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WESLLEY GEREMIAS DOS SANTOS

BULK-EDGE CORRESPONDENCE IN SUPERSYMMETRIC CHERN-SIMONS THEORIES

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2020

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Dissertação apresentada ao Departamento de Física da Universidade Estadual de Londrina, como requisito parcial à obtenção do título de Mestre.
Orientador: Prof. Dr. Pedro Rogério Sérgi Gomes

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Resumo

Teorias topológicas de Chern-Simons em 2+1 dimensões espaço-temporais, possuem uma série de propriedades peculiares que, além de seu interesse intrínseco como uma teoria quântica dos campos, possuem desdobramentos importantes em física da matéria condensada e em matemática. Uma das características mais notáveis é a chamada holografia, em que os graus de liberdade físicos residem somente nas bordas da variedade sobre a qual a teoria é definida. Isso proporciona uma relação entre a física do interior e a física da borda, denominada correspondência bulk-edge. O objetivo principal é investigar a correspondência bulk-edge em teorias de Chern-Simons abeliana com vínculo de supersimetria $\mathcal{N} = 1$ e $\mathcal{N} = 2$.

Palavras-chave: Correspondência Bulk-Edge. Chern-Simons. Supersimetria.

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Abstract

The Chern-Simons theory has many peculiar properties. In addition to their intrinsic interest as a quantum field theory, it has important developments in condensed matter physics and mathematics. One of the most notable features is called holography, where physical degrees of freedom resides only at the edges of the manifold upon which the theory is defined. This provides a relation between interior physics and edge physics, called bulk-edge correspondence. The main goal of this work is to investigate the bulk-edge correspondence in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric Chern-Simons theories.

Keywords: Bulk-Edge Correspondence. Supersymmetry. Chern-Simons.

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1 Introduction

Quantum field theories in 2+1 space-time dimensions present peculiar properties that do not have a counterpart in higher dimensions. For example, besides bosons and fermions, there are also particles with arbitrary charge and statistic, called anyons. These particles appear as emergent particles in strongly coupled quantum matter systems. In general, such systems can not be distinguished by their symmetries, and so can not be described by the Landau-Ginzburg theory [1, 2]. Rather, this new kind of phase, known as topological phases, can be classified according to its topological properties, such as the ground state degeneracy and gapless edge excitations [3, 4, 5, 6, 7].

The characterization of topological phases was first approached from a microscopic point of view, focusing on the electron wavefunction [8]. Subsequent approaches relied on effective low-energy theories to describe their topological properties. The Abelian Quantum Hall states, for example, can be described by an effective Abelian Chern-Simons theory [4, 9]. Such theories have many interesting properties and, in addition to their role in topological phases of matter, arouse interest from a purely theoretical point of view [10, 11, 12, 13]. Among its main properties, one of the most striking is the so-called holography, which arises on bordered manifolds. In the presence of physical boundaries, the gauge symmetry is broken. So, when one restores the symmetry, chiral dynamical excitations emerge at the edge of the manifolds. This provides a relation between bulk and edge physics, called bulk-edge correspondence. Once different bulk structures lead to different structures of edge excitations, we can obtain information about bulk topological properties through edge excitations, providing thus a characterization of such phases [9].

The goal of this work is to study the relation between bulk and edge physics in Abelian Chern-Simons theories subjects to supersymmetry constraints. Since supersymmetry algebra involves space-time translations, on manifolds with physical boundaries both translational symmetry and supersymmetry in general are broken. To restore the supersymmetry, one adds edge contributions to compensates bulk variations [14, 15, 16]. In such cases, dynamical excitations also may arise at the boundary. Our goal is to study the bulk-edge correspondence in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric Abelian Chern-Simons theories. The advantage of this approach is that the edge states emerge more naturally than in the case of bulk-edge correspondence based on gauge invariance restoration. Thus, motivated by the context of gauge symmetry, we would like to investigate whether such theories can also be useful to describe topological phases of matter.

Moreover, there are several reasons to consider supersymmetric theories. One of the most important reason is the cancellation of divergences in loop corrections. In addition, supersymmetric models are generally easier to solve than non-supersymmetric ones since they are more constrained by the higher degree of symmetry. Thus, they can serve as toy models to find certain analytical results that may be useful for qualitative descriptions of realistic theories [17, 18, 19, 20].

The work is organized as follows. In Chapter 2 and 3 we will discuss the main properties of pure Chern-Simons theory, the couplings to matter fields, both in Abelian and non-Abelian cases, as well as the bulk-edge correspondence. In Chapter 4 we will discuss the basics of supersymmetry through the construction of some supersymmetric models. Finally, in Chapter 5 we will discuss $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry with boundaries, focusing on the edge theories that arise from the Abelian Chern-Simons theory. For simplicity, we will use $\hbar \equiv c \equiv 1$ and the Minkowski metric $\eta^{\mu\nu} = (+, -, -)$, exceptionally in the Chapter 5 we will use the metric $\eta^{\mu\nu} = (-, +, +)$. Latin indices $i, j, \cdots$ assigned to coordinates run only over spatial dimensions while Greek indices $\mu, \nu, ...$ run over both space and time dimensions.

2 Abelian Chern-Simons Theory

In this chapter, we will review some properties of pure Abelian Chern-Simons theory, such as the quantization of the level κ, the ground state degeneracy and the discrete symmetry operations. We will also introduce the basic features of anyons and topologically massive gauge theory. Finally, we will discuss the bulk-edge correspondence.

2.1 Discrete Symmetries

In 2+1 space-time dimensions, the existence of the Levi-Civita symbol $\epsilon^{\mu\nu\rho}$ allows us to construct a new theory that is local, and both Lorentz and gauge-invariant, the Chern-Simons theory,

$$S = \frac{\kappa}{4\pi}\int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \tag{2.1}$$

where the dimensionless coefficient κ is called the Chern-Simons level. Such theory breaks both parity $\mathcal{P}$ and time-reversal $\mathcal{T}$ symmetries. In 2+1 dimensions, the parity operation is defined as the reflection of only one spatial coordinate, while the time-reversal operation is defined in the usual way,

$$\mathcal{P}: \quad x^0 \to x^0,\; x^1 \to -x^1,\; x^2 \to x^2 \quad \text{and} \quad \mathcal{T}: \quad x^0 \to -x^0,\; \vec{x} \to \vec{x}.$$

The Maxwell equations must be invariant under such operations. This requirement is fulfilled if the gauge field components transform as

$$\mathcal{P}: \quad A_0 \to A_0,\; A_1 \to -A_1,\; A_2 \to A_2 \quad \text{and} \quad \mathcal{T}: \quad A_0 \to A_0,\; \vec{A} \to -\vec{A},$$

which means that the Chern-Simons theory is odd under both operations. These properties tell us that the Chern-Simons theory can describe physical systems whose parity and time reversal are broken.

2.2 Quantization of the level κ

At first, the Chern-Simons theory does not look gauge invariant, because it depends explicitly on A_μ. However, under a gauge transformation,

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \tag{2.2}$$

the action changes by a total derivative

$$S \to S + \frac{\kappa}{4\pi}\int d^3x \partial_\mu (\Lambda \epsilon^{\mu\nu\rho} \partial_\nu A_\rho). \tag{2.3}$$

Thus, by choosing appropriate boundary conditions, we can neglect the total derivative and the theory will be gauge invariant as long as Λ is single-valued. However, this restriction on Λ leads to inconsistencies in the theory. To see this, recall that there are two kinds of $U(1)$ gauge theory, called compact and non-compact theories. The non-compact theory is also called **R** gauge theory to highlight its non-compactness [1, 11].

In **R** gauge theories, the gauge transformations (2.2) are restrict to those whose Λ is a single-valued function. As a consequence, the electric charges are not quantized and there are no magnetic monopoles. However, it is an experimental fact that electric charges have only integer values. Moreover, Dirac showed that electrons, protons, and neutrons can be consistently coupled with magnetic monopoles since the physical fields are single-valued instead of the function Λ [6].

On the other hand, in $U(1)$ compact gauge theories, the gauge transformations are wider. Since the only gauge-invariant observable is the Wilson loop [1]

$$O_C = e^{-iq\oint_C dx^\mu A_\mu}, \tag{2.4}$$

any transformation that leaves O_C invariant is a gauge transformation. So, in addition to the gauge transformations allowed in the **R** gauge theory, there are also large gauge transformations. Such transformations have multi-valued Λ functions and cannot be smoothly deformed into the identity. In general, any multi-valued Λ that leaves

$$e^{iq\Lambda}, \tag{2.5}$$

single valued will generate valid gauge transformations. For example, a charged field ψ transform as

$$\psi \to e^{iq\Lambda}\psi. \tag{2.6}$$

To make the field single-valued, $e^{iq\Lambda}$ must be single-valued instead of Λ, as mentioned early. In such cases, the Chern-Simons partition function will be gauge invariant instead of the Chern-Simons action, as long as the level κ is an integer. Moreover, the $U(1)$ gauge theory has quantized electric charges and admit magnetic monopoles. So, a compact gauge theory is a good theory to describe the real world, while a non-compact is not. In the following, we will use these properties to proceed with the quantization of the Chern-Simons level.

To see the quantization of the level κ, consider a 2+1-dimensional manifold $S^2 \times S^1$, with the time circle of radius R parametrized by the coordinate $x_0 \in [0, 2\pi R)$, as described by the Figure 1. Through the S^2 we place a unit of Dirac magnetic flux, with $e \equiv 1$,

$$\frac{1}{2\pi}\int_{s^2} d^2x F_{12} = 1, \tag{2.7}$$

and we take a constant gauge field configuration $A_0 = a$. For this particular configuration, in the presence of the above background magnetic flux, the Chern-Simons action can be

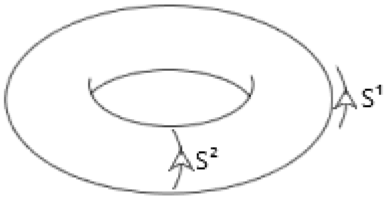

Figure 1 – **A** $2+1$ **dimensional manifold** $S^2 \times S^1$.

written as

$$S = 2\pi\kappa R a. \tag{2.8}$$

Actually this calculation is a little tricky [6, 11]. In the presence of the non-vanishing magnetic flux through S^2, the gauge field A_μ has a Dirac string singularity. In order to avoid ambiguities, it should be carried out in terms of gauge-invariant quantities, as follows

$$\begin{aligned}\frac{\partial S}{\partial a} &= \frac{\kappa}{4\pi}\int d^3x \epsilon^{\mu\nu\rho}\frac{\partial A_\mu}{\partial a}F_{\nu\rho}, \\ &= \frac{\kappa}{4\pi}\int d^3x \epsilon^{\mu\nu\rho}\delta_{\mu 0}F_{\nu\rho}, \\ &= \frac{\kappa}{2\pi}\int d^3x F_{12}, \\ &= 2\pi\kappa R. \end{aligned}\tag{2.9}$$

So, integrating on a and using the fact that $S = 0$ at $a = 0$, we get the result (2.8). The non-trivial background geometry of the Figure 1 allow us to make a large gauge transformation that winds around the time circle, with

$$\Lambda = \frac{x_0}{R}. \tag{2.10}$$

Note that due to the periodicity condition of the time coordinate, the gauge function is not single-valued, but was chosen to make $e^{i\Lambda}$ single-valued. Under this transformation the action (2.8) changes by

$$S \to S + 2\kappa\pi. \tag{2.11}$$

We see that the Chern-Simons action is not gauge-invariant. However in quantum theory, the relevant quantity is the Chern-Simons partition function, which depends only on

$$e^{iS}, \tag{2.12}$$

which is gauge-invariant provided $\kappa \in \mathbf{Z}$, as mentioned early. Although we have shown the quantization of the Chern-Simons level using a particular example, this result is general. In the section (3.1) we will show a more natural quantization condition.

2.3 Maxwell-Chern-Simons Theory

Consider firstly the Maxwell theory,

$$S = -\frac{1}{4e^2}\int d^3x F_{\mu\nu}F^{\mu\nu}, \tag{2.13}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and e^2 is a coupling constant with unity mass dimension. The equation of motion,

$$\partial_\mu F^{\mu\nu} = 0, \tag{2.14}$$

describes the propagation of a single massless degree of freedom, since only a single polarization is allowed. Now, let us see the consequences of adding to the Maxwell action a Chern-Simons term,

$$S = \int d^3x \left(-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho\right). \tag{2.15}$$

The resulting equation of motion,

$$\partial_\mu F^{\mu\nu} + \frac{\kappa e^2}{4\pi}\epsilon^{\nu\rho\sigma}F_{\rho\sigma} = 0, \tag{2.16}$$

no longer describes a massless excitation, but rather the propagation of a single physical degree of freedom with mass

$$m = \frac{\kappa e^2}{2\pi}. \tag{2.17}$$

The most direct way to see the origin of this mass is to rewrite the equation of motion as,

$$\left(\partial^\mu\partial_\mu + \left(\frac{\kappa e^2}{2\pi}\right)^2\right)\tilde{F}^\nu = 0, \tag{2.18}$$

where $\tilde{F}^\mu$ is the dual field defined by

$$\tilde{F}^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho. \tag{2.19}$$

From (2.18) we immediately identify the mass of the excitation. Another useful way to understand the origin of the massive excitations is to compute the propagator of the theory. Firstly, as the action is gauge-invariant, we have to extract out the redundant field configurations via the Faddeev-Popov procedure. However, in this case, the ghost fields decouple from the gauge field and can be integrated out. We will see that in the non-Abelian case this is not true. So, we can simply add a gauge-fixing term to the action,

$$S = \int d^3x \left(-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \frac{1}{2\xi e^2}(\partial^\mu A_\mu)^2\right). \tag{2.20}$$

To proceed with the calculations it is more convenient to rewrite the action as,

$$S = \int d^3x \frac{1}{2e^2}A_\mu\left(\Box\eta^{\mu\nu} - \partial^\mu\partial^\nu\left(1-\frac{1}{\xi}\right) - \frac{\kappa e^2}{2\pi}\epsilon^{\mu\nu\rho}\partial_\rho\right)A_\nu. \tag{2.21}$$

So, we define the propagator $\Delta_{\mu\nu}(x-y)$ of the gauge field as the inverse of the operator between the brackets,

$$\left(\Box\eta^{\mu\nu} - \partial^\mu\partial^\nu\left(1-\frac{1}{\xi}\right) - \frac{\kappa e^2}{2\pi}\epsilon^{\mu\nu\rho}\partial_\rho\right)\Delta_{\nu\sigma}(x-y) = i\delta^\mu_\sigma\delta^{(3)}(x-y). \tag{2.22}$$

or in the momentum space,

$$\left(-p^2\eta^{\mu\nu} + p^\mu p^\nu\left(1-\frac{1}{\xi}\right) + \frac{i\kappa e^2}{2\pi}\epsilon^{\mu\nu\rho}p_\rho\right)\Delta_{\nu\sigma}(p) = i\delta^\mu_\sigma. \tag{2.23}$$

To find the explicit form of the propagator, we write down the most general expression compatible with Lorentz invariance,

$$\Delta_{\nu\sigma}(p) = f(p^2)\eta_{\nu\sigma} + g(p^2)p_\nu p_\sigma + h(p^2)\epsilon_{\nu\sigma\lambda}p^\lambda. \tag{2.24}$$

Substituting this expression into (2.23), we find

$$\Delta_{\nu\sigma}(p) = e^2\left(\frac{p^2 p_\nu p_\sigma - p_\nu p_\sigma - i(\kappa e^2/2\pi)\epsilon_{\nu\sigma\lambda}p^\lambda}{p^2(p^2-(\kappa e^2/2\pi)^2)} - \xi\frac{p_\nu p_\sigma}{(p^2)^2}\right). \tag{2.25}$$

The propagator has one pole at $p^2 = \kappa e^2/2\pi$ and other at $p^2 = 0$. The first pole shows that the excitations are now massive, while the second can be viewed as the pole of Maxwell theory at high energies.

So, the effect of the Chern-Simons is to give mass to the gauge field. This gauge-invariant mechanism of mass generation is completely independent of the standard Higgs mechanism. In fact, we can also consider the Higgs mechanism in a Maxwell-Chern-Simons theory coupled to matter, including a symmetry breaking potential. In this case, we find two independent gauge field masses [10].

The physics described by the pure Chern-Simons theory can be understood by taking the limit $e^2 \to \infty$ with κ kept fixed in the above equations. From (2.18) we see that the photon becomes infinitely massive and no physical excitations are left. In other words, one project the Hilbert space onto the ground state by isolating it from the rest of the spectrum by an infinite gap.

Therefore, for certain systems that has no propagating degrees of freedom at low energies, the Chern-Simons is a good effective theory for describe its low-energy properties, such as the properties of the ground states.

2.4 Topological Theory

Another important property of the Chern-Simons is its topological nature. To see this, let us remember that given a theory defined in a flat space, we can write it in a curved space by replacing

$$\int d^Dx\,\mathcal{L}(\eta_{\mu\nu},\phi,\partial_\mu\phi,\cdots) \to \int d^Dx\,\sqrt{-g}\,\mathcal{L}(g_{\mu\nu},\phi,\partial_\mu\phi,\cdots), \tag{2.26}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, $g_{\mu\nu} = g_{\mu\nu}(x)$ is the space-time dependent metric and $g \equiv det\, g_{\mu\nu}$. Recall that we can construct the energy-momentum tensor of a theory defined in a flat space by deriving the action in the curved space in relation to the arbitrary metric,

$$T_{\mu\nu} \sim \frac{\delta}{\delta g^{\mu\nu}} \int d^D x \sqrt{-g}\, \mathcal{L}(g_{\mu\nu}, \phi, \partial_\mu \phi, \cdots)\Big|_{g_{\mu\nu}=\eta_{\mu\nu}} \tag{2.27}$$

and from $T^{\mu\nu}$ get the conserved quantities,

$$\mathcal{H} = \int d^{D-1}x\, T^{00} \qquad \text{and} \qquad P^i = \int d^{D-1}x\, T^{0i}. \tag{2.28}$$

Looking at the action (2.1), we see that the field indices are contracted with the Levi-Civita symbol, instead of the metric $\eta_{\mu\nu}$. Furthermore, the action is invariant under general coordinate transformations, since the Levi-Civita symbol is a density tensor whose transformation compensates the Jacobian of the integration measure, so switching to a curved space the factor $\sqrt{-g}$ is not necessary. Thus, the theory does not depend on the metric of the background space-time manifold, but only on its topology, as we shall see in the next section.

According to equation (2.27) the energy-momentum tensor of the Chern-Simons theory is null, in particular the Hamiltonian vanishes, which means that there are no degrees of freedom propagating in the bulk or equivalently all quantum states have zero energy, as we have seen in the previous section.

2.5 Ground State Degeneracy

As we have seen above, as a consequence of its topological nature, the Chern-Simons Hamiltonian is null. So, when we quantize the theory, the non-trivial question that arises concerns the degeneracy of the ground state. Since all quantum states have zero energy, we can calculate the degeneracy by counting the number of dimensions of the Hilbert space, which depends on the topology of the compactified two dimensional space [5]. Examples of spatial manifolds are in Figure 2.

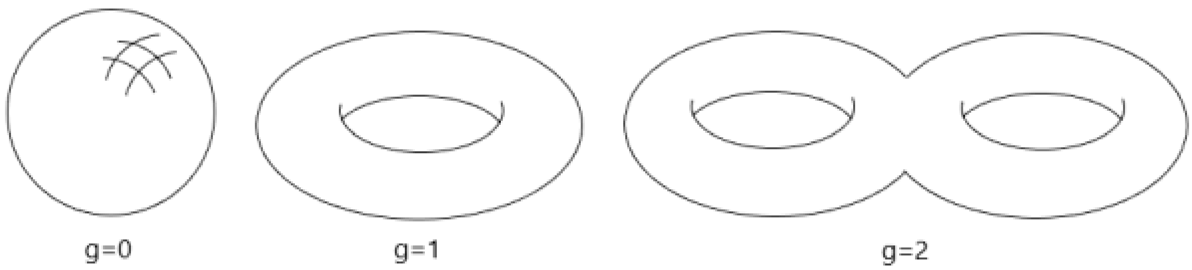

Figure 2 – **Examples of spatial manifolds, $\mathcal{M}^2$, characterized by the genus** g .

We will discuss two different approaches to address this question. For simplicity let us consider the system on a toroidal manifold with genus $g = 1$. In the end, we will generalize the result for a manifold of arbitrary genus g.

The first approach is to map the Chern-Simons action on the Landau problem. In the gauge $A_0 = 0$, the Chern-Simons action becomes,

$$S = -\frac{\kappa}{4\pi}\int d^3x\epsilon^{ij}A_i\dot{A}_j. \tag{2.29}$$

The equation of motion for A_0 should be imposed as a constraint

$$\epsilon^{ij}\partial_i A_j = 0. \tag{2.30}$$

On the torus, the gauge fields that satisfy this equation can be parameterized as

$$A_i = \partial_i\phi + A_i(t). \tag{2.31}$$

The term $A_i(t)$ is interpreted as collective excitations of the ground state. The torus can be parameterized as a rectangle $L_1 \times L_2$ with periodic boundary conditions $x_i \sim x_i + L_i$, as described in Figure 3.

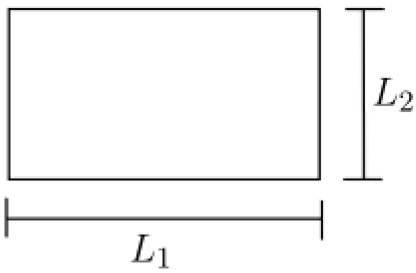

Figure 3 – **Parametrization of the torus in terms of a rectangle of sides** $L_1 \times L_2$.

So, inserting (2.31) into (2.29), we have

$$S = -L_1L_2\frac{\kappa}{4\pi}\int dt\epsilon^{ij}A_i\partial_0 A_j. \tag{2.32}$$

There is still a residual gauge symmetry that does not involve the time coordinate. Such transformation is called large gauge transformation and is given by

$$A_i \to A'_i = A_i - iU^{-1}\partial_i U, \tag{2.33}$$

with

$$U = e^{2\pi i\left(n_1\frac{x^1}{L_1}+n_2\frac{x^2}{L_2}\right)}, \tag{2.34}$$

where n_1 and n_2 are integers to make U single valued on the torus. Thus, the following gauge configurations

$$(A_1, A_2) \qquad \text{and} \qquad (A'_1, A'_2) = \left(A_1 + 2\pi\frac{n_1}{L_1}, A_2 + 2\pi\frac{n_2}{L_2}\right), \tag{2.35}$$

are equivalent. It is convenient introduce new coordinates

$$A_i(t) \equiv \frac{2\pi}{L_i}q_i(t), \tag{2.36}$$

such that the action can be rewritten as

$$S = -\kappa\pi \int dt\, \epsilon^{ij} q_i \dot{q}_j. \tag{2.37}$$

It is also convenient to add a kinetic term to the action to provide a non-trivial dynamics to the system,

$$S = \int dt \left(\frac{1}{2} m \dot{q}_i^2 - \kappa\pi\epsilon^{ij} q_i \dot{q}_j \right). \tag{2.38}$$

At the end we will take the limit $m \to 0$. Note that the action describes the dynamics of a charged particle moving in a torus parameterized by (q_1, q_2), coupled with a vector potential $A_i = -\epsilon^{ij} q_i \dot{q}_j$. The Hamiltonian associated to the action (2.38) is

$$H = \frac{1}{2m} \left[-(\partial_1 - iA_1)^2 - (\partial_2 - iA_2)^2 \right], \tag{2.39}$$

which is exactly the Hamiltonian of the Landau problem, whose ground state degeneracy is well known. The magnetic field can be expressed as

$$B = 2\kappa\pi, \tag{2.40}$$

and the degeneracy of the Landau problem is

$$\begin{aligned} \deg &= \frac{B}{2\pi} \times A, \\ &= \kappa. \end{aligned} \tag{2.41}$$

Note that the degeneracy does not depend on mass m, which means that this is really the ground state degeneracy of the Chern-Simons theory on a torus. All the excited states are proportional to $1/m$. Thus in the limit $m \to 0$, the energy gap between the levels becomes infinite, isolating the ground state from the rest of the spectrum.

The addition of the Maxwell term breaks the topological character of the theory. Nonetheless, it is possible to determine the ground state degeneracy without addition of a regulating term which breaks this character [4]. In the following, we present an alternative way to find the same result as above. Note that, integrating by parts, the action (2.37) may be rewritten as

$$S = \int dt (-2\pi\kappa q_1 \dot{q}_2). \tag{2.42}$$

If we regard q_2 as the position variable, its conjugate momentum is

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = -2\pi\kappa q_1. \tag{2.43}$$

The Hamiltonian is null,

$$H = p_2 \dot{q}_2 - L = 0, \tag{2.44}$$

and the Schrodinger equation reads

$$0 \cdot \psi = E\psi. \tag{2.45}$$

Despite the unusual form of the equation, it is possible to determine the wavefunction $\psi(q_2)$ by imposing some constraints. As the particle lives on a torus, which might be parameterized as a square 1×1 with periodic boundary condition $q_i \sim q_i + 1$, we must require that the wavefunction satisfies the same periodicity condition. For the coordinate q_2,

$$\psi(q_2) = \psi(q_2 + 1). \tag{2.46}$$

This requirement implies that

$$\psi(q_2) = \sum_{n=-\infty}^{\infty} c_n \, e^{i2\pi n q_2}, \tag{2.47}$$

where $n \in \mathbb{Z}$. To impose the periodicity condition in the coordinate q_1, recall that it is related to the canonical momentum p_2, so we have to take the Fourier transforms of the wavefunction $\psi(q_2)$,

$$\begin{aligned} \bar{\psi}(p_2) &= \int dq_2 \, \psi(q_2) e^{ip_2 q_2}, && (2.48) \\ &= \sum_{n=-\infty}^{\infty} c_n \int dq_2 \, e^{iq_2(p_2 + 2n\pi)}, && (2.49) \\ &= \sum_{n=-\infty}^{\infty} c_n \, \delta(p_2 + 2n\pi). && (2.50) \end{aligned}$$

Using the relation (2.43), we have

$$\psi(q_1) = \sum_{n=-\infty}^{\infty} c_n \, \delta(\kappa q_1 - n), \tag{2.51}$$

and imposing the periodicity condition

$$\psi(q_1) = \psi(q_1 + 1), \tag{2.52}$$

we see that

$$\sum_{n=-\infty}^{\infty} c_n \, \delta(\kappa q_1 - n) = \sum_{n=-\infty}^{\infty} c_n \, \delta(\kappa + \kappa q_1 - n). \tag{2.53}$$

With the redefinition of $n \to n + \kappa$, we are left with a relation between the coefficients

$$c_n = c_{n+\kappa}. \tag{2.54}$$

Which shows that exactly κ coefficients $c_\kappa' s$ are independent and consequently the degeneracy of the ground state is κ. For example, choosing $\kappa = 2$,

$$\begin{aligned} c_0 = c_2 = \ \cdots \ &= c_{2n}, \\ c_1 = c_3 = \ \cdots \ &= c_{2n+1}, \end{aligned}$$

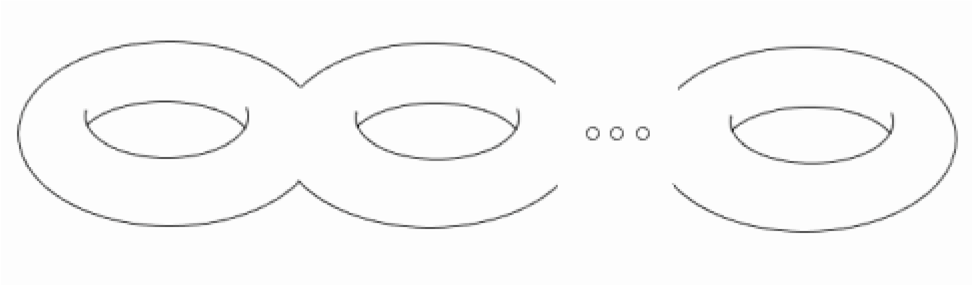

Figure 4 – **Manifold of arbitrary genus** g.

only two coefficients are independent, i.e., two quantum states have zero energy.

The generalization for a manifold of arbitrary genus g is straightforward. We can imagine that for each torus of Figure 4 there is a Chern-Simons term (2.42), such that the full action can be expressed as

$$S = -2\pi \sum_{i,j=1}^{g} \int d^3x \, K_{ij} q_i \dot{q}_j, \tag{2.55}$$

where K_{ij} is the following $g \times g$ diagonal matrix

$$K_{ij} = \begin{bmatrix} \kappa & & \\ & \ddots & \\ & & \kappa \end{bmatrix} \tag{2.56}$$

So, redoing the previous calculations for the action (2.55), the recurrence relation between the coefficients reads

$$c_{n_1,n_2,\cdots,n_g} = c_{n_1+\kappa,n_2+\kappa,\cdots,n_g+\kappa}. \tag{2.57}$$

For example, taking a manifold with genus $g = 3$ and $\kappa = 2$,

$$\begin{aligned}
c_{0,0,0} = c_{2,2,2} = \; & \cdots = c_{2n,2n,2n}, \\
c_{0,0,1} = c_{2,2,3} = \; & \cdots = c_{2n,2n,2n+1}, \\
c_{0,1,0} = c_{2,3,2} = \; & \cdots = c_{2n,2n+1,2n}, \\
c_{1,0,0} = c_{3,2,2} = \; & \cdots = c_{2n+1,2n,2n}, \\
c_{0,1,1} = c_{2,3,3} = \; & \cdots = c_{2n,2n+1,2n+1}, \\
c_{1,0,1} = c_{3,2,3} = \; & \cdots = c_{2n+1,2n,2n+1}, \\
c_{1,1,0} = c_{3,3,2} = \; & \cdots = c_{2n+1,2n+1,2n}, \\
c_{1,1,1} = c_{3,3,3} = \; & \cdots = c_{2n+1,2n+1,2n+1},
\end{aligned}$$

we see that eight coefficients are independent and, consequently, there are eight states with zero energy. Therefore, the generalization for a manifold of arbitrary genus g gives a ground state with degeneracy of

$$\deg = \kappa^g. \tag{2.58}$$

The dependence of the degeneracy with the genus of the manifold is a manifestation of the topological nature of the Chern-Simons theory.

2.6 Chern-Simons Coupled to Matter Fields

In 3+1 dimensional quantum field theories, particles may have either integer or half-integer spins, called bosons or fermions, respectively. On the other hand, in 2+1 dimensions, there are possibilities of particles to have arbitrary spin and statistics. Such particles are called anyons and are important to understand many problems in condensed matter physics [13].

The notion of spin and statistics are usually related to the sign that a many-body wavefunction acquires when any two identical particles are interchanged. Recall that the interchange of two particles can be performed by rotating one of them half-way around the other and then translating both by an appropriated distance. Under a rotation of $\Delta\phi$,

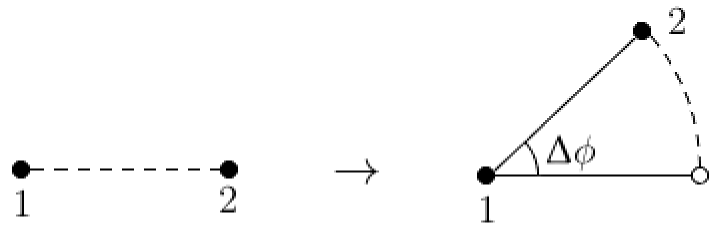

Figure 5 – **Particle 2 rotating around particle 1 by an angle** $\Delta\phi$.

naturally, the wave function must acquire a phase proportional to the rotating angle since the configurations described by Figure 5 differ only by this parameter and must describe the same physics. So,

$$e^{i\nu\Delta\phi}, \tag{2.59}$$

where ν is called statistics parameter. The interchange operation can be performed by a rotation of $\Delta\phi = \pi$ or $\Delta\phi = -\pi$, as showed in the Figures 6 and 7. In 3+1 dimensional case, both rotations are equivalent, since we can continuously deform one to each other. From the Figures 6 and 7 we can imagine that the deformation of the trajectories it is possible due to the extra dimension perpendicular to the plane. Thus,

$$e^{i\nu\pi} = e^{-i\nu\pi}. \tag{2.60}$$

As a consequence, the particles may have only bosonic ($\nu = 0, \mathrm{mod}\, 2$) or fermionic ($\nu = 1, \mathrm{mod}\, 2$) statistic. Actually, this result is true for $d+1$ with $d \geq 3$.

On the other hand, in 2+1 dimensions due to the hard-core interaction between the particles, the configuration space has a non-trivial topology and we cannot continuously

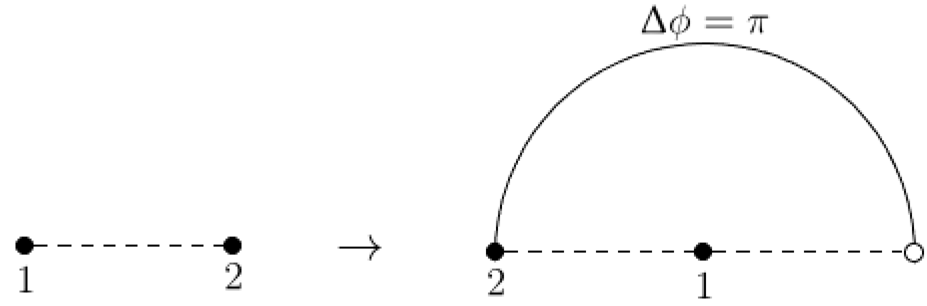

Figure 6 – **Particle 2 rotating counterclockwise around particle 1 by an angle $\Delta\phi = \pi$.**

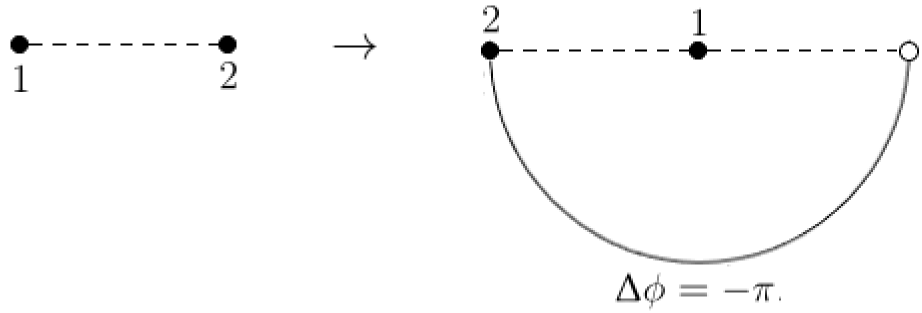

Figure 7 – **Particle 2 rotating clockwise around particle 1 by an angle $\Delta\phi = -\pi$.**

deform the trajectory of the particles. So, the above equality does not hold anymore

$$e^{i\nu\pi} \neq e^{-i\nu\pi}, \tag{2.61}$$

and the statistical parameter ν can be arbitrary. As mentioned previously, particles that have arbitrary statistic are called anyons and The Chern-Simons theory incorporates some anyonic properties. To see these features, consider a matter field coupled to the Chern-Simons gauge field through the conserved current J^μ,

$$S = \int d^3x \frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - A_\mu J^\mu. \tag{2.62}$$

For a collection of identical non-relativistic point particles of unit charge and mass m moving in a plane, the current J^μ is

$$J^0(x) = \sum_{a=1}^{N} \delta^2(\vec{x} - \vec{x}_a(t)) \quad \text{and} \quad J^i(x) = \sum_{a=1}^{N} \dot{x}_a^i(t) \delta^2(\vec{x} - \vec{x}_a(t)), \tag{2.63}$$

where $\vec{x}_a(t)$ describes the trajectory of the a-th particle. From the Chern-Simons equation of motion,

$$\frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho} = J^\mu, \tag{2.64}$$

the zero component of the field equation,

$$\frac{\kappa}{2\pi} F_{12} = \sum_{a=1}^{N} \delta^2(\vec{x} - \vec{x}_a(t)), \tag{2.65}$$

tells us that the Chern-Simons coupling has the effect of endowing the matter charged density a magnetic flux.

Consider a kinetic term for the charged particles along with the action (2.62)

$$S = \frac{m}{2}\sum_{a=1}^{N}\int dt\, \dot{\vec{x}}_a^2 + \int d^3x \frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - A_\mu J^\mu. \tag{2.66}$$

In the following, we may determine the potential A_μ from (2.64) by considering, for example, the Coulomb gauge $A_0 = 0$ and imposing the constraint $\nabla \cdot \vec{A} = 0$. So,

$$A_i(x) = \frac{1}{\kappa}\sum_{a=1}^{N}\epsilon^{ij}\frac{x^j - x_a^j(t)}{|\vec{x} - \vec{x}_a(t)|^2}. \tag{2.67}$$

The Hamiltonian of the system is

$$H = \frac{1}{2m}\sum_{a=1}^{N}[\vec{p}_a^2 - \vec{A}(\vec{x}_a)]^2, \tag{2.68}$$

where

$$A^i(\vec{x}_a) = \frac{1}{\kappa}\sum_{b\neq a}^{N}\epsilon^{ij}\frac{x_a^j - x_b^j(t)}{|\vec{x}_a - \vec{x}_b|^2}. \tag{2.69}$$

As a consequence of the hard-core interaction, the potential is a non-local function and the corresponding magnetic field,

$$B(\vec{x}_a) = \frac{2\pi}{\kappa}\sum_{b\neq a}^{N}\delta^2(\vec{x}_a - \vec{x}_b), \tag{2.70}$$

shows that each particle see the others $N-1$ as a point of flux

$$\Phi = \frac{2\pi}{\kappa}. \tag{2.71}$$

Therefore, the Chern-Simons gauge field plays the role of mediating the long-range interaction between the charged particles, whose effect is to attach flux for each of them, as mentioned early. Classically, the coupling plays no role, which means that the coupling effect is a purely quantum phenomena.

As a result of this flux attachment, when one such particle is transported adiabatically along a curve C around the other, due to the Aharonov-Bohm effect, the wavefunction picks up a phase

$$e^{i\oint_C \vec{A}\cdot d\vec{x}_1} = e^{i2\pi/\kappa}. \tag{2.72}$$

If the adiabatic transporting is interpreted as the exchange of the particles than we get half of this phase

$$e^{i\pi/\kappa}. \tag{2.73}$$

In addition to this phase, the wave function also acquires a phase due to the intrinsic quantum statistic of the particles. Thus, assuming that (2.66) describes bosonic or fermionic particles when we interchange two of them, the wavefunction will pick up a total phase of

$$\pm e^{i\pi/\kappa}. \tag{2.74}$$

Therefore, the effect of the Chern-Simons coupling is to transmute the statistic of the particles. In particular, if we take $\kappa = \pm 1$, bosons become fermions and vice versa. This process is called 3d bosonization [21, 22]. For $\kappa \neq \pm 1$, the particles are neither bosons nor fermions. Instead, they carry anyonic statistics.

2.7 Bulk-Edge Correspondence

As we have seen throughout the text, the Chern-Simons Hamiltonian is null and consequently, there are no degrees of freedom propagating in the bulk. Nonetheless, in a bordered manifold, dynamical degrees of freedom may arise on the edge [1, 2]. The goal of this section is to understand the physics of the edge and how it is related to the physics of the bulk.

According to section 2.2, the Chern-Simons action is not gauge-invariant if the manifold is bordered since its variation is a boundary term. In particular, consider the theory defined in the manifold depicted in the Figure 8.

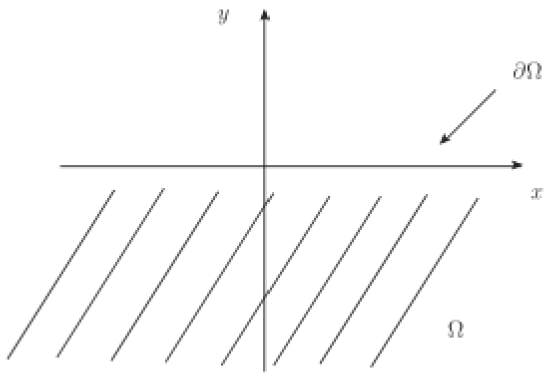

Figure 8 – **A manifold with boundary in the x-axis**.

Under a gauge transformation the action changes by

$$\delta S = \frac{\kappa}{4\pi}\int dtdx\, \Lambda(t,x,0)(\partial_0 A_1 - \partial_1 A_0)|_{y=0}. \tag{2.75}$$

We can guarantee that the theory remains gauge invariant, restricting the transformation such that

$$\Lambda(t,x,0) = 0. \tag{2.76}$$

Due to this restriction, the gauge transformation can not be used to eliminate degrees of freedom at the boundary. Consequently, dynamical edge excitations may arise. On the other hand, in the bulk, we can fix the gauge as long as the transformation respect the above condition.

To obtain an edge theory with non-trivial dynamic, i.e., which describes excitations that propagate with finite velocities, we should treat the velocity as an external parameter, since the bulk does not contain this information. The most convenient way to introduce dynamic to the edge excitations is through the gauge fixing. The most common gauge fixing condition is $A_0 = 0$, but this choice does not introduce dynamic to the edge excitations, so we can chose the following condition

$$A_0 = vA_1. \tag{2.77}$$

Furthermore, the equation of motion of A_0 must be impose as a constraint,

$$\epsilon^{0ij}\partial_i A_j = 0. \tag{2.78}$$

The equation is automatically satisfied choosing

$$A_j = \partial_j \phi. \tag{2.79}$$

Substituting (2.77) and (2.79) into the Chern-Simons action (2.1), we have

$$S_{edge} = \frac{\kappa}{4\pi}\int dtdx\, \phi(\partial_0 - v\partial_1)\partial_1\phi, \tag{2.80}$$

which is a chiral boson theory. The equation of motion,

$$(\partial_0 - v\partial_1)\partial_1\phi = 0, \tag{2.81}$$

describes an edge excitation propagating with velocity v only in one direction. If we had inserted $-v$ in (2.77), the excitation would propagate in the opposite direction.

An alternative and more elegant way to explore the correspondence between bulk and edge physics is modifying the theory including an edge contribution,

$$S_T = S_{CS} + S_{edge}, \tag{2.82}$$

such that, in the presence of the boundary, S_T be gauge invariant. It is possible if

$$\delta S_{edge} = -\delta S_{CS}. \tag{2.83}$$

Since the full action is gauge-invariant, we can explore all symmetries and arbitrariness to determine the form of the S_{edge} and study its properties independently of the bulk physics. As the variation of the Chern-Simons action under gauge transformation is given by (2.75), our first guess to the form of the edge action is

$$S_{edge} = -\frac{\kappa}{4\pi}\int dtdx\, \phi(\partial_0 A_1 - \partial_1 A_0)|_{y=0}, \tag{2.84}$$

where the scalar field ϕ lives at the boundary of the manifold and transform as

$$\phi(x,t) \to \phi(x,t) + \Lambda(t,x,0). \tag{2.85}$$

The S_{edge} contains three fields ϕ, A_0 and A_1, which due to the symmetries of the action not all are independent. The first symmetry is a gauge transformation on the edge,

$$A_0 \to A_0 + \partial_0\chi(x,t) \qquad \text{and} \qquad A_1 \to A_1 + \partial_1\chi(x,t). \tag{2.86}$$

We can use $\chi(t,x)$ to fix one of these fields. As in the first approach, to construct a theory with non-trivial dynamic on the edge, we have to add the velocity of the edge excitations via gauge transformation. Again, we choose

$$A_0 = vA_1. \tag{2.87}$$

So, the action assumes the form

$$S_{edge} = -\frac{\kappa}{4\pi}\int dtdx\, \phi(\partial_0 - v\partial_1)A_1. \tag{2.88}$$

There is also a residual gauge transformation that preserves the gauge choice (2.87),

$$A_0 \to A_0 + \partial_0\xi(x,t) \qquad \text{and} \qquad A_1 \to A_1 + \partial_1\xi(x,t), \tag{2.89}$$

where the function $\xi = \xi(x+vt)$. On the other hand, the field equations

$$(\partial_0 - v\partial_1)A_1 = 0 \qquad \text{and} \qquad (\partial_0 - v\partial_1)\phi = 0, \tag{2.90}$$

also requires that $A_1 = A_1(x+vt)$ and $\phi = \phi(x+vt)$. So, we can use $\xi(x,t)$ to fix one of the remaining fields. For example, the choice

$$A_1 = \partial_1\phi, \tag{2.91}$$

results in the action

$$S_{edge} = -\frac{\kappa}{4\pi}\int dtdx\, \phi(\partial_0 - v\partial_1)\partial_1\phi, \tag{2.92}$$

that also describes a chiral boson. This behavior is compatible with the Chern-Simons action since it breaks both parity and time-reversal symmetries. The boundary action obtained from the gauge-invariance restoration has only 1 gapless bosonic chiral excitation, i.e.,

$$c_{R/L} = 1. \tag{2.93}$$

This edge structure appears in a class of topological phases called Abelian fractional quantum Hall phases [9].

3 Non-Abelian Chern-Simons Theory

So far, we have studied gauge theories described by the Abelian gauge group $U(1)$. From now on, we will generalize to gauge theories based on non-Abelian gauge groups G, more specifically $G = SU(N)$. As in the previous chapter, we will discuss the quantization of the level κ, the basic dynamical features of the non-Abelian topologically massive gauge theory and the edge theory, which in this case is given in term of the WZW model.

3.1 Quantization of the level κ

The first gauge theory we can construct is a non-Abelian version of the Chern-Simons theory (2.1),

$$S = \frac{\kappa}{4\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \, tr \left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right), \tag{3.1}$$

where the Chern-Simons level κ is a dimensionless constant and the gauge fields A_μ are promoted to non-commutating objects that take values in the underlying Lie algebra $\mathcal{G}$,

$$A_\mu = A_\mu^a T^a, \tag{3.2}$$

whose generators T^a are assumed to be Hermitian and so satisfy the commutation relation

$$[T^a, T^b] = i f^{abc} T^c, \tag{3.3}$$

where f^{abc} are the fully anti-symmetric structure constants. In the fundamental representation, the generators are normalized as

$$tr(T^a T^b) = \frac{1}{2} \delta^{ab}, \tag{3.4}$$

with $a, b, c = 1, ...dim(\mathcal{G})$. Under a local gauge transformation by a group element Ω, the gauge field A_μ transform as

$$A_\mu \to \Omega^{-1} A_\mu \Omega + i \Omega^{-1} \partial_\mu \Omega, \tag{3.5}$$

where $\Omega(x) \in SU(N)$. It will also be useful to define an infinitesimal gauge transformation, which to leading order can be written as,

$$\Omega(x) \approx 1 + i\omega^a(x) T^a + \cdots . \tag{3.6}$$

This corresponds to an infinitesimal change in the gauge field,

$$\delta A_\mu^a = (\mathcal{D}_\mu \omega)^a, \tag{3.7}$$

where $\mathcal{D}_\mu{}^a{}_c = \partial_\mu \delta^a{}_c + f^{abc} A^b_\mu$ is the covariant derivative. The non-Abelian equations of motion have the same form as the Abelian one

$$F_{\mu\nu} = 0. \tag{3.8}$$

The solutions are pure gauges $A_\mu = i\Omega^{-1}\partial_\mu\Omega$. However, the equation may have interesting solutions if we take a non-trivial background geometry, as we have seen in the Abelian case where we consider the theory on a torus.

Here, the Chern-Simons level κ also must be an integer. Indeed, under the gauge transformation (3.5), the action change by

$$S \to S + \frac{\kappa}{4\pi}\int d^3x\, \epsilon^{\mu\nu\rho}\left[\partial_\nu tr(\partial_\mu\Omega\,\Omega^{-1}A_\rho) + \frac{1}{3}tr(\Omega^{-1}\partial_\mu\Omega\,\Omega^{-1}\partial_\nu\Omega\,\Omega^{-1}\partial_\rho\Omega)\right]. \tag{3.9}$$

The first term is a total derivative, which vanishes by choosing appropriate boundary conditions. If we take the background geometry to be a compactified manifold S^3, the gauge transformations are characterized by the homotopy group $\Pi_3(SU(N)) \cong \mathbb{Z}$ and the last term can be interpreted as the winding number of the gauge transformation around the space-time [11, 23],

$$\omega(\Omega) = \frac{1}{24\pi^2}\int d^3x\, \epsilon^{\mu\nu\rho} tr(\Omega^{-1}\partial_\mu\Omega\,\Omega^{-1}\partial_\nu\Omega\,\Omega^{-1}\partial_\rho\Omega) \quad \in \mathbb{Z}. \tag{3.10}$$

Thus, the non-Abelian Chern-Simons action changes by an additive term under a gauge transformation

$$S \to S + 2\pi\kappa\omega(\Omega). \tag{3.11}$$

As we have seen earlier, if we look at the weight of the partition function e^{iS}, it will be gauge invariant provided that

$$\kappa \in \mathbb{Z}. \tag{3.12}$$

As in the Abelian case, at the quantum level, the theory is gauge invariant. Although this requirement arises more directly for the non-Abelian theory, we can imagine that the Abelian theory is a subgroup of the non-Abelian. Thus the quantization condition found for the non-Abelian case is also valid for the Abelian one.

3.2 Yang-Mills-Chern-Simons Theory

Dynamical content can be attributed to the Chern-Simons theory by constructing the non-Abelian version of the Maxwell theory (2.13). From the gauge field, we can construct the Lie-algebra valued field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \tag{3.13}$$

that under the gauge transformation (3.5) changes by

$$F_{\mu\nu} \rightarrow \Omega^{-1} F_{\mu\nu} \Omega. \tag{3.14}$$

Although the field strength is not gauge-invariant, due to its simple transformation law, it is possible to construct a gauge invariant theory from $F_{\mu\nu}$ using the properties of the trace. This theory is called Yang-Mills,

$$S_{YM} = -\frac{1}{2e^2} \int d^3x \, tr(F_{\mu\nu} F^{\mu\nu}), \tag{3.15}$$

where e^2 is the Yang-Mills coupling. As in the Maxwell case, the theory has nontrivial dynamical content and propagating degrees of freedom, described by the equation of motion

$$\mathcal{D}_\mu F^{\mu\nu} = 0. \tag{3.16}$$

Note that the non-Abelian equation of motion differs from the Abelian one (2.14) by commutator terms, both inside the covariant derivative and the field strength. The non-linear character of the non-Abelian equation means that the Yang-Mills theory is self-interacting.

As we did in section (2.3), we can combine the two non-Abelian gauge actions into a single one,

$$S = \int d^3x \left[-\frac{1}{4e^2} \, F^a_{\mu\nu} F^{a\mu\nu} + \frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} \left(\frac{1}{2} A^a_\mu \partial_\nu A^a_\rho + \frac{1}{6} f^{abc} A^a_\mu A^b_\nu A^c_\rho \right) \right]. \tag{3.17}$$

The resulting theory is called Yang-Mills-Chern-Simons theory or topologically massive Yang-Mills theory. As in the Abelian case, the presence of the Chern-Simons term results in a new form of mass generation for the gauge field. To see the origin of the mass, let us calculate the propagator of the theory. Firstly, as the action is gauge invariant, we have to fix the gauge via the Faddeev-Popov procedure,

$$\begin{aligned} S &= \int d^3x \left[-\frac{1}{4e^2} \, F^a_{\mu\nu} F^{a\mu\nu} + \frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} \left(\frac{1}{2} A^a_\mu \partial_\nu A^a_\rho + \frac{1}{6} f^{abc} A^a_\mu A^b_\nu A^c_\rho \right) \right. \\ &- \left. \frac{1}{2\xi e^2} (F^a[A_\mu(x)])^2 + \int d^3y \, \bar{C}^a(x) \left. \frac{\delta F^a[A^\theta_\mu(x)]}{\delta \theta^b(y)} \right|_{\theta=0} C^b(y) \right]. \end{aligned} \tag{3.18}$$

Choosing the covariant gauge $F^a[A_\mu(x)] \equiv \partial^\mu A^a_\mu$, we have

$$\begin{aligned} S &= \int d^3x \left[-\frac{1}{4e^2} \, F^a_{\mu\nu} F^{a\mu\nu} + \frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} \left(\frac{1}{2} A^a_\mu \partial_\nu A^a_\rho + \frac{1}{6} f^{abc} A^a_\mu A^b_\nu A^c_\rho \right) \right. \\ &- \left. \frac{1}{2\xi e^2} (\partial^\mu A^a_\mu)^2 - \partial^\mu \bar{C}^a (\mathcal{D}_\mu C)^a \right], \end{aligned} \tag{3.19}$$

where $\bar{C}^a$ and C^b are the Faddeev-Popov ghosts fields. Note that due to the covariant derivative, the ghost fields interact with the gauge field and cannot be ignored in quantum processes. To see the massive pole is enough to investigate the quadratic part of the gauge-fixed action,

$$S_{\Box} = \int d^3x \, \frac{1}{2e^2} A^a_\mu \left[\Box\eta^{\mu\nu} - \partial^\mu\partial^\nu(1 - \frac{1}{\xi}) - \frac{\kappa e^2}{4\pi}\epsilon^{\mu\nu\rho}\partial_\rho \right] A^a_\nu. \tag{3.20}$$

Note that this action is the same as the Abelian one (2.21), with a slight difference of a factor of 2 in the last term due to the normalization of the trace. So, as in the Abelian case, the non-Abelian Chern-Simons term gives a mass to the gauge field.

3.3 Wess-Zumino-Witten model

As in the Abelian case, the non-Abelian Chern-Simons action does not depend on the metric of the space-time background manifold. So, no propagating degrees of freedom arises in the bulk. Nonetheless, on a manifold with a boundary, dynamical degrees of freedom may arise on the edge.

Let us consider the non-Abelian Chern-Simons action (3.1) written in components

$$S = \frac{\kappa}{8\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \left(A^a_\mu \partial_\nu A^a_\rho + \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho \right). \tag{3.21}$$

Under the infinitesimal gauge transformation (3.7) the action changes by a total derivative

$$\delta S = \frac{\kappa}{8\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \partial_\mu (\omega^a \partial_\nu A^a_\rho). \tag{3.22}$$

This result can also be obtained by substituting the infinitesimal gauge transformation (3.6) in the expression (3.9). So, considering a manifold as in Figure 8, the boundary term reads

$$\delta S = \frac{\kappa}{8\pi} \int dxdt \, \omega^a (\partial_0 A^a_1 - \partial_1 A^a_0)|_{y=0}. \tag{3.23}$$

Note that this term has exactly the same form as (2.75). So, to ensure that the theory remains gauge invariant, we must restrict the transformations as

$$\omega^a(t, x, 0) = 0. \tag{3.24}$$

Therefore, dynamical edge excitation can emerge, since we can not use the gauge transformation to gauge away degrees of freedom at the boundary. As in the Abelian case, to obtain a dynamical edge theory, we must insert the velocity of the edge excitation through the fixing condition

$$A_0 = vA_1, \tag{3.25}$$

and impose the equation of motion of A_0 as a constraint

$$F_{ij} = 0. \tag{3.26}$$

As we have seen, the solutions of the equation of motion are pure gauges

$$A_i = i\Omega^{-1}\partial_i\Omega. \tag{3.27}$$

Substituting (3.25) and (3.27) into the non-Abelian Chern-Simons action (3.1), we have

$$\begin{aligned} S &= \frac{\kappa}{4\pi}\int dxdt\,\mathrm{Tr}(\Omega^{-1}(\partial_0 - v\partial_1)\partial_1\Omega) \\ &- \frac{\kappa}{12\pi}\int d^3x\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}(\Omega^{-1}\partial_\mu\Omega\,\Omega^{-1}\partial_\nu\Omega\,\Omega^{-1}\partial_\rho\Omega). \end{aligned} \tag{3.28}$$

This theory is known as Wess-Zumino-Witten theory and describes propagating edge excitations as stated earlier [24, 25]. When we consider $G = U(1)$, the above action reduces to the chiral boson theory (2.80).

4 Supersymmetry

In this chapter, we will present some basic elements of $3D$ minimal supersymmetry ($\mathcal{N} = 1$) in the x-space. We will show how to construct the supersymmetric version of some models and discuss some of its main properties. A detailed treatment can be found in [26].

4.1 General Properties

Consider an action containing a bosonic field $b(x)$ and a fermionic field $f(x)$. Suppose one has an operation that transforms $b(x)$ into $f(x)$ and vice-versa,

$$\delta b(x) \sim f(x)\epsilon \qquad \text{and} \qquad \delta f(x) \sim b(x)\epsilon, \tag{4.1}$$

where ϵ is an anticommuting constant parameter. Our starting point to construct supersymmetry transformations lies in the dimensional analysis of the involved objects. In 2+1 space-time dimensions, the fields mass dimension is,

$$[b(x)] = 1/2 \qquad \text{and} \qquad [f(x)] = 1. \tag{4.2}$$

From (4.1), we see that there is no value for the mass dimension of ϵ that satisfies both relations simultaneously. So, if we choose

$$[\epsilon] \equiv -1/2, \tag{4.3}$$

the first relation holds and we can introduce a derivative in the second one to balance the unit mass dimension gap, i.e.,

$$\delta b(x) \sim f(x)\epsilon \qquad \text{and} \qquad \delta f(x) \sim \partial b(x)\epsilon. \tag{4.4}$$

If the theory has masses and dimensionful coupling constants, these objects can also be used to balance them, as we will see later in this chapter.

One important feature is the matching of bosonic and fermionic degrees of freedom in supersymmetric theories. We can choose the on-shell or off-shell scheme to see this. In the first one, we evoke the equations of motion, while in the other one, we introduce auxiliary degrees of freedom through non-dynamical fields. In the off-shell scheme, the auxiliary fields also enter the supersymmetry transformations. Usually, they appear in the action as F^2, so the transformations become

$$\delta b(x) \sim f(x)\epsilon, \qquad \delta f(x) \sim \partial b(x)\epsilon + F(x)\epsilon \qquad \text{and} \qquad \delta F(x) \sim \epsilon\partial f(x) \tag{4.5}$$

The anticommuting nature of ϵ implies that the commutator of two supersymmetries is non-trivial. In the on-shell scheme, it leads to rigid space-time translations,

$$\delta_1(\delta_2 b(x)) \sim \epsilon_1\epsilon_2\partial b(x) \quad \text{and} \quad \delta_1(\delta_2 f(x)) \sim \epsilon_1\epsilon_2\partial f(x). \tag{4.6}$$

Finally, supersymmetry transformations must always be compatible with Lorentz invariance. With all these ingredients, we are able to construct supersymmetry theories and explore its properties.

4.1.1 Majorana Spinors and Gamma Matrices in 2+1 Dimensions

In what follows, let us discuss some properties of spinors and gamma matrices in 2+1 space-time dimensions. The gamma matrices are 2×2 matrices that satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{4.7}$$

and in the real representation,

$$(\gamma^0)^\alpha{}_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (\gamma^1)^\alpha{}_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\gamma^2)^\alpha{}_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.8}$$

act on two-component Majorana spinors ψ^α, $\alpha = 1, 2$. Spinor indices are raised, lowered and contracted by the antisymmetric matrix

$$C_{\alpha\beta} = -C_{\beta\alpha} = -C^{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad C_{\alpha\beta}C^{\gamma\delta} = \delta^\gamma_{[\alpha}\delta^\delta_{\beta]}, \tag{4.9}$$

accordingly to the north-western convention,

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \psi^\beta C_{\beta\alpha}, \tag{4.10}$$

and

$$\psi\chi = \psi^\alpha\chi_\alpha = \chi^\alpha\psi_\alpha = \chi\psi \quad \text{and} \quad \psi^2 = \frac{1}{2}\psi^\alpha\psi_\alpha. \tag{4.11}$$

In 2+1 space-time dimensions, the parity operation is defined as the reflection of only one spatial coordinate, while the time-reversal operation corresponds to change only the time sign,

$$\mathcal{P}: \quad x^0 \to x^0,\ x^1 \to -x^1,\ x^2 \to x^2 \quad \text{and} \quad \mathcal{T}: \quad x^0 \to -x^0,\ \vec{x} \to \vec{x}.$$

Under x^1 parity operations, Majorana spinors transforms as

$$\mathcal{P}: \quad \psi_\alpha(x) \to \psi^{\mathcal{P}}_\alpha(x) = S^\beta{}_\alpha\psi_\beta(x^0, -x^1, x^2), \tag{4.12}$$

where S^β_α is a unitary matrix that mix up the different components of the spinor. To determine S, we have to assume that both $\psi_\alpha(x)$ and $\psi^{\mathcal{P}}_\alpha(x)$ are solutions of the massless Dirac equation. Firstly, by changing $x^1 \to -x^1$, the equation reads

$$(i\gamma^0\partial_0 - i\gamma^1\partial_1 + i\gamma^2\partial_2)\psi(x^0, -x^1, x^2) = 0. \tag{4.13}$$

On the other hand, $\psi^{\mathcal{P}}$ is solution of the Dirac equation. So,

$$\begin{aligned} i\gamma^\mu\partial_\mu\psi^{\mathcal{P}}(x) &= 0, \\ (i\gamma^0\partial_0 + i\gamma^1\partial_1 + i\gamma^2\partial_2)S\psi(x^0, -x^1, x^2) &= 0, \\ -(i\gamma^0\partial_0 + i\gamma^1\partial_1 + i\gamma^2\partial_2)S\psi(x^0, -x^1, x^2) &= 0. \end{aligned} \tag{4.14}$$

Comparing the two equations, we find

$$S^\beta{}_\alpha = (\gamma^1)^\beta{}_\alpha. \tag{4.15}$$

So, the Majorana spinor parity transformation is

$$\mathcal{P}: \qquad \psi_\alpha(x) \to \psi^{\mathcal{P}}_\alpha(x) = (\gamma^1)^\beta{}_\alpha\psi_\beta(x^0, -x^1, x^2). \tag{4.16}$$

Similarly, for x^2 parity operations, Majorana spinor transforms as

$$\mathcal{P}: \qquad \psi_\alpha(x) \to \psi^{\mathcal{P}}_\alpha(x) = (\gamma^2)^\beta{}_\alpha\psi_\beta(x^0, x^1, -x^2). \tag{4.17}$$

Under time-reversal operation, Majorana spinors transforms as

$$\mathcal{T}: \qquad \psi_\alpha(x) \to \psi^{\mathcal{T}}_\alpha(x) = T^\beta{}_\alpha\psi_\beta(-x^0, \vec{x}), \tag{4.18}$$

where T^β_α is also a unitary matrix that mix up the different components of the spinor. Again, to determine T, we will assume that both $\psi_\alpha(x)$ and $\psi^{\mathcal{T}}_\alpha(x)$ are solutions of the massless Dirac equation. Firstly, by changing $x^0 \to -x^0$ and taking the complex conjugation, the equation changes by

$$(i\gamma^0\partial_0 - i\gamma^i\partial_i)\psi(-x^0, \vec{x}) = 0. \tag{4.19}$$

The transformed spinor $\psi^{\mathcal{T}}$ should also be solution of the Dirac equation

$$\begin{aligned} i\gamma^\mu\partial_\mu\psi^{\mathcal{T}}(x) &= 0, \\ (i\gamma^0\partial_0 + i\gamma^i\partial_i)T\psi(-x^0, \vec{x}) &= 0 \end{aligned} \tag{4.20}$$

So, by comparing the two equations, we find

$$T^\beta{}_\alpha = (\gamma^0)^\beta{}_\alpha. \tag{4.21}$$

Thus,

$$\mathcal{T}: \qquad \psi_\alpha(x) \to \psi^{\mathcal{T}}_\alpha(x) = (\gamma^0)^\beta{}_\alpha\psi_\beta(-x^0, \vec{x}). \tag{4.22}$$

4.2 Wess-Zumino model

Starting with a real scalar field $\phi(x)$, a simple non-trivial model we can construct without introducing masses or interaction terms, is

$$S = \int d^3x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right). \tag{4.23}$$

This theory describes dynamical massless spin-0 particles. So, according to the previous discussion, its supersymmetric extension must also describe dynamical massless spin-1/2 particles. To fulfill this requirement, we add a Majorana field $\psi_\alpha(x)$ to the action,

$$S = \int d^3x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\psi\gamma^\mu\partial_\mu\psi\right). \tag{4.24}$$

This theory is called the Wess-Zumino model. In what follows, note that the action has one bosonic and two fermionic degrees of freedom, since the Majorana field has two components. To match them, as required by supersymmetry, we may use the on-shell or off-shell scheme.

In the first one, we evoke the fermionic equations of motion to constraint its degrees of freedom and reduce them from two to one. In this case, the most general supersymmetry transformation that can be constructed is

$$\begin{aligned} \delta\phi &= \epsilon\psi, \\ \delta\psi_\alpha &= \alpha(\gamma^\mu\epsilon)_\alpha\partial_\mu\phi, \end{aligned} \tag{4.25}$$

where α is a real coefficient to be determined. Under this transformation, the action (4.24) changes by

$$\delta S = \int d^3x(1-\alpha)\,\epsilon\partial_\mu\psi\partial^\mu\phi, \tag{4.26}$$

thus, the invariance requirement fixes $\alpha = 1$. In the off-shell scheme, we add an extra bosonic degree of freedom via a non-dynamical auxiliary field $F(x)$ of mass dimension $[F] = 3/2$,

$$S = \int d^3x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\psi\gamma^\mu\partial_\mu\psi + \frac{1}{2}F^2\right). \tag{4.27}$$

In this case, the new dimensionful object also enters the supersymmetry transformation,

$$\begin{aligned} \delta\phi &= \epsilon\psi, \\ \delta\psi_\alpha &= \beta(\gamma^\mu\epsilon)_\alpha\partial_\mu\phi + \gamma F\epsilon_\alpha, \\ \delta F &= \epsilon\gamma^\mu\partial_\mu\psi, \end{aligned} \tag{4.28}$$

where β and γ are real coefficients to be determined. The variation of the action (4.27) under the above supersymmetry transformation,

$$\delta S = \int d^3x\,(1-\beta)\,\epsilon\partial_\mu\psi\partial^\mu\phi + (1-\gamma)\,\epsilon(\gamma^\mu\partial_\mu\psi)F, \tag{4.29}$$

will vanish if $\beta = \gamma = 1$. An important property of supersymmetry transformations follows from the commutator of two supersymmetries. For both supersymmetry transformations (4.25) and (4.28), the commutator of two supersymmetries acting on the scalar field $\phi(x)$, for example,

$$[\delta_1, \delta_2]\phi = 2(\epsilon_2\gamma^\mu\epsilon_1)\partial_\mu\phi, \tag{4.30}$$

leads to a rigid field translation of parameter $2(\epsilon_2\gamma^\mu\epsilon_1)$. Is straightforward to check, that the same result hold for $\psi_\alpha(x)$ and $F(x)$.

4.3 Supersymmetric Abelian Chern-Simons Theory

The next step is to construct the supersymmetric version of the Abelian Chern-Simons theory (2.1),

$$S = \frac{\kappa}{4\pi}\int d^3x \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho. \tag{4.31}$$

As discussed along the text, the theory does not carry dynamical content, its description is restricted to the topological properties of the system. Therefore, its supersymmetric extension must contain a non-dynamical Majorana field $\lambda_\alpha(x)$. Thus,

$$S = \frac{\kappa}{4\pi}\int d^3x \left(\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \lambda\lambda\right). \tag{4.32}$$

In this case, the bosonic degrees of freedom matches the fermionic ones, since we can gauge away one bosonic field component. So, no auxiliary fields are needed and the most general supersymmetry transformation reads

$$\begin{aligned} \delta A_\mu &= \epsilon\gamma_\mu\lambda, \\ \delta\lambda_\alpha &= \alpha\epsilon_\alpha\partial_\mu A^\mu + \beta(\gamma^\mu\gamma^\nu\epsilon)_\alpha F_{\mu\nu}, \end{aligned} \tag{4.33}$$

where the coefficients α and β are to be determined. Under this transformation, the action (4.32) changes by

$$\delta S = \frac{\kappa}{4\pi}\int d^3x\, 2\left((1+2\beta)\epsilon^{\mu\nu\rho}\epsilon\gamma_\mu\lambda\partial_\nu A_\rho - \alpha\epsilon\lambda\partial_\mu A^\mu\right). \tag{4.34}$$

The requirement of supersymmetry invariance fix the coefficients to be $\beta = -1/2$ and $\alpha = 0$. So, the supersymmetry transformation reduces to,

$$\begin{aligned} \delta A_\mu &= \epsilon\gamma_\mu\lambda, \\ \delta\lambda_\alpha &= -\frac{1}{2}(\gamma^\mu\gamma^\nu\epsilon)_\alpha F_{\mu\nu}, \end{aligned} \tag{4.35}$$

The commutator of two supersymmtries acting on A_μ gives

$$[\delta_1, \delta_2]A_\mu = -(2\epsilon_1\gamma^\nu\epsilon_2)F_{\mu\nu}, \tag{4.36}$$

or, if we rewrite in term of the gauge field,

$$[\delta_1, \delta_2]A_\mu = 2(\epsilon_1\gamma^\nu\epsilon_2)\partial_\nu A_\mu + \partial_\mu(-2(\epsilon_1\gamma^\mu\epsilon_2)A_\mu). \tag{4.37}$$

The first term on the right-hand side is a translation with parameter $2(\epsilon_1\gamma^\nu\epsilon_2)$ and the second term is a gauge transformation of parameter $\Lambda = -2(\epsilon_1\gamma^\mu\epsilon_2)A_\mu$. On the other hand, the commutator acting on the Majorana field,

$$[\delta_1, \delta_2]\lambda_\alpha = 2(\epsilon_1\gamma^\nu\epsilon_2)\partial_\nu\lambda_\alpha, \tag{4.38}$$

leads to only a rigid field translation of parameter $2(\epsilon_1\gamma^\nu\epsilon_2)$.

4.4 Supersymmetric non-Abelian Chern-Simons Theory

Now, we will construct the supersymmetric extension of the non-Abelian Chern-Simons theory (3.1),

$$S = \frac{\kappa}{4\pi}\int d^3x\, \epsilon^{\mu\nu\rho}\left(\frac{1}{2}A^a_\mu\partial_\nu A^a_\rho + \frac{1}{6}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right). \tag{4.39}$$

As in the Abelian case, the theory does not contain dynamical content. So, its supersymmetric version must be constructed by adding a non-dynamical Majorana field $\lambda^a_\alpha(x)$,

$$S = \frac{\kappa}{4\pi}\int d^3x\left[\epsilon^{\mu\nu\rho}\left(\frac{1}{2}A^a_\mu\partial_\nu A^a_\rho + \frac{1}{6}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right) - \frac{1}{2}\lambda^a\lambda^a\right]. \tag{4.40}$$

The counting of degrees of freedom in the on-shell and off-shell scheme is the same as in the Abelian theory. Therefore, the most general supersymmetry transformation is,

$$\begin{aligned}\delta A^a_\mu &= \epsilon\gamma_\mu\lambda^a, \\ \delta\lambda^a_\alpha &= \alpha\epsilon_\alpha\partial^\mu A^a_\mu + \beta(\gamma^\mu\gamma^\nu\epsilon)_\alpha F^a_{\mu\nu},\end{aligned} \tag{4.41}$$

where α and β are constants to be determined. Under these transformations, the action (4.40) changes by

$$\delta S = \frac{\kappa}{4\pi}\int d^3x\left[(1+2\beta)\left(\epsilon^{\mu\nu\rho}\epsilon\gamma_\mu\lambda^a\partial_\nu A^a_\rho + \frac{1}{2}\epsilon^{\mu\nu\rho}f^{abc}\epsilon\gamma_\mu\lambda^a A^b_\nu A^c_\rho\right) - \alpha\epsilon\lambda^a\partial^\mu A^a_\mu\right] \tag{4.42}$$

So, for $\alpha = 0$ and $\beta = -1/2$, the action is invariant and the supersymmetry transformation is rewritten as,

$$\begin{aligned}\delta A^a_\mu &= \epsilon\gamma_\mu\lambda^a, \\ \delta\lambda^a &= -\frac{1}{2}(\gamma^\mu\gamma^\nu\epsilon)_\alpha F^a_{\mu\nu}.\end{aligned} \tag{4.43}$$

Furthermore, is straightforward to check that the supersymmetric action is also invariant under the following gauge transformations,

$$\delta A^a_\mu = (D_\mu\xi)^a \quad \text{and} \quad \delta\lambda^a_\alpha = f^{abc}\lambda^b_\alpha\xi^c. \tag{4.44}$$

As in the previous section, we expect that the commutator of two supersymmetries acting on A^a_μ gives a translation and a gauge transformation. Indeed,

$$[\delta_1, \delta_2]A^a_\mu = -2(\epsilon_1\gamma^\nu\epsilon_2)F^a_{\mu\nu}, \tag{4.45}$$

when written in terms of the gauge field,

$$[\delta_1, \delta_2]A^a_\mu = 2(\epsilon_1\gamma^\nu\epsilon_2)\partial_\nu A^a_\mu - 2(\epsilon_1\gamma^\nu\epsilon_2)(D_\mu A_\nu)^a, \tag{4.46}$$

gives a translation and a gauge transformation with parameter $\xi^a = -2(\epsilon_1\gamma^\nu\epsilon_2)A^a_v$, as expected.

4.5 Supersymmetric Yang-Mills theory

The last example to be considered is the Yang-Mills theory (3.15),

$$S = -\frac{1}{e^2}\int d^3x \left(\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\right). \tag{4.47}$$

Since A^a_μ describes one degree of freedom for a fixed index a, we have to add a real fermionic field λ^a_μ to make the action supersymmetric. Unlike the previous example, the fermionic field must be dynamic,

$$S = -\frac{1}{e^2}\int d^3x \left(\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}\lambda^a(\gamma^\mu D_\mu\lambda)^a\right). \tag{4.48}$$

It is straightforward to check that this action is also invariant under the gauge transformations (4.44). The gauge invariance can be used to gauge away one bosonic component, such that the fermionic degrees of freedom match the bosonic ones. So, no auxiliary field is needed. On the other hand, the dimensionful parameter e^2 enters the supersymmetry transformation, such that its general form is

$$\begin{aligned}
\delta A^a_\mu &= \epsilon\gamma_\mu\lambda^a, \\
\delta\lambda^a_\alpha &= \beta(\gamma^\mu\gamma^\nu\epsilon)_\alpha F^a_{\mu\nu} + \gamma\epsilon_\alpha\partial^\mu A^a_\mu + \delta e^2(\gamma^\mu\epsilon)_\alpha A^a_\mu.
\end{aligned} \tag{4.49}$$

The variation of the action (4.48) under this transformation is,

$$\delta S = -\frac{1}{e^2}\int d^3x \left[F^{a\mu\nu}(D_\mu\delta A_\nu)^a + \lambda^a(\gamma^\mu D_\mu\delta\lambda)^a + \frac{1}{2}f^{abc}\lambda^a(\gamma^\mu\delta A^b_\mu)\lambda^c\right]. \tag{4.50}$$

Looking at the supersymmetry transformation, we see that the first two terms are linear in λ and therefore must cancel each other, while the last term is cubic and must separately vanish, which happens due to the anti-symmetrization of the fermionic field. By explicitly replacing the supersymmetry transformation, the two remaining terms become,

$$\begin{aligned}
\delta S &= \frac{1}{e^2}\int d^3x \left[(D_\mu F^{\mu\nu})^a\epsilon\gamma_\nu\lambda^a - \beta\lambda^a\gamma^\mu\gamma^\nu\gamma^\rho(D_\mu F_{\nu\rho})^a\epsilon\right. \\
&- \left.\gamma(\gamma^\mu D_\mu\partial^\nu A_\nu)^a\epsilon - \delta e^2(\gamma^\mu D_\mu A_\nu)^a\gamma^\nu\epsilon\right].
\end{aligned} \tag{4.51}$$

The two last terms must cancel by themselves, so $\gamma = \delta = 0$. In the second term, we use the identity,

$$\gamma^\mu\gamma^\nu\gamma^\rho = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} + \eta^{\mu\nu}\gamma^\rho + \eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu, \tag{4.52}$$

to simplify the expression. So,

$$\delta S = \frac{1}{e^2}\int d^3x \left[(1+2\beta)(D_\mu F^{\mu\nu})^a \epsilon\gamma_\nu\lambda^a - \beta\lambda^a\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}(D_{[\mu}F_{\nu\rho]})^a\epsilon\right]. \tag{4.53}$$

The last term is null due to the Bianchi identity,

$$D_{[\mu}F_{\nu\rho]} = 0, \tag{4.54}$$

and supersymmetry invariance fixes the remaining coefficient to be $\beta = -1/2$. So, the supersymmetry transformation (4.49) reduces to (4.43), giving the same result for the commutator acting on the fields.

5 Supersymmetric Abelian Chern-Simons Theories

So far, we have studied $3D$ $\mathcal{N}=1$ supersymmetric models in the x-space defined on manifolds without boundaries. From now on, we will generalize our discussion to $3D$ $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry in the superspace, focusing on the Chern-Simons theory defined on a manifold with a boundary at $x^3=0$[1]. In the presence of the boundary, half of the supersymmetry is broken. The goal of the chapter is to show how to restore the supersymmetry and study its implications at the boundary of the system.

5.1 $\mathcal{N}=1$ Supersymmetry Conventions and Definitions

The $\mathcal{N}=1$ superspace is a generalization of ordinary space, labeled by three space-time coordinates x^μ and two new Grassmannian coordinates θ^α. The Grassmann variables satisfy the following differentiation and integration rules

$$\{\partial_\alpha, \theta^\beta\} = \delta_\alpha{}^\beta, \tag{5.1}$$

and

$$\int d\theta = 0 \qquad \text{and} \qquad \int d^2\theta\theta^2 = -1. \tag{5.2}$$

The generators Q_α of supersymmetry transformations and the superspace covariant derivatives D_α are defined as

$$Q_\alpha = \partial_\alpha - (\gamma^\mu\theta)_\alpha\partial_\mu \qquad \text{and} \qquad D_\alpha = \partial_\alpha + (\gamma^\mu\theta)_\alpha\partial_\mu, \tag{5.3}$$

and the corresponding algebra is

$$\{Q_\alpha, Q_\beta\} = -\{D_\alpha, D_\beta\} = 2\gamma^\mu_{\alpha\beta}\partial_\mu, \qquad \{Q_\alpha, D_\beta\} = 0. \tag{5.4}$$

The two basic superfields of $3D$ $\mathcal{N}=1$ superspace are scalar and spinor superfields. The scalar superfield $\mathbf{\Phi}$ is defined as

$$\Phi(x,\theta) = (A, \psi_\alpha, F) = A + \theta\psi + \theta^2 F, \tag{5.5}$$

or in terms of supercovariant derivatives as

$$\Phi \;=\; (\Phi, D_\alpha\Phi, -D^2\Phi)_|, \tag{5.6}$$

[1] Specifically in this Chapter, the Minkowski metric is $\eta^{\mu\nu} = (-,+,+)$, $\mu = 0,1,3$ and $m = 0,1$.

where the symbol "|" means $\theta = 0$. The scalar superfield has as components a scalar field $A(x)$, a Majorana field $\psi_\alpha(x)$ and an auxiliary scalar field $F(x)$. Its supersymmetry transformation is

$$\delta\Phi = \epsilon Q\Phi, \tag{5.7}$$

or explicitly,

$$\delta A + \theta\delta\psi + \theta^2\delta F = \epsilon\psi + \theta(\gamma^\mu\epsilon\partial_\mu A + F\epsilon) + \theta^2\epsilon\gamma^\mu\partial_\mu\psi. \tag{5.8}$$

Comparing the θ-power on both sides, we find the supersymmetry transformation for the component fields

$$\delta A = \epsilon\psi, \qquad \delta\psi = \gamma^\mu\epsilon\partial_\mu A + F\epsilon, \qquad \delta F = \epsilon\gamma^\mu\partial_\mu\psi. \tag{5.9}$$

From these transformation laws, we can show that the commutator of two supersymmetries leads to a rigid field translation. For the scalar field $A(x)$, for example, we have

$$[\delta_1, \delta_2]A(x) = 2(\epsilon_2\gamma^\mu\epsilon_1)\partial_\mu A(x). \tag{5.10}$$

The same result holds for $\psi_\alpha(x)$ and $F(x)$.

The spinor superfield Γ_α is defined as

$$\Gamma_\alpha(x,\theta) = (\chi_\alpha, M, v_\mu, \lambda_\alpha) = \chi_\alpha + \theta_\alpha M + (\gamma^\mu\theta)_\alpha v_\mu + \theta^2\tilde{\lambda}_\alpha, \tag{5.11}$$

where we have a Majorana field $\chi_\alpha(x)$, a gauge field $v_\mu(x)$ and two auxiliary fields, the Majorana field $\tilde{\lambda}_\alpha(x)$ and the scalar field $M(x)$. Nonetheless, the property $\{Q_\alpha, D_\beta\} = 0$, implies that objects like $D_{\alpha_1}...D_{\alpha_n}\Phi$ transforms as superfields, namely,

$$\delta(D_{\alpha_1}...D_{\alpha_n}\Phi) = \epsilon Q(D_{\alpha_1}...D_{\alpha_n}\Phi). \tag{5.12}$$

In particular, we see that $D_\alpha\Phi$ is a spinor superfield like Γ_α. Thus, we define a superfield gauge transformation as

$$\delta\Gamma_\alpha = D_\alpha\Phi, \tag{5.13}$$

which in components reads,

$$\begin{aligned} \delta\chi_\alpha &= \psi_\alpha, \\ \delta M &= F, \\ \delta v_\mu &= \partial_\mu A, \\ \delta\lambda_\alpha &= 0, \end{aligned} \tag{5.14}$$

where $\lambda_\alpha = \tilde{\lambda}_\alpha + (\gamma^\mu\partial_\mu\chi)_\alpha$. So, without loss of generality, we can redefine the spinor superfield as

$$\Gamma_\alpha(x,\theta) = (\chi_\alpha, M, v_\mu, \lambda_\alpha) = \chi_\alpha + \theta_\alpha M + (\gamma^\mu\theta)_\alpha v_\mu + \theta^2(\lambda_\alpha - (\gamma^\mu\partial_\mu\chi)_\alpha), \tag{5.15}$$

which in terms of supercovariant derivatives is

$$\Gamma_\alpha = (\Gamma_\alpha, -\frac{1}{2}D^\alpha\Gamma_\alpha, -\frac{1}{2}D^\alpha(\gamma^\mu\Gamma)_\alpha, -D^2\Gamma_\alpha + (\gamma^\mu\partial_\mu\Gamma)_\alpha)_|. \tag{5.16}$$

As for the scalar superfield, from its supersymmetry transformation,

$$\delta\Gamma_\alpha = \epsilon Q\Gamma_\alpha, \tag{5.17}$$

we find that the spinor superfield components transform as

$$\delta\chi = M\epsilon + \gamma^\mu\epsilon v_\mu, \qquad \delta M = -\tfrac{1}{2}\epsilon\lambda + \epsilon\gamma^\mu\partial_\mu\chi, \tag{5.18}$$

$$\delta v_\mu = -\tfrac{1}{2}\epsilon\gamma_\mu\lambda + \epsilon\partial_\mu\chi, \qquad \delta\lambda = -2\epsilon^{\mu\nu\rho}\gamma_\rho\epsilon\partial_\mu v_\nu. \tag{5.19}$$

In this case, the commutator of two supersymmetries leads to a rigid translation of parameter $2(\epsilon_1\gamma^\mu\epsilon_2)$, for $\chi_\alpha(x)$, $\lambda_\alpha(x)$ and $M(x)$. On the other hand, acting on v_μ,

$$[\delta_1, \delta_2]v_\mu = 2(\epsilon_1\gamma^\nu\epsilon_2)\partial_\nu v_\mu + \partial_\mu(-2(\epsilon_1\gamma^\nu\epsilon_2)v_\nu), \tag{5.20}$$

leads to a rigid translation and a gauge transformation with parameter $\Lambda = -2(\epsilon_1\gamma^\nu\epsilon_2)v_\nu$.

5.1.1 $\mathcal{N}=1$ Supersymmetry with Boundary

To understand how supersymmetry is affected by the presence of a boundary, consider the following simple example

$$S_0 = \int d^3x d^2\theta\, \Phi(x,\theta) = \int d^3x\, F(x). \tag{5.21}$$

Under the supersymmetry transformation (5.9), the above action changes by a boundary term,

$$\delta S_0 = \int d^3x\, \partial_\mu(\epsilon\gamma^\mu\psi). \tag{5.22}$$

So, considering a boundary at $x^3 = 0$, the supersymmetry is broken,

$$\delta S_0 = \int d^3x\, \partial_3(\epsilon\gamma^3\psi). \tag{5.23}$$

Motivated by the gauge-invariance restoration, we can add new degrees of freedom at the boundary to restore the supersymmetry. To see this, first, let us introduce the projectors,

$$(P_\pm)^\alpha{}_\beta = \frac{1}{2}(1 \pm \gamma^3)^\alpha{}_\beta, \tag{5.24}$$

and write the boundary action as

$$\delta S_0 = \int d^3x\, \partial_3(\epsilon_+\psi_- - \epsilon_-\psi_+), \tag{5.25}$$

where

$$\epsilon_{\pm\alpha} \equiv (P_\pm)^\beta{}_\alpha\epsilon_\beta \qquad \text{and} \qquad \psi_{\pm\alpha} \equiv (P_\pm)^\beta{}_\alpha\psi_\beta. \tag{5.26}$$

Look at the supersymmetry transformation (5.9), we see that the action

$$S_1 = \int d^3x\, \partial_3(\Phi|_{\theta=0}) = \int d^3x\, \partial_3(A), \tag{5.27}$$

transform as

$$\delta S_1 = \int d^3x\, \partial_3(\epsilon_+\psi_- + \epsilon_-\psi_+). \tag{5.28}$$

So, adding this action to (5.25), we have

$$\delta S = \delta(S_0 + S_1) = \int d^3x\, 2\partial_3(\epsilon_+\psi_-), \tag{5.29}$$

i.e., if we take ϵ_+ to zero, we restore the half supersymmetry $\mathcal{N} = (0,1)$ generated by $\epsilon_- Q_+$. On the other hand, subtracting the action (5.28) from (5.25),

$$\delta S = \delta(S_0 - S_1) = -\int d^3x\, 2\partial_3(\epsilon_-\psi_+), \tag{5.30}$$

and taking ϵ_- to zero, we restore the half supersymmetry $\mathcal{N} = (1,0)$ generated by $\epsilon_+ Q_-$. In other words, in the presence of a boundary, we can restore only half of the supersymmetry. So, considering a general $3D$ $\mathcal{N} = 1$ action,

$$S_0 = \int d^3x d^2\theta\, \mathcal{L}, \tag{5.31}$$

the action

$$S = S_0 + S_1 = \int d^3x \left(d^2\theta \mathcal{L} \pm \partial_3 \mathcal{L}|_{\theta=0} \right), \tag{5.32}$$

preserves half of the supersymmetry ($\mathcal{N} = (1,0)$ or $\mathcal{N} = (0,1)$) generated by $\epsilon_\pm Q_\mp$. In some cases, it might be convenient to add an extra $2D$ supersymmetric boundary action. Such actions can be constructed systematically using multiplets decomposition, where we split $3D$ $\mathcal{N} = 1$ superfields into $2D$ $\mathcal{N} = (1,0)$ or $\mathcal{N} = (0,1)$ superfields, details are summarized also in Appendix A.

5.2 Bulk-Edge Correspondence

The $\mathcal{N} = 1$ Abelian Chern-Simons action is given by [14, 15, 16]

$$S_0 = -\int d^3x d^2\theta\, \Gamma^\alpha \omega_\alpha, \tag{5.33}$$

where Γ_α is the spinor superfield,

$$\Gamma_\alpha = \chi_\alpha + \theta_\alpha M + (\gamma^\mu\theta)_\alpha v_\mu + \theta^2(\lambda_\alpha - (\gamma^\mu \partial_\mu \chi)_\alpha), \tag{5.34}$$

and ω_α is the gauge-invariant field strength,

$$\omega_\alpha = -D^\beta D_\alpha \Gamma_\beta = \lambda_\alpha - 2\epsilon^{\mu\nu\rho}(\gamma_\rho\theta)_\alpha \partial_\mu v_\nu + \theta^2(\gamma^\mu\partial_\mu\lambda)_\alpha. \tag{5.35}$$

In components, the action (5.33) is written as

$$S_0 = \int d^3x \, \left(4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho + \lambda\lambda + \partial_\mu(\chi\gamma^\mu\lambda)\right). \tag{5.36}$$

Notice that the auxiliary real scalar field M is absent from the action and χ_α enters only as a total derivative. Under the superfield gauge transformation (5.14), the action varies into a boundary term. Thus, on borderless manifolds, such term vanishes and we can impose the Wess-Zumino gauge, $\chi = M = 0$, so that the action reduces to (4.32), up to numerical factors. On a manifold with a boundary at $x^3 = 0$, the variation of the action under the supersymmetry transformation (5.19) reads

$$\delta S_0 = \int d^3x \, \epsilon \partial_3 [\epsilon^{3mn} v_m (2\partial_n \chi + \gamma_n \lambda) + 2\gamma^m \chi F_{3m} + \gamma^3 \lambda M - \lambda v^3]. \tag{5.37}$$

Applying the prescription (5.32), we can use the following boundary action to restore half of the supersymmetry,

$$S_1 = \int d^3x \, \partial_3 (\Gamma^\alpha \omega_\alpha)|_{\theta=0} = \int d^3x \, \partial_3(\chi\lambda). \tag{5.38}$$

Hence we form a $\mathcal{N} = (1, 0)$ action

$$S^{CS}_{(1,0)} = S_0 + S_1 = \int d^3x \, \left(4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho + \lambda\lambda + 2\partial_3(\chi_-\lambda_+)\right), \tag{5.39}$$

and a $\mathcal{N} = (0, 1)$ action,

$$S^{CS}_{(0,1)} = S_0 - S_1 = \int d^3x \, \left(4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho + \lambda\lambda - 2\partial_3(\chi_+\lambda_-)\right), \tag{5.40}$$

where $\chi\lambda = \chi_+\lambda_- + \chi_-\lambda_+$ and $\chi\gamma^3\lambda = -\chi_+\lambda_- + \chi_-\lambda_+$. Notice that, in both cases, the non-propagating field λ_α appears linearly at the boundary coupled to the field χ_α. To remove this coupling without imposing some field boundary conditions, we can separately add supersymmetric actions at the boundary. For the $\mathcal{N} = (1, 0)$ case, we construct the following ϵ_+ invariant boundary action[2],

$$\begin{aligned} S^b_{(1,0)} &= 2\int d^3x \, \partial_3 \int d\theta^\alpha_+ (\gamma^m)^\beta{}_\alpha \hat{X}^-_\beta \hat{\Sigma}^+_m, & (5.41)\\ &= -2\int d^3x \, \partial_3(\chi_-\lambda_+ - \chi_-\gamma^m\partial_m\chi_- - v_m v^m). & (5.42) \end{aligned}$$

Adding this action to (5.39), we have

$$\begin{aligned} S^{total}_{(1,0)} &= S^{CS}_{(1,0)} + S^b_{(1,0)}, \\ &= \int d^3x \, \left(4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho + \lambda\lambda + 2\partial_3(\chi_-\gamma^m\partial_m\chi_- + v_m v^m)\right), & (5.43)\\ &= \int d^3x \, \Big(4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho + \lambda\lambda - 2\partial_3(\chi_-\gamma^1\partial_-\chi_- + v_+v_-)\Big), & (5.44) \end{aligned}$$

[2] See Appendix A.

where $v_\pm = v_0 \pm v_1$ and $\partial_\pm = \partial_0 \pm \partial_1$. The addition of the extra boundary action eliminates the linear term in λ_α and produces a dynamical chiral fermion at the boundary. Analogously, for the $\mathcal{N} = (0,1)$ case, we construct the following ϵ_- invariant action[3],

$$\begin{aligned} S^b_{(0,1)} &= 2\int d^3x\, \partial_3 \int d\theta^\alpha_- (\gamma^m)^\beta{}_\alpha \tilde{X}^+_\beta \tilde{\Sigma}^-_m, &(5.45)\\ &= -2\int d^3x\, \partial_3(\chi_+\lambda_- + \chi_+\gamma^m\partial_m\chi_+ + v_m v^m), &(5.46)\end{aligned}$$

which added to the action (5.40), leads us to

$$\begin{aligned} S^{total}_{(0,1)} &= S^{CS}_{(0,1)} + S^b_{(0,1)}, \\ &= \int d^3x\, \left(4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho + \lambda\lambda + 2\partial_3(\chi_+\gamma^m\partial_m\chi_+ + v_m v^m)\right), &(5.47)\\ &= \int d^3x\, \left(4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho + \lambda\lambda + 2\partial_3(\chi_+\gamma^1\partial_+\chi_+ - v_+ v_-)\right). &(5.48)\end{aligned}$$

Here, as in the previous case, the linear term in λ_α no longer appears at the boundary and a dynamical chiral fermion of opposite chirality emerges at the boundary. So, we expect that $S^{total}_{(1,0)}$ and $S^{total}_{(0,1)}$ are connected by a time-reversal. In what follows, let us write the actions in terms of the spinor components,

$$\begin{aligned} S^{total}_{(1,0)} &= \int d^3x\, \mathcal{L}_{bulk} - 2\partial_3(i\chi_2\partial_-\chi_2 + v_+ v_-), &(5.49)\\ S^{total}_{(0,1)} &= \int d^3x\, \mathcal{L}_{bulk} - 2\partial_3(i\chi_1\partial_+\chi_1 + v_+ v_-), &(5.50)\end{aligned}$$

where $\mathcal{L}_{bulk} = 4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho + \lambda\lambda$. Recall that Majorana spinors transform under a time-reversal operation as [21]

$$\mathcal{T}: \qquad \psi_\alpha(x) \to \psi^{\mathcal{T}}_\alpha(x) = (\gamma^0)^\beta{}_\alpha \psi_\beta(-x_0, \vec{x}), \tag{5.51}$$

or in components

$$\mathcal{T}: \qquad \psi_1(x) \to \psi_2(-x_0, \vec{x}) \qquad \text{and} \qquad \psi_2(x) \to -\psi_1(-x_0, \vec{x}), \tag{5.52}$$

and also that the operation changes $i \to -i$ since it is anti-unitary. Thus, the action (5.49),

$$\begin{aligned} S^{total}_{(1,0)} \to (S^{total}_{(1,0)})^{\mathcal{T}} &= \int d^3x\, \mathcal{L}_{bulk} - 2\partial_3(-i\chi_1(-x_0,\vec{x})\partial_-\chi_1(-x_0,\vec{x}) + v_+ v_-), \\ &= \int d^3x\, \mathcal{L}_{bulk} - 2\partial_3(i\chi_1(x_0,\vec{x})\partial_+\chi_1(x_0,\vec{x}) + v_+ v_-), \\ &= \int d^3x\, \mathcal{L}_{bulk} - 2\partial_3(i\chi_1\partial_+\chi_1 + v_+ v_-), &(5.53)\end{aligned}$$

changes into (5.50) under a time-reversal operation, as stated above.

The boundary actions in (5.49) and (5.50) obtained from the $\mathcal{N} = (1,0)$ and $\mathcal{N} = (0,1)$ half supersymmetry restoration, respectively, has only 1 gapless Majorana chiral excitation, i.e.,

$$c_{R/L} = \frac{1}{2}. \tag{5.54}$$

[3] See Appendix A.

Such edge structures appears in topological superconductors, so we are led to think that this approach might be connected with the description of such topological phases [27]. In the next section, we will consider extended $\mathcal{N}=2$ supersymmetry. In this case, we expect to find two dynamical chiral fermions at the boundary, of equal chirality as well as of opposite chirality.

5.3 $\mathcal{N}=2$ Supersymmetry Conventions and Definitions

The $3D$ $\mathcal{N}=2$ superspace is realized by taking the Grassmann coordinates to be complex. For convenience, we express these coordinates in terms of real $\mathcal{N}=1$ Grassmann coordinates,

$$\theta_\alpha = \frac{1}{\sqrt{2}}(\theta_{1\alpha} + i\theta_{2\alpha}), \qquad \bar{\theta}_\alpha = \frac{1}{\sqrt{2}}(\theta_{1\alpha} - i\theta_{2\alpha}). \tag{5.55}$$

The integration is defined as

$$\int d^4\theta = \int d^2\theta d^2\bar{\theta} = -\int d^2\theta_1 d^2\theta_2. \tag{5.56}$$

In the same way, the $\mathcal{N}=2$ superspace covariant deviratives,

$$D_\alpha = \partial_\alpha + (\gamma^\mu\bar{\theta})_\alpha\partial_\mu, \qquad \bar{D}_\alpha = \bar{\partial}_\alpha + (\gamma^\mu\theta)_\alpha\partial_\mu, \tag{5.57}$$

are decomposed as

$$D_\alpha = \frac{1}{\sqrt{2}}(D_1 - iD_2), \qquad \bar{D}_\alpha = \frac{1}{\sqrt{2}}(D_1 + iD_2), \tag{5.58}$$

where D_i is the usual $\mathcal{N}=1$ superspace covariant derivative (5.3), which satisfy the algebra

$$\{D_{i\alpha}, D_{j\beta}\} = -2\delta_{ij}\gamma^\mu_{\alpha\beta}\partial_\mu. \tag{5.59}$$

The main superfields of $3D$ $\mathcal{N}=2$ superspace are vector and quiral superfields. The vector superfield $V=V^*$ is defined as

$$V(x,\theta_1,\theta_2) = A(x,\theta_1) + \theta_2\Gamma(x,\theta_1) + \theta_2^2(B(x,\theta_1) + D_1^2A(x,\theta_1)), \tag{5.60}$$

where $A(x,\theta_1)$ and $B(x,\theta_1)$ are $\mathcal{N}=1$ scalar superfields and $\Gamma_\alpha(x,\theta_1)$ is a $\mathcal{N}=1$ spinor superfield. The chiral superfield $\mathbf{\Phi}$ is defined such that

$$\bar{D}\mathbf{\Phi} = 0 \to D_2\mathbf{\Phi} = iD_1\mathbf{\Phi}, \tag{5.61}$$

and then be expressed as

$$\mathbf{\Phi}(x,\theta_1,\theta_2) = X(x,\theta_1) + i\theta_2 D_1 X(x,\theta_1) + \theta_2^2 D_1^2 X(x,\theta_1), \tag{5.62}$$

where $X(x,\theta_1)$ is a complex $\mathcal{N}=1$ scalar superfield. Since all $\mathcal{N}=2$ objects were decomposed into $\mathcal{N}=1$, we can simply look at the $\mathcal{N}=1$ superspace conventions and definitions.

5.3.1 $\mathcal{N}=2$ Supersymmetry with Boundary

Let us now consider extended $\mathcal{N}=2$ supersymmetry in the presence of a boundary. We start with a general $3D$ $\mathcal{N}=2$ action

$$S=\int d^3xd^2\theta_1d^2\theta_2\,\mathcal{L}. \tag{5.63}$$

Our goal is to generalize the procedure used in the previous section to restore half of the supersymmetry. In this case, the action

$$S=\int d^3x\left(d^2\theta_1d^2\theta_2\mathcal{L}+d^2\theta_1\partial_3\mathcal{L}|_{\theta_2=0}\pm d^2\theta_2\partial_3\mathcal{L}|_{\theta_1=0}\pm\partial_3\partial_3\mathcal{L}|_{\theta_1=\theta_2=0}\right), \tag{5.64}$$

preserves half of the supersymmetry ($\mathcal{N}=(2,0)$ or $\mathcal{N}=(1,1)$) generated by $\epsilon_{1\pm}Q_{1\mp}$, $\epsilon_{2+}Q_{2-}$. On the other hand, the action

$$S=\int d^3x\left(d^2\theta_1d^2\theta_2\mathcal{L}-d^2\theta_1\partial_3\mathcal{L}|_{\theta_2=0}\mp d^2\theta_2\partial_3\mathcal{L}|_{\theta_1=0}\pm\partial_3\partial_3\mathcal{L}|_{\theta_1=\theta_2=0}\right), \tag{5.65}$$

preserves half of the supersymmetry ($\mathcal{N}=(0,2)$ or $\mathcal{N}=(1,1)$) generated by $\epsilon_{1\mp}Q_{1\pm}$, $\epsilon_{2-}Q_{2+}$. As in the previous section, it might be convenient to add extra $2D$ supersymmetric actions. In this case, such action can be constructed splitting $3D$ $\mathcal{N}=2$ superfields into $2D$ $\mathcal{N}=(2,0)$, $\mathcal{N}=(0,2)$ or $\mathcal{N}=(1,1)$ superfields, details are summarized in Appendix B.

5.4 Bulk-Edge Correspondence

The $\mathcal{N}=2$ Abelian Chern-Simons action is given by [15]

$$S_0=-\int d^3xd^2\theta d^2\bar{\theta}VD^\alpha\bar{D}_\alpha V=\int d^3xd^2\theta_1d^2\theta_2V(D_1^2+D_2^2)V, \tag{5.66}$$

where V is the vector superfield, which can be written as

$$V=A(\theta_1)+\theta_2\Gamma+\theta_2^2(B(\theta_1)+D_1^2A(\theta_1)), \tag{5.67}$$

with components given by $\mathcal{N}=1$ superfields,

$$A=(a,\psi_\alpha,f),\qquad B=(b,\eta_\alpha,g),\qquad \Gamma_\alpha=(\chi_\alpha,M,v_\mu,\lambda_\alpha). \tag{5.68}$$

In terms of the $\mathcal{N}=1$ superfields, the action (5.66) reads

$$S_0=\int d^3xd^2\theta_1(BB+\Gamma^\alpha\omega_\alpha+\frac{1}{2}D_1^\alpha(D_{1\alpha}BA-BD_{1\alpha}A)), \tag{5.69}$$

where ω_α is the gauge-invariant field strength (5.35). Notice that, in absence of boundaries, this action differs from the action (5.33) by an auxiliary superfield. In components, it reduces to

$$S_0=\int d^3x(\eta\eta-2bg-4\epsilon^{\mu\nu\rho}v_\mu\partial_\nu v_\rho-\lambda\lambda+\partial_\mu(a\partial^\mu b-b\partial^\mu a+\lambda\gamma^\mu\chi+\eta\gamma^\mu\psi)). \tag{5.70}$$

As in the $\mathcal{N}=1$ case, on a manifold with a boundary, the variation of the action under a supersymmetry transformation does not vanish. To restore half of the supersymmetry, we can apply the prescription (5.64) or (5.65). In both cases, we have to insert the following actions,

$$\begin{aligned} S_1 &= \int d^3x d^2\theta_1 \partial_3 (V(D_1^2+D_2^2)V)|_{\theta_2=0} = \int d^3x \partial_3 (ag + bf - \psi\eta), && (5.71)\\ S_2 &= \int d^3x d^2\theta_2 \partial_3 (V(D_1^2+D_2^2)V)|_{\theta_1=0} = \int d^3x \partial_3 (ag - bf + bb - \chi\lambda), && (5.72)\\ S_3 &= \int d^3x \partial_3\partial_3 (V(D_1^2+D_2^2)V)|_{\theta_1=\theta_2=0} = \int d^3x \partial_3 (a\partial_3 b + b\partial_3 a). && (5.73) \end{aligned}$$

So, applying the prescription (5.64), we form a $\mathcal{N}=(2,0)$ action

$$\begin{aligned} S^{CS}_{(2,0)} &= S_0 + S_1 + S_2 + S_3,\\ &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(-\lambda_+\chi_- - \eta_+\psi_- + \frac{1}{2}bb + a(g+\partial_3 b)) && (5.74) \end{aligned}$$

and a $\mathcal{N}=(1,1)$ action

$$\begin{aligned} S^{CS}_{(1,1)\mp} &= S_0 + S_1 - S_2 - S_3,\\ &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(\lambda_-\chi_+ - \eta_+\psi_- - \frac{1}{2}bb + b(f-\partial_3 a)). && (5.75) \end{aligned}$$

Applying the prescription (5.65), we form a $\mathcal{N}=(0,2)$ action

$$\begin{aligned} S^{CS}_{(0,2)} &= S_0 - S_1 - S_2 + S_3,\\ &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(\lambda_-\chi_+ + \eta_-\psi_+ - \frac{1}{2}bb - a(g-\partial_3 b)), && (5.76) \end{aligned}$$

and a $\mathcal{N}=(1,1)$ action

$$\begin{aligned} S^{CS}_{(1,1)\pm} &= S_0 - S_1 + S_2 - S_3,\\ &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(-\lambda_+\chi_- + \eta_-\psi_+ + \frac{1}{2}bb - b(f+\partial_3 a)), && (5.77) \end{aligned}$$

where $\mathcal{L}_{bulk} = \eta\eta - 2bg - 4\epsilon^{\mu\nu\rho}v_\mu\partial_\nu v_\rho - \lambda\lambda$ and the subscripts $\mp$ and $\pm$ indicates invariance under $(\epsilon_{1-},\epsilon_{2+})$ and $(\epsilon_{1+},\epsilon_{2-})$ supersymmetry and differentiates the $S_{(1,1)}$ actions. In all cases, the non-propagating fields $\lambda_{\pm\alpha}$ and $\eta_{\pm\alpha}$ appears linearly at the boundary. As in the $\mathcal{N}=1$ case, to remove the couplings without imposing field boundary conditions, we can separately add supersymmetric actions at the boundary. For the $\mathcal{N}=(2,0)$ case, we construct the following $(\epsilon_{1+},\epsilon_{2+})$ invariant action[4]

$$\begin{aligned} S^b_{(2,0)} &= 2\int d^3x\, \partial_3 \int d\theta_{1+}\gamma^m d\theta_{2+} \hat{\hat{V}}\hat{\hat{V}}_m,\\ &= 2\int d^3x\, \partial_3(\psi_-\eta_+ + \chi_-\lambda_+ - \psi_-\gamma^m\partial_m\psi_- - \chi_-\gamma^m\partial_m\chi_- - a(g+\partial_3 b)\\ &+ a\partial_m\partial^m a - v_m v^m), && (5.78) \end{aligned}$$

[4] See Appendix B.

which added to the action (5.74), gives

$$\begin{aligned} S^{total}_{(2,0)} &= S^{CS}_{(2,0)} + S^{b}_{(2,0)}, \\ &= \int d^3x\ \mathcal{L}_{bulk} - 2\partial_3(\psi_-\gamma^m\partial_m\psi_- + \chi_-\gamma^m\partial_m\chi_- + \partial_m a\partial^m a + v_m v^m - \frac{1}{2}bb), \\ &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(\psi_-\gamma^1\partial_-\psi_- + \chi_-\gamma^1\partial_-\chi_- + \partial_+ a\partial_- a + v_+ v_- + \frac{1}{2}bb). \quad (5.79) \end{aligned}$$

The extra boundary action remove the couplings and produces two dynamical fermions of the same chirality at the boundary. For the $\mathcal{N} = (1, 1)$ case, we construct the following $(\epsilon_{1-}, \epsilon_{2+})$ invariant action[5]

$$\begin{aligned} S^{b}_{(1,1)\mp} &= 2\int d^3x\,\partial_3 \int d\theta_{1-}d\theta_{2+}\tilde{\tilde{U}}^{\alpha}\tilde{\tilde{V}}_{\alpha}, \\ &= 2\int d^3x\,\partial_3(\chi_+\lambda_- + \psi_-\eta_+ - \chi_+\gamma^m\partial_m\chi_+ - \psi_-\gamma^m\partial_m\psi_- - (f - \partial_3 a)^2 \\ &- b(f - \partial_3 a) - v_m v^m), \quad (5.80) \end{aligned}$$

which added to the action (5.75), gives

$$\begin{aligned} S^{total}_{(1,1)\mp} &= S^{CS}_{(1,1)\mp} + S^{b}_{(1,1)\mp}, \\ &= \int d^3x\ \mathcal{L}_{bulk} - 2\partial_3(-2\chi_+\lambda_- + \chi_+\gamma^m\partial_m\chi_+ + \psi_-\gamma^m\partial_m\psi_- + v_m v^m \\ &+ (f - \partial_3 a)^2 + \frac{1}{2}bb), \\ &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(2\chi_+\lambda_- + \chi_+\gamma^1\partial_+\chi_+ - \psi_-\gamma^1\partial_-\psi_- + v_+ v_- \\ &- (f - \partial_3 a)^2 - \frac{1}{2}bb). \quad (5.81) \end{aligned}$$

Unlike the previous case, the extra boundary action removes only the term linear in $\eta_{+\alpha}$ and produces two dynamical fermions of opposite chirality. Motivated by the $\mathcal{N} = 1$ case, the $S^{total}_{(0,2)}$ and $S^{total}_{(1,1)\pm}$ actions, can be obtained by applying the time-reversal operation in $S^{total}_{(2,0)}$ and $S^{total}_{(1,1)\mp}$, respectively. Firstly, let us write these actions in terms of the spinor components,

$$\begin{aligned} S^{total}_{(2,0)} &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(i\psi_2\partial_-\psi_2 + i\chi_2\partial_-\chi_2 + \partial_+ a\partial_- a + v_+ v_- + \frac{1}{2}bb), \quad (5.82) \\ S^{total}_{(1,1)\mp} &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(-2i\chi_1\lambda_2 - i\chi_1\partial_+\chi_1 - i\psi_2\partial_-\psi_2 + v_+ v_- - \frac{1}{2}bb \\ &- (f - \partial_3 a)^2). \quad (5.83) \end{aligned}$$

So, by applying the transformation rules (5.52) in (5.82), we have

$$\begin{aligned} S^{total}_{(2,0)} \to (S^{total}_{(2,0)})^{\mathcal{T}} &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(-i\psi_1(-x_0, \vec{x})\partial_-\psi_1(-x_0, \vec{x}) \\ &- i\chi_1(-x_0, \vec{x})\partial_-\chi_1(-x_0, \vec{x}) + \partial_+ a\partial_- a + v_+ v_- + \frac{1}{2}bb), \\ &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(i\psi_1(x_0, \vec{x})\partial_+\psi_1(x_0, \vec{x}) \\ &+ i\chi_1(x_0, \vec{x})\partial_+\chi_1(x_0, \vec{x}) + \partial_+ a\partial_- a + v_+ v_- + \frac{1}{2}bb), \quad (5.84) \end{aligned}$$

[5] See Appendix B.

back to the compact notation, we find the $S^{total}_{(0,2)}$ action

$$S^{total}_{(0,2)} = \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(-\psi_+\gamma^1\partial_+\psi_+ - \chi_+\gamma^1\partial_+\chi_+ + \partial_+a\partial_-a + v_+v_- + \frac{1}{2}bb). \quad (5.85)$$

Note that the $S^{total}_{(0,2)}$ has two dynamical fermions of equal chirality, but opposite chirality as the $S^{total}_{(2,0)}$ action. In what follows, consider a Dirac spinor φ parameterized in terms of the Majorana spinors χ and ψ,

$$\varphi = \chi + i\psi. \quad (5.86)$$

The action (5.82) written in terms of the Dirac spinor reads

$$S^{total}_{(2,0)} = \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(i\varphi_2^\dagger\partial_-\varphi_2 + \partial_+a\partial_-a + v_+v_- + \frac{1}{2}bb), \quad (5.87)$$

i.e., now the theory has a dynamical chiral Dirac fermion at the boundary. Analogously, the action (5.84),

$$S^{total}_{(0,2)} = \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(i\varphi_1^\dagger\partial_+\varphi_1 + \partial_+a\partial_-a + v_+v_- + \frac{1}{2}bb), \quad (5.88)$$

also has a dynamical chiral Dirac fermion at the boundary.

The boundary actions in (5.87) and (5.88) obtained from $\mathcal{N} = (2,0)$ and $\mathcal{N} = (0,2)$ half supersymmetry restoration, respectively, has 1 gapless Dirac chiral excitation and 2 gapless bosonic chiral excitations, such that, effectively one has only 1 gapless chiral excitation, i.e.,

$$c_{R/L} = 1. \quad (5.89)$$

So, the edge structures is similar to that obtained from gauge-invariance restoration, and so we can think that such approach can also be useful to describe Abelian fractional quantum Hall phases.

Similarly, by applying the transformation rules (5.52) in (5.83), we have

$$\begin{aligned} S^{total}_{(1,1)\mp} \to (S^{total}_{(1,1)\mp})^{\mathcal{T}} &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(-2i\chi_2\lambda_1 + i\chi_2(-x_0,\vec{x})\partial_+\chi_2(-x_0,\vec{x}) \\ &+ i\psi_1(-x_0,\vec{x})\partial_-\psi_1(-x_0,\vec{x}) + v_+v_- - (f-\partial_3a)^2 - \frac{1}{2}bb), \\ &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(-2i\chi_2\lambda_1 - i\chi_2(x_0,\vec{x})\partial_-\chi_2(x_0,\vec{x}) \\ &- i\psi_1(x_0,\vec{x})\partial_+\psi_1(x_0,\vec{x}) + v_+v_- - (f-\partial_3a)^2 - \frac{1}{2}bb), \quad (5.90) \end{aligned}$$

back to the compact notation, we find the $S^{total}_{(1,1)\pm}$ action

$$\begin{aligned} S^{total}_{(1,1)\pm} &= \int d^3x\ \mathcal{L}_{bulk} + 2\partial_3(-2\chi_-\lambda_+ - \chi_-\gamma^1\partial_-\chi_- + \psi_+\gamma^1\partial_+\psi_+ \\ &+ v_+v_- - (f-\partial_3a)^2 - \frac{1}{2}bb). \quad (5.91) \end{aligned}$$

The $S^{total}_{(1,1)}$ actions, at first, suggest an invariant system under time-reversal symmetry, since we have two chiral fermions of opposite chirality at the edge, but a closer look shows that this is not true, as there is also a mass term for one of the fermions. So, in both cases, we have effectively only 1 gapless Majorana chiral excitation, i.e.,

$$c_{R/L} = \frac{1}{2}. \tag{5.92}$$

such edge structures are similar to the $\mathcal{N} = (1,0)$ and $\mathcal{N} = (0,1)$ cases, and also can describe topological superconductors.

6 Final Considerations

As we have seen, in 2+1 space-time dimensions, the Chern-Simons theory is responsible for both the statistical transmutation and the generation of mass to gauge fields. In addition, the Chern-Simons theory is topological. The ground-state degeneracy depends on the genus of the manifold upon which it is defined. One of the most remarkable feature of this theory arises on a manifold with physical boundaries. The restoration of gauge symmetry and supersymmetry leads to the emergence of physical excitations at the edge. Such property is called holography and leads to the bulk-edge correspondence.

The restoration of the gauge symmetry gives rise to the emergence of dynamical chiral bosons at the boundary, which are connected to the description of Quantum Hall states [9]. On the other hand, the restoration of $\mathcal{N} = (1, 0)$ or $\mathcal{N} = (0, 1)$ supersymmetry leads to the emergence of one dynamical Majorana chiral fermion at the boundary. In the first case, one has a right-handed fermion, while in the second case, a left-handed fermion. Such fermions are connected by a time-reversal operation. Since the edge theory in these cases is described by Majorana chiral fermions, we are led to think that such theories may be connected with topological superconductors descriptions [27].

In the same way, the restoration of $\mathcal{N} = (2, 0)$ or $\mathcal{N} = (0, 2)$ supersymmetry leads to the emergence of two dynamical Majorana fermions at the boundary. Since we can think the supersymmetry $\mathcal{N} = (2, 0)$ as a combination of two $\mathcal{N} = (1, 0)$ supersymmetries, in this case, we have two right-handed fermions, while in the other case, two left-handed fermions. In both cases, we can combine the Majorana fermions into a single Dirac fermion, which under bosonization are lead to chiral bosons, such that the edge theories suggest also a connection with the description of Quantum Hall states. On the other hand, the restoration of the $\mathcal{N} = (1, 1)$ supersymmetries leads to the emergence of two dynamical Majorana fermions of opposite chirality at the boundary, and naively we could think of an edge theory with time-reversal symmetry, but the presence of a mass term for one of them makes this impossible. So, in both cases, we have effectively only one dynamical fermion at the boundary, such that the gapless content of the edge theories are similar to the $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (0, 1)$ cases.

The future goals include deepening the subject of edge theories, especially in the context of supersymmetries, in order to make a concrete connection with the edge states of topological phases of matter. For example, a possibility of obtaining a system with time-reversal symmetry, and thus obtaining an edge theory with the potential to describe topological insulators, would be to consider BF theories in the bulk, subject to $\mathcal{N} = 4$ supersymmetry constraint, in such way that two chiral Dirac fermions of opposite chirality and no mass terms would emerge at the boundary [28]. In addition, we would like to generalize our discussion for the non-Abelian case. Since its discovery in the context of the quantum Hall effect, the interest in non-Abelian phases of matter has been intensified with the discovery of several topological ordered phases possessing non-Abelian anyonic excitations and, in particular, because of the potential applications in quantum computing [29, 30]. So, it would be interesting to explore the bulk-edge correspondence also in non-Abelian supersymmetric Chern-Simons theories.

A Appendix: $\mathcal{N} = 1$ to $\mathcal{N} = (1,0)$ or $\mathcal{N} = (0,1)$ Multiplet Decomposition

Multiplet decomposition is a systematically procedure in which we decompose the $3D$ superfields $\mathbf{\Phi}$ and Γ_α into $2D$ $\mathcal{N} = (1,0)$ and $\mathcal{N} = (0,1)$ superfields transforming under ϵ_+ and ϵ_- supersymmetry, respectively. To this end, we first introduce the projectors

$$P_\pm = \frac{1}{2}(1 \pm \gamma^3), \tag{A.1}$$

and write the supercharges (5.3) as

$$Q_\pm = Q'_\pm \mp \theta_\pm \partial_3 = e^{\pm\theta_+\theta_-\partial_3} Q'_\pm e^{\mp\theta_+\theta_-\partial_3}, \tag{A.2}$$

where

$$Q'_{\pm\alpha} = \partial_{\pm\alpha} - (\gamma^m \theta_\mp)_\alpha \partial_m, \qquad \partial_{\pm\alpha} \equiv \frac{\partial}{\partial \theta^\alpha_\mp}, \qquad \theta_\pm \equiv P_\pm \theta \tag{A.3}$$

The bulk supercharges, $Q_\pm$, are the generators of $\epsilon_\mp$ supersymmetry transformations on $3D$ $\mathcal{N} = 1$ superfields

$$\delta_\pm \mathbf{\Phi} = \epsilon_\pm Q_\mp \mathbf{\Phi} \qquad \text{and} \qquad \delta_\pm \Gamma_\alpha = \epsilon_\pm Q_\mp \Gamma_\alpha. \tag{A.4}$$

The boundary supercharges, $Q'_\pm$, has the same form as (5.3). Thus, Q'_- can be viewed as the generator of ϵ_+ supersymmetry transformation on $2D$ $\mathcal{N} = (1,0)$ superfields, labeled by a hat,

$$\delta_+ \hat{\mathbf{\Phi}} = \epsilon_+ Q'_- \hat{\mathbf{\Phi}} \qquad \text{and} \qquad \delta_+ \hat{\Gamma}_\alpha = \epsilon_+ Q'_- \hat{\Gamma}_\alpha, \tag{A.5}$$

and Q'_+ as the generator of ϵ_- supersymmetry transformation on $2D$ $\mathcal{N} = (0,1)$ superfields, labeled by a tilde,

$$\delta_- \tilde{\mathbf{\Phi}} = \epsilon_- Q'_+ \tilde{\mathbf{\Phi}} \qquad \text{and} \qquad \delta_- \tilde{\Gamma}_\alpha = \epsilon_- Q'_+ \tilde{\Gamma}_\alpha. \tag{A.6}$$

Since the bulk and boundary supercharges are related by (A.2), a close look at all the above supersymmetry transformations suggests also a relation between bulk and boundary superfields. In fact, by inserting (A.2) into the transformations (A.5) and (A.6), we find that $\hat{\mathbf{\Phi}}$ and $\tilde{\mathbf{\Phi}}$ are related to $\mathbf{\Phi}$ as

$$\mathbf{\Phi} = e^{-\theta_+\theta_-\partial_3} \hat{\mathbf{\Phi}} \qquad \text{and} \qquad \mathbf{\Phi} = e^{+\theta_+\theta_-\partial_3} \tilde{\mathbf{\Phi}}, \tag{A.7}$$

as well as $\hat{\Gamma}_\alpha$ and $\tilde{\Gamma}_\alpha$ are related to Γ_α as

$$\Gamma_\alpha = e^{-\theta_+\theta_-\partial_3} \hat{\Gamma}_\alpha \qquad \text{and} \qquad \Gamma_\alpha = e^{+\theta_+\theta_-\partial_3} \tilde{\Gamma}_\alpha. \tag{A.8}$$

Writing the scalar superfields in terms of the projected coordinate $\theta_\pm$,

$$\begin{aligned}\mathbf{\Phi} &= A+\theta_+\psi_-+\theta_-\psi_++\theta_+\theta_-F, &(A.9)\\ \hat{\mathbf{\Phi}} &= \hat{A}(\theta_+)+\theta_-\hat{A}_+(\theta_+), &(A.10)\\ \tilde{\mathbf{\Phi}} &= \tilde{A}(\theta_-)+\theta_+\tilde{A}_-(\theta_-), &(A.11)\end{aligned}$$

and inserting them into the relation (A.7), we find the $2D$ $\mathcal{N}=(1,0)$ and $\mathcal{N}=(0,1)$ superfields advents from the $3D$ $\mathcal{N}=1$ scalar superfield $\mathbf{\Phi}$,

$$\begin{aligned}\hat{A} &= A+\theta_+\psi_-, &(A.12)\\ \hat{A}_+ &= \psi_++\theta_+(F+\partial_3A), &(A.13)\\ \tilde{A}_- &= \psi_-+\theta_-(F-\partial_3A), &(A.14)\\ \tilde{A} &= A+\theta_-\psi_+. &(A.15)\end{aligned}$$

The $\epsilon_\pm$ supersymmetry transformation of these superfield components is

$$\begin{aligned}\delta A=\epsilon_\pm\psi_\mp,\qquad &\delta\psi_\mp=\gamma^m\epsilon_\pm\partial_mA,\\ \delta\psi_\pm=(F\pm\partial_3A)\epsilon_\pm,\qquad &\delta(F\pm\partial_3A)=\epsilon_\pm\gamma^m\partial_m\psi_\pm. &(A.16)\end{aligned}$$

Similarly, writing the spinor superfields in terms of the projected coordinate $\theta_\pm$,

$$\begin{aligned}\mathbf{\Gamma}^\pm_\alpha &= \chi_{\pm\alpha}+\theta_{\pm\alpha}M+(\gamma^m\theta_\mp)_\alpha v_m\pm\theta_\pm v_3+\theta_+\theta_-\{\lambda_\pm-[(\gamma^m\partial_m\chi_\mp)\pm\partial_3\chi_\pm]\}_\alpha, &(A.17)\\ \hat{\mathbf{\Gamma}}^+_\alpha &= \hat{X}^+_\alpha(\theta_+)-\theta^\beta_+(\gamma^m)_{\beta\alpha}\hat{\mathbf{\Sigma}}^+_m(\theta_+), &(A.18)\\ \hat{\mathbf{\Gamma}}^-_\alpha &= \hat{X}^-_\alpha(\theta_+)+\theta_{-\alpha}\hat{\mathbf{\Sigma}}^-(\theta_+), &(A.19)\\ \tilde{\mathbf{\Gamma}}^+_\alpha &= \tilde{X}^+_\alpha(\theta_-)+\theta_{+\alpha}\tilde{\mathbf{\Sigma}}^+(\theta_-), &(A.20)\\ \tilde{\mathbf{\Gamma}}^-_\alpha &= \tilde{X}^-_\alpha(\theta_-)-\theta^\beta_-(\gamma^m)_{\beta\alpha}\tilde{\mathbf{\Sigma}}^-_m(\theta_-), &(A.21)\end{aligned}$$

and inserting them into the relation (A.8), we find the $2D$ $\mathcal{N}=(1,0)$ and $\mathcal{N}=(0,1)$ superfields advents from the $3D$ $\mathcal{N}=1$ spinor superfield $\mathbf{\Gamma}_\alpha$,

$$\begin{aligned}\hat{X}^+_\alpha &= \chi_{+\alpha}+\theta_{+\alpha}(M+v_3), &(A.22)\\ \hat{\mathbf{\Sigma}}^+_m &= v_m-\theta^\alpha_+(\frac{1}{2}(\gamma_m\lambda_+)_\alpha-\partial_m\chi_{-\alpha}), &(A.23)\\ \hat{X}^-_\alpha &= \chi_{-\alpha}+(\gamma^m\theta_+)_\alpha v_m, &(A.24)\\ \hat{\mathbf{\Sigma}}^- &= (M-v_3)-\theta^\alpha_+(\lambda_{-\alpha}+2\partial_3\chi_{-\alpha}-(\gamma^m\partial_m\chi_+)_\alpha), &(A.25)\\ \tilde{X}^-_\alpha &= \chi_{-\alpha}+\theta_{-\alpha}(M-v_3), &(A.26)\\ \tilde{\mathbf{\Sigma}}^-_m &= v_m-\theta^\alpha_-(\frac{1}{2}(\gamma_m\lambda_-)_\alpha-\partial_m\chi_{+\alpha}), &(A.27)\\ \tilde{X}^+_\alpha &= \chi_{+\alpha}+(\gamma^m\theta_-)_\alpha v_m, &(A.28)\\ \tilde{\mathbf{\Sigma}}^+ &= (M+v_3)-\theta^\alpha_-(\lambda_{+\alpha}-2\partial_3\chi_{+\alpha}-(\gamma^m\partial_m\chi_-)_\alpha). &(A.29)\end{aligned}$$

Finally, the $\epsilon_\pm$ supersymmetry transformation for these superfield components is

$$\begin{gathered}
\delta\chi_\pm = \epsilon_\pm(M \pm v_3), \quad \delta(M \pm v_3) = \epsilon_\pm\gamma^m\partial_m\chi_\pm, \\
\delta v_3 = \epsilon_\pm(\tfrac{1}{2}\lambda_\mp + \partial_3\chi_\mp), \quad \delta(\tfrac{1}{2}\lambda_\mp + \partial_3\chi_\mp) = \gamma^m\epsilon_\pm\partial_m v_3, \\
\delta\chi_\mp = \gamma^m\epsilon_\pm v_m, \quad \delta v_m = -\tfrac{1}{2}\epsilon_\pm\gamma_m\lambda_\pm + \epsilon_\pm\partial_m\chi_\mp, \quad \delta\lambda_\pm = -2\epsilon^{mno}\gamma_o\partial_m v_n\epsilon_\pm.
\end{gathered} \tag{A.30}$$

B Appendix: $\mathcal{N}=2$ to $\mathcal{N}=(2,0)$ or $\mathcal{N}=(1,1)$ Multiplet Decomposition

Here, we will show how to decompose the $3D$ $\mathcal{N}=2$ vector field V into $2D$ $\mathcal{N}=(2,0)$ and $\mathcal{N}=(1,1)$ superfields. Firstly, note that the $\mathcal{N}=(2,0)$ half supersymmetry is a combination of two $\mathcal{N}=(1,0)$ half supersymmetries, generated by (Q'_{1-},Q'_{2-}). Similarly, the $\mathcal{N}=(1,1)$ half supersymmetry is a combination of $\mathcal{N}=(0,1)$ and $\mathcal{N}=(1,0)$ half supersymmetries, generated by (Q'_{1+},Q'_{2-}). According to (A.2), these boundary supercharges are related to the bulk supercharges $(Q_{1\pm},Q_{2-})$, as

$$\begin{aligned} Q_{1\pm} &= Q'_{1\pm}\mp\theta_{1\pm}\partial_3 = e^{\pm\theta_{1+}\theta_{1-}\partial_3}Q'_{1\pm}e^{\mp\theta_{1+}\theta_{1-}\partial_3}, &\text{(B.1)}\\ Q_{2-} &= Q'_{2-}+\theta_{2-}\partial_3 = e^{-\theta_{2+}\theta_{2-}\partial_3}Q'_{2-}e^{+\theta_{2+}\theta_{2-}\partial_3}. &\text{(B.2)}\end{aligned}$$

Starting with the $3D$ $\mathcal{N}=2$ vector superfield (5.60)

$$V = A(\theta_1)+\theta_2\Gamma+\theta_2^2(B(\theta_1)+D_1^2A(\theta_1)), \tag{B.3}$$

where

$$A=(a,\psi_\alpha,f),\qquad B=(b,\eta_\alpha,g),\qquad \Gamma_\alpha=(\chi_\alpha,M,v_\mu,\lambda_\alpha), \tag{B.4}$$

we can simply follow the procedures in Appendix A to construct the boundary superfields. The $2D$ $\mathcal{N}=(2,0)$ superfields are related to V as

$$V = e^{-\theta_{2+}\theta_{2-}\partial_3}e^{-\theta_{1+}\theta_{1-}\partial_3}(\hat{\hat{V}}(\theta_{1+},\theta_{2+})+\theta^\alpha_{1-}\hat{\hat{V}}_\alpha+\theta^\alpha_{2-}\hat{\hat{U}}_\alpha+\theta^\alpha_{2-}(\gamma^m\theta_{1-})_\alpha\hat{\hat{V}}_m), \tag{B.5}$$

and the $2D$ $\mathcal{N}=(1,1)$ superfields as

$$V = e^{-\theta_{2+}\theta_{2-}\partial_3}e^{+\theta_{1+}\theta_{1-}\partial_3}(\tilde{\hat{V}}(\theta_{1-},\theta_{2+})+\theta^\alpha_{1+}\tilde{\hat{V}}_\alpha+\theta^\alpha_{2-}\tilde{\hat{U}}_\alpha+\theta^\alpha_{2-}\theta_{1+\alpha}\tilde{\hat{U}}). \tag{B.6}$$

The $(\theta_{1\pm},\theta_{2\pm})$ expansion of the bulk vector superfield is

$$\begin{aligned} V &= a+\theta_{1+}\psi_-+\theta_{1-}\psi_++\theta_{1+}\theta_{1-}f+\theta_{2+}\left[\chi_-+\theta_{1-}(M-v_3)+(\gamma^m\theta_{1+})v_m\right.\\ &+ \theta_{1+}\theta_{1-}(\lambda_--\gamma^m\partial_m\chi_++\partial_3\chi_-)]+\theta_{2-}[\chi_++\theta_{1+}(M+v_3)+(\gamma^m\theta_{1-})v_m\\ &+ \theta_{1+}\theta_{1-}(\lambda_+-\gamma^m\partial_m\chi_--\partial_3\chi_+)]+\theta_{2+}\theta_{2-}[(b-f)+\theta_{1+}(\eta_--\gamma^m\partial_m\psi_+\\ &+ \partial_3\psi_-)+\theta_{1-}(\eta_+-\gamma^m\partial_m\psi_--\partial_3\psi_+)+\theta_{1+}\theta_{1-}(g-\partial_m\partial^m a-\partial_3\partial^3 a)]. \end{aligned} \tag{B.7}$$

So, back to (B.5) and (B.6), we find that the $\mathcal{N}=(2,0)$ and $\mathcal{N}=(1,1)$ boundary superfields are

$$
\begin{aligned}
\hat{\hat{V}} &= a+\theta_{1+}\psi_{-}+\theta_{2+}\chi_{-}+\theta_{2+}(\gamma^m\theta_{1+})v_m, && \text{(B.8)}\\
\hat{\hat{V}}_\alpha &= \psi_{+\alpha}+\theta_{1+\alpha}(f+\partial_3 a)+\theta_{2+\alpha}(M-v_3)+\theta_{1+\alpha}\theta_{2+}(\lambda_{-}-\gamma^m\partial_m\chi_{+}+2\partial_3\chi_{-}), && \text{(B.9)}\\
\hat{\hat{U}}_\alpha &= \chi_{+\alpha}+\theta_{1+\alpha}(M+v_3)+\theta_{2+\alpha}(b-f+\partial_3 a)\\
&+ \theta_{2+\alpha}\theta_{1+}(\eta_{-}-\gamma^m\partial_m\psi_{+}+2\partial_3\psi_{-}), && \text{(B.10)}\\
\hat{\hat{V}}_m &= v_m+\theta_{1+}(\frac{1}{2}\gamma_m\lambda_{+}-\partial_m\chi_{-})+\theta_{2+}(\frac{1}{2}\gamma_m\eta_{+}-\partial_m\psi_{-})\\
&+ \frac{1}{2}(\theta_{2+}\gamma_m\theta_{1+})(g-\partial_m\partial^m a+\partial_3 b), && \text{(B.11)}\\
\tilde{\tilde{V}} &= a+\theta_{1-}\psi_{+}+\theta_{2+}\chi_{-}+\theta_{2+}\theta_{1-}(M-v_3), && \text{(B.12)}\\
\tilde{\tilde{V}}_\alpha &= \psi_{-\alpha}+\theta_{1-\alpha}(f-\partial_3 a)+(\theta_{2+}\gamma^m)_\alpha v_m+\theta_{1-\alpha}\theta_{2+}(\lambda_{-}-\gamma^m\partial_m\chi_{+}), && \text{(B.13)}\\
\tilde{\tilde{U}}_\alpha &= \chi_{+\alpha}+(\gamma^m\theta_{1-})_\alpha v_m+\theta_{2+\alpha}(b-f+\partial_3 a)+\theta_{2+\alpha}\theta_{1-}(\eta_{+}-\gamma^m\partial_m\psi_{-}), && \text{(B.14)}\\
\tilde{\tilde{U}} &= (M+v_3)+\theta_{1-}(\lambda_{+}-\gamma^m\partial_m\chi_{-}-2\partial_3\chi_{+})+\theta_{2+}(\eta_{-}-\gamma^m\partial_m\psi_{+}+2\partial_3\psi_{-})\\
&+ \theta_{2+}\theta_{1-}(g-\partial_m\partial^m a-2\partial_3\partial_3 a-\partial_3 b+2f). && \text{(B.15)}
\end{aligned}
$$

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