is given by:

$$\begin{aligned}\overline{u}(\vec{p}_f)[\epsilon^{\mu\nu\alpha\beta}\gamma_\nu\gamma_5 q_\alpha l_\beta]u(\vec{p}_i) &\overset{\text{B.53}}{=} -2im\overline{u}(\vec{p}_f)[\epsilon^{\mu\nu\alpha\beta}q_\alpha l_\beta\gamma_\nu\gamma^5]u(\vec{p}_i)\\ &\overset{\text{B.51}}{=} 2im\overline{u}(\vec{p}_f)\Big[-i\Big(q^\mu\not{l} - l^\mu\not{q}\Big) + \Big(q^2-4m^2\Big)\gamma^\mu\\ &+ 2im\Big(l^\mu + q^\mu\Big)\Big]u(\vec{p}_i),\end{aligned}$$

these objects belong to the sets $\mathcal{O}'1$, $\mathcal{O}'2$ and $\mathcal{O}'3$. Therefore, the sets in equations (3.14), (3.15) and (3.16) are all we can use to construct the $\mathcal{O}^\mu(l,\ q)$. Combining all the previous results, we can express the following six independent terms:

$$\begin{aligned}\overline{u}(\vec{p}_f)\mathcal{O}^\mu(l,\ q)u(\vec{p}_i) = \overline{u}(\vec{p}_f)\Big\{&f_1(q^2)q^\mu + f_2(q^2)q^\mu\gamma^5 + f_3(q^2)\gamma^\mu + f_4(q^2)\gamma^\mu\gamma^5\\ &+ f_5(q^2)\sigma^{\mu\nu}q_\nu + f_6(q^2)\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\nu\Big\}\, u(\vec{p}_i),\end{aligned} \tag{3.17}$$

where $f_k(q^2)$ are six Lorentz-invariant form factors $(k = 1, \cdots, 6)$ and q is the four-momentum of the photon. Note that the form factors depend only on q^2, which is the only available Lorentz-invariant kinematical quantity, since $l^2 = 4m^2 - q^2$ and $q \cdot l = 0$. Therefore, we can write $O^\mu(l, q)$ as $O^\mu(q)$[21].

We aim to enforce gauge invariance, $\partial_\mu J^\mu(x) = 0$, and can do so using equation (3.1). This gives us:

$$\begin{aligned} \partial_\mu J^\mu(x) = 0 \iff \partial_\mu \left(e^{-i(p_i - p_f)\cdot x}\right) \overline{u}(\vec{p}_f) O^\mu(q) u(\vec{p}_i) = 0 \\ \Rightarrow q_\mu \overline{u}(\vec{p}_f) O^\mu(q) u(\vec{p}_i) = 0. \end{aligned} \tag{3.18}$$

Using the equation (3.17) we find

$$\begin{aligned} \overline{u}(\vec{p}_f)\Big\{ f_1(q^2)q^2 + f_2(q^2)q^2\gamma^5 + f_3(q^2)\not{q} + f_4(q^2)\not{q}\gamma^5 \\ + f_5(q^2)\sigma^{\mu\nu}q_\mu q_\nu + f_6(q^2)\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\mu q_\nu \Big\} u(\vec{p}_i) = 0, \end{aligned} \tag{3.19}$$

since $q_\mu q_\nu \equiv T_{\mu\nu}$ is a symmetric tensor $\sigma^{\mu\nu}q_\mu q_\nu = \epsilon^{\mu\nu\alpha\beta}q_\mu q_\nu \sigma_{\alpha\beta} = 0$ and the term $\overline{u}(\vec{p}_f) f_3(q^2) \not{q} u(\vec{p}_i)$ vanish due the Dirac equation (A.62). Hence, we can deduce that

$$f_1(q^2)q^2 + f_2(q^2)q^2\gamma^5 + f_4(q^2)2m\gamma^5 = 0, \tag{3.20}$$

where he have used $\overline{u}(\vec{p}_f)\not{q}\gamma^5 u(\vec{p}_i) = \overline{u}(\vec{p}_f)\left(\not{p}' - \not{p}\right)\gamma^5 u(\vec{p}_i) = \overline{u}(\vec{p}_f)2mu(\vec{p}_i)$. Since unity matrix and γ^5 are linearly independenty

$$f_1(q^2) = 0; \tag{3.21}$$

$$f_4(q^2) = -\frac{q^2}{2m} f_2(q^2). \tag{3.22}$$

Thus, the expression in equation (3.17) can be written as follows:

$$\begin{aligned} \overline{u}(\vec{p}_f) O^\mu(q) u(\vec{p}_i) = \overline{u}(\vec{p}_f) \Bigg\{ f_2(q^2)\Big[q^\mu - \frac{q^2}{2m}\gamma^\mu\Big]\gamma^5 + f_3(q^2)\gamma^\mu \\ + f_5(q^2)\sigma^{\mu\nu}q_\nu + f_6(q^2)\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\nu \Bigg\} u(\vec{p}_i), \end{aligned} \tag{3.23}$$

which leave us with four electromagnetic form-factors.Usually, it is common to express the final result through $F_i(i = 1,\ 2,\ 3,\ 4)$ form-factor in the following form:

$$\begin{aligned} \overline{u}(\vec{p}_f) O^\mu(q) u(\vec{p}_i) = \overline{u}(\vec{p}_f) \Bigg\{ F_1(q^2)\gamma^\mu + \frac{i}{2m}\sigma^{\mu\nu}q_\nu F_2(q^2) + \frac{i}{4m}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\nu F_3(q^2) \\ + \frac{1}{2m}\left(q^\mu - \frac{q^2}{2m}\gamma^\mu\right)\gamma^5 F_4(q^2) \Bigg\} u(\vec{p}_i). \end{aligned} \tag{3.24}$$

Implementing the hermiticity condition (3.4) we find:

$$\begin{aligned}\gamma^0 O_\mu^\dagger(q)\gamma^0 &= F_1^*(q^2)\gamma^0\gamma_\mu^\dagger\gamma^0 - \frac{i}{2m}\gamma^0\sigma_{\mu\nu}^\dagger\gamma^0 q^\nu F_2^*(q^2) - \frac{i}{4m}\epsilon_{\mu\nu\alpha\beta}\gamma^0(\sigma^{\alpha\beta})^\dagger\gamma^0 q^\nu F_3^*(q^2)\\ &+ \frac{1}{2m}\left(q^\mu\gamma^0\gamma^5\gamma^0 - \frac{q^2}{2m}\gamma^0\gamma^5\gamma_\mu^\dagger\gamma^0\right)F_4^*(q^2),\end{aligned} \tag{3.25}$$

where we have alredy used that $\gamma^{5\dagger} = \gamma^5$. Using the equation (B.28) we can write

$$\begin{aligned}\gamma^0 O_\mu^\dagger(q)\gamma^0 &= F_1^*(q^2)\gamma_\mu - \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2^*(q^2) - \frac{i}{4m}\epsilon_{\mu\nu\alpha\beta}\sigma^{\alpha\beta}q^\nu F_3^*(q^2)\\ &- \frac{1}{2m}\left(q^\mu + \frac{q^2}{2m}\gamma_\mu\right)\gamma^5 F_4^*(q^2)\\ &= O_\mu(-q).\end{aligned} \tag{3.26}$$

Therefore, the hermiticity impose that

$$F_i^*(q^2) = F_i(q^2). \tag{3.27}$$

Equation (3.24) is the most general relativistic current for the spin-1/2 fermion.

Equation (3.24) corresponds to the diagonal case, i.e., the one that occurs when the side fermions have identical masses. But in principle, we could also have started with the off-diagonal case. In this case, the result is

$$\begin{aligned}\overline{u}(\vec{p}_f)O^\mu(q)u(\vec{p}_i) = \overline{u}(\vec{p}_f)\Big\{ &\left(\gamma^\mu q^2 - \not{q} q^\mu\right)\tilde{F}_1(q^2) + i\sigma^{\mu\nu}q_\nu\tilde{F}_2(q^2) + i\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\nu\tilde{F}_3(q^2)\\ &+\left(q^\mu - \frac{q^2}{m_1+m_2}\gamma^\mu\right)\gamma^5\tilde{F}_4(q^2)\Big\}\, u(\vec{p}_i),\end{aligned} \tag{3.28}$$

which satisfies all the requeriments we have done in the previos current.

The form-factor decomposition given by Equation (3.24) is applicable to both point-like particles, such as electrons, muons and neutrinos, as well as extended particles, such as neutrons and protons. This decomposition holds true for both Dirac and Majorana fermions, although, we will see later that there are differences between these two types of fermions regarding their electromagnetic properties. The decomposition is valid for both charged and neutral fermions, with the latter having a coupling to a photon either due to their extended nature or through their spin-field interaction. In the case of point-like objects, one starts with the coupling $eA^\mu\overline{\psi}\gamma^\mu\psi$ and higher-order corrections in perturbation theory produce the remaining structure. The physical explanation for this phenomenon, is that the "bare"electron is always accompanied by a cloud of virtual particles-antiparticles pairs[4], which effectively makes it an extended object [24].

[4] More precisely, the Coulomb potential can be related to the exchange of a single photon. However, if we consider the 1-loop correction to Coulomb's law, this arises from an e^+e^- loop inside the photon line, which is known as vacuum polarization [19].

3.1.1 Electromagnetic Form Factors

In order to develop an understanding of the meaning of form factors F_i, let us examine the electromagnetic interaction in the non-relativistic limit. The interaction Hamiltonian between the fermionic field $\mathfrak{f}(x)$ and the electromagnetic field $A_\mu(x)$ in the Standard Model is given by the expression:

$$\mathcal{H}_{\text{em}}^{(\mathfrak{f})}(x) = \mathfrak{q}_{\mathfrak{f}}\bar{\mathfrak{f}}(x)\gamma_\mu \mathfrak{f}(x)A^\mu(x), \tag{3.29}$$

where $\bar{\mathfrak{f}}(x)$ is the adjoint spinor of the fermionic field and $\mathfrak{q}_{\mathfrak{f}}$ is the charge of the fermion.

As we have discussed, the decomposition (3.24) can be used to extended body, so we will consider the energy of a non-relativistic hadron with charge e_p in a static external electromagnetic field $A^\mu(x)$. The interaction energy is given by the volume integral

$$\begin{aligned} W &= e_p \int d^3x A^\mu(x) J_\mu(x) = \\ &= \frac{e_p}{V} \int d^3x e^{-i(p_i - p_f)\cdot x} A^\mu(x) \overline{u}(\vec{p}_f) O_\mu(q) u(\vec{p}_i). \end{aligned} \tag{3.30}$$

In the case of a pure electrostatic field $A^0(x)$ it is advantageous to be concerned with the vertex function just with the form-factors $F_1(q^2)$ and $F_2(q^2)$, i.e., we will study

$$\Lambda_\mu(q) = F_1(q^2)\gamma_\mu + \frac{i\sigma_{\mu\nu}}{2m} q^\nu F_2(q^2). \tag{3.31}$$

Using the Gordon identity (Eq. B.33), we can express the equation as:

$$\overline{u}(\vec{p}_f)\Lambda_\mu(q)u(\vec{p}_i) = \overline{u}(\vec{p}_f)\left\{ \Big(F_1(q^2) + F_2(q^2)\Big)\gamma_\mu - \frac{(p_f + p_i)}{2m} F_2(q^2) \right\} u(\vec{p}_i). \tag{3.32}$$

This leads to the following expression for the integrand of Eq. (3.30) in the hadron rest frame:

$$\begin{aligned} A^0(x)\overline{u}(\vec{p}_f)\Lambda_0(q)u(\vec{p}_i) &= A^0(x)\overline{u}(\vec{p}_f)\left\{ \Big(F_1(q^2) + F_2(q^2)\Big)\gamma_0 - \frac{1}{2m}(p_f + p_i)_0 F_2(q^2) \right\} u(\vec{p}_i) \\ &= A^0(x)\overline{u}(\vec{p}_f)\left\{ F_1(q^2) + F_2(q^2)\Big)\gamma_0 - \frac{1}{2m}(p_f + p_i)_0 F_2(q^2) \right\} u(\vec{p}_i). \end{aligned} \tag{3.33}$$

In the non-relativistic limit, the lower component of the Dirac spinors can be neglected. Any vector current, such as $J^\mu = \bar{\psi}\gamma^\mu\psi$, can be written as $\psi^\dagger\psi$. Since the γ^i matrix is off-diagonal, it implies an upper-lower coupling of the Dirac spinor. Therefore, it vanishes. On the other hand, the γ^0 matrix is diagonal, and it implies an upper-upper coupling of the Dirac spinor components. Therefore, in the non-relativistic limit,

$$\bar{\psi}\gamma^\mu\psi \longrightarrow \bar{\psi}\gamma^0\psi = \psi^\dagger\psi,$$

which leads to

$$\begin{aligned} A^0(x)\overline{u}(\vec{p_f})\Lambda_0(q)u(\vec{p_i}) &\approx A^0(x)\overline{u}(\vec{p_f})\left\{ F_1(q^2) + \left(\frac{m - E_{p_f}}{2m}\right) F_2(q^2) \right\} u(\vec{p_i}) \\ &= A^0(x)\overline{u}(\vec{p_f})\left(F_1(q^2) + \frac{q^2}{4m} F_2(q^2) \right) u(\vec{p_i}), \end{aligned} \tag{3.34}$$

where we have used $q^2 = (p_f - p_i)^2 = 2m(m - E_{p_f})$. In the static limit $q^2 \to 0$, i.e., we are considering an external potential which is constant or slowly varying. Therefore, the interaction energy simply becomes

$$W \approx \frac{e_p}{V} \int d^3x e^{-iq\cdot x} \underbrace{A^0(x)}_{\text{static}} F_1(0) = e_p F_1(0) A^0. \tag{3.35}$$

Moving on, we can deduce the classical Hamiltonian for a particle in an electromagnetic field by starting with the free Hamiltonian $H_0 = \frac{p^2}{2m}$ and performing minimal coupling $\vec{p} \to \vec{p} - e\vec{A}$ while adding the electric potential with a charge e. This gives us the following equation:

$$\begin{aligned} H &= \frac{\left(\vec{p} - e\vec{A}\right)^2}{2m} + e\phi \\ &= \frac{p^2}{2m} - e\vec{v}\cdot\vec{A} + e^2\frac{A^2}{2m} + e\phi \\ &= H_0 + H_{int}, \end{aligned} \tag{3.36}$$

where $\vec{A}$ is the vector potential and ϕ is the scalar potential.

Returning to the result found in (3.35), we can interpret it as the electrostatic energy of a particle with charge $e_p F_1(0)$ in a potential A^0, i.e.,

$$\mathcal{H}_{\text{effec}}[F_1(0)] = F_1(0)A^0(x), \tag{3.37}$$

and we can interpret $F_1(0) = \mathbb{q}$ as an effective charge.

We can express the product of the magnetic field described by the vector potential $A^k(x)$ and the spinors $\overline{u}(\vec{p_f})$ and $\overline{u}(\vec{p_i})$ in terms of the form factors $F_1(q^2)$ and $F_2(q^2)$, and the momenta and spin of the particles involved. Specifically, we have:

$$\begin{aligned} A^k(x)\overline{u}(\vec{p_f})\Lambda_k(q)u(\vec{p_i}) &= A^k(x)\overline{u}(\vec{p_f})\left\{ F_1(q^2)\gamma_k + \frac{i\sigma_{kl}q^l}{2m}\left(F_1(q^2) + F_2(q^2)\right) \right\} u(\vec{p_i}) \\ &= \overline{u}(\vec{p_f})\left\{ \frac{F_1(q^2)}{2m}\left(l_k + i\sigma_{kl}q^l\right) + \frac{i\sigma_{kl}q^l}{2m}F_2(q^2) \right\} u(\vec{p_i}) \\ &= \overline{u}(\vec{p_f})\left\{ \frac{A^k p_{f_k}}{m}F_1(q^2) + \frac{iA^k\sigma_{kl}q^l}{2m}\left(F_1(q^2) + F_2(q^2)\right) \right\} u(\vec{p_i}), \end{aligned} \tag{3.38}$$

where $\Lambda_k(q)$ is the vertex function for a photon with momentum q and polarization vector ϵ_k, m is the mass of the particles. In the Dirac representation of the γ-matrices, we can write $\gamma_k = i\sigma_2 \otimes \sigma_k$. Using this representation, we can calculate σ_{kl} as follows:

$$\begin{aligned}
\sigma_{kl} &= \frac{i}{2}[\gamma_k, \gamma_l] \\
&= \frac{i}{2}[i\sigma_2 \otimes \sigma_k, i\sigma_2 \otimes \sigma_l] \\
&= \frac{-i}{2}(\sigma_2)^2 \otimes [\sigma_k, \sigma_l] = 1 \otimes \epsilon_{klm}\sigma_m \\
&= \epsilon_{klm}\begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix} \\
&= \epsilon_{klm}\Sigma^m,
\end{aligned}$$

with $\vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$. We can express the following equation:

$$A^k(x)\overline{u}(\vec{p}_f)\Lambda_k(q)u(\vec{p}_i) \approx -\overline{u}(\vec{p}_f)\left\{ \frac{\vec{A}\cdot\vec{p}_f}{m}F_1(q^2) + \frac{i}{2m}\left(F_1(q^2) + F_2(q^2)\right)\left(\vec{q}\times\vec{A}\right)\cdot\vec{\Sigma} \right\} u(\vec{p}_i). \tag{3.39}$$

Finally, considering that the fields are static and $q^2 \to 0$, the interaction energy can be written as:

$$W \approx -\frac{e_p}{V}\int d^3x e^{-iq\cdot x}u^\dagger(\vec{p}_f)\left\{ \frac{\vec{A}\cdot\vec{p}_f}{m}F_1(0) + \frac{i}{2m}\left(F_1(0) + F_2(0)\right)\left(\vec{q}\times\vec{A}\right)\cdot\vec{\Sigma} \right\} u^\dagger(\vec{p}_f)u(\vec{p}_i). \tag{3.40}$$

As $u^\dagger u = 1$ due to normalization, the first term just describes the interaction of a moving charge with the magnetic field, which can be seen in equation (3.36). We can express the second term as shown below, where $\vec{B}$ is the magnetic field and $\vec{\Sigma} = 2\vec{s}$ with $\vec{s}$ being the spin operator:

$$\begin{aligned}
W &\approx \frac{e_p\left(F_1(0) + F_2(0)\right)}{2m}\overline{u}(\vec{p}_f)\vec{\Sigma}u(\vec{p}_i)\frac{1}{V}\int d^3x e^{-iq\cdot x}\left(-i\vec{q}\times\vec{A}\right) \\
&= \frac{e_p\left(F_1(0) + F_2(0)\right)}{2m}\overline{u}(\vec{p}_f)\vec{\Sigma}u(\vec{p}_i)\frac{1}{V}\int d^3x \left[\underbrace{\vec{\nabla}\times\left(e^{-iq\cdot x}\vec{A}\right)}_{\text{boundary term}\ \to 0} - e^{-iq\cdot x}\vec{\nabla}\times\vec{A}\right] \\
&= \frac{-e_p\left(F_1(0) + F_2(0)\right)}{2m}2\langle\vec{s}\rangle\cdot\vec{B},
\end{aligned} \tag{3.41}$$

with $\vec{\Sigma} = 2\vec{s}$ and $\vec{\nabla}\times\vec{A} = \vec{B}$, where $\vec{B}$ is the magnetic field. We can represent the free Hamiltonian for a spin-1/2 particle as follows:[5]

$$H = \frac{(\vec{\sigma}\cdot\vec{p})^2}{2m}.$$

[5] Here $\vec{p}$ is the momentum quantum operator $\vec{p} = -i\vec{\nabla}$.

Using the minimal coupling, we obtain the Pauli equation as follows:

$$\begin{aligned} H &= \frac{1}{2m}\left[(\vec{\sigma}\cdot\vec{p} - e\vec{\sigma}\cdot\vec{A})^2\right] + e\phi \\ &= \frac{1}{2m}\left[(\vec{p} - e\vec{A})^2 - 2e\vec{\sigma}\cdot\vec{B}\right] + e\phi \\ &= \frac{1}{2m}(\vec{p} - e\vec{A})^2 - \vec{\mu}\cdot\vec{B} + e\phi, \end{aligned} \tag{3.42}$$

where $\vec{\mu} \equiv \frac{e\vec{\sigma}}{m}$ is the intrinsic magnetic moment for a particle satisfying the Dirac equation.

Referring back to the result in equation (3.41), we can interpret it as the energy of a particle with spin-1/2 in a homogeneous magnetic field with magnetic moment $\frac{1}{2m}[F_1(0) + F_2(0)]$, i.e.,

$$\mathcal{H}_{\text{effec}}[F_1(0), F_2(0)] = -\mu\vec{\sigma}\cdot\vec{B}, \tag{3.43}$$

and we can interpret $\frac{1}{2m}[F_1(0) + F_2(0)] = \mu$ as an effective magnetic moment. A similar procedure allow to written [24]

$$\mathcal{H}_{\text{effec}}[F_3(0)] = -\mathbb{d}\vec{\sigma}\cdot\vec{E}; \tag{3.44}$$

$$\mathcal{H}_{\text{effec}}[F_4(0)] \propto F_4(0)\vec{\sigma}\cdot\left[\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t}\right] \tag{3.45}$$

Here $-\frac{1}{2m}F_3(0) = \mathbb{d}$, where d is called electric dipole moment and $F_4(0) = \mathbb{a}$ is called the anapole moment. It is commonlly presented to present the vertex function in the following form:

$$\Lambda^\mu(q) = f_{\mathbb{Q}}(q^2)\gamma^\mu - f_{\mathbb{M}}(q^2)i\sigma^{\mu\nu}q_\nu + f_{\mathbb{E}}(q^2)\sigma^{\mu\nu}q_\nu\gamma^5 + f_{\mathbb{A}}(q^2)\left(q^2\gamma^\mu - q^\mu \not{q}\right)\gamma^5, \tag{3.46}$$

where $f_{\mathbb{Q}}$, $f_{\mathbb{M}}$, $f_{\mathbb{E}}$ and $f_{\mathbb{A}}$ represent the real charge, magnetic dipole, electric dipole, and anapole neutrino form factors, respectively [21].

4 Neutrino Interaction via Magnetic Dipole Moment: Calculation

The electromagnetic properties of neutrinos are a window to physics beyond the Standard Model. The vertex function (3.46) can be used to distinguish between the Dirac Neutrino and Majorana neutrino. In this section we will consider neutrino as a Dirac particle and study the neutrino scattering with a target nuclei. In section (4.1) we will introduce the differential cross section in laboratory frame considering charge-charge interaction, magnetic dipole-charge interaction.

4.1 The differential Cross Section

Considering a beam of particles of type a with flux ϕ_a, crossing a space region in which there are n_b type b particles per volume unity. The interaction rate per target particle r_b will be proportional to the interaction particle flux and can be written as:

$$r_b = \sigma\phi_a. \tag{4.1}$$

The fundamental physics is contained in σ, which area dimension and is termed the interaction cross section[1].

The cross section for any two-body → two-body process is given by

$$\sigma = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}\int d\Omega^* \left|\mathcal{M}_{fi}\right|^2, \tag{4.2}$$

where $p_f^*(p_i^*)$ is the final (initial) momentum in centre-of-mass frame, the solid angle $d\Omega$ represents the angular extent of one of the scattered particles and $\mathcal{M}_{fi}$ is the Lorentz invariant matrix element defined by

$$\mathcal{M}_{fi} = \langle\psi_1', \psi_2', \cdots|\hat{H}'|\psi_a', \psi_b', \cdots\rangle = \left(2E_1 2E_2 2E_a 2E_b\right)T_{fi}, \tag{4.3}$$

where T_{fi} is the transition matrix elements [25]. The wave functions are normalized with a Lorentz-invariant normalisation

$$\psi' = (2E)^{1/2}\psi,$$

with E being the particle energy, and s is the total energy of the two initial particles, sometimes called centre-of-mass energy. The Mandelstam variables are defined by

$$t = (p_1 - p_3)^2 = (p_4 - p_2)^2 \tag{4.4}$$

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{4.5}$$

$$u = (p_1 - p_4)^2 = (p_3 - p_2)^2 \tag{4.6}$$

We can see that the essential physics of the interaction is given by the Lorentz-invariant matrix element $\mathcal{M}_{fi}$. The matrix element is calculated for a given process and we can perform it with the Feynman rules for the given theory.

The diferrential cross section can be writen as:

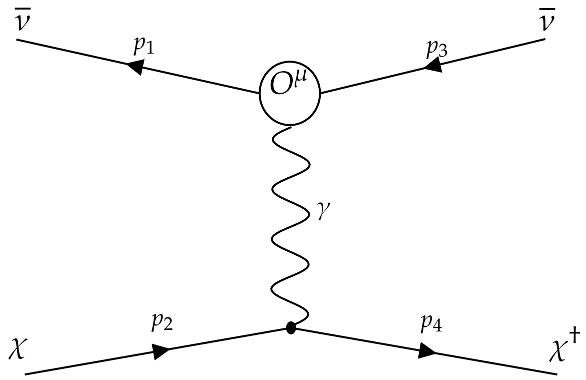
$$d\sigma = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}\left|\mathcal{M}_{fi}\right|^2 d\Omega^* \,. \tag{4.7}$$

We are interested about the following neutrinos scattering process:

$$\overline{\nu} + \chi \longrightarrow \overline{\nu} + \chi^{\dagger}, \tag{4.8}$$

where $\overline{\nu}$ is the antineutrino produced in a weak decay, χ is a target nuclei in the ground state and $\chi^{\dagger}$ is the target nuclei in a excited state. This can be represented using the (3.46) by the following Feynman digram:

Figura 4 – Effective one-photon interaction with a target nuclei χ



Source: Author.

TWhile the differential cross section (4.7) is applicable in the center-of-mass frame, we aim to determine the differential cross section in the laboratory frame. In the rest frame t-channel Mandelstan variable can be calculated by:

$$\begin{aligned} t &= \left(p_1^* - p_3^*\right)^{1/2} \\ &= m_1^2 + m_3^2 - 2p_1^* p_3^* \\ &= m_1^2 + m_3^2 - 2E_1^* E_3^* + 2\vec{p}_1^* \cdot \vec{p}_3^*. \end{aligned} \tag{4.9}$$

In the centre-of-mass frame, the magnetude of the momentum and the final-state particle energies are fixed by energy and momentum conservation and the only free parameter is the scattering angle θ^*, thus

$$dt = 2p_1^* p_3^* \, d\left(\cos\theta^*\right), \tag{4.10}$$

and therefore, $d\Omega^* \equiv d(\cos\theta^*)d\phi^* = \frac{dt\, d\phi^*}{2p_1^* p_3^*}$. Writing p_1^* and p_3^* as p_i^* and p_f^*, respectively, the differential cross section (4.7) can be written as:

$$d\sigma = \frac{1}{128\pi^2 s p_i^{*2}} \left|\mathcal{M}_{fi}\right|^2 d\phi^* dt\,. \tag{4.11}$$

Finnaly, assuming that matrix elements is independent of the azimuthal angle, the integral over $d\phi^*$ just introduce a factor of 2π and therefore

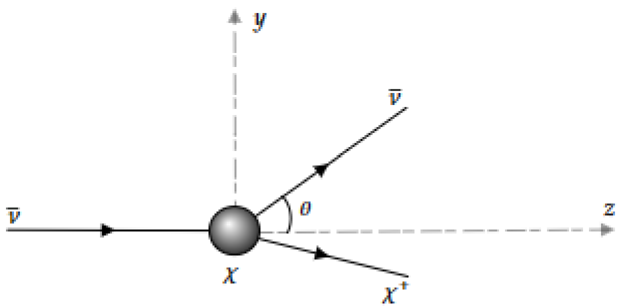
$$\frac{d\sigma}{dt} = \frac{1}{64\pi s p_i^{*2}} \left|\mathcal{M}_{fi}\right|^2. \tag{4.12}$$

We can see that the abovementioned cross section in Lorentz-invariant, since all the constituints are a scalar. The magnetude of the momentum of initial-state particles in the centre-of-mass frame can be expressed as:

$$p_i^{*2} = \frac{1}{4s}\left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right]. \tag{4.13}$$

In the laboratory-frame we have the following description

Figura 5 – $\overline{\nu} + \chi \longrightarrow \overline{\nu} + \chi^{\dagger}$ process in Laboratory Frame.



Source: Author.

With the following kinematic quantities:

$$p_1 = (E_1,\ 0,\ 0,\ E_1) \tag{4.14}$$
$$p_2 = (m_\chi,\ 0,\ 0,\ 0) \tag{4.15}$$
$$p_3 = (E_3,\ 0,\ E_3 \sin\theta,\ E_3 \cos\theta) \tag{4.16}$$
$$p_4 = (E_1 + m_\chi - E_3,\ 0,\ -E_3 \sin\theta,\ E_1 - E_3 \cos\theta). \tag{4.17}$$

Where we have using that $m_1 = m_3 \ll m_\chi$. In this limit the equation (4.13) becomes:

$$p_i^{*2} \approx \frac{\left(s - m_\chi^2\right)^2}{4s}. \tag{4.18}$$

Since

$$\begin{aligned} s = (p_1 + p_2)^{1/2} &= m_\chi^2 + 2p_1 p_2 \\ &= m_\chi^2 + 2E_1 m_\chi \\ \Rightarrow p_i^{*2} &= \frac{E_1^2 m_\chi^2}{s}. \end{aligned} \tag{4.19}$$

The differential cross section can be written in the laboratory frame using that

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi} \frac{dt}{d(\cos\theta)} \frac{d\sigma}{dt}. \tag{4.20}$$

Where we have considered azimuthal symmetry. Substituting (4.19) in (4.12), the Lorentz invariant differential cross section can be written as:

$$\frac{d\sigma}{dt} = \frac{1}{64\pi m_\chi^2 E_1^2} \left|\mathcal{M}_{fi}\right|^2. \tag{4.21}$$

Using the Mandelstam variable $t = (p_4 - p_2)^2$, we can calculate

$$\frac{dt}{d(\cos\theta)} = 2m_\chi \frac{dE_3}{d(\cos\theta)}. \tag{4.22}$$

Since

$$\begin{aligned} (p_1 - p_3)^2 &= (p_4 - p_2)^2 \\ \Rightarrow -2E_1 E_3 (1 - \cos\theta) &= m_\chi^2 + m_{\chi^\dagger}^2 - 2m_\chi (E_1 + m_\chi - E_3) \\ \Rightarrow E_3 &= \frac{m_\chi^2 - m_{\chi^\dagger}^2 + 2m_\chi E_1}{2\left(m_\chi + E_1 (1 - \cos\theta)\right)}. \end{aligned} \tag{4.23}$$

Hence,

$$\frac{dE_3}{d(\cos\theta)} = \frac{E_1 E_3}{m_\chi + E_1 (1 - \cos\theta)}. \tag{4.24}$$

And equation (4.22) becomes

$$\frac{dt}{d(\cos\theta)} = \frac{2m_\chi E_1 E_3}{m_\chi + E_1 (1 - \cos\theta)} = \frac{4m_\chi E_1 E_3^2}{m_\chi^2 - m_{\chi^\dagger}^2 + 2m_\chi E_1} \tag{4.25}$$

Finnaly the differential cross section can be calculated by:

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{64\pi^2} \frac{1}{\left(m_\chi + E_1 (1 - \cos\theta)\right)} \left(\frac{E_3}{m_\chi E_1}\right) \left|\mathcal{M}_{fi}\right|^2. \tag{4.26}$$

Equivalently, using the last equation of (4.25) and (4.21) in (4.20), we find:

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi^2 \left(m_\chi^2 - m_{\chi^\dagger}^2 + 2m_\chi E_1\right)} \left(\frac{E_3^2}{m_\chi E_1}\right) \left|\mathcal{M}_{fi}\right|^2. \tag{4.27}$$

4.1.1 The charge-charge interaction

Now we proceed with the calculation of the matrix element $\left|\mathcal{M}_{fi}\right|$, using Feynman Rules, which employ perturbation theory to evaluate interactions. The Feynman Rules are derived from the Lagrangian of each theory. To illustrate, we will provide a brief explanation of the Feynman Rules for the ϕ^4-theory. This theory is described by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{4.28}$$

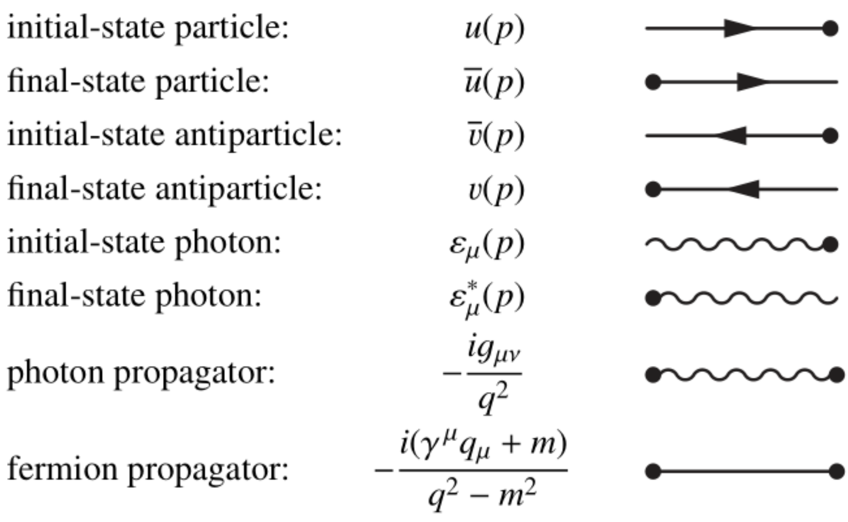
The Feynman vertex for a theory is determined by the interaction Lagrangian. Specifically, in the Feynman vertex, the fields are stripped off, resulting in $i\mathcal{L}_{\text{int}}$. In the case of the ϕ^4-theory, the vertex is $-i\frac{\lambda}{4!}$ without the fields.

The propagator of the field is obtained by taking the inverse of the kinetic operator in momentum space and multiplying it by 'i'. For the ϕ^4-theory, the kinetic term leads to the Klein-Gordon equation. The inverse of the Klein-Gordon equation in momentum space gives the propagator, which is $\frac{i}{p^2-m^2}$ for a scalar field.

Finally, the external line in Feynman diagrams is represented by the appropriate polarization vector or spinor. In the case of the ϕ^4-theory, it is simply the number 1, as it is the most trivial case. With these elements, we can now perform calculations for any Feynman diagram in the ϕ^4-theory.

The Feynman Rules for the QED are depicted in the following figure

Figura 6 – QED Feynman Rules.



Source: Modern Particle Physics [1].

Next, we will proceed with the calculation of the matrix element for the scattering depicted in Figure 4, where the interaction Lagrangian is defined using equation (3.46). The expression for the interaction Lagrangian is as follows:

$$\mathcal{L}_{\text{int}} = \overline{\psi}\Lambda^\mu\psi A_\mu \tag{4.29}$$

Considering the charge-charge interaction we have the usual QED vertex Feynman rule, $(iQ_\nu \gamma^\mu)$, where we have defined $Q_\nu \equiv f_Q(q)$ as the neutrino eletric charge. By employing the Feynman Rules for QED and utilizing the given vertex, we can compute the matrix element for the scattering process $\overline{\nu} + \chi \longrightarrow \overline{\nu} + \chi^\dagger$ as follows:

$$-i\mathcal{M} = \overline{v}\,(p_1)\,(iQ_\nu \gamma^\mu)\, v\,(p_3)\,\frac{(-ig_{\mu\nu})}{(p_1 - p_3)^2}\overline{u}(p_4)\left(iQ_\chi \gamma^\nu\right) u(p_2) \tag{4.30}$$

If the particle is unpolarized we have to sum over the polarizations (spins) of the final particles and the average over the polarization (spins) of the initial ones, so we need replace

$$\left|\mathcal{M}_{fi}\right|^2 \longrightarrow \langle|\mathcal{M}|^2\rangle \equiv \frac{1}{4}\sum_{\text{spins}}\left|\mathcal{M}_{fi}\right|^2.$$

In the calculation of $\langle|\mathcal{M}|^2\rangle = \langle\mathcal{M}^*\mathcal{M}\rangle$, the equation (B.28) is useful, in the case of $\Gamma = \gamma^\mu$, we have

$$\begin{aligned}\left[\overline{\psi}\Gamma\phi\right]^\dagger &= \left[\overline{\psi}\Gamma\phi\right]^* = \phi^\dagger\Gamma^\dagger(\gamma^0)^\dagger\psi \\ &= \overline{\phi}\gamma^0\Gamma^\dagger\gamma^0\psi = \eta_0(\Gamma)\overline{\phi}\Gamma\psi,\end{aligned}$$

Considering the same order of the Γ matrices in $\mathcal{S}$, we find that $\eta_0\,[\Gamma] = (1,\ 1,\ 1,\ 1,\ -1)$. Therefore, for $\Gamma = \gamma^\mu$

$$\left[\overline{\psi}\gamma^\mu\phi\right]^* = \overline{\phi}\gamma^\mu\psi. \tag{4.31}$$

In the current process, the initial-state antineutrino is generated through a weak decay, indicating that it is a right-handed (RH) antineutrino. Consequently, the matrix element needs to be adjusted accordingly, considering the modification:

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\sum_\sigma \mathcal{M}^*\mathcal{M}, \tag{4.32}$$

where σ denotes the spin projection and

$$\begin{aligned}\mathcal{M} &= \frac{Q_\nu Q_\chi}{(p_1 - p_3)^2}\overline{v}_R(p_1)\,\gamma^\mu v\,(p_3)\,\overline{u}(p_4)\gamma_\mu u(p_2) \\ &= \frac{Q_\nu Q_\chi}{q^2}\overline{v}_R(p_1)\,\gamma^\mu v\,(p_3)\,\overline{u}(p_4)\gamma_\mu u(p_2),\end{aligned} \tag{4.33}$$

where $q = p_1 - p_3$ is the momentum of the photon and $\overline{v}_R(p_1)$ is the RH component of the spinor. Using the equation (4.31), we can write:

$$\mathcal{M}^* = \frac{Q_\nu Q_\chi}{q^2}\overline{v}\,(p_3)\,\gamma^\nu v_R(p_1)\,\overline{u}(p_2)\gamma_\nu u(p_4). \tag{4.34}$$

Now we can proceed to the averaged squared matrix element

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\sum_\sigma \frac{Q_\nu^2 Q_\chi^2}{q^4}\overline{v}\,(p_1)\,\gamma^\mu P_R v\,(p_3)\,\overline{u}(p_4)\gamma_\mu u(p_2) \\ \times\overline{v}\,(p_3)\,\gamma^\nu P_R v\,(p_1)\,\overline{u}(p_2)\gamma_\nu u(p_4).\end{aligned} \tag{4.35}$$

We can rearange in the following form:

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{2q^4}\left[\sum_{\sigma_1\sigma_3} Q_\nu^2 \overline{v}\left(p_1,\ \sigma_1\right)\gamma^\mu P_R v\left(p_3,\ \sigma_3\right)\overline{v}\left(p_3,\ \sigma_3\right)\gamma^\nu P_R v\left(p_1,\ \sigma_1\right)\right]\\ &\times\left[\sum_{\sigma_4\sigma_2} Q_\chi^2 \overline{u}\left(p_4,\ \sigma_4\right)\gamma_\mu u\left(p_2,\ \sigma_2\right)\overline{u}\left(p_2,\ \sigma_2\right)\gamma_\nu u\left(p_4,\ \sigma_4\right)\right]\\ &= \frac{1}{2q^4}\mathcal{L}^{\mu\nu}\mathcal{H}_{\mu\nu},\end{aligned} \tag{4.36}$$

where we have make explicit the spin index for each spinor and introduced the fermionic and hadronic tensor, repectively defined by:

$$\mathcal{L}^{\mu\nu} \equiv \sum_{\sigma_1\sigma_3} Q_\nu^2 \overline{v}\left(p_1,\ \sigma_1\right)\gamma^\mu P_R v\left(p_3,\ \sigma_3\right)\overline{v}\left(p_3,\ \sigma_3\right)\gamma^\nu P_R v\left(p_1,\ \sigma_1\right) \tag{4.37}$$

and

$$\mathcal{H}_{\mu\nu} \equiv \sum_{\sigma_4\sigma_2} Q_\chi^2 \overline{u}\left(p_4,\ \sigma_4\right)\gamma_\mu u\left(p_2,\ \sigma_2\right)\overline{u}\left(p_2,\ \sigma_2\right)\gamma_\nu u\left(p_4,\ \sigma_4\right). \tag{4.38}$$

First, let's deal with the fermionic tensor. The equation (4.37) can be rewritten using index notation in the following way:

$$\begin{aligned}\mathcal{L}^{\mu\nu} &= \sum_{\sigma_1\sigma_3} Q_\nu^2 \overline{v_i}\left(p_1,\ \sigma_1\right)\left(\gamma^\mu\right)_{ij}\left(P_R\right)_{jk} v_k\left(p_3,\ \sigma_3\right)\overline{v_l}\left(p_3,\ \sigma_3\right)\left(\gamma^\nu\right)_{lm}\left(P_R\right)_{mo} v_o\left(p_1,\ \sigma_1\right)\\ &= Q_\nu^2 \sum_{\sigma_1} v_o\left(p_1,\ \sigma_1\right)\overline{v_i}\left(p_1,\ \sigma_1\right)\left(\gamma^\mu\right)_{ij}\left(P_R\right)_{jk} \sum_{\sigma_3} v_k\left(p_3,\ \sigma_3\right)\overline{v_l}\left(p_3,\ \sigma_3\right)\left(\gamma^\nu\right)_{lm}\left(P_R\right)_{mo},\end{aligned} \tag{4.39}$$

where $i,\ j,\ k,\ l,\ m,\ o = 1,\ 2,\ 3,\ 4$ all summed over. Using the completeness relation, given by [26]:

$$\begin{aligned}\sum_\sigma u(p,\ \sigma)\overline{u}(p,\ \sigma) &= (\not{p} + m);\\ \sum_\sigma v(p,\ \sigma)\overline{v}(p,\ \sigma) &= (\not{p} - m).\end{aligned} \tag{4.40}$$

We can rewrite the fermionic tensor in the following way:

$$\begin{aligned}\mathcal{L}^{\mu\nu} &= Q_\nu^2 (\not{p}_1 - m_\nu)_{oi}(\gamma^\mu)_{ij}(P_R)_{jk}(\not{p}_3 - m_\nu)_{kl}(\gamma^\nu)_{lm}(P_R)_{mo}\\ &= Q_\nu^2\left[(\not{p}_1 - m_\nu)\gamma^\mu P_R(\not{p}_3 - m_\nu)\gamma^\nu P_R\right]_{oo}\\ &= Q_\nu^2 \mathrm{Tr}\left[(\not{p}_1 - m_\nu)\gamma^\mu P_R(\not{p}_3 - m_\nu)\gamma^\nu P_R\right],\end{aligned} \tag{4.41}$$

where m_ν is the neutrino mass which we will consider vanish since $m_\nu \ll m_\chi$ and the other energies. Using the $P_R = \frac{(1+\gamma^5)}{2}$, we find the fermionic tensor as:

$$\begin{aligned}\mathcal{L}^{\mu\nu} &= \frac{Q_\nu^2}{4}\Big\{\mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_3\gamma^\nu\right] + \mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_3\gamma^\nu\gamma^5\right] \\ &+ \mathrm{Tr}\left[\not{p}_1\gamma^\mu\gamma^5\not{p}_3\gamma^\nu\right] + \mathrm{Tr}\left[\not{p}_1\gamma^\mu\gamma^5\not{p}_3\gamma^\nu\gamma^5\right]\Big\} \\ &= \frac{Q_\nu^2}{4}\Big\{\mathrm{Tr}\left[A\right] + \mathrm{Tr}\left[B\right] + \mathrm{Tr}\left[C\right] + \mathrm{Tr}\left[D\right]\Big\}.\end{aligned} \tag{4.42}$$

Now we need just to take the trace of the γ-matrices (see Apeendix B.2). The first trace can be writeen as:

$$\begin{aligned}\mathrm{Tr}\left[A\right] &= p_{1_\sigma}p_{3_\rho}\mathrm{Tr}\left(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu\right) \\ &\overset{\text{B.40}}{=} 4p_{1_\sigma}p_{3_\rho}\left(g^{\sigma\mu}g^{\rho\nu} - g^{\sigma\rho}g^{\mu\nu} + g^{\sigma\nu}g^{\mu\rho}\right) \\ &= 4\left(p_1^\mu p_3^\nu - (p_1\cdot p_3)g^{\mu\nu} + p_3^\mu p_1^\nu\right),\end{aligned} \tag{4.43}$$

simillarly we can calculate

$$\begin{aligned}\mathrm{Tr}\left[B\right] &= p_{1_\sigma}p_{3_\rho}\mathrm{Tr}\left(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu\gamma^5\right) \\ &\overset{\text{B.43}}{=} -4ip_{1_\sigma}p_{3_\rho}\epsilon^{\sigma\mu\rho\nu},\end{aligned} \tag{4.44}$$

$$\begin{aligned}\mathrm{Tr}\left[C\right] &= p_{1_\sigma}p_{3_\rho}\mathrm{Tr}\left(\gamma^\sigma\gamma^\mu\gamma^5\gamma^\rho\gamma^\nu\right) \\ &= p_{1_\sigma}p_{3_\rho}\mathrm{Tr}\left(\gamma^5\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu\right) \\ &\overset{\text{B.43}}{=} -4ip_{1_\sigma}p_{3_\rho}\epsilon^{\sigma\mu\rho\nu},\end{aligned} \tag{4.45}$$

$$\begin{aligned}\mathrm{Tr}\left[D\right] &= p_{1_\sigma}p_{3_\rho}\mathrm{Tr}\left(\gamma^\sigma\gamma^\mu\gamma^5\gamma^\rho\gamma^\nu\gamma^5\right) \\ &= p_{1_\sigma}p_{3_\rho}\mathrm{Tr}\left(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu\right) \\ &\overset{\text{B.40}}{=} 4\left(p_1^\mu p_3^\nu - (p_1\cdot p_3)g^{\mu\nu} + p_3^\mu p_1^\nu\right).\end{aligned}$$

Finally, equation (4.42) in the following form:

$$\begin{aligned}\mathcal{L}^{\mu\nu} &= 2Q_\nu^2\Big[p_1^\mu p_3^\nu - (p_1\cdot p_3)g^{\mu\nu} + p_3^\mu p_1^\nu - ip_{1_\sigma}p_{3_\rho}\epsilon^{\sigma\mu\rho\nu}\Big] \\ &= T^{\mu\nu} + L^{\mu\nu},\end{aligned} \tag{4.46}$$

with $T^{\mu\nu} = 2Q_\nu^2\left(p_1^\mu p_3^\nu - (p_1\cdot p_3)g^{\mu\nu} + p_3^\mu p_1^\nu\right)$ is a symmetric tensor and $L^{\mu\nu} = -2iQ_\nu^2 p_{1_\sigma}p_{3_\rho}\epsilon^{\sigma\mu\rho\nu}$ is a antisymmetric tensor. Now we need to do the same procedure for the hadronic tensor (4.38). In this case, we need to exercise caution as the mass of the final particle differs from that of the initial particle. To address this, we can make the following simplification: $u\left(p_i,\ \sigma_i\right) \equiv u_i$, the hadronic tensor can be writeen as:

$$\begin{aligned}
\mathcal{H}_{\mu\nu} &= Q_\chi^2 \sum_\sigma \overline{u}_4 \gamma_\mu u_2 \overline{u}_2 \gamma_\nu u_4 \\
&= Q_\chi^2 \sum_\sigma \overline{u}_{4_i} (\gamma_\mu)_{ij} u_{2_j} \overline{u}_{2_k} (\gamma_\nu)_{kl} u_{4_l} \\
&= Q_\chi^2 \sum_\sigma u_{4_l} \overline{u}_{4_i} (\gamma_\mu)_{ij} u_{2_j} \overline{u}_{2_k} (\gamma_\nu)_{kl} \\
&= Q_\chi^2 \mathrm{Tr}\left((\not{p}_4 + m_{\chi^\dagger})\gamma_\mu(\not{p}_2 + m_\chi)\gamma_\nu\right) \\
&= Q_\chi^2 \mathrm{Tr}\Big(\not{p}_4 \gamma_\mu \not{p}_2 \gamma_\nu + m_\chi \not{p}_4 \gamma_\mu \gamma_\nu + \\
&\quad + m_{\chi^\dagger} \gamma_\mu \not{p}_2 \gamma_\nu + m_\chi m_{\chi^\dagger} \gamma_\mu \gamma_\nu\Big) \\
&= Q_\chi^2 \left\{\mathrm{Tr}\left(\tilde{A}\right) + \mathrm{Tr}\left(\tilde{B}\right) + \mathrm{Tr}\left(\tilde{C}\right) + \mathrm{Tr}\left(\tilde{D}\right)\right\}.
\end{aligned} \tag{4.47}$$

Now, as we did in the previous case, we need just take the trace of γ-matrices. We can write this trace as

$$\begin{aligned}
\mathrm{Tr}\left(\tilde{A}\right) &= p_4^\sigma p_2^\rho \mathrm{Tr}\left(\gamma_\sigma \gamma_\mu \gamma_\rho \gamma_\nu\right) \\
&= 4\left(p_{4_\mu} p_{2_\nu} - (p_2 \cdot p_4)\, g_{\mu\nu} + p_{2_\mu} p_{4_\nu}\right)
\end{aligned} \tag{4.48}$$

$$\begin{aligned}
\mathrm{Tr}\left(\tilde{B}\right) &= \mathrm{Tr}\left(m_\chi \not{p}_4 \gamma_\mu \gamma_\nu\right) \\
&= m_\chi p_4^\sigma \mathrm{Tr}\left(\gamma_\sigma \gamma_\mu \gamma_\nu\right) \overset{\text{B.39}}{=} 0
\end{aligned} \tag{4.49}$$

$$\mathrm{Tr}\left(\tilde{C}\right) \overset{\text{B.39}}{=} 0 \tag{4.50}$$

$$\mathrm{Tr}\left(\tilde{D}\right) \overset{\text{B.38}}{=} 4 m_\chi m_{\chi^\dagger} g_{\mu\nu}. \tag{4.51}$$

Finally, the hadronic tensor can be written as:

$$\mathcal{H}_{\mu\nu} = 4Q_\chi^2 \left(p_{4_\mu} p_{2_\nu} - (p_2 \cdot p_4)\, g_{\mu\nu} + p_{2_\mu} p_{4_\nu} + m_\chi m_{\chi^\dagger} g_{\mu\nu}\right), \tag{4.52}$$

Using the fermionic and hadronic tensor (4.46), (4.52), respectivelly, the equation (4.36) can be written as:

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{2q^4} T^{\mu\nu} \mathcal{H}_{\mu\nu} + \frac{1}{2q^4} L^{\mu\nu} \mathcal{H}_{\mu\nu}. \tag{4.53}$$

The second component of the above equation vanish since it is a contraction of a antisymmetric tensor with a symmetric one. Finally, the squared matrix element is written in the following way:

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{4Q_\nu^2 Q_\chi^2}{q^4}\left(p_1^\mu p_3^\nu - (p_1\cdot p_3)g^{\mu\nu} + p_3^\mu p_1^\nu\right)\left(p_{4_\mu}p_{2_\nu} - (p_2\cdot p_4)\,g_{\mu\nu} + p_{2_\mu}p_{4_\nu} + m_\chi m_{\chi^\dagger} g_{\mu\nu}\right)\\ &= \frac{8Q_\nu^2 Q_\chi^2}{q^4}\Big[(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_2)(p_3\cdot p_4) - m_\chi m_{\chi^\dagger}(p_1\cdot p_3)\Big],\end{aligned} \tag{4.54}$$

where we have used $g^{\mu\nu}g_{\mu\nu} = 4$, using that $q^2 = (p_1 - p_3)^2 \approx -2p_1\cdot p_3$, we can write:

$$\langle|\mathcal{M}|^2\rangle = 2Q_\nu^2 Q_\chi^2\left[\frac{(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_2)(p_3\cdot p_4) - m_\chi m_{\chi^\dagger}(p_1\cdot p_3)}{(p_1\cdot p_3)^2}\right]. \tag{4.55}$$

The inner product in the squared matrix element is given by:

$$(p_1\cdot p_2) = m_\chi E_1 \tag{4.56}$$
$$(p_1\cdot p_3) = E_1E_3(1-\cos\theta) \tag{4.57}$$
$$(p_1\cdot p_4) = m_\chi E_1 - E_1E_3(1-\cos\theta) \tag{4.58}$$
$$(p_2\cdot p_3) = m_\chi E_3 \tag{4.59}$$
$$(p_2\cdot p_4) = m_\chi(E_1 + m_\chi - E_3) \tag{4.60}$$
$$(p_3\cdot p_4) = m_\chi E_3 + E_1E_3(1-\cos\theta) \tag{4.61}$$
$$(p_1\cdot q) = -(p_1\cdot p_3) \tag{4.62}$$
$$(p_2\cdot q) = (p_2\cdot p_4) - m_\chi^2 \tag{4.63}$$
$$(p_3\cdot q) = (p_1\cdot p_3) \tag{4.64}$$
$$(p_4\cdot q) = m_{\chi^\dagger}^2 - (p_2\cdot p_4) \tag{4.65}$$

Therefore, equation (4.55) can be written as:

$$\langle|\mathcal{M}|^2\rangle = \frac{2Q_\nu^2 Q_\chi^2 m_\chi}{E_1E_3(1-\cos\theta)^2}\Big[2m_\chi + (E_1 + E_3 + m_{\chi^\dagger})(1-\cos\theta)\Big]. \tag{4.66}$$

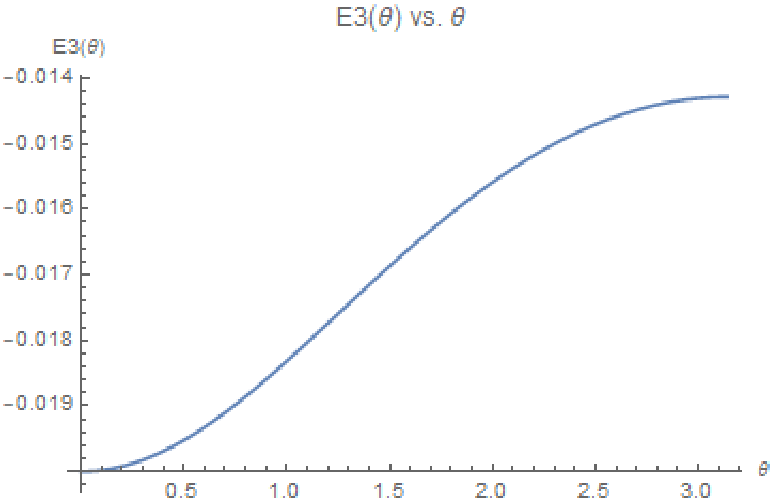
Finnaly, the differential cross-section (4.26) can be written as:

$$\begin{aligned}\frac{d\sigma}{d(\cos\theta)} &= \frac{1}{32\pi^2\left(m_\chi^2 - m_{\chi^\dagger}^2 + 2m_\chi E_1\right)}\left(\frac{E_3^2}{m_\chi E_1}\right)\\ &\times\frac{2Q_\nu^2 Q_\chi^2 m_\chi}{E_1E_3(1-\cos\theta)^2}\Big[2m_\chi + (E_1 + E_3 + m_{\chi^\dagger})(1-\cos\theta)\Big]\\ &= \left(\frac{Q_\nu Q_\chi}{4\pi}\right)^2\left(\frac{E_3}{E_1^2}\right)\frac{\Big[2m_\chi + (E_1 + E_3 + m_{\chi^\dagger})(1-\cos\theta)\Big]}{(1-\cos\theta)^2\left(m_\chi^2 - m_{\chi^\dagger}^2 + 2m_\chi E_1\right)},\end{aligned} \tag{4.67}$$

where $E_3(\theta)$ is given by the equation (4.23).

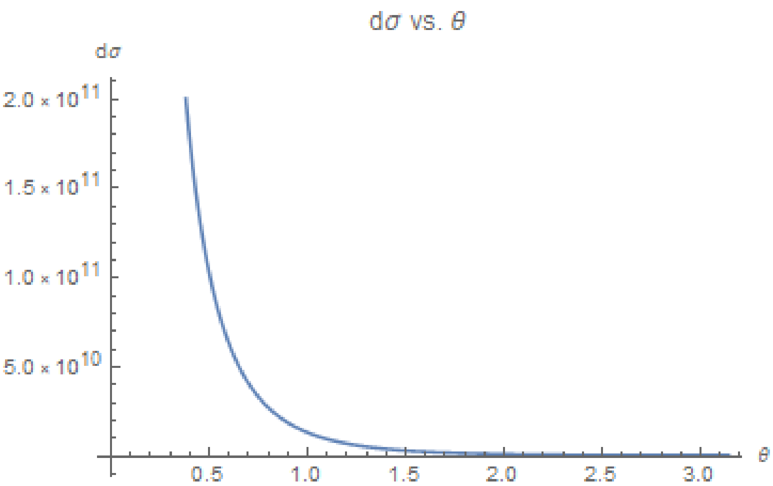
The following figures shows, respectively, the energy of scattered neutrino and the differential cross-section. For simplicity, we set $m_\chi = 1$, $m_{\chi^\dagger} = 1.2$, $E_1 = 0.2$, and $Q_\nu = Q_\chi = 1$.

Figura 7 – Scaterred neutrino energy considering the process $\overline{\nu} + \chi \longrightarrow \overline{\nu} + \chi^\dagger$.



Source: Author.

Figura 8 – Differential cross section as function of the scattering angle considering the process $\overline{\nu} + \chi \longrightarrow \overline{\nu} + \chi^\dagger$.



Source: Author.

We can see that, the differential cross-section diverges for $\theta = 0$.

4.1.2 The dipole-charge interaction

In the case of dipole-charge interaction the vertex Feynman rule must be modified. From equation (3.46) we can see that, the interaction lagrangian is given by

$$\mathcal{L}_{\text{int}} = \mu_\nu \overline{\psi} \sigma^{\mu\alpha} \partial_\alpha A_\mu \psi = \frac{\mu_\nu}{2} \overline{\psi} \sigma^{\mu\alpha} F_{\alpha\mu} \psi, \tag{4.68}$$

where μ_ν is the magnetic dipole moment for neutrino, and $F_{\nu\mu} = (\partial_\nu A_\mu - \partial_\mu A_\nu)$ is the electromagnetic tensor. Using the Feynman rule we can written the matrix element as follows:

$$-i\mathcal{M} = \frac{\mu_\nu Q_\chi}{(p_1 - p_3)^2} \overline{v}_R(p_1)\, \sigma^{\mu\alpha} q_\alpha v\,(p_3)\, \overline{u}(p_4)\gamma_\mu u(p_2). \tag{4.69}$$

Since

$$\overline{\psi_R}\sigma^{\mu\alpha} \overset{\mu\neq\alpha}{=} \psi^\dagger P_R \gamma^0 i\gamma^\mu\gamma^\alpha = \overline{\psi} P_L i\gamma^\mu\gamma^\alpha = \overline{\psi}\sigma^{\mu\alpha} P_L,$$

where we have used that $\{\gamma^5, \gamma^\mu\} = 0$. Therefore, the matrix element can be written as:

$$-i\mathcal{M} = \frac{\mu_\nu Q_\chi}{q^2} \overline{v}\,(p_1)\, \sigma^{\mu\alpha} q_\alpha P_L v\,(p_3)\, \overline{u}(p_4)\gamma_\mu u(p_2). \tag{4.70}$$

Assuming that the antineutrino produced in the electromagnetic interaction must be a left-handed (LH) antineutrino, which is not present in the Standard Model (SM), the equation mentioned above remains valid. Therefore, in our calculations, we assume the existence of the right-handed (RH) neutrino. However, in the case of massive Majorana neutrinos, the interaction current (3.46) indicates that only the anapole moment component remains non-zero [21]. By taking the complex conjugate of equation (4.69), we arrive at the following expression:

$$i\mathcal{M}^* = \frac{\mu_\nu Q_\chi}{q^2} \overline{v}\,(p_3)\, \sigma^{\rho\beta} q_\beta v_R(p_1)\, \overline{u}(p_2)\gamma_\rho u(p_4). \tag{4.71}$$

Using the Casimir trick to reagen the terms, the averaged squared matrix element can be written as[1]:

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{\mu_\nu^2 Q_\chi^2}{2q^4} \sum_\sigma \left\{(v_{1_p}\overline{v}_{1_i})(\sigma^{\mu\alpha}q_\alpha)_{ij}(P_L)_{jk}(v_{3_k}\overline{v}_{3_n})(\sigma^{\rho\beta}q_\beta)_{no}(P_R)_{op}\right\} \\ &\times \sum_\sigma \left\{u_{4_s}\overline{u}_{4_l}(\gamma_\mu)_{lm} u_{2_m}\overline{u}_{2_r}(\gamma_\rho)_{rs}\right\} \\ &= \frac{1}{2q^4}\mathcal{L}^{\mu\rho}\mathcal{H}_{\mu\rho}.\end{aligned} \tag{4.72}$$

Here, we have defined the fermionic tensor as:

$$\begin{aligned}\mathcal{L}^{\mu\rho} &= \mu_\nu^2 \sum_\sigma \left\{(v_{1_p}\overline{v}_{1_i})(\sigma^{\mu\alpha}q_\alpha)_{ij}(P_L)_{jk}(v_{3_k}\overline{v}_{3_n})(\sigma^{\rho\beta}q_\beta)_{no}(P_R)_{op}\right\} \\ &= \mu_\nu^2 \mathrm{Tr}\left[\not{p}_1 \sigma^{\mu\alpha} q_\alpha P_L \not{p}_3 \sigma^{\rho\beta} q_\beta P_R\right],\end{aligned} \tag{4.73}$$

1 Here the sum is over the spin index and we must not be confused with the product of two γ-matrices $\sigma^{\mu\alpha} \equiv \frac{i}{2}[\gamma^\mu,\ \gamma^\alpha]$

using that $P_L \not{p}_3 = \not{p}_3 P_R$ and $[P_R, \sigma^{\mu\alpha}] = 0$, we can write the fermionic tensor in the following way:

$$\begin{aligned}
\mathcal{L}^{\mu\rho} &= \mu_\nu^2 \mathrm{Tr}\left[\not{p}_1 \sigma^{\mu\alpha} q_\alpha \not{p}_3 \sigma^{\rho\beta} q_\beta P_R\right] \\
&= -\frac{\mu_\nu^2}{2}\left\{p_{1_\sigma} p_{3_\kappa} q_\alpha q_\beta \mathrm{Tr}\left[\gamma^\sigma\gamma^\mu\gamma^\alpha\gamma^\kappa\gamma^\rho\gamma^\beta\right] + p_{1_\sigma} p_{3_\kappa} q_\alpha q_\beta \mathrm{Tr}\left[\gamma^\sigma\gamma^\mu\gamma^\alpha\gamma^\kappa\gamma^\rho\gamma^\beta\gamma^5\right]\right\} \\
&= -\frac{\mu_\nu^2}{2} p_{1_\sigma} p_{3_\kappa} q_\alpha q_\beta \left\{\mathrm{Tr}[\bar{A}] + \mathrm{Tr}[\bar{B}]\right\} \\
&= -\frac{\mu_\nu^2}{2} L^{\mu\rho} - \frac{\mu_\nu^2}{2} T^{\mu\rho}
\end{aligned} \tag{4.74}$$

where we have used that $P_R^2 = P_R$. To compute the trace of the product of six γ-matrices, we will utilize our knowledge of taking the trace of the product of four γ-matrices and the anticommutation relation. For the sake of notation simplification, we will denote γ^a as a. Thus, we arrive at the following expression:

$$\begin{aligned}
\mathrm{Tr}\left[\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f\right] &\equiv \mathrm{Tr}(abcdef) \\
&= 2g^{ab}\mathrm{Tr}(cdef) - \mathrm{Tr}(bacdef),
\end{aligned}$$

performing this permitantions until we make the a in the right side of the trace, we get:

$$\begin{aligned}
\mathrm{Tr}(abcdef) &= 2g^{ab}\mathrm{Tr}(cdef) - 2g^{ac}\mathrm{Tr}(bdef) + 2g^{ad}\mathrm{Tr}(bcef) \\
&\quad - 2g^{ae}\mathrm{Tr}(bcdf) + 2g^{af}\mathrm{Tr}(bcdf) - \mathrm{Tr}(bcdefa) \\
\Rightarrow \mathrm{Tr}(abcdef) &= g^{ab}\mathrm{Tr}(cdef) - g^{ac}\mathrm{Tr}(bdef) + g^{ad}\mathrm{Tr}(bcef) - g^{ae}\mathrm{Tr}(bcdf) + g^{af}\mathrm{Tr}(bcdf),
\end{aligned}$$

Since the trace of product of four matrices is given by:

$$\mathrm{Tr}(cdef) = 4\left(g^{cd}g^{ef} - g^{ce}g^{df} + g^{cf}g^{de}\right).$$

We finally, find

$$\begin{aligned}
\mathrm{Tr}(abcdef) = 4\Big[& g^{ab}\left(g^{cd}g^{ef} - g^{ce}g^{df} + g^{cf}g^{de}\right) - g^{ac}\left(g^{bd}g^{ef} - g^{be}g^{df} + g^{bf}g^{de}\right) \\
&+ g^{ad}\left(g^{bc}g^{ef} - g^{be}g^{cf} + g^{bf}g^{ce}\right) - g^{ae}\left(g^{bc}g^{df} - g^{bd}g^{cf} + g^{bf}g^{cd}\right) \\
&+ g^{af}\left(g^{bc}g^{de} - g^{bd}g^{ce} + g^{be}g^{cd}\right)\Big]
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}[\bar{A}] = 4\Big[& g^{\sigma\mu}\left(g^{\alpha\kappa}g^{\rho\beta} - g^{\alpha\rho}g^{\kappa\beta} + g^{\alpha\beta}g^{\kappa\rho}\right) - g^{\sigma\alpha}\left(g^{\mu\kappa}g^{\rho\beta} - g^{\mu\rho}g^{\kappa\beta} + g^{\mu\beta}g^{\kappa\rho}\right) \\
&+ g^{\sigma\kappa}\left(g^{\alpha\mu}g^{\rho\beta} - g^{\mu\rho}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\rho}\right) - g^{\sigma\rho}\left(g^{\alpha\mu}g^{\kappa\beta} - g^{\mu\kappa}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\kappa}\right) \\
&+ g^{\sigma\beta}\left(g^{\alpha\mu}g^{\kappa\rho} - g^{\mu\kappa}g^{\alpha\rho} + g^{\mu\rho}g^{\alpha\kappa}\right)\Big]
\end{aligned} \tag{4.75}$$

Therefore, the first part of the fermionic tensor can be written as:

$$\begin{aligned}L^{\mu\rho} &= 4p_{1_\sigma}p_{3_\kappa}q_\alpha q_\beta\Big[g^{\sigma\mu}\left(g^{\alpha\kappa}g^{\rho\beta} - g^{\alpha\rho}g^{\kappa\beta} + g^{\alpha\beta}g^{\kappa\rho}\right) - g^{\sigma\alpha}\left(g^{\mu\kappa}g^{\rho\beta} - g^{\mu\rho}g^{\kappa\beta} + g^{\mu\beta}g^{\kappa\rho}\right)\\ &+ g^{\sigma\kappa}\left(g^{\alpha\mu}g^{\rho\beta} - g^{\mu\rho}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\rho}\right) - g^{\sigma\rho}\left(g^{\alpha\mu}g^{\kappa\beta} - g^{\mu\kappa}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\kappa}\right)\\ &+ g^{\sigma\beta}\left(g^{\alpha\mu}g^{\kappa\rho} - g^{\mu\kappa}g^{\alpha\rho} + g^{\mu\rho}g^{\alpha\kappa}\right)\Big]\\ &= L_1^{\mu\rho} + L_2^{\mu\rho} + \cdots + L_5^{\mu\rho}\end{aligned} \tag{4.76}$$

$$\begin{aligned}L_1^{\mu\rho} &= 4q^2 p_1^\mu p_3^\rho\\ L_2^{\mu\rho} &= -4(p_1\cdot q)\left((p_3^\mu q^\rho + p_3^\rho q^\mu) - (p_3\cdot q)g^{\mu\rho}\right)\\ L_3^{\mu\rho} &= 4(p_1\cdot p_3)\left(2q^\mu q^\rho - q^2 g^{\mu\rho}\right)\\ L_4^{\mu\rho} &= -4p_1^\rho\left(2(p_3\cdot q)q^\mu - q^2 p_3^\mu\right)\\ L_5^{\mu\rho} &= 4(p_1\cdot q)\Big[\underbrace{(q^\mu p_1^\rho - q^\rho p_1^\mu)}_{\text{antisymmetric}} + g^{\mu\rho}(p_3\cdot q)\Big].\end{aligned}$$

Using that $(p_1\cdot q) = -(p_1\cdot p_3)$ and $(p_3\cdot q) = (p_1\cdot p_3)$, we can simplify the above tensors to:

$$L_1^{\mu\rho} = 4q^2 p_1^\mu p_3^\rho \tag{4.77}$$

$$L_2^{\mu\rho} = 4(p_1\cdot p_3)\left[(p_3^\mu q^\rho + q^\mu p_3^\rho) - (p_1\cdot p_3)g^{\mu\rho}\right] \tag{4.78}$$

$$L_3^{\mu\rho} = 4(p_1\cdot p_3)\left(2q^\mu q^\rho - q^2 g^{\mu\rho}\right) \tag{4.79}$$

$$L_4^{\mu\rho} = -4p_1^\rho\left[2(p_1\cdot p_3)q^\mu - q^2 p_3^\mu\right] \tag{4.80}$$

$$L_5^{\mu\rho} = 4(p_1\cdot p_3)^2 g^{\mu\rho}, \tag{4.81}$$

where we have removed the antisymmetric part since when we contract with the hadronic tensor, which is symmetric, it vanish. Therefore, we can write the equation (4.76) in the following way:

$$L^{\mu\rho} = 2q^2\left\{2\left(p_1^\mu p_3^\rho + p_3^\mu p_1^\rho\right) + 2q^\mu p_1^\rho - \left(p_3^\mu q^\rho + q^\mu p_3^\rho\right) - 2q^\mu q^\rho + q^2 g^{\mu\rho}\right\} \tag{4.82}$$

To calculate the second part of the fermionic tensor, we start by the following equation [27]:

$$\gamma^\sigma\gamma^\mu\gamma^\alpha = g^{\sigma\mu}\gamma^\alpha - g^{\sigma\alpha}\gamma^\mu + g^{\mu\alpha}\gamma^\sigma + i\epsilon^{\sigma\mu\alpha\lambda}\gamma_\lambda\gamma^5, \tag{4.83}$$

multipling by $\gamma^\kappa\gamma^\rho\gamma^\beta\gamma^5$ and taking the trace we find

$$\begin{aligned}\mathrm{Tr}\left(\gamma^\sigma\gamma^\mu\gamma^\alpha\gamma^\kappa\gamma^\rho\gamma^\beta\gamma^5\right) = -4i\Big(&\epsilon^{\alpha\kappa\rho\beta}g^{\sigma\mu} - \epsilon^{\mu\kappa\rho\beta}g^{\sigma\alpha} + \epsilon^{\sigma\kappa\rho\beta}g^{\mu\alpha}\\ &+ \epsilon^{\sigma\mu\alpha\kappa}g^{\rho\beta} - \epsilon^{\sigma\mu\alpha\rho}g^{\kappa\beta} + \epsilon^{\sigma\mu\alpha\beta}g^{\kappa\rho}\Big).\end{aligned} \tag{4.84}$$

When we take the squared matrix element, we need to contract this with the hadronic tensor (4.52), so if the Levi-Civita symbol carry the μ and ρ indexies this will vanish. The same occurs for the indexes α and β, since $T^{\mu\rho} \sim q_\alpha q_\beta$. Finnally, the non vanishing terms will be:

$$\mathrm{Tr}\left(\gamma^\sigma\gamma^\mu\gamma^\alpha\gamma^\kappa\gamma^\rho\gamma^\beta\gamma^5\right) = -4i\left(\epsilon^{\sigma\kappa\rho\beta}g^{\mu\alpha} + \epsilon^{\sigma\mu\alpha\kappa}g^{\rho\beta}\right) \tag{4.85}$$

The second part of the fermionic tensor can be expressed in the form:

$$T^{\mu\rho} = -4ip_{1_\sigma}p_{3_\kappa}q_\alpha\left(q^\mu\epsilon^{\rho\sigma\kappa\alpha} + q^\rho\epsilon^{\mu\sigma\kappa\alpha}\right). \tag{4.86}$$

When $T^{\mu\rho}$ contract with $\mathcal{H}_{\mu\rho}$, the proportional part to $g_{\mu\rho}$ will vanish, since

$$p_{1_\sigma}p_{3_\kappa}q_\alpha q^\mu\epsilon^{\rho\sigma\kappa\alpha} \times g_{\mu\rho} = p_{1_\sigma}p_{3_\kappa}q_\alpha q_\rho\epsilon^{\rho\sigma\kappa\alpha} = 0. \tag{4.87}$$

Therefore, in the contraction we find:

$$\begin{aligned} T^{\mu\rho}\mathcal{H}_{\mu\rho} &= -4ip_{1_\sigma}p_{3_\kappa}q_\alpha\left(q^\mu\epsilon^{\rho\sigma\kappa\alpha} + q^\rho\epsilon^{\mu\sigma\kappa\alpha}\right)4Q_\chi^2\left(p_{4_\mu}p_{2_\rho} - (p_2\cdot p_4)\,g_{\mu\rho} + p_{2_\mu}p_{4_\rho} + m_\chi m_{\chi^\dagger}g_{\mu\rho}\right) \\ &= -16iQ_\chi^2\left((p_4\cdot q)p_{1_\sigma}p_{3_\kappa}q_\alpha p_{2_\rho}\epsilon^{\rho\sigma\kappa\alpha} + (p_2\cdot q)p_{1_\sigma}p_{3_\kappa}q_\alpha p_{4_\rho}\epsilon^{\rho\sigma\kappa\alpha}\right). \end{aligned} \tag{4.88}$$

In the calculation of $T^{\mu\rho}\mathcal{H}\mu\rho$, we observe that any index is contracted between a momentum and an index in the Levi-Civita symbol. However, the kinematic analysis (4.14 - 4.17) reveals the absence of momentum in the x-direction. Since the Levi-Civita symbol requires one index to be equal to 2, it follows that $T^{\mu\rho}\mathcal{H}\mu\rho$ must vanish. This cancellation is desirable because the contraction involves complex numbers. This particular component of the fermionic tensor only contributes in the case of dipole-dipole interaction, where two Levi-Civita symbols are contracted, resulting in the emergence of metric products. Finnaly, the fermionic tensor can be written as:

$$\mathcal{L}^{\mu\rho} = \mu_\nu^2 q^2\left\{\left(p_3^\mu q^\rho + q^\mu p_3^\rho\right) + 2q^\mu q^\rho - 2\left(p_1^\mu p_3^\rho + p_3^\mu p_1^\rho\right) - 2q^\mu p_1^\rho - q^2 g^{\mu\rho}\right\} \tag{4.89}$$

Now we can contract the fermionic tensor (4.89) and the hadronic tensor (4.52). Which is expressed as follow:

$$\begin{aligned} \mathcal{L}^{\mu\rho}\mathcal{H}_{\mu\rho} &= 4\mu_\nu^2 Q_\chi^2 q^2\Big\{2(p_2\cdot q)\left[(p_3\cdot p_4) - (p_1\cdot p_4)\right] + 2(p_4\cdot q)\left[(p_2\cdot p_3) - (p_1\cdot p_2)\right] + 4(p_2\cdot q)(p_4\cdot q) \\ &\quad - 4\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_2\cdot p_3)(p_1\cdot p_4)\right] - 2q^2(p_2\cdot p_4)\Big\} \\ &= 4\mu_\nu^2 Q_\chi^2 q^2\Big\{\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4\Big\}, \end{aligned} \tag{4.90}$$

where we have defined

$$\Gamma_1 \equiv 2(p_2\cdot q)\left[(p_3\cdot p_4) - (p_1\cdot p_4)\right] + 2(p_4\cdot q)\left[(p_2\cdot p_3) - (p_1\cdot p_2)\right] \tag{4.91}$$

$$\Gamma_2 \equiv 4(p_2\cdot q)(p_4\cdot q) \tag{4.92}$$

$$\Gamma_3 \equiv -4\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_2\cdot p_3)(p_1\cdot p_4)\right] \tag{4.93}$$

$$\Gamma_4 \equiv -2q^2(p_2\cdot p_4). \tag{4.94}$$

To check whether this expression is consistent we will take the ultra relativistic limit, which we will consider that the energy of the initicial neutrino E_ν is much greater than the mass of the particles, so we can consider the scattering of the neutrino with an electron in relativistic limite. In this limite we find, we will use the Mandelstan variables and the inner product of the momentum of the particles and the momentum of the photon, can be expressed as:

$$(p_1 \cdot p_2) = (p_3 \cdot p_4) = \frac{s}{2}, \tag{4.95}$$

$$(p_1 \cdot p_3) = (p_2 \cdot p_4) = \frac{-t}{2} \tag{4.96}$$

$$(p_1 \cdot p_4) = (p_2 \cdot p_3) = \frac{-u}{2} \tag{4.97}$$

$$(p_2 \cdot q) = -(p_4 \cdot q) = \frac{-t}{2}. \tag{4.98}$$

Hence, in this limit the Γ coeficient (4.91 - 4.94) can be expressed as:

$$\Gamma_1 = t^2 \tag{4.99}$$

$$\Gamma_2 = -t^2 \tag{4.100}$$

$$\Gamma_3 = -2s^2 - t^2 - 2st \tag{4.101}$$

$$\Gamma_4 = t^2. \tag{4.102}$$

Therefore, the squared matrix element can be written as:

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle = \frac{1}{2q^4} \mathcal{L}^{\mu\rho} \mathcal{H}_{\mu\rho} &= \frac{1}{t^2} 4\mu_\nu^2 e^2 t \left(-2s^2 - 2st\right) \\ &= \frac{-4\mu_\nu^2 e^2}{t} \left(s^2 + st\right) \end{aligned} \tag{4.103}$$

Our goal now is to express the differential cross-section in terms of the energy of the scattered electron (T_e). Since $t = -2(p_2 \cdot p_4) = -2m_e T_e$, where m_e is the electron mass. We can write

$$\begin{aligned} dt &= -2m\, dT_e \\ \Rightarrow \frac{dT_e}{dt} &= -\frac{1}{2m_e}. \end{aligned}$$

The differential cross section can be expressed in terms of electron energy as:

$$\frac{d\sigma}{dt} = \frac{d\sigma}{dT_e} \left| \frac{dT_e}{dt} \right| = \frac{1}{2m_e} \frac{d\sigma}{dT_e} \tag{4.104}$$

Hence, we can express the differential cross section as:

$$\frac{d\sigma}{dT_e} = \frac{1}{32\pi E_\nu^2 m_e} \langle |\mathcal{M}|^2 \rangle, \tag{4.105}$$

using $s = 2mE_\nu$ and $t = -2mT_e$. We find de following differential cross section:

$$\frac{d\sigma}{dT_e} = \frac{\mu_\nu^2 e^2}{4\pi}\left(\frac{1}{T_e} - \frac{1}{E_\nu}\right). \tag{4.106}$$

Finally, using the Bohr magneton $\mu_B \equiv \frac{e}{2m_e}$ and $\alpha \equiv \frac{e^2}{4\pi}$, we find the differential cross section can be written as:

$$\frac{d\sigma}{dT_e} = \frac{\pi\alpha^2}{m_e^2}\left(\frac{1}{T_e} - \frac{1}{E_\nu}\right)\left(\frac{\mu_\nu}{\mu_B}\right)^2, \tag{4.107}$$

which is the same result given by [21].

Using the Mandelstan variables, we can express the Γ's in the following way:

$$\Gamma_1 = -\frac{(\Delta m^2 - t)(\Delta m^2 + t)}{2}, \quad \text{with } \Delta m^2 = m_{\chi^\dagger}^2 - m_\chi^2 \tag{4.108}$$

$$\Gamma_2 = \frac{(\Delta m^2 - t)(\Delta m^2 + t)}{2} \tag{4.109}$$

$$\Gamma_3 = -\left\{(s - m_\chi^2)(s - m_{\chi^\dagger}^2) + (m_\chi^2 - u)(m_{\chi^\dagger}^2 - u)\right\} \tag{4.110}$$

$$\Gamma_4 = -t\left(m_\chi^2 + m_{\chi^\dagger}^2 - t\right). \tag{4.111}$$

In the above equations we can see that, in the zero mass limit we recover the Γ_i's given in relativistic limit which we alredy discussed. Using $u + s + t = m_{\chi^\dagger}^2 + m_\chi^2$, we can express

$$(m_\chi^2 - u)(m_{\chi^\dagger}^2 - u) = \left(s + t - m_{\chi^\dagger}^2\right)\left(s + t - m_\chi^2\right).$$

Hence, the sum of the Γ_i's can be expressed as:

$$\sum_i \Gamma_i = -\left(2s^2 + 2st - 2(m_\chi^2 + m_{\chi^\dagger}^2) + 2(m_\chi m_{\chi^\dagger})\right). \tag{4.112}$$

Therefore, the squared matrix element can be expressed in the following way:

$$\langle|\mathcal{M}|^2\rangle = \frac{-4\mu_\nu^2 Q_\chi^2}{t}\left\{s^2 + st - (m_\chi^2 + m_{\chi^\dagger}^2)s + (m_\chi m_{\chi^\dagger})^2\right\}. \tag{4.113}$$

Again we see that in the massless limit, the equation (4.103) is recovered. Now we can express the squared matrix element in terms of the energy of the initial neutrino, which we will denote by E_ν, the masses of the nuclei in the inital and final state and the energy of the scattered nuclei denoted by T_χ as the follow:

$$\langle|\mathcal{M}|^2\rangle = \frac{-8\mu_\nu^2 Q_\chi^2 m_\chi}{\left(m_\chi^2 + m_{\chi^\dagger}^2 - 2m_\chi^2 T_\chi\right)}\left\{\frac{m_\chi(m_\chi^2 + m_{\chi^\dagger}^2)}{2} + 2m_\chi E_\nu^2\left(1 + \frac{m_\chi}{E_\nu}\right) - m_\chi^2 T_\chi\left(1 + 2\frac{E_\nu}{m_\chi}\right)\right\}. \tag{4.114}$$

Finally, the differential cross section can be expressed as:

$$\frac{d\sigma}{dT_\chi} = -\frac{\pi}{m_e^2}\left(\frac{eQ_\chi}{4\pi}\right)^2 \frac{m_\chi}{\left(m_\chi^2 + m_{\chi^\dagger}^2 - 2m_\chi^2 T_\chi\right)} \times\left\{\frac{(m_\chi^2 + m_{\chi^\dagger}^2)}{2E_\nu^2} + 2\left(1+\frac{m_\chi}{E_\nu}\right) - \frac{m_\chi T_\chi}{E_\nu^2}\left(1+2\frac{E_\nu}{m_\chi}\right)\right\}\left(\frac{\mu_\nu}{\mu_B}\right)^2 \tag{4.115}$$

If we whant to express this differential cross section in terms of the scarreting angle, we will start by considering:

$$\Gamma_1 = 4E_1E_3(1-\cos\theta)\left[(p_2\cdot p_4) - m_\chi^2\right] + 2(m_\chi^3 - m_\chi m_{\chi^\dagger}^2)E_\gamma; \tag{4.116}$$

$$\Gamma_2 = 4\left[\left(m_\chi^2 + m_{\chi^\dagger}^2\right)(p_2\cdot p_4) - (p_2\cdot p_4)^2 - m_\chi^2 m_{\chi^\dagger}^2\right]; \tag{4.117}$$

$$\Gamma_3 = -4m_\chi E_1E_3\left[2m_\chi + E_\gamma(1-\cos\theta)\right]; \tag{4.118}$$

$$\Gamma_4 = -2q^2(p_2\cdot p_4) \tag{4.119}$$

where $E_\gamma = E_1 - E_3$ is the photon energy. The sum of the Γ can be expressed as:

$$\sum_{i=1}^{4}\Gamma_i = 2\left\{m_\chi^4 + 3m_\chi^3 E_\gamma + m_\chi^2\left(2E_\gamma^2 - 2m_{\chi^\dagger}^2 - 4E_1E_3\right) + m_\chi m_{\chi^\dagger}^2 E_\gamma + q^2\left(m_\chi^2 + m_\chi E_\gamma\right)\right\}. \tag{4.120}$$

Therefore, by utilizing equation (4.76), we can express the squared matrix element for the scattering process $\overline{\nu} + \chi \longrightarrow \overline{\nu} + \chi^\dagger$ involving dipole-charge interaction in terms of the scattering angle as follows:

$$\langle|\mathcal{M}|^2\rangle = \frac{-2\mu_\nu^2 Q_\chi^2}{E_1E_3(1-\cos\theta)}\left\{2m_\chi^4 + 3m_\chi^3E_\gamma + 2m_\chi^2\left(E_\gamma^2 - m_{\chi^\dagger}^2 - 2E_1E_3\right) + m_\chi m_{\chi^\dagger}^2 E_\gamma\right\} + 4\mu_\nu^2 Q_\chi^2\left(m_\chi^2 + m_\chi E_\gamma\right). \tag{4.121}$$

The squared matrix element write is this way is all expressed in terms of the initial energy of the particles E_1, m_χ the photon energy E_γ which is a function of the scattering angle and the mass of the excited nuclei $m_{\chi^\dagger}$. Finally, the expression for the differential cross section can be written as:

$$\frac{d\sigma}{d\cos\theta} = \left(\frac{eQ_\chi}{4\pi}\right)^2\left(\frac{E_3}{E_1}\right)\left(\frac{1}{4m_e^2}\right)\left\{\frac{\left(m_\chi + E_\gamma\right)}{\left(m_\chi + E_1(1-\cos\theta)\right)} - \frac{2m_\chi^3 + 3m_\chi^2E_\gamma + 2m_\chi\left(E_\gamma^2 - m_{\chi^\dagger}^2 - 2E_1E_3\right) + m_{\chi^\dagger}^2E_\gamma}{E_1(1-\cos\theta)\left(m_\chi^2 - m_{\chi^\dagger}^2 + 2m_\chi E_1\right)}\right\}\left(\frac{\mu_\nu}{\mu_B}\right)^2. \tag{4.122}$$

Calculating the dipole-dipole differential cross section is not overly challenging, as the main difference lies in the hadronic tensor, which shares similarities with the fermionic tensor used in dipole-charge interactions. However, a detailed explanation of this is beyond the scope of this discussion. In summary, we have obtained the differential cross section for electromagnetic scattering of neutrinos off a nucleus, where the final state of the nucleus is in an excited state.

5 Current Experimental Determination of Neutrino Magnetic Dipole Moment and Prospects for ^{6}Li as a Detector

The Minimally Extended Standard Model (MSM) predicts a very small magnetic moment value for massive neutrino ($\mu_\nu \sim 10^{-19}\mu_B$), which is currently beyond the reach of experimental observation [28]. However, several theoretical extensions beyond the MSM allow for neutrino magnetic moments on the order of $10^{-(10-12)}\mu_B$ for Majorana neutrinos [29, 30]. In the case of Dirac neutrinos, the neutrino magnetic moment is constrained to be below $10^{-14}\mu_B$. Thus, the detection of a neutrino magnetic moment exceeding $10^{-14}\mu_B$ would provide strong evidence for new physics and unequivocally indicate that the neutrino is a Majorana particle.

Moreover, the existence of new lepton-number violation physics responsible for generating the neutrino magnetic moment implies the presence of a scale Λ well below the seesaw scale. For instance, for $\mu_\nu \sim 10^{-11}\mu_B$ and neutrino mass $m_\nu = 0.3$eV we can infer that $\Lambda \leq 100$TeV [31].

The Institute for Theoretical and Experimental Physics (ITEP) in Moscow and the Joint Institute for Nuclear Research (JINR) collaborated on a research project focused on investigating the neutrino magnetic moment. In this study, a high-purity germanium detector weighing 1.5kg was employed by the researchers. The detector was strategically positioned at a distance of 13.9m from a 3GWth reactor core. The detector was exposed to an antineutrino flux of $\phi_{\overline{\nu}} = 2.7 \times 10^{13}\text{cm}\cdot\text{s}^{-1}$. The research methodology involved comparing the recoil electron spectra during the reactor's ON and OFF periods.

The laboratory measurement of the neutrino magnetic moment is conducted by considering its contribution to neutrino-electron scattering. In the presence of a nonzero neutrino magnetic moment, the differential cross section $\frac{d\sigma}{dT}$ for $\nu - e$ scattering is the sum of the weak interaction cross section and the electromagnetic cross section [28]

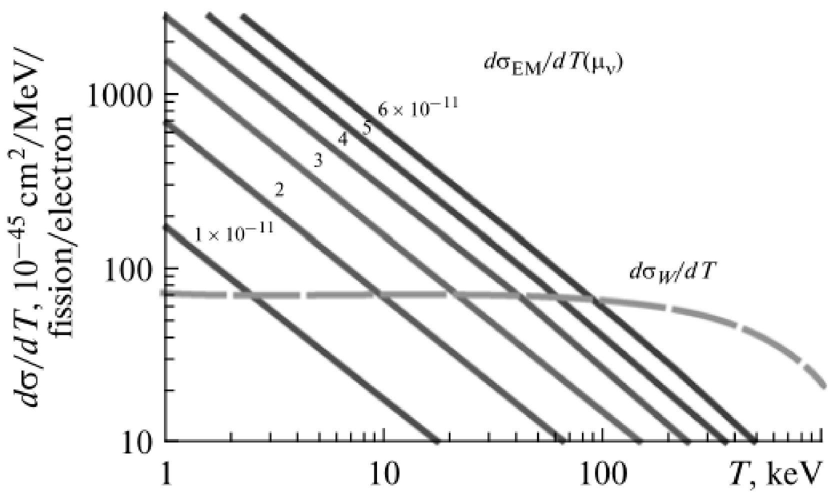
$$\begin{aligned}\frac{d\sigma}{dT} &= \frac{d\sigma_W}{dT} + \frac{d\sigma_{EM}}{dT} \\ &= G_F^2\left(\frac{m_e}{2\pi}\right)\left[4x^2 + (1+2x^2)^2\left(1-\frac{T}{E}\right)^2 - 2x^2(1+x^2)\frac{m_eT}{E^2}\right] \\ &+ \pi r_0^2\left(\frac{1}{T_e} - \frac{1}{E_\nu}\right)\left(\frac{\mu_\nu}{\mu_B}\right)^2,\end{aligned} \tag{5.1}$$

where E is the incident neutrino energy, T is the electron recoil energy; $x^2 = \sin^2\theta_W$ is a

Weinberg parameter; and r_0 is a classical electron radius.

In the ultrarelativistic limit, the interaction of the neutrino magnetic moment alters the neutrino helicity, whereas the Standard Model weak interaction conserves helicity. In the cross section, these two contributions add incoherently. The figure below displays the differential cross section, considering a value of $\mu_\nu \sim 10^{-11}\mu_B$

Figura 9 – Weak (W) and electromagnetic (EM) cross sections calculated for several μ_ν values in terms of μ_B.



Source: Gemma Experiment: The Results of Neutrino Magnetic Moment Search [28].

Observing the figure, it is evident that at low recoil energy $(T \ll E_\nu))$ the value of $\frac{d\sigma_W}{dT}$ remains almost constant, while $\frac{d\sigma_{EM}}{dT}$ increases as T^{-1}. It is evident that at lower energy the effect of the neutrino magnetic moment becomes stronger compared to the unavoidable weak contribution.

The Gemma Experiment reported an upper limit for the neutrino magnetic moment as [28]:

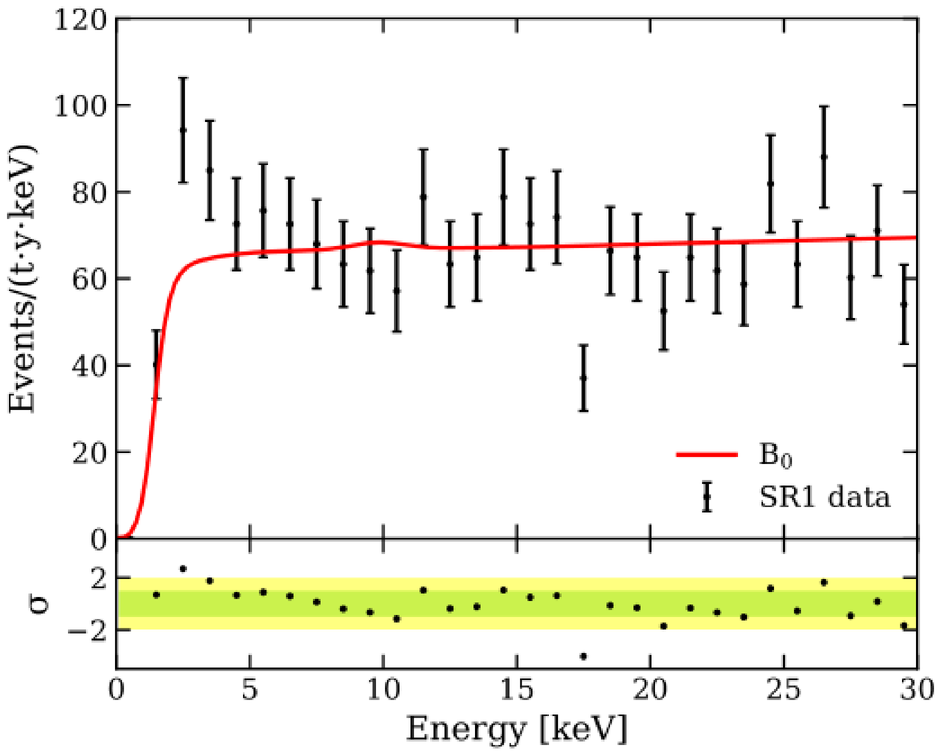
$$\mu_\nu < 2.9 \times 10^{-11}\mu_B \quad (90\% \quad \text{CL}). \tag{5.2}$$

The XENON1T detector, located in Gran Sasso National Laboratory, is specifically designed to detect rare events such as the two-neutrino double electron capture [32]. It is composed of a substantial quantity of ultra radio-pure liquid xenon, amounting to 3.2 tons. The detector boasts a remarkably low background rate of 76 ± 2 events/(tonne-year-KeV). One of the primary objectives of the XENON1T experiment is to search for potential dark matter candidates, particularly Weakly Interacting Massive Particles (WIMPs).

The XENON Collaboration reported as excess of electron recoil and raised three possible hypotheses for its explanation, the sollar axions with 3.4σ of significance, the neutrino magnetic moment signal with 3.2σ of significance and the β-decay of trituim which are in the xenon with the concentration of $(6.2 \pm 2.0) \times 10^{-25}\ \frac{\text{mol}}{\text{mol}}$. In the research for the neutrino magnetic moment the XENON Collaboration used the solar neutrino, predominantly those from the proton-proton (pp) interaction [33].

The following data was reported

Figura 10 – Excess eletronic recoil events in XENON1T.

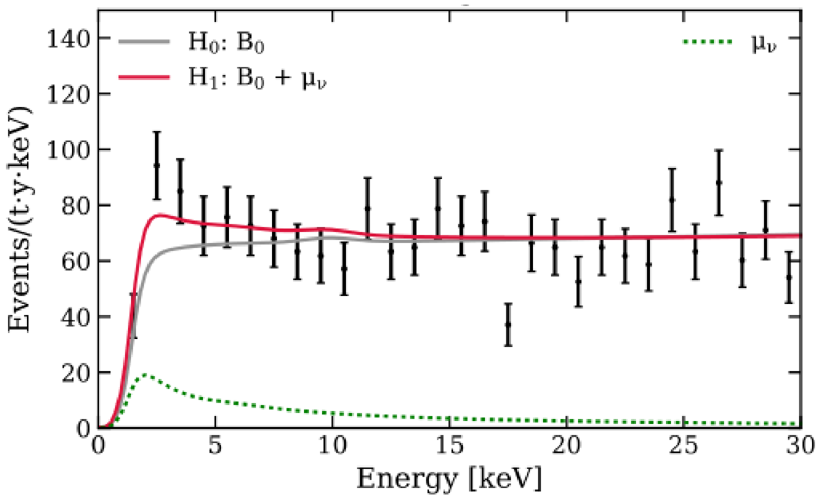


Source: Excess Electronic Recoil Events in XENON1T [34].

The figure above illustrates that the excess slightly deviates from the background model around 7KeV, exhibiting an increase as the energy decreases and peaking around $2-3$KeV. Subsequently, it diminishes to within $\pm 1\sigma$ of the background model near $1-2$KeV. Within the energy range of $1-7$KeV, the data records 285 observed events, while the expected number of events from the background-only fit is 232 ± 15, indicating a 3.3σ Poissonian fluctuation [34].

Upon comparison with the neutrino magnetic moment signal model, the background model B_0 is rejected at a significance level of 3.2σ. The following figure displays the best fits for the null hypothesis (B_0) and the alternative hypothesis ($B_0 + \mu_\nu$) in this search.

Figura 11 – Comparation of the background model B_0 and the neutrino magnetic moment μ_ν plus the background model.



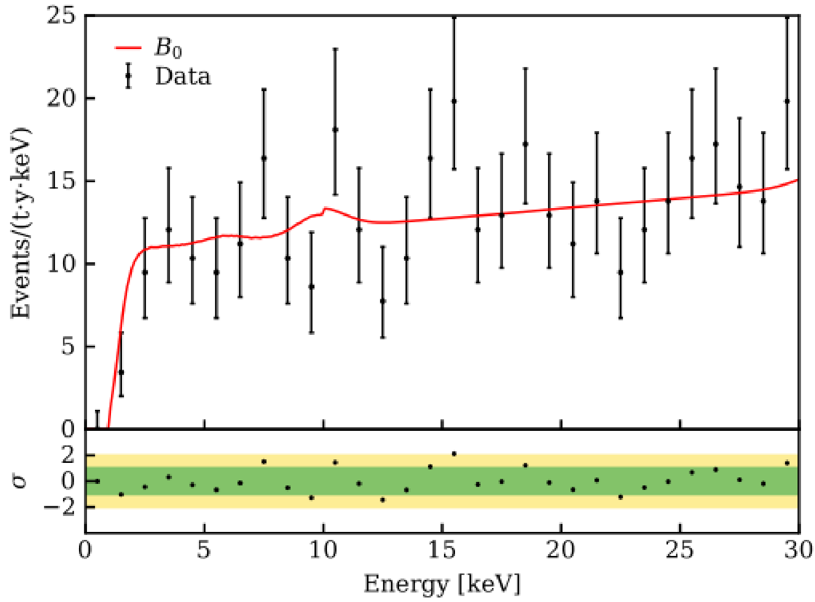
Source: Excess Electronic Recoil Events in XENON1T [34].

After conducting the data analysis, the collaboration reported a 90% confidence interval for μ_ν as follows:

$$\mu_\nu \in (1.4,\ 2.9) \times 10^{-11} \mu_B.$$

The XENONnT experiment was developed as a rapid upgrade to its predecessor, XENON1T. In XENON1T, the main source of low-energy background was the presence of ^{214}Pb. To address this issue in XENONnT, extensive material radio assay campaigns were conducted, and a novel high-flow random removal system was implemented to further diminish this background. The experiment employed the collection of two signals, namely $S1$ and $S2$, which enabled a more precise analysis and identification of events. The results obtained from the experiment are as follows:

Figura 12 – Comparation of the background model B_0 and the events number for electron recoil in XENONnT experiment.

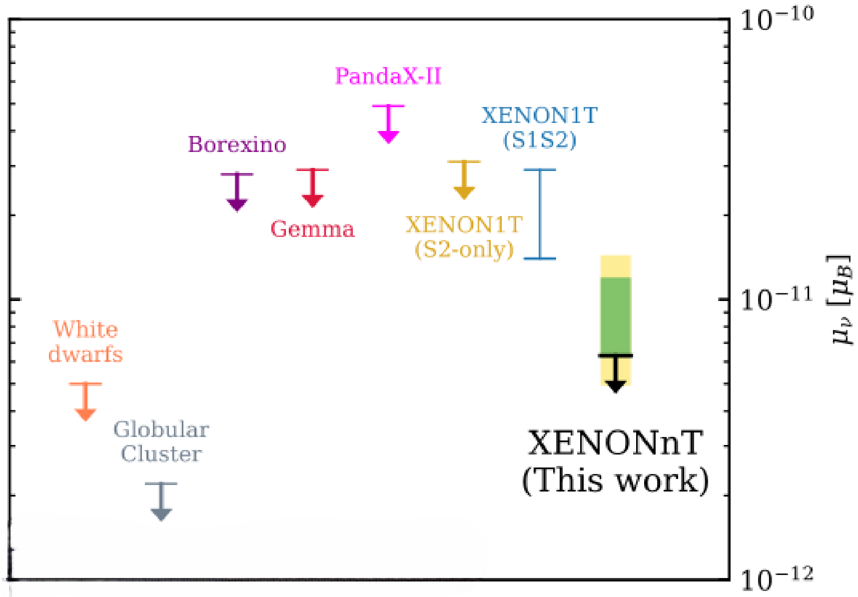


Source: Search for New Physics in Electronic Recoil Data from XENONnT [35].

However, the blind analysis conducted revealed no significant excess beyond the background, thereby excluding our previous interpretations of the XENON1T excess within the framework of beyond the Standard Model (BSM) physics.

The observed excess in the XENON1T experiment, which was modeled as a mono-energetic peak at 2.3KeV, is excluded with a statistical significance of approximately 4σ. Furthermore, a 90% confidence level (C.L.) upper limit on solar neutrinos with an enhanced magnetic moment is established as $\mu_\nu < 6.4 \times 10^{-12} \mu_B$. Lastly, the following upper limit on the neutrino magnetic moment was reported. The followin image shows the constraint in neutrino magnetico moment presented in many experiments:

Figura 13 – Constraints on solar neutrinos with an enhanced magnetic moment with 90% C.L..



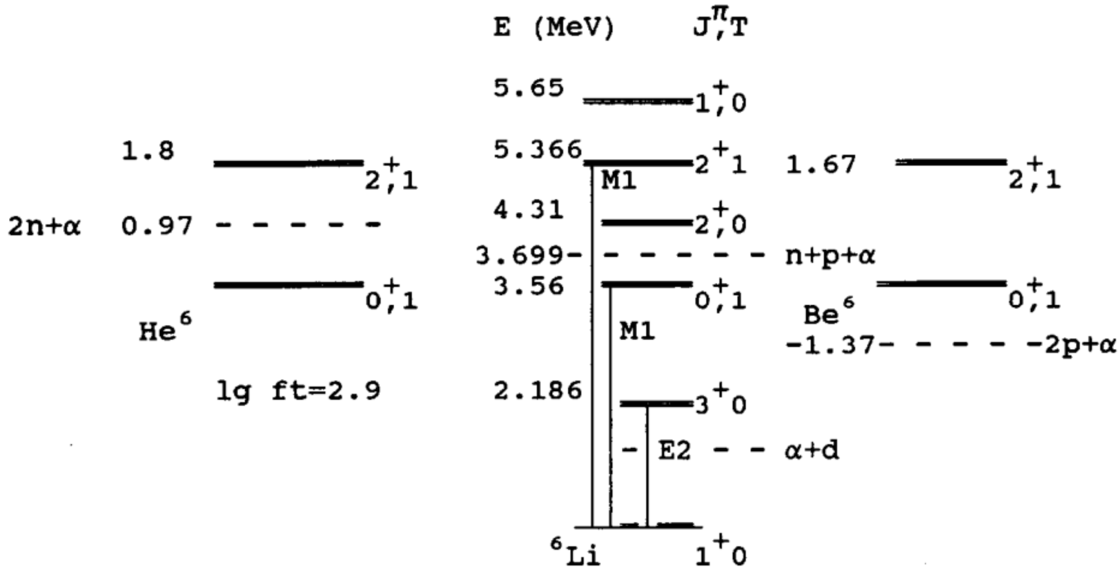
Source: Search for New Physics in Electronic Recoil Data from XENONnT [35].

5.1 ^{6}Li as a possible target detector

Experiments aimed at measuring the magnetic dipole moment of neutrinos investigate their scattering at low energies, which is influenced by the fascinating nature of their interaction. By employing ^{6}Li, a lightweight nucleus, these experiments successfully reduce the complexity from a many-body problem to a more manageable few-body problem, benefiting from established treatment methods [36]. Furthermore, the intriguing nature of this nucleus makes it particularly captivating for investigating the scattering $\overline{\nu} + \chi \longrightarrow \overline{\nu} + \chi^\dagger$ process, owing to the potential interactions involved.

Considering the ^{6}Li, as a target in the interaction with a low-energy neutrino. If we consider only allowed β-transitions there are four possible interactions of ^{6}Li$(\overline{\nu},\ e)$ and ^{6}Li$(\overline{\nu},\ \nu')$ for charged-current (CC) and neutral-current (NC), respectively [36]. The below figure shows the low-energy levels for $A = 6$ nuclei, where A is the mass number:

Figura 14 – Low-energy levels for ^{6}Li, ^{6}He and ^{6}Be.



Source: Charged- and neutral-current disintegration of the ^{6}Li nucleus by solar neutrino and reactor antineutrino [36].

In the case of allowed β-transitions, we have the following interactions:

$$(1) \quad \bar{\nu} + {}^{6}\text{Li}(g.s) \longrightarrow {}^{6}\text{He}(g.s) + e^{+}.$$
$$(2) \quad \bar{\nu} + {}^{6}\text{Li}(g.s) \longrightarrow 2n + \alpha + e^{+}.$$
$$(3) \quad \bar{\nu} + {}^{6}\text{Li}(g.s) \longrightarrow {}^{6}\text{Li}(0^{+}) + \bar{\nu}.$$
$$(4) \quad \bar{\nu} + {}^{6}\text{Li}(g.s) \longrightarrow n + p + \alpha + \bar{\nu}$$

The first interaction is a β^{+} decay, requiring a minimum energy threshold of 4.53MeV. This decay process manifests two distinct signal responses. The first signal corresponds to the emission of 2γ photons, traveling in opposite directions, which result from the annihilation of a positron and carry an energy of 0.5MeV. The second signal arises from the decay of the unstable nucleus $^{6}\text{He}(g.s)$, which possesses a half-life of $t_{\frac{1}{2}} = 0.8067$s [37].

In the second interaction, we observe another β^{+} decay process. The energy threshold for this decay is 5.505MeV, obtained by adding the previous interaction threshold of 4.53MeV to an additional energy of 0.97MeV. This extra energy contributes to the disintegration of ^{6}He. As a result of this decay, the signal response comprises two neutrons, an α-particle, and a positron.

In the third interaction, we observe a neutral current interaction, which gives rise to a signal response in the form of a photon with an energy of 3.56MeV. Notably, this interaction allows for contributions from the electromagnetic properties of neutrinos, which can influence the overall number of events.

In the last interaction, we once again observe a neutral current interaction, resulting in the disintegration of the daughter nucleus that was formed in the previous interaction. The signal response from this interaction includes the emission of a neutron, a proton, and an α-particle.

It is interesting to note that the neutrons produced in the theses decays can be detected by ^{6}Li itself via the following interaction:

$$^{6}\text{Li} + n \longrightarrow \alpha + t + 4.8\text{MeV}$$

The (1) and (3) are Gamow–Teller transitions, whereas the (2) and (4) interactions are continuun spectrum transition. The cross section are alredy presented in the literature, for more detail see [36].

6 Conclusion

Neutrino oscillation occurs because neutrinos possess mass, a phenomenon confirmed through various experiments including the Sudbury Neutrino Observatory, which received the 2015 Nobel Prize in Physics. While the Standard Model imposes restrictions on the number of active neutrinos, it does not restrict the existence of sterile neutrinos, which are non-active. Supermassive sterile neutrinos attracted interest in extensions of the Standard Model and were proposed as candidates for dark matter. However, the mechanism responsible for neutrino mass generation remains unknown. It is still an open question whether neutrinos are Dirac or Majorana particles.

The small masses of neutrinos imply that the mechanism responsible for their mass generation may differ from that of other fermions in the Standard Model. One proposed mechanism, called the Weinberg mechanism, takes into account physics beyond the Standard Model and introduces Majorana masses for neutrinos. This mechanism incorporates a suppression factor Λ, which imposes limits on the neutrino masses and potentially explains their small values. Following this mechanism, a Lagrangian was introduced to describe neutrino oscillations from first principles, taking into consideration the presence of Majorana neutrinos.

While neutrinos are electrically neutral in the Standard Model, quantum loops allow them to acquire electromagnetic properties. As a result, they can possess effective charge, magnetic and electric dipole moments, as well as anapole dipole moments. In some extensions of the standard model, neutrinos are considered to have very small charges known as milicharge. Ever since Pauli postulated it, there has been a consideration that neutrinos may possess magnetic dipole moments. Furthermore, the magnitude of this magnetic moment can vary depending on whether the neutrinos are Majorana or Dirac particles. The presence of a magnetic dipole moment allows for interactions between neutrinos and electrons. Moreover, this type of interaction can also occur between neutrinos and any target nucleus.The significance of the elastic scattering between neutrinos and electrons through dipole-charge interaction becomes most pronounced at low energies. Therefore, when studying neutrino dipole moments, it is optimal to focus on low-energy neutrinos.

The Toy Model presented suggests that antineutrino-nucleus scattering exhibits a similar behavior, when the final state of the nucleus is in an excited state. Consequently, investigating neutrino-nucleus scattering with an excited final state proves advantageous, particularly at low energies. However, for a more comprehensive analysis, it is essential to utilize more detailed studies and realistic models.

Among the different target nuclei, ^{6}Li holds promise as a detector for these measurements. This is due to its light nucleus nature, which reduces the many-body problem to a few-body problem. Additionally, antineutrino reactors can induce four possible interactions with ^{6}Li. These interactions produce response signals that can be measured, providing valuable insights into neutral current interactions involving electromagnetic interaction or the exchange of the Z_0 boson, as well as charged current interaction. Such information is invaluable for testing the Standard Model and identifying more suitable avenues for various extensions of this theory.

Referências

[1] THOMSON, M. *Modern particle physics*. [S.l.]: Cambridge University Press, 2013.

[2] XING, Z.; ZHOU, S. *Neutrinos in particle physics, astronomy and cosmology*. [S.l.]: Springer Science & Business Media, 2011.

[3] IGNATIEV, A. Y.; JOSHI, G. Neutrino electric charge and the possible anisotropy of the solar neutrino flux. *Physical Review D*, APS, v. 51, n. 5, p. 2411, 1995.

[4] GOLDHABER, M.; GRODZINS, L.; SUNYAR, A. W. Helicity of neutrinos. *Physical review*, APS, v. 109, n. 3, p. 1015, 1958.

[5] BARDON, M.; FRANZINI, P.; LEE, J. Helicity of μ- mesons; mott scattering of polarized muons. *Physical Review Letters*, APS, v. 7, n. 1, p. 23, 1961.

[6] HEISTER, A.; COLLABORATION, A. Measurement of the tau polarisation at lep. *The European Physical Journal C-Particles and Fields*, Springer, v. 20, n. 3, p. 401–430, 2001.

[7] ESTEBAN, I. et al. The fate of hints: updated global analysis of three-flavor neutrino oscillations. *Journal of High Energy Physics*, Springer, v. 2020, n. 9, p. 1–22, 2020.

[8] DELL'ORO, S. et al. Neutrinoless double beta decay: 2015 review. *Advances in High Energy Physics*, Hindawi Limited, v. 2016, p. 1–37, 2016.

[9] DIRECT neutrino-mass measurement with sub-electronvolt sensitivity. *Nature Physics*, Nature Publishing Group UK London, v. 18, n. 2, p. 160–166, 2022.

[10] WEINBERG, S. *Cosmology*. [S.l.]: OUP Oxford, 2008.

[11] HANNESTAD, S. Primordial neutrinos. *Annu. Rev. Nucl. Part. Sci.*, Annual Reviews, v. 56, p. 137–161, 2006.

[12] FUKUGITA, M.; YANAGIDA, T. Barygenesis without grand unification. *Physics Letters B*, Elsevier, v. 174, n. 1, p. 45–47, 1986.

[13] XING, Z.-z. Theoretical overview of neutrino properties. *International Journal of Modern Physics A*, World Scientific, v. 23, n. 26, p. 4255–4272, 2008.

[14] SOLER, F. P.; FROGGATT, C. D.; MUHEIM, F. *Neutrinos in particle physics, astrophysics and cosmology*. [S.l.]: CRC Press, 2008.

[15] TANABASHI, M. et al. Review of particle physics: particle data groups. *Physical Review D*, v. 98, n. 3, p. 1–1898, 2018.

[16] AKHMEDOV, E. Majorana neutrinos and other majorana particles: Theory and experiment. *arXiv preprint arXiv:1412.3320*, 2014.

[17] WEINBERG, S. Baryon-and lepton-nonconserving processes. *Physical Review Letters*, APS, v. 43, n. 21, p. 1566, 1979.

[18] GRZADKOWSKI, B. et al. Dimension-six terms in the standard model lagrangian. *Journal of High Energy Physics*, Springer, v. 2010, n. 10, p. 1–18, 2010.

[19] SCHWARTZ, M. D. *Quantum field theory and the standard model*. [S.l.]: Cambridge university press, 2014.

[20] FELIX, O. et al. Neutrino masses and mixings in a minimal s3-invariant extension of the standard model. *Revista mexicana de física*, Sociedad Mexicana de Física, v. 52, p. 67–73, 2006.

[21] GIUNTI, C.; STUDENIKIN, A. Neutrino electromagnetic interactions: a window to new physics. *Reviews of Modern Physics*, APS, v. 87, n. 2, p. 531, 2015.

[22] SCHECHTER, J.; VALLE, J. W. Majorana neutrinos and magnetic fields. *Physical Review D*, APS, v. 24, n. 7, p. 1883, 1981.

[23] SHROCK, R. E. Electromagnetic properties and decays of dirac and majorana neutrinos in a general class of gauge theories. *Nuclear Physics B*, Elsevier, v. 206, n. 3, p. 359–379, 1982.

[24] NOWAKOWSKI, M.; PASCHOS, E.; RODRIGUEZ, J. All electromagnetic form factors. *European journal of physics*, IOP Publishing, v. 26, n. 4, p. 545, 2005.

[25] SAKURAI, J. J.; COMMINS, E. D. *Modern quantum mechanics, revised edition*. [S.l.]: American Association of Physics Teachers, 1995.

[26] RYDER, L. H. *Quantum field theory*. [S.l.]: Cambridge university press, 1996.

[27] PAL, P. B. Representation-independent manipulations with dirac matrices and spinors. *arXiv preprint physics/0703214*, 2007.

[28] BEDA, A. et al. Gemma experiment: The results of neutrino magnetic moment search. *Physics of Particles and Nuclei Letters*, Springer, v. 10, p. 139–143, 2013.

[29] FUKUGITA, M.; YANAGIDA, T. Particle-physics model for voloshin-vysotsky-okun solution to the solar-neutrino problem. *Physical Review Letters*, APS, v. 58, n. 18, p. 1807, 1987.

[30] BELL, N. F. et al. Model independent bounds on magnetic moments of majorana neutrinos. *Physics Letters B*, Elsevier, v. 642, n. 4, p. 377–383, 2006.

[31] BELL, N. F. et al. How magnetic is the dirac neutrino? *Physical Review Letters*, APS, v. 95, n. 15, p. 151802, 2005.

[32] MASBOU, J. First observation of two-neutrino double electron capture in 124xe with xenon1t. In: *IRN Terascale 2019: International Research Network on the experimental and theorical search for new physics at the tev scale*. [S.l.: s.n.], 2019.

[33] BAHCALL, J. N.; PINSONNEAULT, M. What do we (not) know theoretically about solar neutrino fluxes? *Physical Review Letters*, APS, v. 92, n. 12, p. 121301, 2004.

[34] APRILE, E. et al. Excess electronic recoil events in xenon1t. *Physical Review D*, APS, v. 102, n. 7, p. 072004, 2020.

[35] APRILE, E. et al. Search for new physics in electronic recoil data from xenonnt. *Physical Review Letters*, APS, v. 129, n. 16, p. 161805, 2022.

[36] SHUL'GINA, N.; DANILIN, B. Charged-and neutral-current disintegration of the 6li nucleus by solar neutrino and reactor antineutrino. *Nuclear Physics A*, Elsevier, v. 554, n. 1, p. 137–157, 1993.

[37] IAEA Live Chart of Nuclides. `https://www-nds.iaea.org/relnsd/vcharthtml/VChartHTML.html`. Accessed: july 01, 2023.

[38] GURALNIK, G. S.; HAGEN, C. R.; KIBBLE, T. W. Broken symmetries and the goldstone theorem. *Advances in particle physics*, John Wiley & Sons New York, NY, USA, v. 2, p. 567–708, 1968.

[39] EREDITATO, A. State of the art of neutrino physics, the: A tutorial for graduate students and young researchers. World Scientific, 2018.

[40] SCHWICHTENBERG, J. *Physics from symmetry*. [S.l.]: Springer, 2018.

[41] GREINER, W.; REINHARDT, J. *Quantum electrodynamics*. [S.l.]: Springer Science & Business Media, 2008.

[42] GIUNTI, C.; KIM, C. W. *Fundamentals of neutrino physics and astrophysics*. [S.l.]: Oxford university press, 2007.

[43] WIGNER, E. On unitary representations of the inhomogeneous lorentz group. *Annals of mathematics*, JSTOR, p. 149–204, 1939.

[44] STREATER, R. F.; WIGHTMAN, A. S. *PCT, spin and statistics, and all that*. [S.l.]: Princeton University Press, 2000. v. 52.

[45] BERTLE, H.; GODAZGAR, M. Goldstone boson and higgs mechanism. *Proseminar Theoretical Physics, ETH Zurich*, 2018.

A Single-particle relativistic wave equation

Any theory of the fundamental nature of matter must be consistent with relativity, as well as with quantum theory. Special relativity is constructed considering the Minkowski space-time. In this particular spacetime, there exists a position four-vector represented by

$$x^{\mu} = (x^0, x^1, x^2, x^3) = (ct, x, y, z)\,,$$

given c as the speed of light. If we consider an event in $(ct + cdt, x + dx, y + dy, z + dz)$, the interval given by

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2,$$

is the same for all (inertial) frames. By utilizing this definition, we can distinguish events that are separated by a *timelike* interval $ds^2 < 0$, *spacelike* interval $ds^2 > 0$ and *lightlike* $ds^2 = 0$. In order to define the Lorentz invariant interval, two types of four vectors are necessary - covariant and contravariant. Therefore, we define the covariant vector as follows:

$$dx_{\mu} = (cdt, -dx, -dy, -dz)$$

and the contravariant one as

$$dx^{\mu} = (cdt, dx, dy, dz).$$

By utilizing these two types of four-vectors, we are able to calculate the Lorentz invariant interval

$$ds^2 = \sum_{\mu=0}^{3} dx^{\mu} dx_{\mu}.$$

To simplify the notation, we adopt the summation convention: an index appearing once in an upper and once in a lower position is automatically summed from 0 to 3:

$$\sum_{\mu=0}^{3} V^{\mu} V_{\mu} \rightarrow V^{\mu} V_{\mu}.$$

The relation between a contravariant four-vector and a covariant one is given by the metric tensor $g_{\mu\nu}$:

$$x_{\mu} = g_{\mu\nu} x^{\nu},$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

We define the diferential operator as $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. In particle physics, it is customary to use units in which the speed of light $c = 1$. Therefore, $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, and $\partial_\mu = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = (\frac{\partial}{\partial t}, \vec{\nabla})$. The d'Alambertian operator is defined by

$$\Box \equiv \partial^\mu \partial_\mu = \partial_t^2 - \vec{\nabla}^2.$$

We will frequently utilize two common invariants, known as the dispersion relation or the mass-shell condition, which can be expressed as follows:

$$p^\mu p_\mu = E^2 - \vec{p} \cdot \vec{p} = m^2, \tag{A.2}$$

and

$$p \cdot x = Et - \vec{p} \cdot \vec{r}. \tag{A.3}$$

A.1 Klein-Gordon Equation

The Klein-Gordon equation is a wave equation for a particle with no spin, i.e., a scalar particle. Since this is a scalar particle, it has only one component denoted by ϕ. To ensure that a particle is inaccordance to special relativity, it is necessary to satisfy the dispersion relation. In quantum mechanics, the energy and momentum operators are expressed as:

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \vec{p} \to -i\hbar\vec{\nabla},$$

therefore the equation A.2 becomes

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi = 0. \tag{A.4}$$

Here, we are utilizing a system of units which $\hbar = c = 1$. The above equation can be written as

$$\left(\Box + m^2\right)\phi = 0, \tag{A.5}$$

called the Klein-Gordon equation. The statistical interpretation of this equation is problematic, we define the probability densinty as:

$$\rho = \frac{i}{2m}\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right). \tag{A.6}$$

The probability density defined above is not positive definite. This problem will be solved when we reinterpret it as a field equation for a quantum field. For the moment, we will write the four-current for Klein-Gordon equation, as

$$j_\mu = \frac{-i}{m}\left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right). \tag{A.7}$$

There is another problem related to the Klein-Gordon equation, when it is interpreted as an energy equation it yields both negative and positive energy solutions, as shown by:

$$E = \pm\sqrt{\vec{p}^2 + m^2},$$

This presents a challenge in an interaction theory given that in such context the interacting particle can transfer energy to its surroundings. Without any restrictions, the system may cascade to infinite negative energy states, thereby emitting an infinite amount of energy.

A.2 Dirac Equation

Dirac equation holds for particles of spin half, since Klein-Gordon equation expresses nothing more than the relativistic relation between energy, momentum and mass, it must be applicable to particles of every spin. The Dirac equation, may be derived from spinor transformation under the Lorentz group. We will see that we can define the spinors by a representation of the Lorentz group and, with it in hands, we can cronstruct a scalar field, a vector field and other fields, therefore the fields can be defined as the objects that transform in a certain way for a given representation of the Lorentz group. It is common to refer to the fields as representations of the Lorentz group.

The Lorentz group is defined as the transformation that leaves the inner product between two four-vector invariant, i.e., considering the 4-vector of position, we have

$$x^\mu x_\mu = x'^\mu x'_\mu \tag{A.8}$$

where $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. From equation A.8 the Lorentz transformation must satisfy

$$g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}. \tag{A.9}$$

Equation A.9 provides the definition of the Lorentz group and can be expressed in matrix notation as follows:

$$\Lambda^T g \Lambda = g. \tag{A.10}$$

Taking the determinant of A.9, we find

$$\det(\Lambda) = \pm 1. \tag{A.11}$$

Therefore, the Lorentz transformation has $\det(\Lambda) = \pm 1$. Furthermore, imposing $\rho = \sigma = 0$ in equation A.9 we find

$$\Lambda^0_0 = \pm\sqrt{1 + \sum_i (\Lambda^i_0)^2}. \tag{A.12}$$

The Lorentz group can be divided into four parts, forming equivalent classes based on the signs in equations A.11 and A.12, as shown below:

$$
\begin{aligned}
L_+^\uparrow : &\quad \det(\Lambda) = +1; \quad \Lambda^0_0 \geq 1;\\
L_-^\uparrow : &\quad \det(\Lambda) = -1; \quad \Lambda^0_0 \geq 1;\\
L_+^\downarrow : &\quad \det(\Lambda) = +1; \quad \Lambda^0_0 \leq 1;\\
L_-^\downarrow : &\quad \det(\Lambda) = -1; \quad \Lambda^0_0 \leq 1.
\end{aligned}
$$

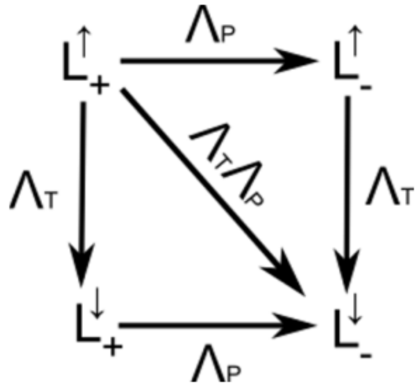
This is important because only transformations in the $L_+^\uparrow$ class can be generated through infinitesimal transformations, and therefore we can use Lie theory to learn about the Lie algebra of the Lorentz group.[1] The transformations in the other categories are a combination of the transformations in the $L_+^\uparrow$ category and one or both of two special transformations known as time-reversal Λ_T and space-inversion Λ_P, where

$$
\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathrm{diag}(1, -1, -1, -1); \tag{A.13}
$$

$$
\Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{diag}(-1,\ 1,\ 1,\ 1). \tag{A.14}
$$

Figure 15 illustrates how the other classes of the Lorentz group can be obtained from the proper orthochronous one.

Figura 15 – The four parts of the Lorentz group are connected through Λ_T and Λ_P transformations.



Source: Physics From Symmetry [40].

[1] The technical term for this most important category $L_+^\uparrow$ is proper orthochronous Lorentz group.

The set $O(1,3) = \{L^{\uparrow}_{+}, \Lambda_T L^{\uparrow}_{+}, \Lambda_P L^{\uparrow}_{+}, \Lambda_T \Lambda_P L^{\uparrow}_{+}\}$ can be understood as the complete Lorentz group.

A.2.1 The Generators of the Lorentz Group

The Lorentz group is a group of transformations with six degrees of freedom, consisting of three velocities and three rotation angles. The group is generated by six corresponding generators denoted by

$$K^i \longrightarrow \text{boosts}$$
$$J^i \longrightarrow \text{rotations}, \quad \text{with } (i = 1, 2, 3.).$$

Since the rotation part of the Lorentz group are given by

$$\Lambda_{\text{Rot}} = \begin{pmatrix} 1 & \\ & R_{3\text{dim}} \end{pmatrix},$$

i.e., it rotates the spatial coordinates while leaving the time coordinate unchanged. The generators of rotations are given by

$$\Lambda_{\text{Rot}} = \begin{pmatrix} 0 & \\ & j_i^{3\text{dim}} \end{pmatrix},$$

where $j_i^{3\text{dim}}$ are the generators of the $SO(3)$ group. For example

$$J_1 = \begin{pmatrix} 0 & \\ & j_1^{3\text{dim}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \tag{A.15}$$

Given the definition of the Lorentz group (equation A.10), we can obtain the generators K_i through an infinitesimal boost transformation given by

$$\Lambda^{\mu}{}_{\rho} \approx \delta^{\mu}{}_{\rho} + \epsilon K^{\mu}{}_{\rho},$$

which implies that

$$K^{\mu}{}_{\rho} g_{\mu\sigma} = -g_{\rho\nu} K^{\nu}{}_{\sigma}$$

or, in matrix notation,

$$K^T g = -gK. \tag{A.16}$$

Thus, to obtain the generators for a boost, let us consider a scenario where we are interested in a boost along the x-direction. In this case, we have $y' = y$ and $z' = z$, which means that the generator K_1 takes the form:

$$K_1 = \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \\ & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}, \tag{A.17}$$

By applying equation A.16, we determine the conditions $a = d = 0$ and $b = c$. Based on these conditions, we can express the generator for a boost in the x-direction as:

$$k_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.18}$$

We can obtain the remaining generators using a similar procedure. However, we will not present their matrix representation here. Instead, we will provide the commutation relation between these generators in a subsequent section. In his work, Jakob Schwichtenberg [40] engages in a more in-depth discourse regarding this subject.

To obtain the generators for the transformations of the remaining components of the Lorentz group[2] we can apply the parity operator Λ_P and time reversal operator Λ_T to the matrices J_i and K_i. This will help us understand what these generators look like.

Under parity transformation we find

$$J_i \underbrace{\longrightarrow}_{\Lambda_P} J_i \tag{A.19}$$

$$K_i \underbrace{\longrightarrow}_{\Lambda_P} -K_i. \tag{A.20}$$

For a time reversal tranformation we find

$$J_i \underbrace{\longrightarrow}_{\Lambda_T} J_i \tag{A.21}$$

$$K_i \underbrace{\longrightarrow}_{\Lambda_T} -K_i. \tag{A.22}$$

A.2.2 The Lie Algebra of the Proper Orthochronous Lorentz Group

The corresponding Lie algebra can be obtained by utilizing the explicit matrix form of the generators for $L_+^\uparrow$

$$[J_i, J_j] = i\epsilon_{ijk} J_k; \tag{A.23}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k; \tag{A.24}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{A.25}$$

A general Lorentz transformation is of the form

$$\Lambda(\vec{\theta}, \vec{\phi}) = e^{i\vec{J}\cdot\vec{\theta} + i\vec{K}\cdot\vec{\phi}}. \tag{A.26}$$

[2] Note that we have only found the generator for proper orthochronous Lorentz group.

By examining the commutation relation between the generators, it becomes apparent that the rotation generators exhibit closure under commutation. Conversely, the boost generators do not demonstrate closure under commutation, which implies that the Lorentz boost does not constitute a subgroup of the Lorentz group.

We can define two new operators, which are closed under commutation and commute with each other, as follows:

$$N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i). \tag{A.27}$$

Computing the commutation relations yields:

$$[N_i^+, N_j^+] = i\epsilon_{ijk}N_k^+; \tag{A.28}$$

$$[N_i^-, N_j^-] = i\epsilon_{ijk}N_k^-; \tag{A.29}$$

$$[N_i^+, N_j^-] = 0. \tag{A.30}$$

These relations precisely match the commutation relations for the Lie algebra of $SU(2)$, which means that the Lorentz group can be viewed as a product of $SU(2) \otimes SU(2)$.[3] We can compact the Lorentz algebra introducing a new symbol $M_{\mu\nu}$, which is defined through the equations

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk} \tag{A.31}$$

$$K_i = M_{0i}. \tag{A.32}$$

Using this definition, the Lorentz algebra can be expressed as

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\sigma}M_{\mu\rho}\right). \tag{A.33}$$

Here, $g_{\mu\nu}$ denotes the components of the metric tensor.

A.2.3 The generator of the Casimir invariant for $SU(2)$.

Casimir operators can be constructed from a given set of generators to label the representations of a Lie algebra. These operators have the special property of commuting with all generators and being proportional to the identity operator. For the Lie algebra $su(2)$ there is one such operator

$$J^2 = J_1^2 + J_2^2 + J_3^2. \tag{A.34}$$

In the 3-dimensional representation, the Casimir operator is expressed as:

$$J^2_{3-\text{dim}} = 2 \times \mathbb{1}_{3\times 3}, \tag{A.35}$$

[3] To be more precise, the complexification of the Lorentz group's Lie algebra is $so(1,3)_{\mathbb{C}} \cong sl(2,\mathbb{C}) \bigoplus sl(2,\mathbb{C})$, where $sl(2,\mathbb{C})$ is the Lie algebra of the special linear group. It's worth noting that whenever we use lowercase letters, we are referring to the Lie algebra of some group, such as $su(2)$ being the Lie algebra of the group $SU(2)$.

where $\mathbb{1}_{3\times 3}$ represents the identity matrix in the 3-dimensional space. In contrast, for the 2-dimensional representation $J_i = \frac{1}{2}\sigma^i$, where σ^i is the Pauli matrices, we get

$$J^2_{2-\text{dim}} = \frac{3}{4} \times \mathbb{1}_{2\times 2} = \frac{3}{4}. \tag{A.36}$$

Note that the last equation in A.36 implies an implicit identity matrix.

We use the proportionality number to label different representations. The basis vectors on which our operators act are labeled with another label, given by the Cartan operator. The Cartan elements are those generators that can be diagonalized simultaneously. For $su(2)$, there is only one such element, and it is convenient to choose J_3. In the 2-dimensional representation, it is expressed as:

$$J_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \tag{A.37}$$

The eigenstates of J^2 and J_3 has two labels j and m, with

$$J^2|j,m\rangle = j(j+1)|j,m\rangle; \qquad \text{with} \begin{cases} j = \frac{\text{integer}}{2} = 0,\ 1/2,\ 1, \cdots \\ m = -j, -j+1, \cdots, j. \end{cases} \tag{A.38}$$

$$J_3|j,m\rangle = m|j,m\rangle. \tag{A.39}$$

A.2.4 Lorentz Group Representations

The Lorentz group can be represented as a product of $SU(2) \otimes SU(2)$, as previously shown. The different fundamental representations of $SU(2)$ are labeled by the j number given by the J^2 operator. The fundamental representations of the Lorentz group are characterized by two labels, $(j,\ j')$, one for each $SU(2)$ group.

A.2.4.1 The $(0,\ 0)$ Representation

The lowest dimensional representation of $SU(2) \otimes SU(2)$ is a one-dimensional vector space[4]. Our generators must therefore be 1×1 matrices and the only 1×1 "matrices" that fulfills the commutation relation is the number 0: $N_i^+ = N_i^- = 0 \implies e^{iN_i^\pm} = e^0 = 1$. Therefore, we conclude that the $(0,\ 0)$ representation acts on objects do not change under Lorentz transformation. This representation is called the scalar representation. In mathematical language, the objects acted upon by a representation are commonly identified with the representation itself. Accordingly, we can refer to the fundamental particles described by the objects upon which the Lorentz group acts as representations. For instance, the scalar field is identified as a $(0,\ 0)$ representation of the Lorentz group, or more precisely, a fundamental representation of the Poincaré group that will be introduced shortly.

[4] The dimension of the vector space which the $SU(2) \otimes SU(2)$ group act is given by $(2j+1)(2j^{'}+1)$.

A.2.4.2 The $(\frac{1}{2},\ 0)$ Representation

We utilize a 2-dimensional representation for N_i^+, which is one copy of the $SU(2)$ Lie algebra, represented by $\frac{\sigma_i}{2}$. On the other hand, for N_i^-, we apply a 1-dimensional representation, represented by $N_i^- = 0$. From equation A.27, we can deduce that:

$$J_i = \frac{\sigma_i}{2}; \qquad K_i = -i\frac{\sigma_i}{2}. \tag{A.40}$$

Hence, the Lorentz transformation can be expressed as follows:

$$\Lambda(\vec{\theta},\vec{\phi})_{(\frac{1}{2},\ 0)} = \exp\left[i\frac{\vec{\sigma}}{2}\cdot\left(\vec{\theta}-i\vec{\phi}\right)\right]. \tag{A.41}$$

This transformation acts on a two-component object known as the left-chiral spinor, denoted by χ_L:

$$\chi_L = \begin{pmatrix}(\chi_L)_1 \\ (\chi_L)_2\end{pmatrix}. \tag{A.42}$$

A.2.4.3 The $(0,\ \frac{1}{2})$ Representation

Analogous to the $(\frac{1}{2},\ 0)$ representation, we can construct this version with $N_i^+ = 0$ and $N_i^- = \frac{\sigma_i}{2}$. As a result, we obtain

$$J_i = \frac{\sigma_i}{2}; \qquad k_i = i\frac{\sigma_i}{2}. \tag{A.43}$$

A $(0,\ \frac{1}{2})$ Lorentz transformation is expressed as

$$\Lambda(\vec{\theta},\vec{\phi})_{(0,\ \frac{1}{2})} = \exp\left[i\frac{\vec{\sigma}}{2}\cdot\left(\vec{\theta}+i\vec{\phi}\right)\right]. \tag{A.44}$$

This operates on a two-component entity known as a right-chiral spinor

$$\xi_R = \begin{pmatrix}(\xi_R)_1 \\ (\xi_R)_2\end{pmatrix}. \tag{A.45}$$

Left- and right-chiral spinors are commonly referred to as Weyl spinors.

The $(\frac{1}{2},\ 0)$ and $(0,\ \frac{1}{2})$ representations are related by a parity transformation, which maps K_i to $-K_i$ and leaves J_i invariant. Hence, when considering parity, it is no longer sufficient to treat the 2-spinors separately, but we must consider the 4-spinor defined by

$$\psi = \begin{pmatrix}\chi_L \\ \xi_R\end{pmatrix}, \tag{A.46}$$

which transforms as

$$\psi \longrightarrow \psi' = \begin{pmatrix}\exp i\left[\frac{\vec{\sigma}}{2}\cdot\left(\vec{\theta}-i\vec{\phi}\right)\right] & 0 \\ 0 & \exp i\left[\frac{\vec{\sigma}}{2}\cdot\left(\vec{\theta}+i\vec{\phi}\right)\right]\end{pmatrix}\begin{pmatrix}\chi_L \\ \xi_R\end{pmatrix}. \tag{A.47}$$

The 4-spinor[5] ψ, which consists of two 2-component spinors, is an irreducible representation of the extended Lorentz group, $(\frac{1}{2},\ 0) \oplus (0,\ \frac{1}{2})$, which includes parity transformations[6]. These 4-spinors are commonly referred to as Dirac spinors.

A.2.5 The Dirac Equation

We are expressing the boost as a function of ϕ, which is known as rapidity and can be related to velocity by

$$\tanh\phi = \frac{v}{c}, \qquad c \text{ is the speed of light.}$$

From this, we can infer that

$$\gamma = \cosh\phi; \qquad\qquad \gamma\beta = \sinh\phi. \tag{A.48}$$

By using the identity $\cosh\phi^2 - \sinh\phi^2 = 1$, we can derive the following equations:

$$\cosh\frac{\phi}{2} = (\frac{\gamma+1}{2})^{\frac{1}{2}} \tag{A.49}$$

$$\sinh\frac{\phi}{2} = (\frac{\gamma-1}{2})^{\frac{1}{2}}. \tag{A.50}$$

A pure boost can be calculated using its exponential series, for instance:

$$\Lambda(0,\ \phi)_{(0,\ \frac{1}{2})} = e^{\vec{\sigma}\cdot\frac{\vec{\phi}}{2}} = \sum_{n=0}^{\infty}\frac{1}{2n!}(\frac{\phi}{2})^{2n}(\vec{\sigma}\cdot\hat{n})^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(\frac{\phi}{2})^{2n+1}(\vec{\sigma}\cdot\hat{n})^{2n+1}, \tag{A.51}$$

where $\hat{n}$ is a unit vector in the direction of the boost and since $\sigma^2 = 1$, the boost can be expressed as

$$\Lambda(0,\ \phi)_{(0,\ \frac{1}{2})} \equiv B(\phi)_{(0,\ \frac{1}{2})} = \cosh\frac{\phi}{2} + \vec{\sigma}\cdot\hat{n}\sinh\frac{\phi}{2}. \tag{A.52}$$

Thus,

$$\xi_R \longrightarrow \xi_R' = \left[(\frac{\gamma+1}{2})^{\frac{1}{2}} + \vec{\sigma}\cdot\hat{p}(\frac{\gamma-1}{2})^{\frac{1}{2}}\right]\xi_R(0), \tag{A.53}$$

where $\xi_R(0)$ is the spinor in a rest frame and $\hat{p}$ denotes the direction of the boost. Using $\gamma = \frac{E}{m}$ and $E^2 - m^2 = \vec{p}^2$ we can write

$$\xi_R(\vec{p}) = \left[\frac{E + m + \vec{\sigma}\cdot\vec{p}}{[2m(E+m)]^{\frac{1}{2}}}\right]\xi_R(0). \tag{A.54}$$

---

[5] Note that no a priori connection between χ and ξ is assumed in order to maintain generality. However, there exists a special case where these two spinors are connected.

[6] In the previous subsections, we discussed representations of the proper orthochronous Lorentz group called homogeneous Lorentz group. However, to fully describe the behavior of spinors under Lorentz transformations, we also need to consider the parity transformations.

Similarly,

$$\chi_L(\vec{p}) = \left[\frac{E + m - \vec{\sigma}\cdot\vec{p}}{[2m(E+m)]^{\frac{1}{2}}}\right]\chi_L(0). \tag{A.55}$$

When a particle is at rest, its chirality cannot be defined as left- or right-handed, resulting in $\xi_R(0) = \chi_L(0)$. As a consequence, we can express

$$\xi_R(\vec{p}) = \frac{E + m + \vec{\sigma}\cdot\vec{p}}{E + m - \vec{\sigma}\cdot\vec{p}}\chi_L(\vec{p}), \tag{A.56}$$

By defining $\epsilon \equiv E + m$ and $\pi \equiv \vec{\sigma}\cdot\vec{p}$, and performing some algebraic manipulations, we can obtain

$$\xi_R(\vec{p}) = \frac{E + \vec{\sigma}\cdot\vec{p}}{m}\chi_L(\vec{p}), \tag{A.57}$$

and similarly,

$$\chi_L(\vec{p}) = \frac{E - \vec{\sigma}\cdot\vec{p}}{m}\xi_R(\vec{p}). \tag{A.58}$$

The equations A.57 and A.58 can be written in the 4×4 matrix form

$$\begin{pmatrix} -m & p_0 + \vec{\sigma}\cdot\vec{p} \\ p_0 - \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} \xi_R(\vec{p}) \\ \chi_L(\vec{p}) \end{pmatrix} = 0. \tag{A.59}$$

We can define $u(\vec{p}) = \begin{pmatrix} \xi_R(\vec{p}) \\ \chi_L(\vec{p}), \end{pmatrix}$, and the 4×4 matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{A.60}$$

Substituting these definitions into Eq. A.59, we obtain

$$\left(\gamma^0 p_0 + \gamma^i p_i - m\right)u(p) = 0, \tag{A.61}$$

or equivalently,

$$\left(\gamma^\mu p_\mu - m\right)u(p) = \left(\not{p} - m\right)u(p) = 0, \tag{A.62}$$

where we define $\not{p} \equiv \gamma^\mu p_\mu$. Hence, we are now able to represent the equation known as the Dirac equation, though it is important to note that the derivation presented here differs from Dirac's original approach. Dirac's objective was to find an equation that did not suffer from the issues observed with the Klein-Gordon equation discussed earlier.

When dealing with massless particles, we obtain two independent equations, each involving a 2-component spinor. These equations are referred to as the Weyl equations:

$$(p_0 + \vec{\sigma}\cdot\vec{p})\chi_L(\vec{p}) = 0; \tag{A.63}$$

$$(p_0 - \vec{\sigma}\cdot\vec{p})\xi_R(\vec{p}) = 0. \tag{A.64}$$

For a massless particle, $p_0 = |\vec{p}|$, which yields the following expressions:

$$(\vec{\sigma} \cdot \hat{p})\chi_L(\vec{p}) = -\chi_L(\vec{p}); \qquad (\vec{\sigma} \cdot \hat{p})\xi_R(\vec{p}) = +\xi_R(\vec{p}). \tag{A.65}$$

The operator $\vec{\sigma} \cdot \hat{p}$ is referred to as the helicity operator, and thus, the Weyl spinors are the eigenvectors of the helicity.

Substituting in Dirac equation (A.62) $p_\mu \longrightarrow i\partial_\mu$ we have

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \tag{A.66}$$

This equation is the Dirac equation in coordinate space and $\psi(x)$ is the Fourier transform of the spinor $u(p)$ in position space. Applying $i\gamma^\nu \partial_\nu$ again in this equation we can write this as

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu \psi(x) + m^2 \psi(x) = 0. \tag{A.67}$$

We previously noted that the Klein-Gordon equation expresses the energy-momentum-mass relationship and necessitates that all components of the Dirac spinors satisfy it. Therefore, it follows that the γ matrices must satisfy the equation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{A.68}$$

where $\{A, B\} \equiv AB + BA$ is the anticomutator. These matrices are known as the Dirac matrices, and their properties will be explored in a subsequent appendix. It is important to note that there exist various representations of the Dirac matrices[7]; the Weyl representation, which is presented in equation (A.60), is one such example.

One can deduce certain properties of the γ matrices immediately from the anticommutation relation (A.68), such as:

$$(\gamma^0)^2 = 1; \qquad (\gamma^i)^2 = -1. \tag{A.69}$$

Similar to the derivation of a conserved current for the Klein-Gordon equation, we can derive one for the Dirac equation. By taking the Hermitian operation of the Dirac equation (A.62), we obtain:

$$i(\partial_\mu \overline{\psi})\gamma^0(\gamma^\mu)^\dagger \gamma^0 \psi + m\overline{\psi}\psi = 0, \qquad \text{where} \quad \psi(\bar{x}) \equiv \psi^\dagger(x)\gamma^0. \tag{A.70}$$

Multiplying on right-hand side of the Dirac equation by $\overline{\psi}$ yields

$$i\overline{\psi}\gamma^\mu(\partial_\mu \psi) - m\overline{\psi}\psi = 0. \tag{A.71}$$

Combining equations (A.70) and (A.71), we obtain:

$$(\partial_\mu \overline{\psi})\gamma^0(\gamma^\mu)^\dagger \gamma^0 \psi + \overline{\psi}\gamma^\mu(\partial_\mu \psi) = 0.$$

[7] Matrices that fulfill the anticommutation relationship of the Dirac matrices and equation (A.72) can be regarded as a representation of them. Notably, satisfying condition (A.72) is crucial to deriving a continuity equation from the Dirac equation with a quantum mechanical density equivalent to the Schrödinger case.

By imposing that the γ matrices satisfy

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu, \tag{A.72}$$

we obtain:

$$\partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0, \tag{A.73}$$

which leads to the definition of the conserved current, given by

$$j^\mu = \overline{\psi}\gamma^\mu\psi. \tag{A.74}$$

The density J^0 can be computed as

$$j^0 = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2, \tag{A.75}$$

which is positive. Therefore, j^0 can be interpreted as a probability density, and this issue with the Klein-Gordon equation has been resolved.

A.3 The Poincaré Group

In the previous section, we explored the Lorentz group, which encompasses symmetries related to spacetime. However, there is another symmetry present in spacetime known as translation symmetry. When considering elementary particles, properties such as spin and mass are considered "kinematic". It is important to describe these properties using quantities that remain invariant under relativistic transformations.

Mass is expressed as $m^2 = p_\mu p^\mu$, where P_μ represents the four-momentum operator. It is worth noting that this operator does not appear in the analysis of the (homogeneous) Lorentz group. Instead, P_μ serves as the generator of spacetime translations, which can be represented as:

$$x^\mu \longrightarrow x'^\mu = x^\mu + a^\mu.$$

What we need to do is to combine these transformations with those of the Lorentz group. This results in the inhomogeneous Lorentz group, often referred to as the Poincaré group. The first comprehensive analysis of this group was conducted by Eugene Paul Wigner [43]. His findings revealed that mass and spin are the two fundamental properties that characterize systems invariant under the Poincaré group.

To determine the structure of the Poincaré group, we can derive the generator corresponding to a parameter a^α using the following equation:

$$X_\alpha = i\frac{\partial x'^\mu}{\partial a^\alpha}\bigg|_{a=0}\frac{\partial}{\partial x^\mu}. \tag{A.76}$$

For instance, let's consider a boost in the x-direction where $x' = \gamma(x + vt)$, $y' = y, z' = z$, and $t' = \gamma(t + vx)$. Applying this to the equation, we find:

$$k_x = i\left(t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}\right). \tag{A.77}$$

Consequently, P_μ can be expressed as:

$$P_\mu = i\frac{\partial}{\partial x^\mu}. \tag{A.78}$$

The generators of translations correspond to the energy-momentum operators. The Poincaré group exhibits the following commutation relations:

$$\left[P_\mu, P_\nu\right] = 0; \tag{A.79}$$

$$\left[P_\mu, M_{\rho\sigma}\right] = i\left(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho\right), \tag{A.80}$$

where $M_{\rho\sigma}$ was defined in Equation A.31.

A general inhomogeneous Lorentz transformation is

$$x'^\mu = \Lambda^\mu_{\ \nu}x^\nu + a^\mu,$$

if we perform a second transformation we find

$$\bar{x}^\mu = \bar{\Lambda}^\mu_{\ \nu}x'^\nu + \bar{a}^\mu$$
$$\Rightarrow \bar{x}^\mu = \bar{\Lambda}^\mu_{\ \nu}\Lambda^\nu_{\ \alpha}x^\alpha + \bar{\Lambda}^\mu_{\ \alpha}a^\alpha + \bar{a}^\mu.$$

The group law can be expressed as

$$\{\bar{\Lambda}, \bar{a}\}\{\Lambda, a\} = \{\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a}\}. \tag{A.81}$$

Now, let's discuss the Wigner method, which relies on the fact that for a state with momentum p^μ, a Lorentz transformation changes p^μ while leaving $p^\mu p_\mu$ unchanged. A state $|p\rangle$ with

$$P^\mu|p\rangle = p^\mu|p\rangle,$$

is converted, by a transformation (Λ, a) into

$$U(\Lambda, a)\,|p\rangle = |\Lambda p\rangle, \tag{A.82}$$

where $U(\Lambda, a)$ is the unitary representation of the Poincaré transformation. Based on this transformation, we have

$$\begin{aligned} &P^\mu|\Lambda p\rangle = (\Lambda p)^\mu\,|\Lambda p\rangle \\ &\Rightarrow P^\mu P_\mu|\Lambda p\rangle = (\Lambda p)^\mu\,(\Lambda p)_\mu\,|\Lambda p\rangle \\ &\Rightarrow \Lambda^\mu_{\ \alpha}\Lambda^\beta_{\ \mu}p^\alpha p_\beta|\Lambda p\rangle = p^\mu p_\mu|\Lambda p\rangle. \end{aligned} \tag{A.83}$$

This is due to the fact that $P^\mu P_\mu$ commutes with all generators, making P^2 a Casimir invariant of the Poincaré group, known as the first Casimir invariant.

As a consequence, all the states obtained by Lorentz transformations from an initial state have the same value of p^2. Moreover, since the sign of p^0 is unchanged by a homogeneous Lorentz group transformation, the complete set of states forming a basis for representations of the group can be divided into six distinct classes:

$$
\begin{aligned}
&(i) && p^2 = m^2 > 0, && p^0 > 0;\\
&(ii) && p^2 = m^2 > 0, && p^0 < 0;\\
&(iii) && p^2 = 0, && p^0 > 0;\\
&(iv) && p^2 = 0, && p^0 < 0;\\
&(v) && p^\mu \equiv 0; && \\
&(vi) && p^2 < 0. &&
\end{aligned}
$$

The first and third classes correspond to physical massive and massless particles, respectively. The fifth class represents the vacuum state, while the sixth class is likely associated with virtual particles. The other classes are probably unphysical[26].

Having choosen a particular p^μ, belonging to a particular class $\{p^\mu\}$, an importante observation, is that the subgroup of the Poincaré group that leaves p^μ invariant- which is called little group of p^μ - has the same structure for all momentum in $\{p^\mu\}$. For example, consider the first class, i, e., $p^2 = m^2$, with $p^0 > 0$. A particular p^μ is the particle rest-frame, let us denote it as

$$k^\mu = (m,\ 0,\ 0,\ 0)\,.$$

It is clear that the rotation group is the little group of k^μ. Now consider the following transformations:

$$k^\mu \xrightarrow[L(p)]{} p^\mu; \tag{A.84}$$

$$p^\mu \xrightarrow[\Lambda]{} p'^\mu; \tag{A.85}$$

$$k^\mu \xrightarrow[L(\Lambda p)]{} p'^\mu. \tag{A.86}$$

We denote the state in Hilbert-space as:

$$|k,\ \sigma\rangle; \quad |p,\ \sigma\rangle \quad \text{and} \quad |p',\ \sigma\rangle : \quad \sigma \text{ is an spin index.}$$

Corresponding to above transformation in spacetime, we have in Hilbert-space the relation

$$|p,\ \sigma\rangle = U(L(p))|k,\ \sigma\rangle; \tag{A.87}$$

$$|p',\ \sigma\rangle = U(\Lambda)U(L(p))|k,\ \sigma\rangle. \tag{A.88}$$

Here $U(L(p))$ is an unitary operator. We can written the first transformation as follows:

$$\begin{aligned} |p',\, \sigma\rangle &= U(L(\Lambda p))U^{-1}(L(\Lambda p)U(\Lambda L(p))|k,\, \sigma\rangle \\ &= U(L(\Lambda p))U\left[L^{-1}(\Lambda p)\Lambda L(p)\right]|k,\, \sigma\rangle, \end{aligned} \tag{A.89}$$

where we have used the group law $U(A)U(B) = U(AB)$ and $U^{-1}(A) = U(A^{-1})$. From the transformation (A.86) we see that

$$U\left[L^{-1}(\Lambda p)\Lambda L(p)\right]|k,\, \sigma\rangle = |k,\, \sigma\rangle, \tag{A.90}$$

i. e., $U\left[L^{-1}(\Lambda p)\Lambda L(p)\right]$ is an element of the little group of k^μ. Therefore it is a matrix of the form $\exp\left(i\vec{J}\cdot\vec{\theta}\right)$, whose elements we denote by $\underset{\sigma'\sigma}{D}(R)$, so

$$\begin{aligned} U(\Lambda)|p,\, \sigma\rangle &= \sum_{\sigma'} \underset{\sigma'\sigma}{D}(R)U(L(\Lambda p))|k,\, \sigma'\rangle \\ &= \sum_{\sigma'} \underset{\sigma'\sigma}{D}(R)|\Lambda p,\, \sigma'\rangle. \end{aligned} \tag{A.91}$$

We can conclude that, to determine the representation of the Lorentz group for a timelike state, we only need to know the representation of the rotation group. Therefore, the attribute of spin, which characterizes the additional property that states may possess and is influenced by Lorentz transformations, is determined by the rotation group. This applies to all timelike momenta. However, when dealing with the third class of states, those with lightlike momenta, we find that this is no longer true. Spin is no longer determined by the rotation group, but by the $E(2)$ group, also known as the Euclidean group [26].

It can be observed that mass corresponds to the Casimir operator P^2. From the commutation relation of the Poincaré group, we can deduce that J^2, as defined in (A.34), is not a Casimir operator for the Poincaré group. The second Casimir operator of the Poincaré group can be obtained by defining the Pauli-Lubanski pseudovector W_μ as follows:

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma. \tag{A.92}$$

From this definition, it is immediately clear that $P_\mu W^\mu = 0$. Therefore, in the rest frame of a particle, W_μ is spacelike, given by $W_\mu = (0, \vec{W})$ in the case of massive particle, where

$$W_i = -\frac{1}{2}\epsilon_{i\nu\rho\sigma}M^{\nu\rho}P^\sigma = -\frac{m}{2}\epsilon_{ijk0}M^{jk} = -m\Sigma_i, \tag{A.93}$$

and $\Sigma_i = \begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}$, which we have already encountered in non-relativistic quantum mechanics. The square of Pauli-Lubanski vector can be write as

$$W^2|p,\, s\rangle = -m^2 s(s+1)|p,\, s\rangle, \tag{A.94}$$

where s is the spin of the particle.

Let us briefly consider the case of lightlike particles, since $p^2 = 0$. Let us choose it in the rest frame

$$k^{\mu} = (K,\ 0,\ 0,\ K),$$

which describe a massless particle moving along the z-direction. In this case the little group is $E(2)$, with generators $j_3,\ L_1$ and L_2, respectively, the generator of rotations on a plane and translations. In the case of a lightlike particle, the following conditions hold:

$$W^{\mu} W_{\mu} |k,\ s\rangle = 0; \tag{A.95}$$

$$P^{\mu} P_{\mu} |k,\ s\rangle = 0; \tag{A.96}$$

$$W^{\mu} P_{\mu} |k,\ s\rangle = 0, \tag{A.97}$$

These equations imply that W^{μ} and P^{μ} are proportional:

$$\left(W^{\mu} - \lambda P^{\mu}\right) |k,\ s\rangle = 0. \tag{A.98}$$

Here, we introduce the parameter λ, which characterizes massless particles. It represents the ratio of W^{μ} to P^{μ} and has the dimension of angular momentum. This parameter is referred to as helicity. If we consider parity, helicity can take on two values, λ and $-\lambda$. This description aligns with our understanding of neutrinos (when considering them as massless) and photons. Photons exhibit both right and left polarization states with $\lambda = \pm 1$, but they do not possess $\lambda = 0$, which would be the case if photons had mass, as seen with the weak interaction bosons.

In summary, these labels are crucial for characterizing fundamental particles. Additional labels are provided by internal symmetries, such as the $U(1)$ symmetry, which leads to charge conservation and introduces a new label for a particle.

B The Dirac Algebra and Gordon Identities

B.1 The Dirac Algebra and Γ matrices

As we have seen, the Dirac equation is given by

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0,$$

to which the positive plane wave solution can be written as

$$\psi(x) = u(\vec{p})e^{-ipx}, \tag{B.1}$$

which amounts to solving the Dirac equation in momentum space

$$\left(\not{p} - m\right) u(p) = 0.$$

We will now explore some characteristics of the γ algebra. The γ matrices are given by the equation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}; \tag{B.2}$$

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu. \tag{B.3}$$

Additionally, we define γ^5 as:

$$\gamma^5 \equiv \gamma_5 \equiv \frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{B.4}$$

The γ^5 matrix has the following properties:

$$(\gamma^5)^2 = 1; \tag{B.5}$$

$$(\gamma^5)^\dagger = \gamma^5. \tag{B.6}$$

From equations (A.68) and (A.72), it is immediate to deduce the following properties of γ matrices:

$$(\gamma^0)^\dagger = \gamma^0; \qquad (\gamma^i)^\dagger = -\gamma^i. \tag{B.7}$$

The product of a γ^5 with other γ matrices can be calculated as follows:

$$\gamma^0\gamma^5 = i\gamma^1\gamma^2\gamma^3;$$
$$\gamma^1\gamma^5 = i\gamma^0\gamma^2\gamma^3;$$
$$\gamma^2\gamma^5 = -i\gamma^0\gamma^1\gamma^3;$$
$$\gamma^3\gamma^5 = i\gamma^0\gamma^1\gamma^2.$$

This can be expressed in a more compact form as:

$$\gamma^{\alpha}\gamma^{5} = \frac{i}{3!}g^{\alpha\beta}\epsilon_{\beta\mu\nu\rho}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}. \tag{B.8}$$

We can introduce the rank-2 anti-symmetric tensor $\sigma^{\mu\nu}$ as:

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]. \tag{B.9}$$

The relation between the $\sigma^{\mu\nu}$ tensor and the γ^5 matrix is given by:

$$\gamma^{5}\sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}, \qquad \left[\gamma^{5}, \sigma^{\mu\nu}\right] = 0. \tag{B.10}$$

For instance, the first equation in B.10 can be shown as:

$$\begin{aligned}\gamma^{5}\sigma^{01} &= -(i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3})\gamma^{0}\gamma^{1} \\ &= \gamma^{1}\gamma^{2}\gamma^{3}\gamma^{1} \\ &= -\gamma^{2}\gamma^{3},\end{aligned} \tag{B.11}$$

which is equal to the right-hand side of the first equation B.10 for $\mu = 0$ and $\nu = 1$. The set

$$\mathcal{S} = \{1, \gamma^{\mu},\ \sigma^{\mu\nu},\ \gamma^{\mu}\gamma^{5},\ \gamma^{5}\}, \tag{B.12}$$

contains 16 matrix linearly independent $\Gamma^{a}(a = 1, \cdots, 16)$ that we will see in a moment. We represent the Γ^a as follows:

$$\Gamma^{1} \equiv 1 \qquad \text{(no } \gamma^{\mu} \text{ matrices);} \tag{B.13}$$

$$\Gamma^{2} - \Gamma^{5} \equiv \gamma^{\mu} \qquad \text{(one } \gamma^{\mu} \text{ matrix);} \tag{B.14}$$

$$\Gamma^{6} - \Gamma^{11} \equiv \sigma^{\mu\nu} \qquad \text{(products of two } \gamma^{\mu} \text{ matrices);} \tag{B.15}$$

$$\Gamma^{12} - \Gamma^{15} \equiv \gamma^{\mu}\gamma^{5} \qquad \text{(products of three } \gamma^{\mu} \text{ matrices);} \tag{B.16}$$

$$\Gamma^{16} \equiv \gamma^{5} \qquad \text{(products of four } \gamma^{\mu} \text{ matrices).} \tag{B.17}$$

Every product of γ matrices is proportional to one of the 16 $\Gamma \in \mathcal{S}$ with a coefficient of proportionality equal to ± 1 or $\pm i$. Any product of Γ matrices is proportional to a Γ matrix because we can simply permute any γ matrix and use the fact that $(\gamma^0)^2 = 1$ and/or $(\gamma^k)^2 = -1$. Furthermore, if two Γ matrices are different, their product is different from the identity, since at least one γ matrix must remain. This relationship can be expressed as follows:

$$\Gamma^{a}\Gamma^{b} \propto \Gamma^{c} \qquad \text{with} \quad \Gamma^{c} \neq 1 \qquad \text{for} \quad a \neq b. \tag{B.18}$$

The square of all Γ matrices is equato to ± 1:

$$(\Gamma^{a})^{2} = s^{a}, \qquad \text{with} \quad s^{a} = \frac{1}{4}\mathrm{Tr}\left[(\Gamma^{a})^{2}\right] \tag{B.19}$$

Tabela 2 – Order of the matrices Γ^a and the corresponding values of s^a.

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Γ^a	1	γ^0	γ^1	γ^2	γ^3	σ^{01}	σ^{02}	σ^{03}	σ^{12}	σ^{23}	σ^{31}	$\gamma^0\gamma^5$	$\gamma^1\gamma^5$	$\gamma^2\gamma^5$	$\gamma^3\gamma^5$	γ^5
s^a	1	1	-1	-1	-1	-1	-1	-1	1	1	1	-1	1	1	1	1

Source: Fundamentals of neutrino physics and astrophysics [42].

At least one Γ^b exists for every Γ^a where $a > 1$, such that it anticommutes with Γ^a,

$$\Gamma^a\Gamma^b = -\Gamma^b\Gamma^a \quad \Longleftrightarrow \quad \{\Gamma^a, \Gamma^b\} = 0$$

$$\Gamma^a = \gamma^0 \qquad (a = 2) \quad \Longrightarrow \quad \Gamma^b = \gamma^k, \gamma^5 \qquad (b = 3,\ 4,\ 5,\ 16); \tag{B.20}$$

$$\Gamma^a = \gamma^k \qquad (a = 3,\ 4,\ 5) \quad \Longrightarrow \quad \Gamma^b = \gamma_0, \gamma^5 \qquad (b = 1,\ 16); \tag{B.21}$$

$$\Gamma^a = \sigma^{\mu\nu} \qquad (a = 6-11) \quad \Longrightarrow \quad \Gamma^b = \sigma^{\mu\rho} \qquad (\rho \neq \nu); \tag{B.22}$$

$$\Gamma^a = \gamma^0\gamma^5 \qquad (a = 12) \quad \Longrightarrow \quad \Gamma^b = \gamma^k\gamma^5 \qquad (b = 13, 14, 15); \tag{B.23}$$

$$\Gamma^a = \gamma^k\gamma^5 \qquad (a = 13,\ 14,\ 15) \quad \Longrightarrow \quad \Gamma^b = \gamma^0\gamma^5 \qquad (b = 12); \tag{B.24}$$

$$\Gamma^a = \gamma^5 \qquad (a = 16) \quad \Longrightarrow \quad \Gamma^b = \gamma^\mu \qquad (b = 2,\ 3,\ 4,\ 5). \tag{B.25}$$

The matrices Γ^a with $a > 1$ are traceless, since

$$\mathrm{Tr}\left[(\Gamma^b)^2\right] = s^b \Rightarrow 1 = \frac{1}{s^b}\mathrm{Tr}\left[(\Gamma^b)^2\right]$$
$$\Rightarrow \mathrm{Tr}(\Gamma^a) = \frac{1}{s^b}\mathrm{Tr}\left[\Gamma^a(\Gamma^b)^2\right],$$

using Γ^b such that $\{\Gamma^a, \Gamma^b\} = 0$ we deduce:

$$\mathrm{Tr}(\Gamma^a) = \frac{1}{s^b}\mathrm{Tr}\left(\Gamma^a\Gamma^b\Gamma^b\right) = \frac{-1}{s^b}\mathrm{Tr}\left(\Gamma^b\Gamma^a\Gamma^b\right).$$

Using the cyclic permutation invariance of the trace, denoted by $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$, and the identity $\mathrm{Tr}\left[(\Gamma^b)^2\right] = s^b$ we find

$$\mathrm{Tr}(\Gamma^a) = -\mathrm{Tr}(\Gamma^a) \Rightarrow \mathrm{Tr}(\Gamma^a) = 0 \qquad \text{for} \quad a > 1. \tag{B.26}$$

Using the equations (B.18), (B.19) and (B.26) we can write

$$\mathrm{Tr}(\Gamma^a\Gamma^b) = 4s^a \qquad \text{if} \quad (a = b);$$
$$\mathrm{Tr}(\Gamma^a\Gamma^b) = \alpha\mathrm{Tr}(\Gamma^c), \qquad \text{but} \quad \mathrm{Tr}(\Gamma^c) = 0 \quad \text{if} \quad (a \neq b);$$
$$\Rightarrow \mathrm{Tr}(\Gamma^a\Gamma^b) = 4s^a\delta_{ab}$$

Finally, we can make the statement that the Γ matrices are linearly independent, meaning that:

$$\sum_a c_a \Gamma^a = 0 \quad \text{implies} \quad c_a = 0 \quad \text{for all} \quad a = 1,\ 2, \cdots,\ 16.$$

This statement can be initially proven by supposing that:

$$\begin{aligned}
&\sum_a c_a \Gamma^a = 0 \\
&\Rightarrow \mathrm{Tr}(\sum_a c_a \Gamma^a) = 0 \\
&\sum_a c_a \mathrm{Tr}(\Gamma^a) = 0 \\
&\Rightarrow c_1 = 0,
\end{aligned}$$

where in the last equation, we have used equation (B.26). Since $\sum_a c_a \Gamma^a = 0$ we can write

$$\begin{aligned}
&\mathrm{Tr}\left[\Gamma^b (\sum_a c_a \Gamma^a)\right] = 0, \qquad \text{for} \quad b = 2,\ 3, \cdots,\ 16. \\
&\Rightarrow \sum_a c_a \mathrm{Tr}(\Gamma^b \Gamma^a) = 0; \\
&\Rightarrow 4 \sum_a c_a s^b \delta_{ba} = 0; \\
&\Rightarrow c_b s^b = 0, \qquad (s^b \neq 0); \\
&\Rightarrow c_b = 0.
\end{aligned}$$

To summarize, we assume that $\sum c_a \Gamma^a = 0$ and deduce that $c_a = 0,$ for $(a = 1, \cdots,\ 16)$, implying that the Γ matrices are linearly independent.

Finally, we can express any 4×4 matrix as a linear combination of the Γ^a matrices:

$$X = \sum_a x_a \Gamma^a, \qquad \text{with} \quad x_a = \frac{s_a}{4} \mathrm{Tr}(X \Gamma^a). \tag{B.27}$$

B.2 Gordon Identity

For any $\Gamma \in \mathcal{S}$, using equation (A.72) we can show

$$\Gamma^\dagger = \eta_0 [\Gamma] \gamma^0 \Gamma \gamma^0, \tag{B.28}$$

with the same order of the Γ matrices in $\mathcal{S}$ we have $\eta_0 [\Gamma] = (1,\ 1,\ 1,\ 1,\ -1)$.

We can use the positive plane wave solution (B.1) along with the Dirac equation to obtain:

$$\begin{aligned}
\gamma^\mu p_\mu u(\vec{p}) &= m u(\vec{p}); \\
\overline{u}(\vec{p}) \gamma^\mu p_\mu &= m \overline{u}(\vec{p}).
\end{aligned} \tag{B.29}$$

Furthermore, the anti-symmetric tensor $\sigma^{\mu\nu}$ can be expressed as:

$$i\sigma^{\mu\nu} = g^{\mu\nu} - \gamma^{\mu}\gamma^{\nu} = \gamma^{\nu}\gamma^{\mu} - g^{\mu\nu}. \tag{B.30}$$

Using equations (B.29) and (B.30), we can calculate the following expression:

$$\begin{aligned}
\overline{u}(p')i\sigma^{\mu\nu}(p'_{\nu} - p_{\nu})u(p) &= \overline{u}(p')\left[(\gamma^{\nu}\gamma^{\mu} - g^{\mu\nu})p'_{\nu} - (g^{\mu\nu} - \gamma^{\mu}\gamma^{\nu})p_{\nu}\right]u(p) \\
&= \overline{u}(p')\left[p'_{\nu}\gamma^{\nu}\gamma^{\mu} - (p'^{\mu} + p^{\mu}) + \gamma^{\mu}\gamma^{\nu}p_{\nu}\right]u(p) \\
&= \overline{u}(p')\left[2m\gamma^{\mu} - l^{\mu}\right]u(p).
\end{aligned}$$

Therefore, we can express:

$$\overline{u}(p')\gamma^{\mu}u(p) = \frac{1}{2m}\overline{u}(p')\left[l^{\mu} + i\sigma^{\mu\nu}q_{\nu}\right]u(p). \tag{B.31}$$

Similarly, we can evaluate:

$$\begin{aligned}
\overline{u}(p')i\sigma^{\mu\nu}l_{\nu}\gamma^{5}u(p) &= \overline{u}(p')\left[\not{p}\gamma^{\mu}\gamma^{5} - q^{\mu}\gamma^{5} - \gamma^{\mu}\not{p}\gamma^{5}\right]u(p) \\
&= \overline{u}(p')\left[2m\gamma^{\mu}\gamma^{5} - q^{\mu}\gamma^{5}\right]u(p).
\end{aligned}$$

Hence, we can express:

$$\overline{u}(p')\gamma^{\mu}\gamma^{5}u(p) = \frac{1}{2m}\overline{u}(p')\left[\gamma^{5}q^{\mu} + i\gamma^{5}\sigma^{\mu\nu}l_{\nu}\right]u(p). \tag{B.32}$$

There are two more Gordon identities that can be deduced analogously to the above, to summarize we will present them here:

$$\overline{u}(p')\gamma^{\mu}u(p) = \frac{1}{2m}\overline{u}(p')\left[l^{\mu} + i\sigma^{\mu\nu}q_{\nu}\right]u(p); \tag{B.33}$$

$$\overline{u}(p')\gamma^{\mu}\gamma^{5}u(p) = \frac{1}{2m}\overline{u}(p')\left[\gamma^{5}q^{\mu} + i\gamma^{5}\sigma^{\mu\nu}l_{\nu}\right]u(p); \tag{B.34}$$

$$\overline{u}(p')i\sigma^{\mu\nu}l_{\nu}u(p) = -\overline{u}(p')q^{\mu}u(p); \tag{B.35}$$

$$\overline{u}(p')i\sigma^{\mu\nu}q^{\nu}u(p) = \overline{u}(p')[2m\gamma^{\mu} + l^{\mu}]u(p), \tag{B.36}$$

where $l^{\mu} \equiv p'^{\mu} + p^{\mu}$ and $q^{\mu} = p'^{\mu} - p^{\mu}$.

Another set of related identities involves the Levi-Civita tensor. Where here, we will use the following convention:

$$\epsilon^{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma} = \begin{cases} 1; & \mu\nu\rho\sigma = \text{even permutation of } 0123 \\ -1; & \mu\nu\rho\sigma = \text{even permutation of } 0123 \\ 0; & \text{if any two index are the same.} \end{cases}$$

We will appraise the trace of the $\gamma-$matrices' product. By utilizing the anticommutation relation and the trace's cyclic permutation property, we obtain:

$$\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) = 4g^{\mu\nu}.$$

Any odd number of $\gamma-$matrices has a trace of zero. This fact can be observed by utilizing the equation $\mathrm{Tr}(\Gamma^a\Gamma^b) = 4s^a\delta_{ab}$, where $a \in \{2, \cdots, 5\}$ and $b \in \{12, \cdots, 15\}$.

The trace of four $\gamma-$matrices can be obtained using $\gamma^\mu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu$ by:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma^\rho\gamma^\sigma \\ &= 2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu.\end{aligned}$$

Hence

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\left[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right]$$

The trace of the product of four $\gamma-$matrices and a γ^5 can be calculated as

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}, \tag{B.37}$$

since for two equal indexes, e.g., $\mu = \sigma$ follows

$$\begin{aligned}\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\mu) \\ &= -i(-1)^n\mathrm{Tr}(\gamma^5\sigma^{\nu\rho}) = 0.\end{aligned}$$

Therefore $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 0$ if we have two or more equal indexes. On the other hand, if all the indices are different, for example, $\mu = 0,\ \nu = 1,\ \rho = 2,\ \sigma = 3$, follows

$$\begin{aligned}\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= -i\mathrm{Tr}(\gamma^5 i\gamma^0\gamma^1\gamma^2\gamma^3) \\ &= -i\mathrm{Tr}\left[(\gamma^5)^2\right] \\ &= -i4.\end{aligned}$$

If the indices are in a different order, we need to perform permutations to rearrange them to the previous order. This permutation can be described by the Levi-Civita symbol, which recovers equation (B.37).

To summarize we will present here the values of trace of products of $\gamma-$matrices:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}; \tag{B.38}$$

$$\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = 0 \qquad \text{for} \quad n \quad \text{odd}; \tag{B.39}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\left[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right]; \tag{B.40}$$

$$\mathrm{Tr}(\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = 0 \qquad \text{for} \quad n \quad \text{odd}; \tag{B.41}$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0; \tag{B.42}$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}. \tag{B.43}$$

By having the $\gamma-$matrices' traces at our disposal, we can infer one of Gordon's identities that establish a connection between the Levi-Civita symbol. We start by evaluating $\gamma^\alpha\sigma^{\mu\nu}$, this matrix can be writen as a linear combination of the $\Gamma-$matrices $\in \mathcal{S}$. Since

$$\gamma^\alpha \in \{\Gamma^2, \cdots, \Gamma^5\} \qquad \text{and}$$
$$\sigma^{\mu\nu} \in \{\Gamma^6, \cdots, \Gamma^{11}\}.$$

We can express the product $\gamma^\alpha\sigma^{\mu\nu}$ as a linear combination of Γ^a with $a \in \{2, \cdots, 5\}$ and Γ^b with $b \in \{12, \cdots, 15\}$, i.e.,

$$\gamma^\alpha\sigma^{\mu\nu} = \sum_{a=2}^{5} c^{\alpha\mu\nu a}\Gamma^a + \sum_{b=12}^{15} d^{\alpha\mu\nu b}\Gamma^b. \tag{B.44}$$

We can determine the coefficients of this expansion by taking the trace of the above equation:

$$\mathrm{Tr}(\gamma^\alpha\sigma^{\mu\nu}\Gamma^c) = \sum_{a=2}^{5} c^{\alpha\mu\nu a}\mathrm{Tr}(\Gamma^a\Gamma^c) + \sum_{b=12}^{15} d^{\alpha\mu\nu b}\mathrm{Tr}(\Gamma^b\Gamma^c). \tag{B.45}$$

For $c \in \{2, \cdots, 5\}$:

$$c^{\alpha\mu\nu c} = \frac{1}{4s^c}\mathrm{Tr}(\gamma^\alpha\sigma^{\mu\nu}\Gamma^c). \tag{B.46}$$

Similarly, for $c \in \{12, \cdots, 15\}$ we find:

$$d^{\alpha\mu\nu c} = \frac{1}{4s^c}\mathrm{Tr}(\gamma^\alpha\sigma^{\mu\nu}\Gamma^c). \tag{B.47}$$

Evaluating equation (B.46) yields $c \in \{2, \cdots, 5\}$, which is equivalent to evaluating:

$$\mathrm{Tr}(\gamma^\alpha\sigma^{\mu\nu}\gamma^\beta) = 4i\left[g^{\alpha\mu}g^{\nu\beta} - g^{\alpha\nu}g^{\mu\beta}\right].$$

By employing equations (B.40) and (B.43), we can derive the expansion of (B.44) as shown in the following relation:[1]

$$\gamma^\alpha\sigma^{\mu\nu} = i\left[g^{\alpha\mu}g^{\nu\beta} - g^{\alpha\nu}g^{\mu\beta}\right]\gamma_\beta + \epsilon^{\alpha\mu\nu\beta}\gamma^5\gamma_\beta. \tag{B.48}$$

Now we can evaluate

$$\begin{aligned}\overline{u}(p')\epsilon^{\alpha\mu\nu\beta}\gamma^5\gamma_\beta q_\mu l_\nu u(p) &= \overline{u}(p')\gamma^\alpha\sigma^{\mu\nu}q_\mu l_\nu u(p) - i\overline{u}(p')(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu})\gamma_\beta q_\mu l_\nu u(p) \\ &= \overline{u}(p')\gamma^\alpha\sigma^{\mu\nu}q_\mu l_\nu u(p) - i\overline{u}(p')\left[q^\alpha \not{l} - l^\alpha \not{q}\right]u(p).\end{aligned}$$

The last step is noting that

$$-2i\sigma^{\mu\nu}q_\mu l_\nu = -2(g^{\mu\nu} - \gamma^\mu\nu)q_\mu l_\nu = -2q\cdot l + 2\not{q}\not{l} = 2\not{q}\not{l},$$

where we have used that $q \cdot l = 0$. We calculate $\not{q}\not{l}$, noting that

$$\not{a}\not{b} - \not{b}\not{a} = \not{a}\not{b} - b_\mu a_\nu(2g^{\mu\nu} - \gamma^\nu\gamma^\mu) = 2\not{a}\not{b} - 2b\cdot a,$$

[1] We have used s^c as the metric tensor coefficient to lower the index β.

hence

$$\not{q}\,\not{l} = \not{p}'\not{p} - \not{p}\not{p}' = 2(-p'\cdot p + \not{p}'\not{p}) = 2\left(\frac{q^2-2m^2}{2} + \not{p}'\not{p}\right). \tag{B.49}$$

Using the equation (B.50) we can write

$$\begin{aligned}\overline{u}(p')\gamma^\alpha\sigma^{\mu\nu}q_\mu l_\nu u(p) &= i\overline{u}(p')\gamma^\alpha\left(q^2-2m^2+2\not{p}'\not{p}\right)u(p)\\ &= i\overline{u}(p')\left(q^2-2m^2\right)\gamma^\alpha u(p) + 2im\overline{u}(p')\left(2p'^\alpha - \gamma^\alpha\not{p}'\right)u(p)\\ &= i\overline{u}(p')\left(q^2-4m^2\right)\gamma^\alpha u(p) + 2im\overline{u}(p')\left(l^\alpha + q^\alpha\right)u(p)\end{aligned}$$

Finally, we can express

$$\overline{u}(p')\epsilon^{\alpha\mu\nu\beta}\gamma^5\gamma_\beta q_\mu l_\nu u(p) = \overline{u}(p')\left\{-i\left[q^\alpha\not{l} - l^\alpha\not{q}\right] + i\left(q^2-4m^2\right)\gamma^\alpha + 2im\left(l^\alpha+q^\alpha\right)\right\}u(p)$$

There are four more similar identities, but the calculations are cumbersome and we will just present it here[2]

$$\overline{u}(p')i\sigma^{\mu\nu}\gamma^5 q_\nu u(p) = -\overline{u}(p')l^\mu\gamma^5 u(p) \tag{B.50}$$

$$\begin{aligned}\overline{u}(p')[\epsilon^{\alpha\mu\nu\beta}\gamma^5\gamma_\beta q_\mu l_\nu]u(p) &= \overline{u}(p')\Big[-i(q^\alpha\not{l} - l^\alpha\not{q}) + i(q^2-4m^2)\gamma^\alpha\\ &\quad + 2im(l^\alpha+q^\alpha)\Big]u(p)\end{aligned} \tag{B.51}$$

$$\begin{aligned}\overline{u}(p')[\epsilon^{\alpha\mu\nu\beta}\gamma_\beta q_\mu l_\nu]u(p) &= \overline{u}(p')\Big[i(q^\alpha\not{l} - l^\alpha\not{q})\gamma^5 + iq^2\gamma^5\gamma^\alpha\\ &\quad - 2im(l^\alpha+q^\alpha)\gamma^5\Big]u(p)\end{aligned} \tag{B.52}$$

$$\overline{u}(p')[\epsilon^{\alpha\nu\mu\beta}q_\mu l_\beta\gamma_\nu\gamma^5]u(p) = \frac{i}{2m}\overline{u}(p')[\epsilon^{\alpha\nu\mu\beta}q_\mu l_\beta\sigma_{\nu\rho}q^\rho]u(p) \tag{B.53}$$

$$\overline{u}(p')[\epsilon^{\alpha\nu\mu\beta}q_\mu l_\beta\sigma_{\nu\rho}l^\rho]u(p) = 0. \tag{B.54}$$

[2] The author acknowledges that the previous calculations may be somewhat cumbersome. However, the calculations previously deemed it instructive since they led to the discovery of the trace of the product of γ-matrices and an understanding of how to expand a 4×4 matrix in the S basis.

C Classical field theory

Classical field theory can be seen as a mechanical system that has an infinite number of degrees of freedom. The Field theory can be defined by either, the Hamiltonian formulation or the Lagrangian formulation. This two objects are defined as the integration over space of his respective density

$$H = \int d^3x\mathcal{H} \tag{C.1}$$

$$L = \int d^3x\mathcal{L}. \tag{C.2}$$

The word "density"is almost always omitted. Formaly the Hamiltonian (density) is a functional of the field and their conjugate momenta $\mathcal{H}[\phi, \pi]$, and the Lagrangian is the Legendre transformation of the Hamiltonian, defined as

$$\mathcal{L} = \pi[\phi, \dot{\phi}]\dot{\phi} - \mathcal{H}[\phi, \pi[\phi, \dot{\phi}]], \tag{C.3}$$

which $\dot{\phi} = \partial_t\phi$ is the temporal partial derivative of the field and $\dot{\phi}$ is defined implicitly by $\dot{\phi} = \frac{\partial\mathcal{H}[\phi,\pi]}{\partial\pi}$. The inverse transformation is

$$\mathcal{H}[\phi, \pi] = \dot{\phi}[\phi, \pi]\pi - \mathcal{L}[\phi, \pi], \tag{C.4}$$

where $\dot{\phi}[\phi, \pi]$ is implicity defined by $\pi = \frac{\partial\mathcal{L}[\phi,\dot{\phi}]}{\partial\dot{\phi}}$.
Similarly to classical mechanics, the Hamiltonian in classical field theory is regarded as the sum of both kinetic and potential energy

$$\mathcal{H} = \mathcal{K} + \mathcal{V}$$

Unlike the usage of kinetic and potential energy, field theory utilizes kinetic and interaction terms. The kinetic term is bilinear in the fields or their derivatives, so

$$\mathcal{L}_{\mathcal{K}} \subset \frac{1}{2}\phi\Box\phi, \quad \overline{\psi}\partial\!\!\!/\psi, \quad \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad \frac{1}{2}m^2\phi^2, \quad \phi_1\partial_\mu A^\mu, \quad \cdots$$

The kinetic term conveys information about the behavior of the free field. Sometimes it is best to think of a term like $m^2\phi^2$ an interaction term instead a kinetic term, this will be more clear when we talk about spontaneous symmetry break. The remaining terms are called terms of interaction,

$$\mathcal{L}_{int} \subset \lambda\phi^3, \quad g\overline{\psi}A\!\!\!/\psi, \quad g\partial_\mu\phi A^\mu\phi^*, \quad gA^\mu A_\mu A^\nu A_\nu, \quad \cdots$$

so we can write

$$\mathcal{L} = \mathcal{L}_{\mathcal{K}} + \mathcal{L}_{int} \tag{C.5}$$

C.1 Euler-Lagrange Equation

In quantum field theory, we generally utilize the Lagrangian. The simplest reason for this is that the Lagrangian is a Lorentz invariant, in contrast, the Hamitonian transforms as the time component of the energy-momentum vector. The dynamics of the system is governed by the principle of least action. The action is defined as the integral over time of the Lagrangian

$$S = \int dtL = \int d^4x\mathcal{L} \tag{C.6}$$

Say we have a Lagrangian as a functional of the field and his first derivative, $\mathcal{L} = \mathcal{L}[\phi, \partial_\mu\phi]$, so the least action priciple tell us that, the action must be stationary, i.e., $\frac{\delta S}{\delta\phi} = 0$, and we can deduce the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0. \tag{C.7}$$

In particular if we consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{C.8}$$

the Euler-Lagrange equation lead to

$$\left(\Box + m^2\right)\phi = 0. \tag{C.9}$$

This is the well-know Klein-Gordon equation. The Klein-Gordon discussed in the previous chapter. This equation was proposed by Oskar Klein and Walter Gordon in 1926 as a relativistic extension of the Schrödinger equation. This equation plays an important role in Quantum Field Theory, since it imposes the fields or particles must satisfy the dispersion relation

$$p^\mu p_\mu = m^2,$$

also called mass-shell.

C.2 Noether Theorem

It may happen that the Lagrangian is invariant under a field transformation $\phi \to \phi + \delta\phi$, so we say that this Lagrangian is symmetric under this kind of transformation. The Lagrangian of a scalar complex field, given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi^* - \frac{1}{2}m^2\phi\phi^*, \tag{C.10}$$

is unchanged under the transformation $\phi \to e^{-i\alpha}\phi$ where $\alpha \in \mathbb{R}$. If a symmetry depends on a parameter α, which can be taken as small, i.e., is continuous, then we find

$$\frac{\delta\mathcal{L}}{\delta\alpha} = \sum_n \left\{\left[\frac{\partial\mathcal{L}}{\partial\phi_n} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_n\right)}\right)\right]\frac{\delta\phi_n}{\delta\alpha} + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_n\right)}\frac{\delta\phi_n}{\delta\alpha}\right]\right\}, \tag{C.11}$$

and, since the Lagrangian is symmetric under this transformation, equation C.11 vanishes, therefore if the field satisfies the equation of motion, we find

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi_n \right)} \frac{\delta \phi_n}{\delta \alpha} \right] = 0. \tag{C.12}$$

Hence, we can define the Noether Current J^μ as:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi_n \right)} \frac{\delta \phi_n}{\delta \alpha}, \tag{C.13}$$

in which $\partial_\mu J^\mu = 0$. For example, if we consider the Lagrangian in equation C.10 we find

$$J_\mu = -i \left(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi \right)$$

is a Noether Current, we often say this is a conserved current, because the charge defined as,

$$Q = \int d^3 x J_o, \tag{C.14}$$

is conserved in time, i.e., $\partial_t Q = 0$.

Let us consider the Dirac Lagrangian[1]:

$$\mathcal{L} = \overline{\psi} \left(i \gamma^\mu \overleftrightarrow{\partial}_\mu - m \right) \psi, \tag{C.15}$$

where

$$\overleftrightarrow{\partial}_\mu \equiv \frac{\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu}{2},$$

with $\overrightarrow{\partial}_\mu$ denoting the usual derivative operator, and

$$\overline{\psi} \overleftarrow{\partial}_\mu \equiv \partial_\mu \overline{\psi}.$$

The Dirac Lagrangian (C.15) exhibits global $U(1)$ invariance, meaning it remains unchanged under the following transformation:

$$\begin{aligned} \psi(x) &\longrightarrow \psi'(x) = e^{-iL\theta} \psi(x) \\ &\Rightarrow \delta\psi(x) = -iL\theta\psi(x) \\ \overline{\psi}(x) &\longrightarrow \overline{\psi}'(x) = e^{iL\theta} \overline{\psi}(x) \\ &\Rightarrow \delta\overline{\psi}(x) = iL\theta\overline{\psi}(x). \end{aligned}$$

This leads to the Noether current:

$$J^\mu = iL\overline{\psi}(x)\gamma^\mu\psi(x). \tag{C.16}$$

The generator L is referred to as lepton number, and we associate $L = \pm 1$ for fermions (antifermions). Therefore, the global $U(1)$ symmetry results in the conservation of lepton number[2] [19].

1 The Dirac Lagrangian is often presented as $\mathcal{L} = \overline{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi$, but this differs from the one presented below by a total derivative.

2 In the quantized Dirac field, which we won't delve into here, the Noether current relies on the fermion

number operator $\hat{n}_f \equiv \hat{a}\hat{a}^\dagger$. This operator quantifies the number of fermions and, consequently, the current characterizes the fermion count. Similarly, the current demonstrates an analogous dependency on the antifermion number operator.

D Spontaneous Symmetry Breaking

The Fermi Model for weak interactions has the following Lagrangian

$$\mathcal{L} = \frac{-G_F}{\sqrt{2}} j_\mu^\dagger j^\mu, \tag{D.1}$$

where G_F is the Fermi constant and j_μ is the weak V-A current, the V-A current must be introduced since the weak interaction violates the parity symmetry, and we can write it as

$$j^\mu = \bar{\nu}_e \gamma^\mu (1-\gamma^5) e + \bar{\nu}_\mu \gamma^\mu (1-\gamma^5)\mu + \bar{\nu}_\tau \gamma^\mu (1-\gamma^5)\tau, \tag{D.2}$$

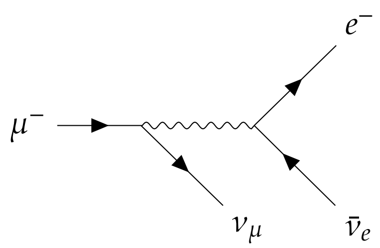
which the ν_l and l are, respectively, the neutrino of the lepton l and the lepton l, e.g., ν_e is the eletron-neutrino and e is the electron.

The dimension of GF being $[\mathrm{m}]^{-2}$ implies that loop corrections diverge with respect to the cut-off, making the 4-fermion theory incapable of being renormalized.

One possible solution to address the issue of non-renormalizability is to propose the inclusion of an intermediate vector boson that has a sufficiently large mass to account for the short range nature of the interaction.

The exchange boson interaction is represented in the following diagram, where we consider the decay $\mu \longrightarrow e^- + \bar{\nu}_e + \nu_\mu$.

Figura 16 – Tree-level Feynman diagram for μ^- decay.



Source: Author.

In such picture, the dimensionful coupling constant G_F is an effective coupling

$$G_F \longrightarrow \frac{g^2}{k^2 - m^2},$$

where g is a dimensionless coupling constant, m is the mass of the boson and k is the momentum flowing through the propagator. However, this does not yet solve the problem of non-renormalizability. The propagator of a massive vector boson is given by

$$\frac{-i}{k^2 - m^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right). \tag{D.3}$$

For large momenta the propagator becomes $\sim \frac{ik^{\mu}k^{\nu}}{k^2m^2}$, instead of being to proportional $\frac{1}{k^2}$ as is the case of scalar bosons or those of massless vector boson. To maintain renormalizability, a gauge fixing term is introduced in the Lagrangian. The achievement of renormalizability and consistency with spontaneous symmetry breaking was the groundbreaking work of t'Hooft and Veltman, which earned them the Nobel Prize in 1999 [14].

D.1 Spontaneous Symmetry Breaking: A Simple Example for a Real Scalar Field

If we consider a model of a real scalar field ϕ with discrete symmetry, called parity symmetry or a Z_2 symmetry, with the following potential

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4, \tag{D.4}$$

we have therefore the Lagrangian below

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{D.5}$$

and Hamiltonian

$$\mathcal{H} = \partial_\mu\phi\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{D.6}$$

This theory is invariant under $\phi \longrightarrow -\phi$.

It is important to note that, unlike in Klein-Gordon theory, we cannot interpret the term$+\frac{\mu^2}{2}\phi^2$ as a mass term, because it has the wrong sign.

The lowest-energy can be obtained through

$$\frac{\partial\mathcal{H}}{\partial\phi} = 0,$$

so we find

$$\phi\left(\frac{\lambda}{3!}\phi^2 - \mu^2\right) = 0, \tag{D.7}$$

such equation has three solutions

$$\phi_0 = 0,$$
$$\phi_0 = \pm\sqrt{\frac{6}{\lambda}}\mu.$$

Calculating $V(\phi_0)$ in both cases, we see that the ground state is

$$\phi_0 = \pm v = \pm\sqrt{\frac{6}{\lambda}}\mu,$$

this is called vaccum spectation value. To interpret this theory imagine quantum fluctuations close to one of the minima, $+v$ say.

$$\phi(x) = v + \sigma(x) \tag{D.8}$$

Now, the vaccum theory is

$$\mathcal{L} = \partial^{\mu}\sigma\partial_{\mu}\sigma + \left(\mu^2 v - \frac{\lambda v^3}{6}\right) + \left(\frac{\mu^2}{2} - \frac{\lambda}{4}v^2\right)\sigma^2 + \frac{\lambda}{6}\sigma^3 - \frac{\lambda}{4!}\sigma^4 + \text{constants}, \tag{D.9}$$

the linear term in σ is zero, since we are talking about the minimun of the potential, write it in terms of the parameters μ and λ we find

$$\mathcal{L} = \partial^{\mu}\sigma\partial_{\mu}\sigma - \mu^2\sigma^2 - \frac{\lambda v}{6}v^3 - \frac{\lambda}{4!}\sigma^4, \tag{D.10}$$

wich is a theory for a scalar field with mass μ and self-interactions for the σ field. And we can see that the $\phi \longrightarrow -\phi$ symmetry of the Lagrangian is not a symmetry of the vaccum .

D.2 The Abelian Higgs Model

We want to consider now gauge theories. Consider aa theory for a complex scalar field coupled to the electromagnetic field,

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_{\mu}\phi|^2 - V(\phi), \tag{D.11}$$

in this case

$$D_{\mu} = \partial_{\mu} + ieA_{\mu}$$
$$V(\phi) = -\mu^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2.$$

$\mathcal{L}$ is invariant under local $U(1)$ gauge transformation

$$\phi(x) \longrightarrow e^{i\alpha(x)}\phi(x)$$
$$A_{\mu}(x) \longrightarrow A_{\mu}(x) - \frac{1}{e}\partial_{\mu}\alpha(x)$$
$$D_{\mu}\phi(x) \longrightarrow e^{i\alpha(x)}D_{\mu}\phi(x).$$

The last transformation is a consequence of the gauge transformation of the fields $\phi(x)$ and $A_{\mu}(x)$.

When $\mu^2 < 0$, the model corresponds to the quantum electrodynamics of a charged scalar boson. However, when $\mu^2 > 0$, the $U(1)$ symmetry is spontaneously breaking.

In the case of $\mu^2 > 0$, the minimum of the potential occurs at

$$\frac{\partial V(\phi)}{\partial \phi} = 0,$$

hence the field $\phi(x)$ musty satisfy

$$\phi^* \left(-\mu^2 + \lambda(\phi^*\phi)\right) = 0, \tag{D.12}$$

to which the solutions are

$$\phi(x) = 0 \tag{D.13}$$

$$|\phi(x)| = \frac{\mu^2}{\lambda} \tag{D.14}$$

Therefore, the minimum is

$$\langle|\phi|\rangle = |\phi_0| = \left(\frac{\mu^2}{\lambda}\right)^{\frac{1}{2}}. \tag{D.15}$$

Now, we choose the minimum to be in the positive direction, i.e., ϕ_0 to be real positive, and define the shifted field

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right). \tag{D.16}$$

Using this field expansion around the vaccum, the potential can be write as

$$V(\phi) = \frac{1}{2}(2\mu^2)\phi_1^2 + O(\phi_i^3) + \text{constats}. \tag{D.17}$$

The term $O(\phi_i^3)$ refers to interactions between scalar and vector fields. According to equation (D.17) there is now a mass term for one of the scalar fields, i.e., the scalar field $\phi_1(x)$ is a massive scalar field, in contrast, $\phi_2(x)$ does not has a mass term, therefore it is a massless scalar field.

The interaction of the gauge field is given by

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + e^2\phi_0^2 A_\mu A^\mu + \cdots. \tag{D.18}$$

The quadratic term in the vector field, $e^2\phi_0^2 A_\mu A^\mu$ implies that the gauge boson has acquired a mass

$$m_A^2 = 2e^2\phi_0^2.$$

The original theory consisted of two massive scalar fields and a massless photon, providing us a total of $2 + 2 = 4$ degrees of freedom, as a massless vector boson has only two polarization modes, both transverse. The current theory, on the other hand, involves one massive scalar field and one massive vector boson, yielding $1 + 3 = 4$ degrees of freedom since a massive vector field can have three polarizations - two transverse and one longitudinal. The Goldstone boson (ϕ_2) has been absorbed as the longitudinal degree of freedom of the massive vector boson, which is why it is often said that the Goldstone boson has been *eaten* by the vector [45].

D.3 The Non-Abelian $SU(2) \times U(1)$ Higgs Model

In the bosonic sector of the Standard electroweak theory of Glashow-Weinberg-Salam, the gauge theory is $SU(2) \times U(1)$, with $A^a_\mu (a = 1,\ 2,\ 3)$ are the gauge field of $SU(2)$ and B_μ for the $U(1)$. The field ϕ is a doublet of scalar fields and has a $U(1)$ charge, also called weak hypercharge of $Y = \frac{1}{2}$. The Lagrangian of this theory is:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \left(D_\mu \phi\right)^\dagger (D^\mu \phi) - \lambda \left(\phi^\dagger \phi - \frac{v^2}{2}\right)^2, \tag{D.19}$$

for

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \qquad \text{and} \\ D_\mu \phi &= \partial_\mu \phi - i\frac{g}{2} \tau^a A^a_\mu \phi - i\frac{g'}{2} B_\mu \phi. \end{aligned}$$

For the field ϕ the chosen ground state will be

$$\phi^{(v)} = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}}, \end{pmatrix} \tag{D.20}$$

for a constante v. As for the generators of the electroweak theory, they consist of the Hermitian matrices $T^a = \frac{\tau^a}{2}$, where τ^a are the Pauli matrices. In order to find the unbroken generators Q, the gauge fields will be ignored and Q should also be Hermitian and satisfy

$$Q\phi^{(v)} = 0, \tag{D.21}$$

therefore Q is of the form:

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \equiv T^3 + Y. \tag{D.22}$$

This Q is the generator of an unbroken $U(1)$ symmetry, left over after the symmetry $SU(2) \times U(1)$ breaks for the ground state defined above. It corresponds to a massless gauge field, which will be identified as the electromagnetic field. The electromagnetic potential A_μ is a linear combination of the fields A^a_μ and B_μ. For small pertubations of the fields about the vaccum state, in unitary gauge and $\chi(x)$ a real scalar field, we find:

$$\tilde{\phi} = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \frac{\chi(x)}{\sqrt{2}} \end{pmatrix} \tag{D.23}$$

The covariant derivative results in the following expression:

$$D_\mu \tilde{\phi} = \begin{pmatrix} -i\frac{g}{2} W^-_\mu (v + \chi(x)) \\ \frac{\partial_\mu \chi(x)}{\sqrt{2}} + \frac{\sqrt{g^2 + g'^2}}{2\sqrt{2}} Z_\mu (v + \chi(x)) \end{pmatrix} \tag{D.24}$$

where we have introduced the complex fields:

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(A_\mu^1 \mp iA_\mu^2\right), \tag{D.25}$$

and two real fields:

$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(gA_\mu^3 - g'B_\mu\right) \tag{D.26}$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(gB_\mu + g'A_\mu^3\right). \tag{D.27}$$

The second order of the covariant derivative term in the Lagrangian D.19, becomes

$$(D_\mu\tilde{\phi})^2 = \frac{1}{2}(\partial_\mu\chi(x))^2 + \frac{g^2v^2}{4}W_\mu^+W^{-\mu} + \frac{1}{2}\left(\frac{(g^2+g'^2)}{4}v^2Z_\mu^2\right) + \cdots \tag{D.28}$$

Similarly, the kinetic terms of the vector fields up to quadratic order results in

$$-\frac{1}{4}F_{\mu\nu}^aF^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} = -\frac{1}{2}W_{\mu\nu}^+W^{-\mu\nu} - \frac{1}{4}F_{\mu\nu}^3F^{3\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}, \tag{D.29}$$

where $W_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm$. From this we can see that

$$Z_\mu^2 + A_\mu^2 = (A_\mu^3)^2 + B_\mu^2. \tag{D.30}$$

Which we can use to determine the quadratic term of the gauge fields as:

$$-\frac{1}{4}F_{\mu\nu}^aF^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} = -\frac{1}{2}W_{\mu\nu}^+W^{-\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{D.31}$$

where we have defined

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \tag{D.32}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{D.33}$$

Finally the quadratic order in the potential becomes

$$\lambda\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2 = \lambda v^2\chi^2(x) + \cdots \tag{D.34}$$

The full new Lagrangian of the quadratic terms becomes

$$\begin{aligned}\mathcal{L}^{(2)} =& -\frac{1}{2}W_{\mu\nu}^+W^{-\mu\nu} + m_W^2W_\mu^+W^{-\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + m_Z^2Z_\mu Z^\mu \\ &+ \frac{1}{\sqrt{2}}(\partial_\mu\chi(x))^2 + \frac{m_\chi^2}{2}\chi^2(x),\end{aligned} \tag{D.35}$$

where we have identified

$$\begin{aligned} m_W &= \frac{gv}{2} \\ m_Z &= \frac{\sqrt{g^2+g'^2}v}{2} \\ m_\chi &= \sqrt{2}\lambda v^2. \end{aligned} \tag{D.36}$$

This Lagrangian contains a massive complex vector field $W^{\pm}_{\mu}$ with mass m_W, which is identified as the W boson of the weak interaction, a massless vector field A_μ that is identified as the photon field and a massive real vector field Z_μ which is the Z^0 boson of the weak interaction and a massive real scalar field $\chi(x)$ called the Higgs field.

In particle physics, there also exists the concept of the weak mixing angle θ_W, defined by

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}} \tag{D.37}$$

$$\sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}. \tag{D.38}$$

Using these relations, the fields defined become

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix}. \tag{D.39}$$

The θ_W is commonly called the Weinberg angle [14].

E Neutrino Oscillation

The phenomenon of neutrino oscillation can be explained by examining the connection between the flavor eigenstates (ν_e, ν_μ, and ν_τ) associated with weak interactions and the mass eigenstates (ν_1, ν_2, and ν_3) of the free-particle Hamiltonian.

When charged leptons move through matter, they create a visible trail of ionized atoms. In contrast, neutrinos are never directly observed; their presence is only detected through their weak interactions. Therefore, different neutrino flavors can only be distinguished by the flavors of charged leptons produced in charged-current weak interactions. Consequently, the electron neutrino ν_e is defined as the neutrino state generated in a charged-current weak interaction alongside an electron. Initially, it was believed that ν_e, ν_μ, and ν_τ were massless fundamental particles. The physical states of particles, known as the mass eigenstates, are stationary states of the free particle Hamiltonian and satisfy the equation:

$$\hat{H}\psi = i\frac{\partial \psi}{\partial t} = E\psi. \tag{E.1}$$

The time evolution of the mass eigenstates are labeled as ν_1, ν_2, and ν_3. There is no inherent reason to assume that the weak eigenstates correspond to the mass eigenstates. Any of the three mass eigenstates can be produced in conjunction with the electron during the initial weak interaction. Since it is impossible to determine which mass eigenstate was produced, the system must be described by a coherent linear combination of ν_1, ν_2, and ν_3 states. In quantum mechanics, the basis of weak eigenstates can be related to the basis of mass eigenstates using an unitary matrix U:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e_1} & U_{e_2} & U_{e_3} \\ U_{\mu_1} & U_{\mu_2} & U_{\mu_3} \\ U_{\tau_1} & U_{\tau_2} & U_{\tau_3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{E.2}$$

Consequently, the electron neutrino is a linear combination of the defined mass eigenstates and can be expressed as:

$$|\psi\rangle = U_{e_1}|\nu_1\rangle + U_{e_2}|\nu_2\rangle + U_{e_3}|\nu_3\rangle. \tag{E.3}$$

As the neutrino state travels, it exists as a coherent linear combination of the three mass eigenstates. However, when it interacts, the wavefunction collapses into a specific weak eigenstate, resulting in the production of an observable charged lepton with a particular flavor. In the case where the masses of ν_1, ν_2, and ν_3 are not equal, phase differences emerge among the various components of the wave function. These phase differences give rise to the phenomenon of neutrino oscillation.

E.1 Two Flavours Oscillation

Let us consider the weak eigenstates ν_e and ν_μ, which are regarded here as coherent linear combinations of the mass eigenstates ν_1 and ν_2. The mass eigenstates propagate as plane waves given by:

$$|\nu_1(t)\rangle = |\nu_1\rangle e^{-ip_1 x} \tag{E.4}$$

$$|\nu_2(t)\rangle = |\nu_2\rangle e^{-ip_2 x}, \tag{E.5}$$

here, $p_1 x = Et - \vec{p}\cdot\vec{x}$ represents the Lorentz-invariant phase. In the context of the two-flavor treatment of neutrino oscillations, the weak eigenstates are connected to the mass eigenstates by a 2×2 unitary matrix, which can be expressed in terms of a single angle,

$$\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}. \tag{E.6}$$

Suppose at time $t = 0$ a neutrino is a weak eigenstate

$$|\psi(0)\rangle \equiv |\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle. \tag{E.7}$$

The state subsequently evolves to

$$|\psi(x,t)\rangle = \cos\theta e^{-ip_1 x}|\nu_1\rangle + \sin\theta e^{-ip_2 x}|\nu_2\rangle. \tag{E.8}$$

If the neutrino then interacts at time T and distance L along its path, the neutrino state at this space-time point is

$$|\psi(L,T)\rangle = \cos\theta e^{-i\phi_1}|\nu_1\rangle + \sin\theta e^{-i\phi_2}|\nu_2\rangle, \tag{E.9}$$

here, $\phi_i \equiv E_i T - p_i L$. The mass eigenstates are written as:

$$\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}. \tag{E.10}$$

This leads to:

$$|\psi(L,T)\rangle = e^{-i\phi_1}\left\{\left(\cos^2\theta + \sin^2\theta e^{i\Delta\phi_{12}}\right)|\nu_e\rangle - \left(1 - e^{i\Delta\phi_{12}}\right)\cos\theta\sin\theta|\nu_\mu\rangle\right\} \tag{E.11}$$

The probability that the neutrino, initially produced as ν_e, will interact to produce a muon is given by

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= \left|\left(1 - e^{i\Delta\phi_{12}}\cos\theta\sin\theta\right)\right|^2 \\ &= 2\cos^2\theta\sin^2\theta\left(1 - \cos\left(\Delta\phi_{12}\right)\right) \\ &= \sin^2 2\theta \sin^2\left(\frac{\Delta\phi_{12}}{2}\right). \end{aligned} \tag{E.12}$$

Hence, the $\nu_e \rightarrow \nu_\mu$ oscillation probability depends on the mixing angle θ and the phase difference $\Delta\phi_{12}$. Assuming $p_1 = p_2 = p$, although it should be noted that this treatment overlooks the fact that the different mass eigenstates will propagate at different velocities and therefore travel the distance L in different times. To address this concern, a proper wave-packet treatment of the coherent state propagation can be applied. This treatment leads to the same expression as given in the following equation

$$\Delta\phi_{12} \approx \left[p\left(1 + \frac{m_1^2}{p^2}\right)^{\frac{1}{2}} - p\left(1 + \frac{m_2^2}{p^2}\right)^{\frac{1}{2}} \right] \approx \frac{m_1^2 - m_2^2}{2p^2}. \tag{E.13}$$

In the above equation, we have used $T = L$ (in natural units), which follows from $\beta_\nu \approx 1$. Finally, the probability of oscillation can be expressed as

$$P(\nu_e \rightarrow \nu_\mu) = \sin^2 2\theta \sin^2 \left(\frac{\left(m_1^2 - m_2^2\right) L}{4E_\nu} \right). \tag{E.14}$$

E.2 Three Flavours Oscillation

In the context of three-flavor neutrino oscillation, the relationship between the three weak eigenstates and the mass eigenstates is described by a 3×3 unitary matrix known as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, as shown in equation (E.2). By utilizing the unitarity of the PMNS matrix, the mass eigenstates can be expressed in terms of the weak eigenstates,

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U_{e_1}^* & U_{\mu_1}^* & U_{\tau_1}^* \\ U_{e_2}^* & U_{\mu_2}^* & U_{\mu_2}^* \\ U_{e_3}^* & U_{\mu_3}^* & U_{\mu_3}^* \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}. \tag{E.15}$$

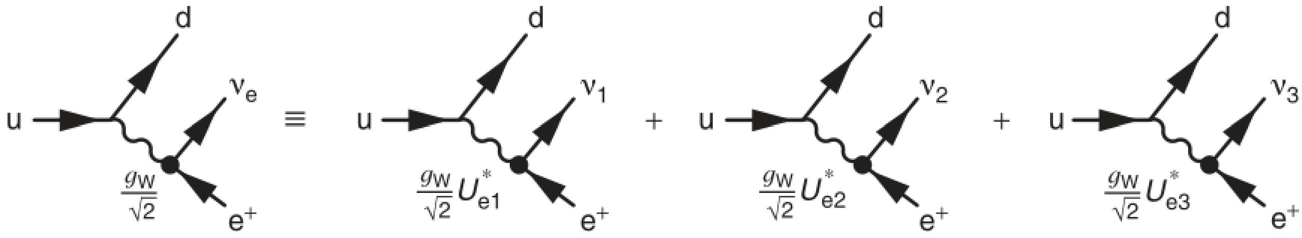
The unitarity of the PMNS matrix implies that:

$$U_{e_1} U_{e_1}^* + U_{e_2} U_{e_2}^* + U_{e_3} U_{e_3}^* = 1; \tag{E.16}$$

$$U_{e_1} U_{\mu_1}^* + U_{e_2} U_{\mu_2}^* + U_{e_3} U_{\mu_3}^* = 0. \tag{E.17}$$

Now consider the neutrino state that is produced in a charged-current weak interaction along with an electron. The neutrino, which enters the weak interaction vertex as the adjoint spinor as indicated in the following figure

Figura 17 – The contributions of the different mass eigenstates to the β-decay process.



Source: Modern Particle Physics [1].

The adjoint spinor correspond to a coherent linear superposition of mass eigenstates with a wavefunction at time $t = 0$ given by:

$$|\psi(0)\rangle \equiv U^*_{e_1}|\nu_1\rangle + U^*_{e_2}|\nu_2\rangle + U^*_{e_3}|\nu_3\rangle. \tag{E.18}$$

The time evolution of the wavefunction is determined by the time evolution of the mass eigenstates and following the same steps of the two flavours case we can write

$$\begin{aligned} |\psi(x,t)\rangle = &\left(U^*_{e_1}U_{e_1}e^{-i\phi_1} + U^*_{e_2}U_{e_2}e^{-i\phi_2} + U^*_{e_3}U_{e_3}e^{-i\phi_3}\right)|\nu_e\rangle \\ &+ \left(U^*_{e_1}U_{\mu_1}e^{-i\phi_1} + U^*_{e_2}U_{\mu_2}e^{-i\phi_2} + U^*_{e_3}U_{\mu_3}e^{-i\phi_3}\right)|\nu_\mu\rangle \\ &+ \left(U^*_{e_1}U_{\tau_1}e^{-i\phi_1} + U^*_{e_2}U_{\tau_2}e^{-i\phi_2} + U^*_{e_3}U_{\tau_3}e^{-i\phi_3}\right)|\nu_\tau\rangle. \end{aligned} \tag{E.19}$$

This can be expressed in the form $|\psi(x,t)\rangle = c_e|\nu_e\rangle + c_\mu|\nu_\mu\rangle + c_\tau|\nu_\tau\rangle$, from which the oscillation probability can be obtained, for example

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= c_\mu c^*_\mu \\ &= \left|U^*_{e_1}U_{\mu_1}e^{-i\phi_1} + U^*_{e_2}U_{\mu_2}e^{-i\phi_2} + U^*_{e_3}U_{\mu_3}e^{-i\phi_3}\right|^2. \end{aligned} \tag{E.20}$$

If the phases were all the same, i. e., $\phi_1 = \phi_2 = \phi_3$, the probability $P(\nu_e \to \nu_\mu) = 0$, which follows from equation (E.17), and, as before, neutrino flavour oscillations only occur if the neutrinos have mass and the masses are not all the same.

Using the identity

$$\begin{aligned} |z_1 + z_2 + z_3 +|^2 &= |z_1|^2 + |z_2|^2 + |z_3|^2 \left(z_1 z_2^* + z_1^* z_2\right) + \left(z_1 z_3^* + z_1^* z_3\right) + \left(z_2 z_3^* + z_2^* z_3\right) \\ &= |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\Re\mathrm{e}\left\{z_1 z_2^* + z_1 z_3^* + z_2 z_3^*\right\}, \end{aligned} \tag{E.21}$$

where $\Re\mathrm{e}\{z\}$ is the real part of z, we can write the probability as:

$$\begin{aligned} P(\nu_e \to \nu_\mu) = &\left|U^*_{e_1}U_{\mu_1}\right|^2 + \left|U^*_{e_2}U_{\mu_2}\right|^2 + \left|U^*_{e_3}U_{\mu_3}\right|^2 + \\ &+ 2\Re\mathrm{e}\Big\{ U^*_{e_1}U_{\mu_1}U_{e_2}U^*_{\mu_2}e^{-i(\phi_1-\phi_3)} + U^*_{e_1}U_{\mu_1}U_{e_3}U^*_{\mu_3}e^{-i(\phi_1-\phi_3)} + \\ &+ U^*_{e_2}U_{\mu_2}U_{e_3}U^*_{\mu_3}e^{-i(\phi_2-\phi_3)}\Big\} \end{aligned} \tag{E.22}$$

The unitarity condition of U imples that

$$\begin{aligned}
&\left|U^*_{e_1}U_{\mu_1}U^*_{e_1}U_{\mu_1}\right|^2 = 0 \\
\Rightarrow &\left|U^*_{e_1}U_{\mu_1}\right|^2 + \left|U^*_{e_2}U_{\mu_2}\right|^2 + \left|U^*_{e_3}U_{\mu_3}\right|^2 = -2\Re\mathrm{e}\Big\{ U^*_{e_1}U_{\mu_1}U_{e_2}U^*_{\mu_2} + U^*_{e_1}U_{\mu_1}U_{e_3}U^*_{\mu_3} + \\
&\qquad + U^*_{e_2}U_{\mu_2}U_{e_3}U^*_{\mu_3}\Big\} .
\end{aligned} \tag{E.23}$$

The probability can be expressed in the following form:

$$\begin{aligned}
P(\nu_e \to \nu_\mu) = 2\Re\mathrm{e}\Big\{ &U^*_{e_1}U_{\mu_1}U_{e_2}U^*_{\mu_2}\left[e^{-i(\phi_1-\phi_2)} - 1\right] + \\
&+ U^*_{e_1}U_{\mu_1}U_{e_3}U^*_{\mu_3}\left[e^{-i(\phi_1-\phi_3)} - 1\right] + U^*_{e_2}U_{\mu_2}U_{e_3}U^*_{\mu_3}\left[e^{-i(\phi_2-\phi_3)} - 1\right]\Big\} .
\end{aligned} \tag{E.24}$$

The probability of the electron neutrino survive can be obtained by the exchange $\mu \to e$ in equation (E.22) and using the unitarity relation (E.16), we can write:

$$\begin{aligned}
P(\nu_e \to \nu_e) = &\left|U_{e_1}\right|^4 + \left|U_{e_2}\right|^4 + \left|U_{e_3}\right|^4 + \\
&+ 2\Re\mathrm{e}\Big\{ \left|U_{e_1}\right|^2\left|U_{e_2}\right|^2 e^{-i(\phi_1-\phi_2)} + \left|U_{e_1}\right|^2\left|U_{e_3}\right|^2 e^{-i(\phi_1-\phi_3)} + \\
&+ \left|U_{e_2}\right|^2\left|U_{e_3}\right|^2 e^{-i(\phi_2-\phi_3)}\Big\} \\
&= 1 + 2\left|U_{e_1}\right|^2\left|U_{e_2}\right|^2 \Re\mathrm{e}\Big\{ \left[e^{i(\phi_2-\phi_1)} - 1\right]\Big\} + 2\left|U_{e_1}\right|^2\left|U_{e_3}\right|^2 \Re\mathrm{e}\Big\{ \left[e^{i(\phi_3-\phi_1)} - 1\right]\Big\} + \\
&+ 2\left|U_{e_2}\right|^2\left|U_{e_3}\right|^2 \Re\mathrm{e}\Big\{ \left[e^{i(\phi_3-\phi_2)} - 1\right]\Big\} .
\end{aligned} \tag{E.25}$$

This can be simplified using the following equation:

$$\begin{aligned}
\Re\mathrm{e}\Big\{ \left[e^{i(\phi_j-\phi_i)} - 1\right]\Big\} &= \cos\left(\phi_j - \phi_i\right) - 1 \\
&= -2\sin^2\Delta_{ji},
\end{aligned} \tag{E.26}$$

where $\Delta_{ji} \equiv \frac{\phi_j-\phi_i}{2} = \left(\frac{m_j^2-m_i^2}{4E_\nu}\right)L$. Hence we can written

$$\begin{aligned}
P(\nu_e \to \nu_e) = 1 &- 4\left|U_{e_1}\right|^2\left|U_{e_2}\right|^2 \sin^2\Delta_{21} - 4\left|U_{e_1}\right|^2\left|U_{e_3}\right|^2 \sin^2\Delta_{31} \\
&- 4\left|U_{e_2}\right|^2\left|U_{e_3}\right|^2 \sin^2\Delta_{32}.
\end{aligned} \tag{E.27}$$

Therefore, the electron probability survival depends on three differences of squared masses.

E.3 PMNS Matrix

In the case of three flavours, the neutrino mixing matrix U can be parametrized in terms of 3 mixing angles, one cp violating phase and, if neutrinos are Majorana particles, 2 additional "Majorana phases". In the most general case, i.e., in the case of n flavours, there are $\frac{n(n-1)}{2}$ angles and $\frac{(n-1)(n-2)}{2}$ "Dirac phases"and possibly $n-1$ additional Majorana phases, totaling $n(n-1)$ degree of freedons (dof). The most general $n \times n$ comples matrix depends on $2n^2$ real parameters. If a matrix is unitary, then there are n^2 dof, since in the diagonal there are n dof, and off diagonal there are $n(n-1)$. The n^2 remaining parameters can be divided into $\frac{n(n-1)}{2}$ angles and $\frac{n(n+1)}{2}$ phases. If neutrinos are a Dirac particle, $2n-1$ phases can be removed with a proper rephasing of the left handed fields. This is easy to see, considering the set of equations:

$$|\nu_e\rangle = U^*_{e_1}|\nu_1\rangle + U^*_{e_2}|\nu_2\rangle + U^*_{e_3}|\nu_3\rangle \tag{E.28}$$

$$|\nu_\mu\rangle = U^*_{\mu_1}|\nu_1\rangle + U^*_{\mu_2}|\nu_2\rangle + U^*_{\mu_3}|\nu_3\rangle \tag{E.29}$$

$$|\nu_\tau\rangle = U^*_{\tau_1}|\nu_1\rangle + U^*_{\tau_2}|\nu_2\rangle + U^*_{\tau_3}|\nu_3\rangle, \tag{E.30}$$

we can make real matrix element rephasing a flavor eigenstate. For example, the value $U_{e_1} = \left|U_{e_1}\right| e^{i\varphi_{e1}}$ can be made real positive redefining the electron neutrino state

$$|\nu_e\rangle \longrightarrow |\nu_e\rangle e^{-i\varphi_{e1}}$$
$$\Rightarrow |\nu_e\rangle = \left|U_{e_1}\right| |\nu_1\rangle + U^*_{e_2}|\nu_2\rangle + U^*_{e_3}|\nu_3\rangle,$$

and similarly an entire collumn can be made real redefining the three flavour eigenstates and eliminating 3, or in the general case n phases, from the U matrix. If the neutrino is a Dirac particle we are also allowed to rephase the mass eigenstates, for example

$$|\nu_2\rangle \longrightarrow e^{i\varphi_{e2}}|\nu_2\rangle$$
$$|\nu_3\rangle \longrightarrow e^{i\varphi_{e3}}|\nu_3\rangle.$$

Hence, in the 3 flavours case we can write

$$|\nu_e\rangle = \left|U_{e_1}\right| |\nu_1\rangle + \left|U_{e_2}\right| |\nu_2\rangle + \left|U_{e_3}\right| |\nu_3\rangle \tag{E.31}$$

$$|\nu_\mu\rangle = \left|U_{\mu_1}\right| |\nu_1\rangle + \left|U_{\mu_2}\right| e^{-i\left(\varphi_{\mu 2}-\varphi_{e2}\right)}|\nu_2\rangle + \left|U_{\mu_3}\right| e^{-i\left(\varphi_{\mu 3}-\varphi_{e3}\right)}|\nu_3\rangle \tag{E.32}$$

$$|\nu_\tau\rangle = \left|U_{\tau_1}\right| |\nu_1\rangle + \left|U_{\tau_2}\right| e^{-i(\varphi_{\tau 2}-\varphi_{e2})}|\nu_2\rangle + \left|U_{\tau_3}\right| e^{-i(\varphi_{\tau 3}-\varphi_{e3})}|\nu_3\rangle. \tag{E.33}$$

Therefore, we can eliminate two phases in the three flavours case and $n-1$ phases in the n flavours case. Finnaly, for Dirac neutrinos the rephasing eliminates $(2n-1)$ phases from the mixing matrix leaving a total of $N^{\text{Dirac}}_{\delta} = \frac{(n-1)(n-2)}{2}$ phases. For example, in the case of two flavours, there is no phase δ. For $n=3$ there is one phase, called phase of charge-parity (CP) violation δ_{CP}.

If neutrinos are Majorana particles, there is less freedom to rephase the fields, since one can not arbitrarily change the phase of the matter fields $|\nu_i\rangle, \quad i = 1,\ 2,\ \cdots, n.$. This can be understood by observing that the Majorana mass term in the Lagrangian is of the form $\nu^T C^{-1}\nu + h.c$ rather than $\overline{\nu}_R \nu_L + h.c$, hence not invariant under $U(1)$

transformations. Therefore, n phases can be eliminated from the mixture matrix and one remains with an additional$(n-1)$ Majorana phases.

In the case of three flavours, the mixing matrix can then be parametrized with three mixing angles and one CP violation phase, the additional Majorana phases have no influence on the flavor evolution. The most commonlly used parametrization has the form:

$$\begin{aligned} U &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{\mathrm{CP}}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{\mathrm{CP}}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\mathrm{CP}}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{\mathrm{CP}}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{\mathrm{CP}}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{\mathrm{CP}}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{\mathrm{CP}}} & c_{13}c_{23} \end{pmatrix}, \end{aligned} \tag{E.34}$$

where the mixing angles are denoted by $\theta_{12},\ \theta_{13},\ \theta_{23}$ and we have used the notation $c_{jk} \equiv \cos\theta_{jk}$ and $s_{jk} \equiv \sin\theta_{jk}$.

E.3.1 CP violation in neutrinos oscillations

The $V-A$ chiral structure of the weak interaction implies that parity is maximally violated. It also implies that charge-conjugation symmetry is maximally violated. This can be seen by considereding the $\pi^- \longrightarrow \mu^- + \bar{\nu}_\mu$ decay. Because neutrino masses are extremelly small compared to the energies involved, the antineutrino is effectively always emitted is a right-handed (RH) helicity state, i. e., the decay $\pi^- \longrightarrow \mu^- + \bar{\nu}_\mu$ produces an antineutrino RH and a left-hand (LH) muon. The parity operation (P) in this decay, will lead to a production of a LH antineutrino and a RH muon, the LH antineutrino is forbidden in the weak decay because of the chiral structure of the weak interaction. The charge-conjugation (C) in this decay will produce a RH neutrino and a LH antimuon, the RH neutrino is also forbidden in the weak interaction. But the CP combined will lead to an LH neutrino and a RH antimuon, this is a process allowed in the weak interaction.

It is known that CP violation is needed to account for the excess of matter over antimatter in the universe today. The Quantum Electrodynamics (QED) and Quantum Chrommodynamics (QCD) conserves C and P separetely and therefore conserve the CP. The only possible place in the Standard Model (SM) where CP-violating effects can occur is in the weak interaction.

Another astonishing thing is that all local Lorentz-invariant Quantum Field Theories can be shown to be invariant under the combined operation of C, P and T, this is called CPT-theorem, a formal proof of this theorem is given in [44]. Therefore, *CP*-violation implies the T-reversal symmetry is also violated.

If time reversal symmetry applies, then the oscilattion probability for $P(\nu_e \to \nu_\mu)$ will be equal to $P(\nu_\mu \to \nu_e)$. The oscillation probability $P(\nu_e \to \nu_\mu)$ is given by the equation (E.24), the elements of the PMNS matrix that appear in the expression for $P(\nu_\mu \to \nu_e)$ are the complex conjugates of those in $P(\nu_e \to \nu_\mu)$, since

$$\begin{aligned} |\psi(0)\rangle \equiv |\nu_\nu\rangle &= U^*_{\mu_1}|\nu_1\rangle + U^*_{\mu_2}|\nu_2\rangle + U^*_{\mu_3}|\nu_3\rangle \\ \Rightarrow |\psi(x,\ t)\rangle &= \left(U^*_{\mu_1}U_{e_1}e^{-i\phi_1} + U^*_{\mu_2}U_{e_2}e^{-i\phi_2} + U^*_{\mu_3}U_{e_3}e^{-i\phi_3} + \cdots\right), \qquad \text{hence} \\ \Rightarrow P(\nu_\mu \to \nu_e) &= \left|U^*_{\mu_1}U_{e_1}e^{-i\phi_1} + U^*_{\mu_2}U_{e_2}e^{-i\phi_2} + U^*_{\mu_3}U_{e_3}e^{-i\phi_3}\right|^2. \end{aligned} \tag{E.35}$$

Proceding as in the case of $P(\nu_e \to \nu_\mu)$, we find

$$P(\nu_\mu \to \nu_e) = 2\Re\mathrm{e}\left\{ U^*_{\mu_1}U_{e_1}U_{\mu_2}U^*_{e_2}\left[e^{i(\phi_2-\phi_1)} - 1\right] + \cdots\right\}. \tag{E.36}$$

By comparing the elements of the PMNS matrix in the expressions for $P(\nu_\mu \to \nu_e)$ and $P(\nu_e \to \nu_\mu)$, it becomes evident that they are complex conjugates of each other. Thus, if any of the elements U_{e_i} and U_{μ_j} are non-real, it implies that time reversal symmetry may not hold in neutrino oscillation. As we have previously discussed when examining the parametrization of the PMNS matrix and its degrees of freedom, the presence of three flavours introduces a complex phase, resulting in reverse time violation during neutrino oscillation. This violation ultimately implies CP violation.

Lastly, let us examine the combined CPT operation. When analyzing the Feynman diagram in a process similar to the one depicted in Figure E.2, the CP transformation yields the exchange $U \leftrightarrow U$, following the conventional Feynman rules. Similarly, the T operation results in the exchange $U \leftrightarrow U$. Consequently, neutrino oscillation aligns with the principles of the CPT theorem. Moreover, the imaginary components of the PMNS matrix offer a potential origin for CP violation within the Standard Model [1].