Universidade Estadual de Londrina

Weslei Bernardino Fontana

S-AdS Black Hole Thermodynamics

Londrina
2017

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Dissertação de mestrado apresentada ao Departamento de Física da Universidade Estadual de Londrina, como requisito parcial à obtenção do título de Mestre.
Orientador: Prof. Dr. Mario César Baldiotti

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BANCA EXAMINADORA

____
Orientador: Prof. Dr. Mario Cesar Baldiotti
Universidade Estadual de Londrina - UEL

____
Prof. Dr. Pedro Rogerio Sergi Gomes
Universidade Estadual de Londrina - UEL

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Prof. Dr. Carlos André Hernaski
Universidade Estadual de Londrina - UEL

Londrina, _____ de ________________ de _______.

À minha mãe, Derli Bernardino.

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"Bear in mind that the wonderful things that you learn in your schools are the work of many generations. All this is put into your hands as your inheritance in order that you may receive it, honor it, add to it, and one day faithfully hand it on to your children."

Albert Einstein

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Resumo

Durante os últimos 50 anos, a pesquisa acerca da teoria de buracos negros em relatividade geral trouxe luz para a existência de uma forte relação entre gravitação, termodinâmica e teoria de campos. Uma das partes mais importantes dessa conexão entre as teorias é a termodinâmica de buracos negros, que nos diz que algumas das leis que regem a mecânica de buracos negros são simplesmente as leis usuais da termodinâmica aplicada à um sistema com um buraco negro. O intuito deste trabalho é fazer uma revisão sobre o que se conhece atualmente acerca do assunto. Em especial, aplicado à teoria de buracos negros em espaços-tempo assintóticamente Anti de-Sitter. Daremos um enfoque especial ao formalismo Hamiltoniano da termodinâmica, o qual nos permite estender a descrição termodinâmica de buracos negros, ao se introduzir a constante cosmológica como um parâmetro no espaço de fase da teoria termodinâmica. O que nos fornece uma nova teoria termodinâmica com um grau de liberdade extra.

Palavras-chave: Buraco Negro. Geometria Anti de-Sitter. Termodinâmica. Relatividade Geral. Formalismo Hamiltoniano.

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Abstract

For the last 50 years, the research around the theory of black holes in general relativity brought to light the existence of a strong relationship between gravitation, thermodynamics and field theory. One of the most important parts of this relationship between the theories, is the thermodynamics of black holes, which tell us that some of the laws that dictates the behavior of black holes are, in fact, the usual thermodynamical laws applied to a system containing a black hole. The goal of this work is to make a review about what is known in the present days about the subject, in special, applied to black holes in asymptotically Anti de-Sitter spacetimes. We shall focus on the Hamiltonian formalism to thermodynamics, which allow us to find a consistent way of extending the thermodynamic description for the black hole, by introducing the cosmological constant as a parameter in the phase space of the thermodynamical theory, giving birth to a new description with a new degree of freedom.

Keywords: Black Hole. Anti de-Sitter Geometry. Thermodynamics. General Relativity. Hamiltonian Formalism.

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Sumário

1 Schwarzchild Black hole and the minimal thermodynamics

In this chapter, we will introduce the Schwarzchild solution for the Einstein's equations. We assume some basic knowledge of general relativity, if the reader is unfamiliar with the theory we present a brief review of the general aspects in the appendix A. In the end of the first section we generalize our notions to the Schwarzchild black hole in the AdS spacetime. In the second section we introduce the notions of the minimal thermodynamics for black holes where, by minimal we mean the usual theory that is discussed in the most of the literature.

This chapter will be based on the references [8, 17, 11, 1].

1.1 Schwarzchild solution

The simplest solution for the Einstein's equation is know as *Schwarzchild geometry.* The Schwarzchild metric is the static spherically solution for the Einstein's equations. In order to see that, let us first consider the Minkowski spacetime written in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2, \tag{1.1}$$

Where,

$$d\Omega_2 = d\phi^2 + \sin^2\theta d\theta^2.$$

The idea is the following, we want to generalize the metric written in (1.1) in such a way to introduce a curvature, without breaking up the static and spherically symmetry conditions. A static metric is the one that does not have time dependency and is time reversal invariant, and the spherical symmetry tell us that the metric also should not depends on the angular variables θ and ϕ. Thus, the only option we have is to introduce a term in (1.1) that is a function of the radial variable r. One can write the generalized curved metric in an appropriate form

$$ds^2 = -e^{-2f(r)}dt^2 + e^{2h(r)}dr^2 + r^2 d\Omega_2^2, \tag{1.2}$$

Where the functions $f(r)$ and $h(r)$ are to be determined by the Einstein's equations. With (1.2) one can calculate the Christoffel symbols (A.12) for this metric and find

$$\Gamma^t_{tr} = \frac{df}{dr}, \quad \Gamma^r_{tt} = \frac{dh}{dr}e^{2(f-h)}, \quad \Gamma^r_{rr} = \frac{dh}{dr}, \quad \Gamma^r_{\theta\theta} = -re^{-2h},$$
$$\Gamma^r_{\phi\phi} = -r\sin^2\theta\, e^{-2h}, \quad \Gamma^\theta_{r\theta} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta,$$
$$\Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \cot\theta. \tag{1.3}$$

We can now calculate the components of the Ricci tensor and find,

$$R_{tt} = e^{2(f-h)}\left(\frac{d^2f}{dr^2} + \left(\frac{df}{dr}\right)^2 - \frac{df}{dr}\frac{dh}{dr} + \frac{2}{r}\frac{df}{dr}\right), \quad R_{rr} = -\left(\frac{d^2f}{dr^2} + \left(\frac{df}{dr}\right)^2 - \frac{df}{dr}\frac{dh}{dr} - \frac{2}{r}\frac{dh}{dr}\right),$$
$$R_{\theta\theta} = 1 - e^{-2h}\left(1 + r\frac{df}{dr} - r\frac{dh}{dr}\right), \quad R_{\phi\phi} = R_{\theta\theta}\sin^2\theta. \tag{1.4}$$

From (1.4) one can obtain the Ricci scalar,

$$R = -2e^{-2h}\left[\frac{d^2f}{dr^2} + \left(\frac{df}{dr} + \frac{2}{r}\right)^2\left(\frac{df}{dr} - \frac{dh}{dr}\right) + \frac{1}{r^2}\left(1 - e^{2h}\right)\right]. \tag{1.5}$$

Thus, one can finally obtain the Einstein tensor,

$$G_{tt} = \frac{1}{r^2}e^{2f}\frac{d}{dr}\left[r\left(1 - e^{2h}\right)\right], \quad G_{rr} = \frac{1}{r^2}\left(1 + 2r\frac{df}{dr} - e^{2h}\right),$$
$$G_{\theta\theta} = r^2e^{-2h}\left[\frac{d^2f}{dr^2} + \left(\frac{df}{dr} + \frac{1}{r}\right)\left(\frac{df}{dr} - \frac{dh}{dr}\right)\right], \quad G_{\phi\phi} = G_{\theta\theta}\sin^2\theta. \tag{1.6}$$

So far we explored only the left side of the Einstein's equations, now we want to look at right side. Simplicity has been our best friend and with this in mind, we want to work with the simplest model to the stress energy tensor, which is the one for a perfect fluid. What we want to do now is to write down this tensor in order to represent a static sphere with radius R, density $\rho(r)$ and pressure p. Thus we have,

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}. \tag{1.7}$$

For a static fluid $u_r = u_\theta = u_\phi = 0$. Thus, this stress energy tensor can be written, using our metric (1.2), as

$$T_{tt} = \rho e^{2f}, \quad T_{rr} = pe^{2h}, \quad T_{\theta\theta} = pr^2, \quad T_{\phi\phi} = pr^2\sin^2\theta. \tag{1.8}$$

Due to spherical symmetry, the density and pressure of the fluid can depend only on r, and for any value $R > 0$ it must vanish. Thus, using (1.6) and (1.8), one can obtain the following differential equations,

$$\frac{1}{r^2}e^{2f}\frac{d}{dr}\left[r\left(1-e^{2h}\right)\right] = 8\pi e^{2f}\rho\left(r\right), \tag{1.9}$$

$$\frac{1}{r^2}\left(1+2r\frac{df}{dr}-e^{2h}\right) = 8\pi e^{2h}p\left(r\right), \tag{1.10}$$

$$r^2e^{-2h}\left[\frac{d^2f}{dr^2}+\left(\frac{df}{dr}+\frac{1}{r}\right)\left(\frac{df}{dr}-\frac{dh}{dr}\right)\right] = 8\pi r^2p\left(r\right). \tag{1.11}$$

Straightforward integration of (1.9) gives

$$h\left(r\right) = -\frac{1}{2}\ln\left(1-\frac{2m\left(r\right)}{r}\right), \quad m\left(r\right)\equiv 4\pi\int dr\, r^2\rho\left(r\right). \tag{1.12}$$

Choosing the condition where $m\left(0\right)=0$ allow us to interpret $m\left(r\right)$ as the mass inside the radius r. Integration of (1.10) gives

$$f\left(r\right) = \int dr\,\frac{m\left(r\right)+4\pi r^3p\left(r\right)}{r\left(r-2m\left(r\right)\right)}. \tag{1.13}$$

For any value of r inside the sphere, the above integral is very complicated and for our interests this region will not be important. So, from now on, we will discuss only the region outside the mass-energy distribution. With the approximation $r>R$. We note that $m\left(r\right)=M$, which is the total mass of our distribution, besides $p\left(r\right)=0$ because of the symmetry. Integration on (1.13) gives,

$$f\left(r\right) = \frac{1}{2}\ln\left(1-\frac{2M}{r}\right). \tag{1.14}$$

The equation (1.11) will only give information about the pressure distribution of matter within the sphere. But now we are focusing only in regions outside the sphere. So, for our purposes, the solution of (1.11) is not interesting. Thus, for the region outside the matter distribution, the spacetime can be described by the metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{\left(1-\frac{2M}{r}\right)}+r^2d\Omega_2^2. \tag{1.15}$$

Eq.(1.15) it is the so called *Schwarzchild solution.* It is worth to know that this is the only time independent spherically symmetric solution to Einstein's equations in the vacuum, this is stated by the *Birkhoff's theorem.*

In the AdS spacetime the idea is pretty much the same, the only change is the appearance of the cosmological constant in the functions above. Thus, the Schwarzchild-AdS metric has the form,

$$ds^2 = -\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)dt^2+\frac{dr^2}{\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)}+r^2d\Omega_2^2. \tag{1.16}$$

One can generalize the line element in (1.16) to D dimensions and obtain

$$ds^2 = -\left(1 - \frac{\tilde{M}}{r^{D-3}} - \tilde{\Lambda} r^2\right) dt^2 + \frac{dr^2}{\left(1 - \frac{\tilde{M}}{r^{D-3}} - \tilde{\Lambda} r^2\right)} + r^2 d\Omega_{D-2}^2. \tag{1.17}$$

Where,

$$d\Omega_{D-2}^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_1^2 + \ldots + \sin^2\theta_1 \ldots \sin^2\theta_{D-2} d\theta_{D-2}^2. \tag{1.18}$$

Is the line element over the unit sphere $\mathbb{S}^{D-2}$. The other quantities that appear in (1.16) are defined as

$$\tilde{M} = \frac{16\pi}{(D-2)\, B_{D-2}} M, \quad B_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}, \quad \tilde{\Lambda} = \frac{2}{(D-1)(D-2)} \Lambda. \tag{1.19}$$

B_{D-2} can be seen as the "area" of the $\mathbb{S}^{D-2}$.

1.2 Minimal Thermodynamics for black holes

During the past 50 years, the work in the theory of black holes in general relativity has brought to light hints of a fundamental relationship between gravitation, thermodynamics, and quantum theory. The cornerstone of this relationship is black hole thermodynamics, where it appears that certain laws of black hole mechanics are, in fact, simply the ordinary laws of thermodynamics applied to a system containing a black hole. Indeed, the discovery of the thermodynamic behavior of black holes—achieved primarily by classical and semi classical analyses—has given rise to most of our present physical insights into the nature of quantum phenomena occurring in strong gravitational fields.

We can understand a black hole as a thermodynamic system, by the following analogy. Thermodynamics is the theory of many molecules or atoms, but in order to describe the thermodynamic system one does not have to specify, for example, the position nor the momentum of each atom or molecule, the system can be described by a few macroscopic variables such as pressure and temperature. This prescription of going from microscopic variables to macroscopic variables is often called as coarse-grained description.

Now, let us take a look at a stationary black hole, thanks to the *no-hair theorem[17]* we know that a black hole can be described only by a few parameters: the mass, angular momentum and charge. Thus, the black hole does not depends on the properties of the original stars, such as shape, composition and etc. From this point of view, this peculiar property of black holes, suggests that somehow the black hole is a coarse-grained description. In the following sections we must investigate how this description is formulated and, in this work, we will call this description *the minimal thermodynamics for black holes.*

First we will consider the minimal description for the Schwarzchild black hole and, later, the minimal description for the Schwarzchild-AdS black hole.

1.2.1 Zeroth law for black holes

In the thermodynamic context we want to describe systems in a very special state called *equilibrium state,* in this particular state the temperature of the system becomes constant everywhere. The zeroth law of thermodynamics state the existence of such equilibrium states and, with that, the existence of a temperature function. With black holes we have a similar situation.

When a black hole is formed, it can have angular momentum and charge, besides of course, its mass. So, essentially the black hole will be asymmetric when it is formed, but, eventually it will become spherical symmetric Schwarzchild solution, which have a constant surface gravity everywhere over the horizon. A stationary black hole, whose horizon gravity becomes constant is, in a sense, a equilibrium state. This will make more sense in the future, when we identify the temperature of the black hole with this surface gravity. For the time, it suffices to see that, in the black hole context, the zeroth law of thermodynamics translates in this way.

1.2.1.1 Surface gravity

Consider a static metric of the form

$$ds^2 = -f\left(r\right)dt^2 + \frac{dr^2}{f\left(r\right)} + \ldots . \tag{1.20}$$

Where the "..." represents the line element over the horizon which will be irrelevant for our discussion. The surface gravity is defined as the force per unit mass $a_\infty\left(r_0\right)$ that is necessary to hold the particle at the horizon by an asymptotic observer. One can use the equation (A.16) that gives the components of the 4 acceleration and then find that the magnitude of this acceleration is given by

$$a^2 := g_{\mu\nu}a^\mu a^\nu = \frac{f'^2}{4f}, \tag{1.21}$$

$$a = \frac{f'}{2f^{1/2}}. \tag{1.22}$$

Where the $' = \partial_r$. This acceleration is the proper acceleration for an observer in the region near the horizon. To find the surface gravity we must obtain the acceleration as measured by the asymptotic observer. To do this we can imagine that the asymptotic observer pulls a particle at some position r, using a massless "string", if the observer pulls the particle by some proper distance δs, the work done is given by

$$W_\infty = a_\infty \delta s, \quad (asymptotic\,infinity)\,, \tag{1.23}$$

$$W_r = a\delta s, \quad (at\,position\,r)\,. \tag{1.24}$$

We can convert the work into energy and collect this energy at the infinity. The energy measured by the asymptotic observer will suffer the effects of the gravitational red shift, which is given by the formula

$$\frac{E_B}{E_A} = \sqrt{\frac{g_{00}(A)}{g_{00}(B)}}. \tag{1.25}$$

So, the energy that will be measured at infinity is given by

$$E_\infty = \sqrt{\frac{f(r)}{f(\infty)}} E_r = f(r)^{1/2}\, a\delta s. \tag{1.26}$$

Where we used the fact that $f(\infty) = 1$. From (1.26) one can get that

$$\kappa := a_\infty(r_0) = \frac{f'(r_0)}{2}. \tag{1.27}$$

Where κ is the surface gravity.

1.2.2 Black holes have temperature

In 1974, Hawking made the startling discovery that the physical temperature of a black hole is not absolutely zero, as a result of a quantum particle creation effects (Hawking radiation), a black hole radiates to infinity all species of particles with a perfect black body spectrum, with a temperature given by

$$T = \frac{\kappa}{2\pi}. \tag{1.28}$$

Where κ is the surface gravity derived in the last section. In the modern days there are several ways to derive this temperature. The original calculation, did by Hawking, considered a classical spacetime $(\mathbb{M}, g)$ describing the gravitational collapse to a Schwarzchild black hole. Then he considered a free quantum field propagating in this background spacetime, which is initially in its vacuum state prior the collapse, and he computed the particle content of this field at infinity at late times. His calculation revealed that at late times, the expected number of particles at infinity corresponds to the emission from a perfect black body, with finite size, at the Hawking temperature (1.28). It should be noted that this calculation relies only in the analysis of quantum fields in the region exterior to the black hole. Thus, it does not make use of any gravitational field equations.

Nowadays we have simpler ways of finding this temperature, and it is one of these simpler ways that we will use in this section. From quantum statistical mechanics we learn that the periodicity in the imaginary time is naturally associated with the inverse temperature by $t_E \to t_E + \beta$, where t_E is the periodic time and β the inverse temperature. We are going to mimic this technique and apply to black holes.

Let us start by making the analytically continuation of the Schwarzchild metric to the Euclidean signature with $t \to i\tau$,

$$ds_E^2 = f(r)\, dt_E^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2. \tag{1.29}$$

Consider the region near the horizon $r \approxeq r_0$. In this region one can expand the function $f(r)$ up to first order in the quantity $(r - r_0)$ and obtain

$$ds_E^2 = f'(r_0)(r - r_0)\, dt_E^2 + \frac{dr^2}{f'(r_0)(r - r_0)} + r^2 d\Omega_2^2. \tag{1.30}$$

Now, make the coordinate transformation $\rho = 2\sqrt{(r - r_0)/f'(r_0)}$. With this new coordinate the metric in (1.30) assumes the form

$$ds_E^2 = d\rho^2 + \rho^2 d\theta^2 + r^2 d\Omega_2^2, \quad \theta = \frac{f'(r_0)}{2} t_E. \tag{1.31}$$

The metric (1.31) has the topology $\mathbb{R}^2 \times \mathbb{S}^2$. One can easily see that the first two terms in (1.31) describes the polar coordinates in an euclidean manifold and have a conical singularity at $\rho = 0$ unless the coordinate θ have a 2π period. The requirement that the black hole metric should be smooth in the Euclidean signature gives the condition

$$\beta = \frac{4\pi}{f'(r_0)}. \tag{1.32}$$

And the temperature follows from (1.32) immediately and coincides with (1.28). For the Schwarzchild case, one can find that the temperature is given by

$$k_B T = \frac{\hbar c^3}{8\pi G M}. \tag{1.33}$$

Where we have introduced all the constants again in equation (1.33).

A few remarks: This is the temperature felt by an asymptotic observer. We have made the identification with the inverse temperature using the proper time t_E of the asymptotic observer. Another observer located , at some position r in the spacetime, will feel another temperature and one can obtain this using the same procedure, but making the identification $t \to t + \beta$, where t is the proper time of this other observer.

In the particular case of the Schwarzchild metric, the observer at the position r will feel a temperature given by,

$$T_{local}(r) = T_\infty f^{-1/2}(r). \tag{1.34}$$

Where T_∞ is the temperature of the asymptotic observer. As one approach the horizon, $f(r)$ goes to zero and, this temperature goes to infinity. Thus, a stationary observer near the horizon feels a huge temperature. The region near the horizon of the black hole is a very hot place!

1.2.3 First law

As was shown by Wald in [23] the first law of black hole thermodynamics follows from considering the Lagrangian of the gravitational theory as a $n - form$ $\boldsymbol{L}$ rather than a scalar density. $\boldsymbol{L}$ is required to be a function of the spacetime metric $g_{\mu\nu}$ and differentiable at each point p of the spacetime[1]. In this approach, the Lagrangian is invariant under diffeomorphisms, which means that for any diffeomorphism $\psi : M \to M$, we have,

$$\boldsymbol{L}[\psi^*(\phi)] = \psi^* \boldsymbol{L}[\phi]. \tag{1.35}$$

Where ψ^* is the pullback and ϕ represents all the dynamical fields of the theory, including the metric. ψ^* is applied only to the dynamical quantities. Wald in his work considered arbitrary variations of the dynamical fields ϕ on the Noether current that arise from the diffeomorphism symmetry. And, after a not so straightforward calculation, he was able to show that with this procedure the result was the first law of the black hole thermodynamics,

$$dM = \frac{\kappa}{8\pi} dA. \tag{1.36}$$

Where M is the black hole mass, κ the surface gravity and A the area of the black hole. We shall explore more this law in the following sections. The terms proportional to the charge Q and angular momentum J arises naturally when one consider a more complicated scenario.

1.2.4 Area law

To derive the area law let us first introduce the partition function for the gravitational theory, which can be written as,

$$Z(\beta) = \int Dg\, e^{-S[g]}. \tag{1.37}$$

[1] The number of derivatives is left undefined, thus, higher derivatives theories are also included in this framework.

Where the integral is performed over all the Riemannian metrics satisfying certain asymptotic fall off conditions. Using the saddle point approximation, (1.37) can be written as

$$Z(\beta) \approxeq e^{-S_{cl}}. \tag{1.38}$$

Where S_{cl} denotes the value of the classical action for the saddle point geometries. It is worth to say that the saddle point geometries are also solutions of the classical equations of motion. For our purposes, the action appearing in (1.38) can be schematically written as

$$S_{cl} = S_{bulk} + S_{GH} + S_{CT}. \tag{1.39}$$

We shall give some attention now for each one of the terms in (1.39). The bulk action[2] above is just the usual Einstein-Hilbert action, already mentioned in the first chapter, that for the AdS spacetime is written as,

$$S_{Bulk} = S_{EH} = -\frac{1}{16\pi G_D}\int d^D x\sqrt{g}\,(R-2\Lambda). \tag{1.40}$$

The second term in (1.39) is the Gibbons-Hawking action, which is written as,

$$S_{GH} = -\frac{1}{8\pi G_D}\int d^{D-1}x\sqrt{\gamma}K. \tag{1.41}$$

The Gibbons-Hawking action is a surface term which is needed in order to have a well defined variational problem. γ is the determinant of the induced metric at the boundary, and K is the trace of the extrinsic curvature, both follows from the decomposition of the metric[3],

$$ds_D^2 = g_{rr}dr^2 + \gamma_{\mu\nu}dx^\mu dx^\nu, \tag{1.42}$$

$$K = n^r\frac{\partial_r\sqrt{\gamma}}{\sqrt{\gamma}}, \quad n^r = \frac{1}{\sqrt{g_{rr}}}. \tag{1.43}$$

The diagonal form in (1.42) is just for simplicity. n^r is the unit normal vector of the surface $r = const$ that points in the direction of increasing r, outwards the boundary surface. Let us now motivate the introduction of the Gibbon-Hawking term. Consider the usual Einstein-Hilbert action, the Ricci tensor involves two derivatives acting on the metric component $g_{\mu\nu}$. However, in general, the Lagrangian contains only first derivatives (and we are not looking to introduce derivatives of higher orders here), one can perform a integration by parts in the Einstein-Hilbert action and, the boundary

2 We are evaluating all the integrals in the Euclidean signature where our time coordinate have the periodicity β, that is why the volume element is proportional to $\sqrt{g}$ and not $\sqrt{-g}$.

3 This decomposition is know as the ADM - decomposition.

term that arises from this process, is exactly the Gibbons-Hawking action. In other words, we introduce the Gibbons-Hawking action in (1.39) to cancel the boundary term that arises from the variation of the Einstein-Hilbert term, and, with that, we have a well defined variational problem for the gravity theory. Substituting (1.42) and (1.43) in (1.41), we obtain,

$$S_{GH} = -\frac{1}{8\pi G_D}\int d^{D-1}x\, n^r \partial_r \sqrt{\gamma}. \tag{1.44}$$

The last term in (1.39) is the counter term action. When we calculate the Einstein-Hilbert and the Gibbons-Hawking actions, we will see that both of them diverges asymptotically. The counter term is added to the action to deal with those divergences. The addition of this term is necessary to have a finite gravitational partition function[4]. In general, the form of the counter term will depends upon the particular problem we are dealing with, and it is written with the quantities at the boundary. For our purposes, the following counter term will be enough,

$$S_{CT} = -\frac{1}{8\pi G_D}\int d^{D-1}x \sqrt{\gamma_M} K_M. \tag{1.45}$$

The index M stands for quantities evaluated with the Minkowski metric. The counter term (1.45) play the role of an action for a reference background, in this case Minkowski, which we are subtracting from the other actions to regularize the solution.

Consider now the Schwarzchild case, the bulk term goes to zero and the only contributions arises from the boundary terms,

$$S_{cl} = -\frac{1}{8\pi G_D}\int d^{D-1}x \left(\sqrt{\gamma}K - \sqrt{\gamma_M}K_M\right). \tag{1.46}$$

For the second term in (1.46), the temperature is arbitrary, but we will fix it such that, asymptotically, the metric matches with the Schwarzchild metric. In other words, the following equalities must hold,

$$\int_0^{\beta} d\tau \sqrt{\gamma_{\tau\tau}} = \int_0^{\beta_0} d\tau \sqrt{\gamma_{\tau\tau}}_M, \tag{1.47}$$

$$\beta\left(1 - \frac{2GM}{R}\right) = \beta_0. \tag{1.48}$$

For some large R. Calculating the action (1.46), substituting the value of β_0 found in (1.48) and taking the limit $R \to \infty$, we find,

$$S_{cl} = \frac{M}{2T} = \frac{\hbar c^5}{16\pi G k_B T^2}. \tag{1.49}$$

[4] This procedure also have the name of *Holographic renormalization*

Where we have used equation (1.33). From (1.49) we find that the partition function is given by,

$$Z = e^{-\frac{\hbar c^5}{16\pi G k_B T^2}}. \tag{1.50}$$

Which gives the free energy,

$$F = -\frac{1}{\beta}\ln(Z) = \frac{\hbar c^5 \beta}{16\pi G}. \tag{1.51}$$

And the entropy follows immediately,

$$S = k_B \beta^2 \frac{\partial F}{\partial \beta} = \frac{\hbar c^5}{16\pi G k_B T^2} = \frac{k_B A}{4G\hbar} c^3. \tag{1.52}$$

S in (1.52) is the Bekenstein-Hawking entropy. As we can see, the entropy for the Schwarzchild black hole is proportional to it horizon area.

1.3 Minimal thermodynamics for the S-AdS black hole

The thermodynamic description for the Schwarzchild-AdS spacetime postulates an ensemble of AdS black holes with no charge, nor angular momentum, each one in equilibrium with its thermal Hawking atmosphere, as described in [24, 15, 17].

From the previous sections we obtained the results that characterizes the thermodynamics of the Schwarzchild black holes. But, even though we have made the calculations for this particular case, the following results are general for the minimal description

$$U = M, \quad S = \frac{A}{4}, \quad T = \frac{\kappa}{2\pi}. \tag{1.53}$$

In the minimal description, the black hole thermodynamics is characterized by the identification made in (1.53). Some important features are worth to mention. In the minimal description we are promoting the mass of the black hole as an thermodynamical variable, this means that we are allowing the mass to vary with, for example, the radius of the horizon. But the geometry is completely characterized by Einstein's equations! The metric for the Schwarzchild Anti de-Sitter black hole will be a solution of those equations, only and if only, the mass is a constant. Thus, the question that arises is how do we compatibilize the fact that the mass of the black hole is a constant in the geometry but not in the thermodynamic description?

The answer is quite simple, but have to be told in order to fully characterize the description. Imagine that we have a black hole with mass M, and the geometry of the spacetime is characterized by Einstein's equations due to the presence of the black hole and, the total mass of the black hole is a constant in the theory. Now, we promote the mass to a equation of state, thus, the mass is allowed to vary, but this variation is given in a very specific manner. If initially we have a black hole with mass M, we can build a black hole with mass $M' = M + m$ by dropping, for example, a box containing an amount of matter m, the black would absorb this matter and the area of the horizon would increase a bit. In this new configuration the geometry is still given by Eintein's equations but now, for the black hole with mass $M' = M + m$.

The spacetime is foliated in a family of $t = const$ space-like hypersurfaces, in each hypersurface we have a "photo" of the black hole with a given mass, in each hypersurface the Eintein's equations are satisfied and the geometry fully characterized. But in each hypersurface the mass of the black hole will be different (because matter fell into the black hole or because the black hole evaporated). The variation of the mass in the thermodynamic theory is given in this way, this kind of variation usually have the name *quasi static evolution* (for a "thermodynamicist") or an *adiabatic evolution* (for a "quantum mechanicist").

For the Schwarzchild case we can write the quantities M, A and κ as,

$$M = \frac{(D-2)\, B_{D-2}}{16\pi} r_+^{D-3} \left(1 - \tilde{\Lambda} r_+^2\right), \tag{1.54}$$

$$2\kappa = \frac{(D-3)}{r_+} - (D-1)\, \tilde{\Lambda} r_+, \tag{1.55}$$

$$A = r_+^{D-2} B_{D-2}. \tag{1.56}$$

Using the expression for the area, we can rewrite the equations (1.54) and (1.55) as

$$M = \frac{(D-2)\, B_{D-2}}{16\pi} \left(\frac{A}{B_{D-2}}\right)^{\frac{D-3}{D-2}} \left[1 - \tilde{\Lambda} \left(\frac{A}{B_{D-2}}\right)^{\frac{2}{D-2}}\right], \tag{1.57}$$

$$2\kappa = (D-3) \left(\frac{A}{B_{D-2}}\right)^{-\frac{1}{D-2}} - (D-1)\, \tilde{\Lambda} \left(\frac{A}{B_{D-2}}\right)^{\frac{1}{D-2}}. \tag{1.58}$$

Taking the differential from (1.57), one can obtain,

$$dM = \frac{1}{4\pi} \left[(D-3) \left(\frac{A}{B_{D-2}}\right)^{-\frac{1}{D-2}} - (D-1)\, \tilde{\Lambda} \left(\frac{A}{B_{D-2}}\right)^{\frac{1}{N-2}}\right] d\left(\frac{A}{4}\right). \tag{1.59}$$

Comparing with (1.58) and (1.53), it is easy to see that (1.59) just represent the differential form for the first law, and thus, our black hole obeys $dU = TdS$, where $T = \partial U / \partial S$. Combining (1.57) and (1.58), we can obtain an expression for the mass in terms of the area (entropy), and the surface gravity (temperature)

$$8\pi \left(\frac{D-1}{D-2}\right) M = A \left(\frac{B_{D-2}}{A}\right)^{\frac{1}{D-2}} + \kappa A. \tag{1.60}$$

One important feature of (1.60) is that it does not depend on the cosmological constant explicitly. Now, introducing the thermodynamical variables on (1.60), we obtain,

$$\left(\frac{D-1}{D-2}\right) U - TS = \frac{1}{2\pi}\left(\frac{B_{D-2}}{4} S^{D-3}\right)^{\frac{1}{D-2}}. \tag{1.61}$$

For the minimal description, this is the only equation of state for the S-AdS ensemble. But, in this description our system is very simple, it has just one degree of freedom. But the theory presented here is not homogeneous, and homogeneity is required for extensivity [15].

Another feature that we may observe, is that the temperature defined here is not the *thermodynamical (absolute) temperature* in the same sense that one have in the usual Thermodynamics, i.e., the temperature is not the one that is obtained from a Carnot cycle, nor the integrating factor for a reversible heat exchange [5], nor the temperature that is selected due to extensivity of the metrical entropy. The conclusion then is, the thermodynamics described here is not strictly a thermodynamic theory.

In the following chapter we shall introduce a new formalism to treat the Thermodynamic theory, and by means of this new formalism we will generalize the minimal approach for the S-AdS thermodynamics.

1.4 An Attempt to generalize the minimal description of S-AdS black holes

The minimal thermodynamical description of the S-AdS black hole is too simple. It only have one equation of state, the internal energy U is not a first order homogeneous function, the system does not have a Carnot cycle, i.e., the isotherms and the adiabatic curves of the system are the same curve.

That suggests that the temperature associated to this system is not a well defined temperature in the thermodynamic sense (unique). In this section we shall see a first attempt to generalize this thermodynamics by introducing more structure to it, unfortunately (or maybe fortunately) this description also is not entirely consistent, as we will see in the following.

The starting point is to realize the cosmological constant, Λ, as a new thermodynamic variable [20, 19, 22]. The techniques presented in [15, 14, 17, 19, 22] allow us to extend the Smarr formula to the D dimensional S-AdS black hole. And for the case where there is no charge or angular momentum, one has the Smarr formula,

$$(D-3)\, M = (D-2)\,\frac{\kappa A}{8\pi} - 2\frac{\theta\Lambda}{8\pi}. \tag{1.62}$$

In the Smarr formula (1.62), θ represents a new thermodynamic variable, conjugated to Λ. Substituting the expressions (1.60) and (1.61) in the Smarr formula (1.62), one can find that,

$$\theta = -\frac{B_{D-2}}{(D-1)}\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}} = -\frac{B_{D-2}}{(D-1)}r_+^{D-1}. \tag{1.63}$$

It is easy to see that equation (1.63) can be interpreted as the volume that is extracted from the spacetime due to the presence of the black hole. This identification suggests to us that the cosmological constant Λ should be interpreted as a pressure. With this identification, the mass M of the black hole will no longer be identified with the internal energy, but now, it shall be identified with its enthalpy function H. In this generalized approach, the thermodynamics of the S-AdS black hole have two independent thermodynamic variables, and the dictionary can be written as,

$$H \equiv M, \quad S \equiv \frac{A}{4}, \quad T \equiv \frac{\kappa}{2\pi}, \quad P \equiv -\frac{\Lambda}{8\pi}, \quad V \equiv -\theta. \tag{1.64}$$

The P is the thermodynamic pressure associated to V.

However, the definition of the enthalpy, H, with the mass, M, leads to an inconsistency in the thermodynamic description. From the thermodynamic dictionary defined in (1.64), it follows,

$$\left(\frac{\partial H}{\partial P}\right)_S = V. \tag{1.65}$$

With this new prescription, the internal energy can be obtained as usual by,

$$U = H - PV = \frac{(D-2)\,B_{D-2}}{16\pi}\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-3}{D-2}}. \tag{1.66}$$

As a result,

$$\frac{\kappa}{2\pi} = T \neq \frac{\partial U}{\partial S}. \tag{1.67}$$

And the thermodynamics described by the dictionary (1.64) is not consistent. Indeed, this problem is a consequence of the singularity in the Legendre transformation of the pair (P, V), due to the fact that V has no dependence on P.

Although this approach is not consistent, because the definition of internal energy is not consistent with the definition of temperature, we can argue in favor of a thermodynamic volume. A more robust theoretical framework is needed, and we shall investigate this in the following chapter.

2 Hamiltonian approach for thermodynamics

In this chapter we will introduce the general theory of the Hamiltonian approach for the thermodynamic theory by a extension of the usual phase space, where the thermodynamics equations of state can be identified as constraints in this extended phase space.

The main references for this chapter will be [2, 3].

2.1 Poisson brackets and Maxwell relations

A Hamiltonian system is composed of the triple (M, ω, X_H), where M is a smooth manifold, ω the canonical symplectic form on M and X_H is the Hamiltonian vector field. Let us take a moment to present the Darboux theorem:

Theorem. Let ω^2 be a closed non degenerate differential 2-form in a neighborhood of a point x in the space $\mathbb{R}^{2n}$. Then, in some neighborhood of x, one can choose a coordinate system $\left(p_i, q^i\right)$, $i = 1, \ldots, n$ such that the form ω^2 has the standard form

$$\omega^2 = dp_i \wedge dq^i.$$

Where we are adopting the convention that repeated index indicates a sum over them.

This local system of coordinates where the 2-form ω^2 can be written in this standard form is called *Darboux coordinates.*

Consider now that we have a set of local Darboux coordinates, and a canonical transformation $g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $g\,(p, q) = (P, Q)$. It is always possible to find among the $2n$ variables P_i and Q^i a set of n independent variables $\{\, y \,\}_{i=1}^{n}$ such that the Jacobian:

$$\frac{\partial^2 g\,(q, y)}{\partial y^i \partial q^j} \neq 0. \tag{2.1}$$

Where the function $g\,(q, y)$ is called *generating function* of the canonical transformation g.

To illustrate, let us consider a set of independent components q^i, Q^i and, a generating function $g\,(q, Q)$ that satisfies the relation $dg = p_i dq^i - P_i dQ^i$. Then, is trivial to see that

$$\frac{\partial g}{\partial q^i} = p_i(q, Q), \quad -\frac{\partial g}{\partial Q^i} = P_i(q, Q). \tag{2.2}$$

Therefore, one can consider the Poisson brackets (PB) between the variables p_i, once the $p_i's$ are functions of the set (q, Q), one can obtain

$$\{p_i, p_j\}_{p,q} = \frac{\partial p_i}{\partial q^j} - \frac{\partial p_j}{\partial q^i} \equiv 0. \tag{2.3}$$

Which expresses a subset of integrability conditions for the generating function g, that arises from $d^2 g \equiv 0$. The subscripts are just to indicate in which variables the PB are being evaluated. Other integrability conditions can be obtained from a Legendre transformation in the generating function. For instance, consider $g' = g - p_i q^i$, that is a generating function for the canonical transformation g such that $dg' = -q^i dp_i - P^i dQ_i$, where the q^i's and the P^i's are given by

$$q^i(p, Q) = -\frac{\partial g'}{\partial p_i}, \quad P_i(p, Q) = -\frac{\partial g'}{\partial Q^i}. \tag{2.4}$$

The PB, in this case, gives

$$\left\{q^i, q^j\right\}_{p,q} = \frac{\partial q_i}{\partial p_j} - \frac{\partial q^j}{\partial p^i} \equiv 0. \tag{2.5}$$

To fix ideas and start the analogy with the Thermodynamic theory, let us consider a system described by the canonical coordinates $\left(q_i, p^i\right)_{i=1,2}$ in $\mathbb{R}^4$ and a canonical transformation $g : \mathbb{R}^4 \to \mathbb{R}^4$ that makes $\left(q_i, p^i\right) \to \left(q'_i, p'^i\right)$ which is generated by the following generating function

$$F\left(q^1, p_2, q'^1, q^2\right) = g(q, q') - q^2 p_2, \tag{2.6}$$

$$dF = p_1 dq^1 - q^2 dp_2 - p'_1 dq'^1 - p'_2 dq'^2. \tag{2.7}$$

Where the function $g(q, q')$ is the one we have defined to find equations (2.4). From (2.7) is easy to see that, in particular, the coordinates $p_1 = p_1\left(q^1, p_2, q'^1, q^2\right)$ and $q^2 = q^2\left(q^1, p_2, q'^1, q^2\right)$. Then the PB between them gives the condition

$$\left\{p_1, q^2\right\}_{p,q} = -\frac{\partial q^2}{\partial q^1} - \frac{\partial p_1}{\partial p_2} \equiv 0. \tag{2.8}$$

We can compare with Thermodynamics now. Consider a Thermodynamic system described by the set of coordinates and momenta (S, T, P, V) with the internal energy $dU = TdS - PdV$. If one makes the identifications $q^1 = S$, $p_1 = T$, $q^2 = V$, $p_2 = -P$, putting this into the equation (2.8) one achieves the Maxwell relation

$$\left(\frac{\partial V}{\partial S}\right)_P = \left(\frac{\partial T}{\partial P}\right)_S. \tag{2.9}$$

All the Maxwell relations in Thermodynamics can be found with this procedure, one just have to evaluate different generating functions that are connected to each other by a Legendre transformation.

2.2 Extended phase space

Let us start by introducing a new canonical pair (τ, π) in the usual phase space $(T^*\mathbb{R}, dp \wedge dq)$. With the introduction of this new canonical pair the PB can be written as

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial f}{\partial \tau}\frac{\partial g}{\partial \pi} + \frac{\partial g}{\partial \tau}\frac{\partial f}{\partial \pi}. \tag{2.10}$$

It is straightforward to check that the following relations are satisfied

$$\{q, p\} = \{\pi, \tau\} = 1, \tag{2.11}$$

$$\{p, \pi\} = \{q, \tau\} = 0. \tag{2.12}$$

We have the freedom to choose any non-canonical pair as independent variables. For instance, take the pair $\{q, \tau\}$ as the independent variables, which implies that $p = p(q, \tau)$ and $\pi = \pi(q, \tau)$. With this choice the PB takes the form

$$\{p, \pi\} = \frac{\partial p}{\partial q}\frac{\partial \pi}{\partial p} - \frac{\partial \pi}{\partial q}\frac{\partial p}{\partial p} - \frac{\partial p}{\partial \tau},$$

$$\{p, \pi\} = \{p, \pi\}_{q,p} - \frac{\partial p}{\partial \tau}. \tag{2.13}$$

Where $\{q, \pi\}_{q,p}$ is the usual PB in the reduced phase space. Using the relation (2.12) the equation (2.13) can be rewritten as

$$\{p, \pi\}_{q,p} = \frac{\partial p}{\partial \tau}. \tag{2.14}$$

If $(\partial p / \partial q) = 0$ and we can make the identification of the Hamiltonian H as the momenta π, then, the equation (2.14) becomes one of the Hamilton's equations.

Now, consider the symplectic form $dp_i \wedge dq^i$ in local coordinates in an open set of the cotangent bundle T^*Q of the configuration space manifold Q of n dimensions. Also consider the configuration space $N = Q \times \mathbb{R}$ with the cotangent bundle T^*N with the symplectic form given by

$$\omega = dp_i \wedge dq^i + d\pi \wedge d\tau. \tag{2.15}$$

Let $H : T^{*}N \to \mathbb{R}$ be some function in the extended phase space, that in local coordinates can be written as

$$H = \pi + h\left(q,\, p,\, \tau\right). \tag{2.16}$$

Where $h\left(q,\, p,\, \tau\right)$ is just another function in the phase space $T^{*}N$. In these coordinates the PB's have the form (2.10) with a summation in all the q^{i}'s and p_i's, with the two functions f and g now defined on $T^{*}N$.

Let $\left(p\left(t\right),\, q\left(t\right)\right)$ be trajectories in the phase space $T^{*}N$. A integral curve γ in this space is one that obeys $\dot{\gamma} = X_H\left(\gamma\left(t\right)\right)$, where the field X_H is the field who generates the Hamiltonian phase flux and is given by

$$X_H = \frac{\partial}{\partial \tau} + \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^{i}} - \frac{\partial h}{\partial q^{i}}\frac{\partial}{\partial p_i} - \frac{\partial}{\partial \tau}\frac{\partial}{\partial \pi}. \tag{2.17}$$

If $\left(p\left(t\right),\, q\left(t\right)\right)$ are integral curves on $T^{*}N$, then we have

$$\begin{aligned} \frac{dq^{i}}{dt} &= X_H\left(q\right) = \frac{\partial h}{\partial p_i}, \quad \frac{dp_i}{dt} = X_H\left(p\right) = -\frac{\partial h}{\partial q^{i}}, \\ \frac{d\pi}{dt} &= X_H\left(\pi\right) = -\frac{\partial h}{\partial \tau}, \quad \frac{d\tau}{dt} = X_H\left(\tau\right) = 1. \end{aligned} \tag{2.18}$$

Which is just the Hamilton equations with Hamiltonian h. Consider now some function f that does not have any explicit dependency on π. The time evolution for f is given by

$$\frac{df}{dt} = X_H\left(f\right) = \left\{f,\, H\right\} = \frac{\partial f}{\partial \tau} + \frac{\partial h}{\partial p_i}\frac{\partial f}{\partial q^{i}} - \frac{\partial h}{\partial q^{i}}\frac{\partial f}{\partial p_i}. \tag{2.19}$$

Just a remark, in the framework of analytical mechanics, one can extend the phase space - usually to introduce the time as a canonical variable - but, this extension comes with a cost! That is a constraint surface in the extended phase space. In our case this constraint surface is defined by the equation (2.16) taken to be zero. This constraint, together with the relation $d\tau = dt$, shows that the tautological form in the extended phase space reduces to the usual Poincaré-Cartan form in the contact space $T^{*}Q \times \mathbb{R}$ with the Hamiltonian h,

$$p_i dq^{i} + \pi d\tau|_{H=0} = p_i dq^{i} - hdt, \tag{2.20}$$

Therefore, this is saying to us that we can formulate mechanics on the contact space $T^{*}Q \times \mathbb{R}$ in the same way as mechanics on the extended phase space $T^{*}N$ at the cost to introduce a constraint $H = 0$. With the introduction of these constraints, the canonical Hamiltonian in the contact phase space $T^{*}Q \times \mathbb{R}$ can be written as

$$H_c = \lambda H + c_m \phi_m. \tag{2.21}$$

Where the c_m can be any functions of the dynamical variables, the ϕ_m's are the primary constraints of the theory[1]. Now, we want to impose the chronological constraint

$$\psi = \tau - t. \tag{2.22}$$

This constraint formalizes the idea of time as a phase space coordinate. Introducing this constraint lead us to a second class constraint theory[2], since $\{\psi, H\} = 1$. The equation for τ in (2.18) tell us that ψ is conserved in time and, with this, one can find that

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \{\psi, H\} \Rightarrow \lambda = 1. \tag{2.23}$$

Which means that our canonical Hamiltonian is simply H. So far, we have just worked in the extended phase space. Our intuit now is to show that this description in the extended phase space with the introduction of constraints is equivalent to a description in the reduced with no constraints. To do this, consider the following canonical transformation

$$q'^i = q^i, \quad p'_i = p_i, \quad \tau' = \tau - t, \quad \pi' = \pi, \tag{2.24}$$

$$W\left(q, \tau, p', \pi', t\right) = q^i p'_i + (\tau - t)\,\pi'. \tag{2.25}$$

Where, the function W in (2.25) is the generating function for the canonical transformation (2.24). With this transformation, the constraints surfaces (2.16) and (2.22) becomes

$$\phi_1 = \pi + h\left(q, p, \tau' + t\right) = 0, \quad \phi_2 = \tau' = 0, \tag{2.26}$$

$$H' = H + \frac{\partial W}{\partial t} = h. \tag{2.27}$$

Where ϕ_1, ϕ_2 expresses the new constraint surfaces and H' represents the new Hamiltonian function in the new coordinates. The time evolution of the quantities $\eta = (q, p, \tau\prime, \pi)$ is given by the Dirac Hamiltonian theory of constraints[10] and is given by the so called Dirac brackets,

$$\{f, g\}_D M_{ab} = \{f, g\}_{PB} + \sum_{a,b} \{f, \phi_a\}_{PB} M^{-1}_{ab} \{\phi_b, g\}_{PB}, M_{ab} \quad = \{\phi_a, \phi_b\}_{PB}. \tag{2.28}$$

[1] A primary constraint ϕ is a relation between the coordinates and momenta that holds without using the equations of motion.

[2] A first class constraint is the one whose all PB with all the other constraints vanishes on the constraint surface. A second class constraint is the one whose PB with at least one constraint is non vanishing.

Where the ϕ's are the second class constraints. With this, the time evolution of the set η can be written as,

$$\frac{d\eta}{dt} = \{\eta, h\}_D . \tag{2.29}$$

However, in this particular case, the constraint relations between the constraints and the reduced Hamiltonian h, show us that the Dirac brackets reduce to usual Poisson brackets, and thus we have

$$\begin{aligned} &\frac{dq^i}{dt} = \left\{q^i, h\right\} = \frac{\partial h}{\partial p_i}, \frac{dp_i}{dt} = \{p_i, h\} = -\frac{\partial h}{\partial q^i}, \\ &\frac{d\pi}{dt} = 0, \frac{d\tau\prime}{dt} = 0, \phi_1 = \phi_2 = 0. \end{aligned} \tag{2.30}$$

Therefore, the evolution of some arbitrary dynamical function f is given by

$$\frac{d}{dt} f\left(q, p, t\right) = \{f, h\}_{\phi=0} = \frac{\partial f}{\partial t} + \{f, h\}_{q,p} . \tag{2.31}$$

Which is just the time evolution in the reduced phase space with coordinates and momenta (q, p) and reduced Hamiltonian h. In this sense, the dynamics in the extended phase space with Hamiltonian H and gauge fixing condition $\tau = t$ is equivalent to the dynamics in the reduced phase space with the reduced Hamiltonian h.

2.3 Mechanical setup for the ideal gas

In this section we just want to give a simple example of the application of the formalism. The ideal gas is described by the following equations of state[3],

$$T\left(u, v\right) = \frac{2u}{3}, \quad P\left(T, v\right) = \frac{T}{v}. \tag{2.32}$$

Where T is the temperature, u is the internal energy, P is the pressure and v is the volume. From (2.32) one can see that the system is described by the pairs of conjugate variables (s, T) and $(v, -P)$ such that the first law is obeyed $du = Tds - Pdv$. Now, consider the phase space $\mathbb{R}^4$ with coordinates (q, p, τ, π) such that the PB are given by (2.10) with Hamiltonian $H = \pi + h\left(q, p, \tau\right)$, where h, is the reduced Hamiltonian to be determined. The idea is to apply the method discussed in the previous section to find the reduced Hamiltonian in the reduced phase space and, from that, build the fundamental relation for the thermodynamic system.

[3] Lower case letter indicate specific quantities, eg. $x_i = X_i/N$. X_i is some extensive variable and N is the number of particles.

To map the Thermodynamic variables into Mechanical variables, we identify the tautological form $\theta = pdq + \pi d\tau$ with the internal energy form $du = Tds - Pdv$, and, from that, one can impose the following dictionary between the variables

$$\tau = s,\ \pi = T,\ q = v,\ p = -P. \tag{2.33}$$

And the equations of state (2.32) becomes

$$\pi = -pq, \quad \pi = \frac{2u}{3}. \tag{2.34}$$

Taking the exterior derivative of the second relation in (2.34) we have

$$d\pi = \frac{2}{3}\left(pdq + \pi d\tau\right). \tag{2.35}$$

We can see (2.34) as constraints in the phase space. And by using them, one can integrate equation (2.35) to obtain

$$p = -Ae^{\frac{2}{3}\tau}q^{-\frac{5}{3}}. \tag{2.36}$$

Where A is a constant of integration. Eq.(2.36) gives the constraint,

$$\phi\left(p, q, \tau\right) = p + Ae^{\frac{2}{3}\tau}q^{-\frac{5}{3}}. \tag{2.37}$$

From the first equation in (2.34) we obtain that $\pi + qp = 0$ and, using (2.37) to eliminate p, one can rewrite the constraint as,

$$H = \pi - Ae^{\frac{2}{3}\tau}q^{-\frac{2}{3}}. \tag{2.38}$$

Which gives the reduced Hamiltonian,

$$h = -Ae^{\frac{2}{3}\tau}q^{-\frac{2}{3}}. \tag{2.39}$$

So far, we obtained two constraints ϕ and H, which means that our canonical Hamiltonian H_c is given by,

$$H_c = \sigma H + \lambda\phi. \tag{2.40}$$

The two Lagrange multipliers, σ and λ, can be found by using the conservation equations for the constraints H and ϕ, which gives:

$$\frac{dH}{dt} = \{H, H_c\} = \lambda\{H, \phi\} = \lambda\phi, \tag{2.41}$$

$$\frac{d\phi}{dt} = \{\phi, H_c\} = \sigma\{\phi, H\} = -\sigma\phi. \tag{2.42}$$

Thus, the constraints are conserved on the constraint surface they define. Also, one can see that the total set of constraints is first class. The next step is to introduce the chronological gauge $\psi = \tau - t$. With this one can find that $\sigma = 1$ and $H_c = H + \lambda(t)\,\phi$, where the arbitrary function of time $\lambda(t)$ embodies the gauge freedom of the model.

Returning to the Thermodynamic variables, one can see that the constraints $H = 0$ and $\phi = 0$ gives the two equations of state

$$T - vP = 0, \quad P - A\exp\left(\frac{2}{3}s\right)v^{-\frac{5}{3}} = 0. \tag{2.43}$$

The two constraints in (2.43) together with the second equation in (2.34) gives the fundamental equation in the entropy representation

$$u(s, v) = \frac{3}{2}\frac{A}{v^{2/3}}\exp\left(\frac{2}{3}s\right). \tag{2.44}$$

The constant A in (2.44) expresses the freedom in the zero value of the entropy, that can be fixed by the Nernst theorem for example. Its worth to remind that the system has no degrees of freedom whatsoever, for each degree of freedom of the problem we find a constraint.

2.4 From the ideal gas to the Van der Waals gas

Before we start with the description of how one can go from the ideal gas to the Van der Waals gas, we will talk about the Lagrangian structure in the approach we have been describing.

2.4.1 Constraint structure and Lagrangian

Let $\{q^i\}_{i=1}^n$ denote a set of extensive parameters of a thermodynamic system. The internal energy u is a first order homogeneous function of the q^i's, i.e., $u = u(q^1, \ldots, q^n)$. In order to completely characterizes the thermodynamic system, we also need n equations of state, of the form,

$$p_i = \frac{\partial}{\partial q^i}u\left(q^1, \ldots, q^n\right) = f_i\left(q^1, \ldots, q^n\right). \tag{2.45}$$

p_i are the intensive parameters of the system. The equations (2.45) gives n constraints in the Hamiltonian formalism, where q^i and p_i are the coordinates and conjugate momenta, respectively. These corresponds precisely, to a system with n degrees of freedom and n primary constraints $\Phi = p_i - f_i(q)$.

Furthermore, given two states in the thermodynamic configuration space any trajectory connecting them must be a valid thermodynamic path, there are no physical degrees of freedom in the mechanical analog. As result, either the number of first class constraints is n, or it will be k, in case there are m second class constraints, such that $n = k + m/2$.

Now, consider the following Lagrangian,

$$L\,(q\,\dot{q}) = p\dot{q} - H_c|_{p=p(q,\dot{q})}. \tag{2.46}$$

Where H_c is the canonical Hamiltonian, which is a linear combination of the primary constraints Φ_i. One can show that the Lagrangian in (2.46) is a total derivative. To see that, consider a time independent set of first class constraints $\{\Phi_i\}_{i=1}^{n}$. The fact that their PB commute, gives,

$$\frac{\partial f_i}{\partial q^j} - \frac{\partial f_j}{\partial q^i} = 0. \tag{2.47}$$

Which implies that f_i is a gradient, i.e., $f_i = \partial\phi/\partial q^i$. Since there are n first class constraints, there are no degrees of freedom, and the Hamiltonian H_c is given by,

$$H_c = \lambda^i \Phi_i. \tag{2.48}$$

Where the Lagrange multipliers λ^i are understand as undetermined velocities, $\lambda^i = \dot{q}^i$. Thus, the Lagrangian (2.46),

$$L\,(q\,\dot{q}) = p_i\dot{q}^i - \dot{q}^i\Phi_i = \dot{q}^i\frac{\partial\phi}{\partial\dot{q}^i} = \frac{d\phi}{dt}. \tag{2.49}$$

The same can be shown to be truth for the case where there is m second class constraints, $\{\chi_i\}_{i=1}^{m}$, provided that the total number of degrees of freedom remains the same, i.e., $n = k + m/2$. In this case, the canonical Hamiltonian can be written as,

$$H_c = \sum_{i=1}^{k}\lambda^i\Phi_i + \sum_{i=1}^{m}\beta^i\chi_i. \tag{2.50}$$

The consistency relation that the second class constraints χ_i must be preserved in time gives $\beta^i = 0$. Therefore, we have the same situation as before and, thus, the Lagrangian is a total derivative.

The fact that the Hamiltonian function vanishes in the constraint surface implies the Lagrangian is a first order homogeneous function in the velocities. In addition, since all mechanical systems have vanishing physical degrees of freedom, one can always find a canonical transformation connecting two systems of the same dimension.

2.4.2 Van der Waals gas

In this last example, our goal is to build the fundamental equations for the Van der Waals gas from the ideal gas via a canonical transformation. We can start this procedure, by noting that the Lagrangian for the ideal gas can be written using (2.46) as,

$$L(q, \dot{q}, \tau) = Ae^{\frac{2}{3}\tau}q^{-\frac{5}{3}}(\dot{\tau}q - \dot{q}). \tag{2.51}$$

As we would expect, the Lagrangian is first-order homogeneous function in the velocities, and a total derivative, as one can check that,

$$L(q, \dot{q}, \tau) = \frac{du}{dt}. \tag{2.52}$$

Where u is the internal energy (2.44). Thus, once we build the Lagrangian and use the Dirac's theory for constrained systems we can find the fundamental equation for the ideal gas.

We show that there is a canonical transformation that connects the ideal gas to the Van der Waals gas. We refer to the Van der Waals systems as primed quantities $(u', \pi', q', p' \tau')$. The Van der Waals equation can be written using the dictionary (2.33) as,

$$\pi = (q - b)\left(\frac{a}{q^2} - p\right). \tag{2.53}$$

Now, we can compare with the first equation in (2.34), and we find that the coordinates and momenta in the ideal gas are connected with the coordinates and momenta in the Van der Waals gas as,

$$\begin{aligned} q &= q' - b, \quad p = p' - \frac{a}{q'^2}, \\ \pi' &= \pi, \quad \tau = \tau'. \end{aligned} \tag{2.54}$$

Thus, the constraints can be directly build for the Van der Waals gas as,

$$H'(\eta') = H(\eta(\eta')) = \pi' - Ae^{\frac{2}{3}\tau'}(q' - b)^{-\frac{2}{3}}, \tag{2.55}$$

$$\phi'(\eta') = \phi(\eta(\eta')) = p' - \frac{a}{q'^2} + \frac{1}{(q' - b)^{\frac{5}{3}}}Ae^{\frac{2}{3}\tau'}. \tag{2.56}$$

Where η represents the set of variables we are dealing with. Since the canonical transformation is time independent, the transformed Hamiltonian is simply,

$$H'_c = \sigma H' + \lambda\phi'. \tag{2.57}$$

Now, substitute the canonical transformation in the tautological form $du = pdq + \pi d\tau$, we can find that,

$$du = \left(p' - \frac{a}{q'^2}\right)dq' + \pi' d\tau. \tag{2.58}$$

As the transformation is canonical, the transformed tautological form must be $du' = p'dq' + \pi'd\tau'$, and then, the difference between the transformed tautological form and (2.58) is an exact differential,

$$du - du'u = ad\left(q'^{-1}\right) \Rightarrow = u' + \frac{a}{q'}. \tag{2.59}$$

We can use the fundamental equation for the ideal gas (2.44) and find that,

$$u' = \frac{3}{2}\frac{A}{(v-b)^{2/3}}\exp\left(\frac{2}{3}s\right) - \frac{a}{v}. \tag{2.60}$$

Which is the result we were looking for.

2.5 Thermodynamics mechanical analog for black holes

Our goal in this section is to use the formalism discussed in the section 4.2 to study the thermodynamics of the Schwarzchild black hole in the AdS background. We already have discussed a bit about the Schwarzchild solution of black holes in the AdS spacetime in section 3.1 and about the minimal thermodynamical description for these black holes in section 3.3. At this point we shall remember the important features we have discussed earlier (just to make the discussion easier), and point some other important results we will need in what follows.

We want to study the thermodynamics of asymptotically AdS black holes in equilibrium with a thermal atmosphere generated by the Hawking radiation, just as before. The S-AdS metric is the one we obtained in equation (1.16),

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2, \tag{2.61}$$

Where,

$$f(r) = \left(1 - \frac{\tilde{M}}{r^{D-3}} - \tilde{\Lambda}r^2\right), \tag{2.62}$$

$$\tilde{M} = \frac{16\pi}{(D-2)\,B_{D-2}}M, \quad B_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}, \quad \tilde{\Lambda} = \frac{2}{(D-1)(D-2)}\Lambda. \tag{2.63}$$

We can express the mass and the surface gravity in terms of the horizon area,

$$M = \frac{(D-2)\,B_{D-2}}{16\pi}\left(\frac{A}{B_{D-2}}\right)^{\frac{D-3}{D-2}}\left[1 - \tilde{\Lambda}\left(\frac{A}{B_{D-2}}\right)^{\frac{2}{D-2}}\right], \tag{2.64}$$

$$2\kappa = (D-3)\left(\frac{A}{B_{D-2}}\right)^{-\frac{1}{D-2}} - (D-1)\,\tilde{\Lambda}\left(\frac{A}{B_{D-2}}\right)^{\frac{1}{D-2}}. \tag{2.65}$$

And for the minimal thermodynamics description, we have the following identifications,

$$U \equiv M, \quad S \equiv \frac{A}{4}, \quad T \equiv \frac{\kappa}{2\pi}. \tag{2.66}$$

In this description we have just one equation of state that completely characterizes the black hole,

$$\left(\frac{D-1}{D-2}\right) U - TS = \frac{1}{2\pi}\left(\frac{B_{D-2}}{4} S^{D-3}\right)^{\frac{1}{D-2}}. \tag{2.67}$$

In the following, we want to generalize this description by using the formalism described in sections 2.1 and 2.2, by introducing a new thermodynamical variable in our phase space. We already know that given some thermodynamic potential, let us say M, its differential dM is given by the canonical tautological form pdq on the constraint surface. We want to introduce a new canonical pair (ξ, τ), given rise to a extended phase space, such that the tautological form in the extended space $pdq + \xi d\tau$ reduces to the usual Poincaré-Cartan $pdq - hd\tau$ on the constraint surface $H = \xi + h(q, p, \tau)$.

Thus, with this prescription, one obtain a description in the reduced phase space (p, q) as well in the extended phase space (p, q, ξ, τ). This way, all thermodynamic potentials are related by canonical transformations, giving equivalent representations.

Let us start the mechanical analog by introducing the mechanical variables (q, p),

$$q = \frac{4S}{B_{D-2}}, \quad p = \pi T = \frac{\kappa}{2}. \tag{2.68}$$

Just as before, we interpret the thermodynamical equations of state as constraints in the extended phase space. In this case, let us realize the equation (2.65) as a constraint. In terms of the mechanical variables, it takes the form,

$$4p = (D-3)\, q^{-\frac{1}{D-2}} - \frac{2\Lambda q^{\frac{1}{D-2}}}{(D-2)}. \tag{2.69}$$

For future convenience it is good to write the equation (2.64) in terms of the mechanical variables,

$$M = \frac{B_{D-2}}{16\pi} q^{\frac{D-3}{D-2}} \left[(D-2) - \frac{2\Lambda q^{\frac{2}{D-2}}}{(D-1)}\right]. \tag{2.70}$$

By differentiation of (2.70), considering that Λ is a function of the mechanical variables and using (2.69), we obtain,

$$dM = \frac{B_{D-2}}{4\pi}\left(pdq - \frac{1}{2(D-1)} q^{\frac{D-1}{D-2}} d\Lambda\right). \tag{2.71}$$

Now, if $\Lambda = \Lambda(q)$, we can express (2.71) as,

$$dM = \varpi dq, \tag{2.72}$$

Where,

$$\varpi = \frac{B_{D-2}}{4\pi}\left(p - \frac{1}{2(D-1)}q^{\frac{D-1}{D-2}}\frac{\partial\Lambda}{\partial q}\right). \tag{2.73}$$

The one form (2.72) is the tautological form for the description in the reduced phase space in a natural set of coordinates, restricted to the constraint surface defined by (2.69),

$$\phi = p - \frac{(D-3)}{4}q^{-\frac{1}{D-2}} + \frac{\Lambda q^{\frac{1}{D-2}}}{2(D-2)} = 0. \tag{2.74}$$

To make the statement that $dM = \varpi dq$ is indeed the tautological form, one can introduce the symplectic form ω, that follows from the exterior derivative of the tautological form,

$$\omega = \left(\frac{\partial\varpi}{\partial p}\right) dq \wedge dp = \frac{B_{D-2}}{4\pi} dq \wedge dp. \tag{2.75}$$

Thus we see that the transformation $(q, p) \to (q, \varpi)$ is indeed canonical because it preserves the canonical symplectic 2-form up to a constant. Now, the idea is to introduce more structure to this description extending the phase space by introducing a new canonical pair (ξ, τ), such that the tautological form dM becomes,

$$dM = \varpi dq + \xi d\tau. \tag{2.76}$$

Since the beginning, Λ is a function on the phase space, thus, a priori $\Lambda = \Lambda(\varpi, q; \xi, \tau)$. However, by comparison between equations (2.76) and (2.71) it is straightforward to see that Λ can only be a function of (q, τ), i.e., $\Lambda = \Lambda(q, \tau)$, and, with this information, we can write (2.71) as,

$$dM = \varpi dq - \frac{B_{D-2}}{8\pi(D-1)}q^{\frac{D-1}{D-2}}\frac{\partial\Lambda}{\partial\tau}d\tau. \tag{2.77}$$

And, by means of (2.77), we arrive at the additional constraint,

$$\chi = \xi + \frac{B_{D-2}}{8\pi(D-1)}q^{\frac{D-1}{D-2}}\frac{\partial\Lambda}{\partial\tau}. \tag{2.78}$$

The constraint $\chi = 0$ reduces the description in the extended phase space to the description in the reduced phase space. In other words, the tautological form $pdq + \xi d\tau$

in the extended space becomes the Poincaré-Cartan form $pdq - hdt$ in the reduced phase space, where the function h is identified as,

$$h(q,\tau) = \frac{B_{D-2}}{8\pi(D-1)} q^{\frac{D-1}{D-2}} \frac{\partial\Lambda}{\partial\tau}. \tag{2.79}$$

The canonical symplectic form (2.75), in the extended phase space, assumes the form,

$$\tilde{\omega} = \frac{\partial\varpi}{\partial p} dq \wedge dp + \frac{\partial\varpi}{\partial\tau} dq \wedge d\tau + d\tau \wedge d\xi. \tag{2.80}$$

The symplectic 2-form $\tilde{\omega}$ gives rise to the canonical Poisson bracket,

$$\{f(\eta), g(\eta)\} = 1 + \frac{2|\Lambda|}{(D-2)(D-3)} \left(\frac{4S}{B_{D-2}}\right)^{\frac{2}{D-2}} \frac{\partial f}{\partial q}\frac{\partial g}{\partial\varpi} + \frac{\partial f}{\partial\tau}\frac{\partial g}{\partial\xi} - f \leftrightarrow g. \tag{2.81}$$

Where $\eta = (q, \varpi, \tau, \xi)$ denotes all the coordinates in the extended phase space. With the PB (2.81), one can see that,

$$\{\chi, \phi\} = 0. \tag{2.82}$$

What tell us that the constraints ϕ and χ are first class, and so, the number of degrees of freedom is zero.

Now, back to the thermodynamics, we are taking the mass of the black hole as a thermodynamic potential, which give us $\varpi dq \equiv T_{eff} dS$, where the effective thermodynamic temperature is given by (2.73), which can be rewritten in terms of the thermodynamic variables as,

$$T_{eff} = T - \frac{B_{D-2}}{8\pi(D-1)} \left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}} \frac{\partial\Lambda}{\partial S}. \tag{2.83}$$

If we make the identification of $-\tau$ with the pressure P leads to $dM = T_{eff} dS$ being the exchanged heat in an isobaric process which is the enthalpy. And, from the constraint (2.78), one can find that,

$$V = -\frac{B_{D-2}}{8\pi(D-1)} q^{\frac{D-1}{D-2}} \frac{\partial\Lambda}{\partial P}. \tag{2.84}$$

Where the conjugate variable ξ is the thermodynamic volume. It is worth to note that the identification $\tau = -P$, or any identification of the mechanical variables with the thermodynamic ones, has nothing fundamental behind it. This is done just to make comparisons with previous works. The only physical statement is that M is a thermodynamic potential and T_{eff} is the integrating factor for entropy, i.e., T_{eff} is the thermodynamical temperature.

Now, we are able to make the connection with the Schwarzchild solution. To do that, we rewrite the dM in (2.71) as,

$$dM = \varpi' dq - \frac{B_{D-2}}{8\pi (D-1)} d\left(q^{\frac{D-1}{D-2}}\Lambda\right), \tag{2.85}$$

$$\varpi' = \frac{B_{D-2}}{4\pi}\left(p + \frac{q^{\frac{1}{D-2}}\Lambda}{2(D-2)}\right). \tag{2.86}$$

The equation (2.85) suggests that there is a τ-dependent canonical transformation $(q, \varpi) \to (q, \varpi')$ that maps the reduced phase space with coordinates (q, ϖ) to a reduced phase space with coordinates (q, ϖ'). The tautological form $dM = \varpi dq$ in this new phase space assumes the form,

$$dM' = \varpi' dq, \tag{2.87}$$

$$M' = M - E_\Lambda, \quad E_\Lambda = -\frac{B_{D-2}}{8\pi (D-1)}\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}}\Lambda. \tag{2.88}$$

Consider the second type generating function,

$$F_\Lambda(\varpi', q, \tau) = -\frac{B_{D-2}}{8\pi (D-1)} q^{\frac{D-1}{D-2}}\Lambda + \varpi' q. \tag{2.89}$$

For the transformation $(\varpi, q) \to (\varpi', q')$,

$$\begin{aligned}\varpi = \frac{\partial F_\Lambda}{\partial q} &= -\frac{\Lambda}{8\pi}\frac{B_{D-2}}{D-2} q^{\frac{1}{D-2}} - \frac{B_{D-2}}{8\pi (D-1)} q^{\frac{D-1}{D-2}}\frac{\partial \Lambda}{\partial q} + \varpi' \\ &= \frac{B_{D-2}}{4\pi}\left(p - \frac{q^{\frac{D-1}{D-2}}}{2(D-1)}\frac{\partial \Lambda}{\partial q}\right).\end{aligned} \tag{2.90}$$

Which is the result presented in (2.73). Moreover,

$$q' = \frac{\partial F_\Lambda}{\partial \varpi'} = q, \quad h' = h + \frac{\partial F_\Lambda}{\partial \tau} = 0. \tag{2.91}$$

Transformations of the type (2.89) guarantee that $q' = q$ and preserve the Bekenstein's notion of entropy and the second law of Thermodynamics, once the transformation does not change the q coordinate.

In the above description, Λ is not a thermodynamic variable but a parametrizing function for all extended phase spaces, all of which are in the same equivalence class of the reduced phase space with coordinates (ϖ', q') modulo time-dependent canonical transformation, the energy E_Λ. In this sense, one has a family of constraints $\chi(\Lambda)$ and $\phi(\Lambda)$, but, for each choice of Λ the constraints χ and ϕ lead to different equations of state and, thus, to a physically different thermodynamic system. For example, the choice $\Lambda = constant$ corresponds to the minimal description we presented earlier. The choice $\Lambda = 8\pi\tau$ corresponds to the introduction of a thermodynamic volume, and so on.

The constraint (2.74) and the mass becomes, in the primed coordinate system,

$$\begin{aligned}\phi' &= \frac{4\pi}{B_{D-2}}\varpi' - \frac{1}{4}(D-3)\,q'^{-\frac{1}{D-2}},\\ M' &= \frac{B_{D-2}}{16\pi}(D-2)\,q'^{\frac{D-3}{D-2}}, \quad q' = q = \frac{4S}{B_{D-2}}.\end{aligned} \tag{2.92}$$

It is easy to see that (2.92) represents the mass of a D dimensional Schwarzchild black hole described by the metric (2.61) with $\Lambda = 0$. Thus, the reduced phase space corresponds to the asymptotic flat case. Furthermore, the quantity $\Lambda/8\pi$ in (2.88) can be seen as a spacetime energy per unit volume [14] and the $-\theta$ is the volume of a sphere of radius r_+. So, we are able to obtain the S-AdS solution from the asymptotically flat Schwarzchild case by adding to the mass of the Schwarzchild black hole the energy E_Λ in (2.88).

It is worth to say, it not always easy to find the canonical transformations that fits for our problem, but once the canonical transformation is found one can obtain the solution for the problem in much simpler way. In this last case, one can infer the form of the canonical transformation (2.89) by noting that all the effects due to a negative cosmological constant can be included to the problem by adding the energy E_Λ in (2.88) to the mass M' of the Schwarzchild black hole.

Therefore, all thermodynamic results discussed in section 1.3 can be obtained from the primed system discussed above, without ever solving the Einsteins equations, just using the canonical transformation F_Λ. Another important result we can obtain from the canonical transformation (2.89) is the Smarr formula for a general cosmological constant. This is possible once that the mass (2.92) is homogeneous in q', unlike the S-AdS case. Euler's theorem for homogeneous functions [15] tell us that, if $G\left(\lambda^\alpha x_1, \lambda^\beta x_2\right) = \lambda^r G\left(x_1, x_2\right)$, then

$$rG = \alpha\left(\frac{\partial G}{\partial x_1}\right)x_1 + \beta\left(\frac{\partial G}{\partial x_2}\right)x_2. \tag{2.93}$$

Now, consider that $G = M'$, $x_1 = q'$ and $x_2 = 0$, applying this into (2.93), one obtains,

$$(D-3)\,M' = (D-2)\,\varpi' q'. \tag{2.94}$$

Substituting (2.86) and (2.88) in (2.94), we obtain,

$$(D-3)\,M = (D-2)\,\frac{B_{D-2}}{4\pi}pq + \frac{B_{D-2}}{4\pi}\frac{q^{\frac{D-1}{D-2}}\Lambda}{(D-1)}. \tag{2.95}$$

Note that the Poincaré-Cartan form $dM = pdq - hd\tau$ allow us to write the equation (2.79) as,

$$h(q, \tau) = \frac{B_{D-2}}{8\pi(D-1)} q^{\frac{D-1}{D-2}} \frac{\partial \Lambda}{\partial \tau} = -\frac{\partial M}{\partial \tau} = -\frac{\partial M}{\partial \Lambda}\frac{\partial \Lambda}{\partial \tau}. \tag{2.96}$$

Equation (2.96) together with (2.91) and the canonical transformation (2.89), gives the relation,

$$-2\frac{\partial M}{\partial \Lambda} = \frac{B_{D-2}}{4\pi(D-1)} q^{\frac{D-1}{D-2}}. \tag{2.97}$$

Substituting (2.97) and rewriting the $q's$ and $p's$ in terms of the entropy and temperature, the relation (2.95) becomes,

$$(D-3)\, M = (D-2)\, TS - 2\Lambda \frac{\partial M}{\partial \Lambda}. \tag{2.98}$$

The relation (2.98) is the Smarr formula for the S-AdS spacetime in D dimensions. It is important to remind that this relation is not an homogeneity relation (indeed, we did not require homogeneity of the equations of state yet, this will be done in the next section). The relation (2.98) can be understood as the image of the Euler relation for the Schwarzchild solution by the canonical transformation F_Λ.

2.6 Euler relation and homogeneity

The homogeneity of the equations of state is a necessary requirement in order to have a consistent black hole thermodynamic description, as was pointed out by [5, 4]. The requirement of homogeneity does not completely fix the dependence of Λ in the phase space, but will give strong restriction on his functional form.

The requirement of homogeneity also will give us a a better way of choosing the family of theories that can arise from the canonical transformation F_Λ discussed before. Because of this requirement we focus only in the subset of theories that respects the homogeneity condition. This subset will be parametrized by some phenomenological constants, as will be discussed in the following.

We start the discussion by requiring that the entropy is a homogeneous function with respect to the re-scaling $S \to \lambda S$ in(1.58). This is satisfied once the temperature obeys,

$$T \to \lambda^{-\frac{1}{D-2}} T. \tag{2.99}$$

and also,

$$\Lambda \to \lambda^{\frac{2}{2-D}} \Lambda,\; V \to \lambda^{\frac{D-3}{D-2} - c} V,\; P \to \lambda^{c} P,\; M \to \lambda^{\frac{D-3}{D-2}} M. \tag{2.100}$$

Where c is an arbitrary constant. At principle, this constant c parametrizes the set of homogeneous thermodynamic descriptions. Applying the Euler's formula (2.93) setting $G = M$, $x_1 = S$ and $x_2 = P$ we get $\alpha = 1$ and $\beta = c$ and $r = (D-3)/(D-2)$ and, then, the homogeneity relation reads,

$$(D-3)\,M = (D-2)\left[T_{eff}S + cVP\right]. \tag{2.101}$$

Where, T_{eff} is the temperature we have found in (2.83) that respects the homogeneity condition. Analogously, the homogeneity condition of the equation of state for $\Lambda = \Lambda(S, P)$, i.e., setting $G = \Lambda$, $x_1 = S$ and $x_2 = P$ gives $\alpha = 1$, $\beta = c$ and $r = 2/(2-D)$, and then, the relation reads,

$$\Lambda = \frac{D-2}{2}\left(-c\frac{\partial\Lambda}{\partial P}P - \frac{\partial\Lambda}{\partial S}S\right). \tag{2.102}$$

We can study the cases for $c \neq 0$ and for $c = 0$, both cases gives a different thermodynamic theory. For $c = 0$ the expression (2.102) can be solved easily, and we find that the cosmological constant can be written as,

$$\Lambda = S^{\frac{2}{2-D}} f(P). \tag{2.103}$$

Where $f(P)$ is an arbitrary function of the pressure. So, it is easy to see, that only the requirement of homogeneity, in this present case, do not completely fix the functional form of the cosmological constant. Let us turn our attention to the heat capacity for a moment. As usual, the heat capacity can be evaluated using $C_P = T(\partial S/\partial T)_P$. Taking the temperature derivative of both sides in (2.65) and using the expression for the cosmological constant (2.103), we can obtain,

$$C_P = -\pi T_{eff}\frac{(D-2)}{(D-3)}\frac{B_{D-2}}{g_D(\Lambda, S)}\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}}, \tag{2.104}$$

$$g_D(\Lambda, S) = 1 + \frac{2|\Lambda|}{(D-2)(D-3)}\left(\frac{4S}{B_{D-2}}\right)^{\frac{2}{D-2}}. \tag{2.105}$$

We observe that the function $g_D(\Lambda, S)$ is always positive, hence, the heat capacity C_P is always negative, which tell us that the system is always unstable. That is, the S-AdS thermodynamics with $c = 0$ it is similar to the Schwarzchild counterpart, in the sense that it can not describe stability or phase transitions.

We now turn our attention to the case where $c \neq 0$. For this case, the general solution of (2.102) can be found as,

$$\Lambda(S, P) = P^{\frac{1}{c}\frac{2}{2-D}} f\left(SP^{-\frac{1}{c}}\right). \tag{2.106}$$

Where f is an arbitrary function. f can be fixed by demanding that the zeroth law of the black hole thermodynamics is preserved, we use (2.106) to rewrite the effective temperature (2.83) as,

$$T_{eff} = \frac{D-3}{4\pi}\left(\frac{4S}{B_{D-2}}\right)^{\frac{1}{2-D}} + \frac{\tilde{\Lambda}}{4\pi}\left[\frac{2-D}{f}X\frac{df}{dX} - D + 1\right]\left(\frac{4S}{B_{D-2}}\right)^{\frac{1}{D-2}}, \qquad (2.107)$$

Where,

$$X = SP^{-\frac{1}{c}}.$$

Our intention here is to identify this new temperature with the surface gravity in order to have a well defined thermodynamic description. To guarantee the consistency between T_{eff} and κ, we require that they have the same functional dependence on S and Λ, and, consequently, the expressions for the temperature and surface gravity must have the same powers of S and Λ. Additionally, we demand that when we take the limit $\Lambda \to 0$ we should recover the minimal description.

Now, by comparison between (2.107) with (2.65), and with all the considerations we have made in the last paragraph, we find that the terms inside the brackets in (2.107) must be a constant,

$$\frac{2-D}{f}X\frac{df}{dX} = 2a. \qquad (2.108)$$

Where a is an arbitrary constant. Solving for X and using the result (2.106), we can find that the cosmological constant is given by,

$$\Lambda = -K\left[\left(\frac{4S}{B_{D-2}}\right)^{a} P^{b/c}\right]^{\frac{2}{2-D}}, \quad a + b = 1, \, 2b \neq (2-D)\,c. \qquad (2.109)$$

K is a integration constant that will be set to 1 from this point on. The condition $2b \neq (2-D)\,c$ is necessary to have a well defined Legendre transformation $(S, V) \to (S, P)$.

By requiring homogeneity of the cosmological constant, and demanding that the zeroth law is obeyed, we were able to find an explicit expression for the cosmological constant. We can now find an equation of state $T_{eff} = T_{eff}\,(S, P)$. Substituting (2.69) in (2.83) we can find that the effective temperature is given by,

$$T_{eff} = \frac{1}{4\pi}\left[(D-3)\left(\frac{4S}{B_{N-2}}\right)^{-\frac{1}{D-2}} - \frac{2\Lambda\,(D-2a-1)}{(D-2)\,(D-1)}\left(\frac{4S}{B_{D-2}}\right)^{\frac{1}{D-2}}\right]. \qquad (2.110)$$

This temperature differs from the Hawking temperature by an additional term proportional to the derivative of Λ. Thus, the Hamiltonian approach used here lead us to different physical descriptions. It is important to note that the geometric interpretation is preserved, i.e., the effective temperature can be written as $T_{eff} = \kappa_{eff}/2\pi$, where the effective surface gravity κ_{eff} is given by (2.65), but with the cosmological constant Λ being substituted by an effective version given by,

$$\Lambda_{eff} = \left(1 - \frac{2a}{D-1}\right)\Lambda. \tag{2.111}$$

Thus, given the solution for f of the equation (2.108), we find an effective temperature T_{eff} that is proportional to an effective surface gravity κ_{eff} in the S-AdS background which in turn is proportional to an effective cosmological constant Λ_{eff}. In other words, the geometrical interpretation of the black hole temperature remains valid in this Hamiltonian approach, provided that the spacetime has an effective cosmological constant that is given by (2.111).

Now, substituting (2.109) in the expression (2.84) for V and in the expression (2.110) for the temperature, we are able to find to equations of state,

$$P^{\gamma^{-1}}V = \frac{B_{N-2}}{8\pi}\frac{\left(1-\gamma^{-1}\right)}{D-1}\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1-2a}{D-2}}, \quad \gamma^{-1} = 1 + \frac{2b}{(D-2)\,c}. \tag{2.112}$$

$$T_{eff} = \frac{1}{4\pi}\left(\frac{4S}{B_{D-2}}\right)^{-\frac{1}{D-2}}\left[D-3+\frac{2\,(D-2a-1)}{(D-2)\,(D-1)}\left(\frac{4SP^{-\frac{1}{c}}}{B_{D-2}}\right)^{\frac{2b}{D-2}}\right]. \tag{2.113}$$

The new equation of state (2.112) is a consequence of the additional thermodynamic variable. Since the thermodynamical system has two degrees of freedom, the equations of state (2.112) and (2.113) completely characterizes the Thermodynamic description.

It is easy to see from equation (2.112), that the quantity PV^{γ} is constant in an isentropic process. Thus, at principle, one can compare the factor γ with the isentropic expansion factor for the ideal gas $\left(\gamma_{gas} = C_P/C_V\right)$. In the case of the ideal gas, γ is related with the internal structure of the system, i.e., microscopic degrees of freedom, in the present case this suggests that we can think of the constants b and c as related with the microscopic features of the theory.

As before, let us take a look at the heat capacity at constant pressure of our black hole,

$$C_P = -\pi T_{eff}\frac{(D-2)}{f_{a,D}\,(\Lambda,\, S)}\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}}, \tag{2.114}$$

$$f_{a,D}\,(\Lambda,\, S) = D-3-2\,|\Lambda|\,\frac{(1-2a)\,(D-1-2a)}{(D-2)\,(D-1)}\left(\frac{4S}{B_{D-2}}\right)^{\frac{2}{D-2}}. \tag{2.115}$$

A negative heat capacity implies a thermodynamic instability, and this happens when the function $f_{a,D}\,(\Lambda, S)$ is positive. This gives the condition $\frac{1}{2} \leq a \leq a_{crit}$, where,

$$a_{crit} = \frac{D-1}{2}. \tag{2.116}$$

If a assumes any value that satisfies this condition the system is unstable for any value of Λ. In particular, if $a = a_{crit}$, $\Lambda_{eff} = 0$ and the AdS black hole behaves as its Schwarzchild counterpart.

It is worth to stress that a is a phenomenological parameter, hence, consistency relations alone does not fix a, a microscopic theory is necessary. For instance, the instability in the threshold $a \in [1/2, a_{crit}]$ is a microscopic theory whose boundary conditions are not reflexive. On the other hand, if we set boundary condition that allow energy to escape, i.e., the boundary is not reflexive anymore, then, the thermodynamics can not be stable and thus, $a \in [1/2, a_{crit}]$.

Still about the stability, if $a \notin [1/2, a_{crit}]$, we observe from the heat capacity (2.114) that stability can only be acquired if $f_{a,D} < 0$, which can be translated in the following inequality,

$$|\Lambda| > \frac{(D-3)(D-2)(D-1)}{2(1-2a)(D-1-2a)}\left(\frac{4S}{B_{D-2}}\right)^{-\frac{2}{D-2}}. \tag{2.117}$$

In particular, we observe that the parameter a can be fixed for the 4 - dimensional case, once we impose that our black hole must have the behavior predicted by the Hawking-Page theory [13]. In this approach, the spacetime is stable for $|\Lambda| > \pi/S$. Combining $|\Lambda| > \pi/S$ with the inequality (2.117), we can find that for $D = 4$, stability will occur for $a = 0$ and for $a = 2$.

As a new result, we are able to fix the value for the constant a, demanding that the black hole described by the two equations of state (2.112) and (2.113) we can show that the divergent point of the heat capacity (2.114) coincides with the minima of the surface gravity (2.65), this is know as the Small-Large black hole phase transition [13]. thus, analyzing the heat capacity,

$$C_P = -\pi T_{eff}\frac{(D-2)}{f_{a,D}(\Lambda, S)}\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}}, \tag{2.118}$$

$$f_{a,D}(\Lambda, S) = D-3-2|\Lambda|\frac{(1-2a)(D-1-2a)}{(D-2)(D-1)}\left(\frac{4S}{B_{D-2}}\right)^{\frac{2}{D-2}}. \tag{2.119}$$

Which diverges at the minima of the black hole temperature,

$$T = \frac{1}{4\pi}\left[\frac{N-3}{r_+} - (N-1)\tilde{\Lambda}r_+\right]. \tag{2.120}$$

The minima of (2.120) is at,

$$r_+^2 = \frac{(N-3)(N-2)}{2|\Lambda|}. \tag{2.121}$$

When we take the function $f_{a,N}(\Lambda, S) = 0$ and substitute (2.121) in it, we can find that,

$$a = \frac{N}{2}. \tag{2.122}$$

3 Quasilocal energy and conserved charges

In this chapter we shall discuss another fruitful approach to the thermodynamics of asymptotic anti de-Sitter black holes. From this point of view, the temperature found in the previous chapter must be, somehow, generalized to fix the problem with the surface gravity that arises in spacetimes that are not asymptotically flat.

3.1 Non uniqueness of the surface gravity

To begin the discussion we must introduce the notion of a Killing horizon. Consider that we have a Killing vector field χ^{μ}, i.e., a vector field that satisfies the Killing equation,

$$D_{\mu}\chi_{\nu} + D_{\nu}\chi_{\mu} = 0. \quad (3.1)$$

Where D_{μ} is the covariant derivative. Then, we say that if a Killing vector χ^{μ} is null along some null hypersurface Σ, it tell us that Σ is a *Killing horizon.*

The notion of a Killing horizon, in general, has nothing to do with the notion of an event horizon. But, in spacetime that have time-translation symmetry the two are closely related. Under some conditions, we can make the following classification,

1. Every event horizon Σ in a stationary, asymptotic flat spacetime is a Killing horizon for some Killing vector χ^{μ}.

2. If the spacetime is static, χ^{μ} will be the Killing vector field $K^{\mu} = (\partial_t)^{\mu}$ representing time translations at infinity.

3. If the spacetime is stationary but not static, it will be axisymmetric with a rotational Killing vector field $R^{\mu} = \left(\partial_{\phi}\right)^{\mu}$, and χ^{μ} will be a linear combination $K^{\mu} + \Omega R^{\mu}$ for some constant Ω.

All these assumptions are valid due to Carter and Hawking. The two independent results, usually referred to as *rigidity theorems,* are those who guarantee to us that these assumptions are valid. The first one, due to Carter [9], states that the 3 items are true once we have a static black hole. The Carter's result does not rely on the field equations, it is only based in geometric facts. But that leaves the possibility that could exist non static black holes whose event horizons are not Killing horizons. But then comes the second result, due to Hawking [12], that directly proves that, in vacuum or eletro-vacuum general relativity, the event horizon of any stationary black hole must be a Killing horizon. Although the Hawking's theorem does not make any assumptions of symmetries beyond the stationarity, it does rely on the properties of the field equations.

Consider an arbitrary Killing horizon Σ with a normal Killing vector χ^μ. Since $D^\mu\left(\chi^\nu\chi_\nu\right)$ also is normal to Σ, these vector must be proportional at every point on Σ. Hence, there exist a function κ, on Σ, know as *surface gravity* on Σ which is defined as,

$$D^\mu\left(\chi^\nu\chi_\nu\right) = -2\kappa\chi^\mu. \tag{3.2}$$

κ is a constant along each null geodesic generator of Σ, but in general it can vary from generator to generator. By using the Killing equation (3.1) and the fact that $\chi_{[\mu}D_\nu\chi_{\sigma]} = 0$ (since χ^μ is normal to Σ), one can find an expression for the surface gravity κ,

$$\kappa^2 = -\frac{1}{2}\left(D_\mu\chi_\nu\right)\left(D^\mu\chi^\nu\right). \tag{3.3}$$

The surface gravity associated with a Killing horizon is in principle arbitrary, since one can always scale the Killing field by a real constant and obtain another Killing field. In static, asymptotically flat spacetime the time translation Killing vector $K = \partial_t$ can be normalized by setting,

$$\lim_{r\to\infty} K_\mu K^\mu = -1. \tag{3.4}$$

This condition guarantee to us that surface gravity is unique. The problem of non uniqueness of the surface gravity arises in non flat asymptotically spacetimes, like AdS. In such spacetimes the Killing vector can not, in general, be normalized and thus the surface gravity defined by such vector is not unique. This means that, in the thermodynamic description, the temperature function is not well defined, since it can assume two distinct values at the same point.

3.2 Gravitational theory with finite boundary conditions

As we have saw in the previous section, for non flat asymptotic spacetimes we do not have a well defined surface gravity and hence, at principle, a not well defined temperature. Now we want to discuss the ideas presented in [6, 7].

We want to consider a theory that is built on a finite spatial region, in such a way that the asymptotic behavior becomes irrelevant. This approach will allow us to have well defined quantities for non flat asymptotically black holes.

To begin the discussion, consider a spacetime manifold $\mathcal{M}$ with dimension D, which is topologically the product $\Sigma \times I$, where Σ is a spacelike hypersurface and I a real line interval. The boundary of Σ is denoted by $\partial\Sigma = B$. The spacetime metric is $g_{\mu\nu}$ and the covariant derivative associated with it will be denoted by ∇_μ. The boundary of $\mathcal{M}$ will consist of two initial and final spacelike hypersurfaces t' and t'' that are connected by a timelike hypersurface $\mathcal{B} = B \times I$. The metric that is induced over the initial and final hypersurfaces t' and t'' is denoted by h_{ij}, and the induced metric over the hypersurface $\mathcal{B}$ will be denoted by γ_{ij}.

The appropriate gravitational action is given by[25],

$$S^1 = \frac{1}{2\alpha}\int_{\mathcal{M}} d^D x\sqrt{-g}\,(R-2\Lambda) + \frac{1}{\alpha}\int_{t'}^{t''} d^{D-1}x\sqrt{h}K - \frac{1}{\alpha}\int_{\mathcal{B}} d^{D-1}x\sqrt{-\gamma}\Theta. \tag{3.5}$$

α is a coupling constant and, as usual, Λ is the cosmological constant. We can also add to (3.5) a action for the matter contribution, but for now, it will not make any difference for the present discussion. The integral $\int_{t'}^{t''} d^{D-1}x$ represents a integral over the boundary element t'' minus a integral over the t'. K is the trace of the extrinsic curvature K_{ij} for the boundary elements t' and t'' defined with respect to a future pointing unit normal and Θ is the trace of the extrinsic curvature Θ_{ij} of the boundary $\mathcal{B}$, defined with respect to an outward pointing unit normal.

Under variations of the metric, the action (3.5) assumes the form,

$$\delta S^1 = (...) + \int_{t'}^{t''} d^{D-1}x P^{ij}\delta h_{ij} + \int_{\mathcal{B}} d^{D-1}x\pi^{ij}\delta\gamma_{ij}, \tag{3.6}$$

$$P^{ij} = \frac{1}{2\alpha}\sqrt{h}\left(Kh^{ij} - K^{ij}\right), \tag{3.7}$$

$$\pi^{ij} = -\frac{1}{2\alpha}\sqrt{-\gamma}\left(\Theta\gamma^{ij} - \Theta^{ij}\right). \tag{3.8}$$

Where(...) represents the contributions for the variation that vanishes when the equations of motion hold. P^{ij} and π^{ij} are the gravitational momentums. Equation (3.6) also includes contributions over the regions $t'' \cap \mathcal{B}$ and $t' \cap \mathcal{B}$, but those contributions will also not be needed in the present discussion.

The action (3.5) yields the equations of motion when the induced metric on the boundary B is fixed. We have the freedom to include an action S^0 that does not change the equations of motion, which is, in general, a functional of the metric on the boundary B. This is exactly the same freedom we have in the usual Hamilton-Jacobi theory. For simplicity we will define S^0 to be a functional of γ_{ij} only, and thus the only modification we will have in the variation of S will be the replacement of π^{ij} by $\pi^{ij} - \left(\delta S^0/\delta\gamma^{ij}\right)$.

Now, foliate the boundary elements $\mathcal{B}$ into $(D-2)$- dimensional hypersurfaces B with induced metrics σ_{ab}. The $(D-1)$-dimensional metric γ_{ij} can be written according to the ADM decomposition,

$$\gamma_{ij}dx^i dx^j = -N^2dt^2 + \sigma_{ab}\left(dx^a + V^a dt\right)\left(dx^b + V^b dt\right). \tag{3.9}$$

N is the lapse function and V^a is the shift vector. The variation of γ_{ij} can then be written as,

$$\delta\gamma_{ij} = -\frac{2u_i u_j}{N}\delta N - \frac{2\sigma_{a(i}u_{j)}}{N}\delta V^a + \sigma^a_{(i}\sigma^b_{j)}\delta\sigma_{ab}. \tag{3.10}$$

u_i is the unit normal to the slices B and $\sigma^i_a = \delta^i_a$ projects covariant tensors from $\mathcal{B}$ to the slices B. With (3.10), the contribution to the variation of the action S from the boundary $\mathcal{B}$ assumes the form,

$$\begin{aligned} \delta S|_{\mathcal{B}} &= \int_{\mathcal{B}} d^{D-1}x \left(\pi^{ij} - \left(\delta S^0/\delta\gamma^{ij}\right)\right)\delta\gamma_{ij}, \\ &= \int_{\mathcal{B}} d^{D-1}x\sqrt{\sigma}\left(-\varepsilon\delta N + j_a\delta V^a + \left(\frac{N}{2}\right)s^{ab}\delta\sigma_{ab}\right). \end{aligned} \tag{3.11}$$

Where the coefficients ε, j_a, s^{ab} are defined as,

$$\varepsilon = \frac{2}{N\sqrt{\sigma}}u_i\pi^{ij}u_j + \frac{1}{\sqrt{\sigma}}\frac{\delta S^0}{\delta N}, \tag{3.12}$$

$$j_a = -\frac{2}{N\sqrt{\sigma}}\sigma_{ai}\pi^{ij}u_j - \frac{1}{\sqrt{\sigma}}\frac{\delta S^0}{\delta V^a}, \tag{3.13}$$

$$s^{ab} = \frac{2}{N\sqrt{\sigma}}\sigma^a_i\pi^{ij}\sigma^b_j - \frac{2}{N\sqrt{\sigma}}\frac{\delta S^0}{\delta\sigma^{ab}}. \tag{3.14}$$

The equations (3.12), (3.13) and (3.14) can be recast in terms of the extrinsic curvature k_{ab}, defined by the parallel transport of the unit normal n to $\mathcal{B}$ across a $(D-2)$-dimensional slice B. Let $a_\mu = u^\nu\nabla_\nu u_\mu$ denote the acceleration of the unit normal u_μ for the family of hypersurfaces Σ. Thus, the equations for ε, j_a and s^{ab} can be written as,

$$\varepsilon = \frac{1}{\alpha}k - \varepsilon_0, \tag{3.15}$$

$$j_i = -\frac{2}{\sqrt{h}}\sigma_{ij}P^{jk}n_k - (j_o)_i, \quad j_i = j_a\sigma^a_i. \tag{3.16}$$

$$s^{ab} = \frac{1}{\alpha}\left(k^{ab} + \left(n_\mu a^\mu - k\right)\sigma^{ab}\right) - (s_0)^{ab}. \tag{3.17}$$

The quantities with a 0 index are representing the terms proportional to the functional derivatives of S^0. Now, we want to point out that the quantity $-\sqrt{\sigma}\varepsilon$ is equal to the time rate change of the action, as one can see from the equation (3.11), where the changes on time are controlled by the lapse function N on $\mathcal{B}$. Thus, ε can be seen as an energy density for the system and, the total quasilocal energy[1] is defined by integration of ε over a $(D-2)$-dimensional hypersurface B,

$$E = \int_B d^{D-2}x\sqrt{\sigma}\varepsilon. \tag{3.18}$$

When there is a Killing vector ξ on the boundary $\mathcal{B}$, we can also define an associated charge as [7],

$$Q_\xi = \int_B d^{D-2}x\sqrt{\sigma}\left(\varepsilon u^i + j^i\right)\xi_i. \tag{3.19}$$

If there is no matter stress-energy in the neighborhood of $\mathcal{B}$ then, Q_ξ is conserved, in the sense that it is independent on the particular hypersurface B that is chosen for its evaluation. If the Killing vector ξ is time like, then the negative of (3.19) is defined as the conserved mass for the system. If the Killing vector ξ is also surface forming i.e., it generates a hypersurface, then the mass can be evaluated on a hypersurface B whose unit normal is proportional to ξ, in this case the mass can be written as,

$$M = \int_B d^{D-2}x\sqrt{\sigma}N\varepsilon. \tag{3.20}$$

And the relation between the Killing vector ξ and the lapse function N is defined as $\xi = Nu$. Thus, if ξ restricted to B can be normalized in such way that it has unit norm, then $N = 1$ and the conserved mass M coincides with the energy (3.18) of the hypersurface Σ whose boundary is B. However, if ξ can not be normalized to unit at B, then the mass M and the energy E will be two different things, i.e., the energy E evaluated on other slices of $\mathcal{B}$ will not, in general, be equal to the conserved mass M.

As an example, in spacetimes that are asymptotically Anti-de-Sitter, the time like Killing vector diverges as it approaches the infinity. Thus, it is impossible to normalize it and the mass M and the energy E will not coincide.

Another result that is known [6], is that, if we take the limit where B is taken to infinity, the charge Q_ξ is the negative of the ADM charge associated with ξ. Specifically, the ADM mass agree with the mass M in (3.20), in the limit that B is taken to infinity.

We want to stress two ambiguities that appear in this whole process. The first is what we already knew from the previous section, the charges defined here depends on the normalization of the Killing vector on the asymptotic limit. Thus, the charge associated with the Killing vector $c\xi$, where c is a constant, is equal to the product of c and the charge associated with ξ, and both ξ or $c\xi$ are equally good Killing vectors. Another ambiguity that rises in this process is due to the arbitrary action S^0 that makes the terms ε_0, $(j_0)_i$, $(s_0)_{ab}$ appears.

[1] In analogy with the expression that gives the energy in the Hamilton-Jacobi theory, note that ε is minus the functional derivative of the gravitational action with respect to the lapse function.

3.3 S-AdS black holes with finite boundary

In 4-dimensions, the metric for the Schwarzchild Anti-de-Sitter black hole can be written as ($\alpha = 8\pi$, *thus* $G = 1$),

$$ds^2 = -N^2(r)\,dt^2 + \frac{dr^2}{f^2(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.21}$$

Where,

$$N^2(r) = f^2(r) = 1 - \frac{2m}{r} + \frac{r^2}{L^2}, \quad \Lambda = -\frac{3}{L^2}. \tag{3.22}$$

Let Σ be the interior of a hypersurface defined as the slice $t = const$ with a 2-boundary B specified by $r = R = const$. The term ε_0 will be a function of R and the term $(j_0)_a$ will be set to zero. The trace of the extrinsic curvature over the boundary can be found as,

$$k = -\frac{2f(R)}{R} = -\frac{2}{R}\sqrt{1 - \frac{2m}{R} + \frac{R^2}{L^2}}. \tag{3.23}$$

Then, the energy surface density (3.15) can be written as,

$$\varepsilon = -\frac{1}{4\pi R}\sqrt{1 - \frac{2m}{R} + \frac{R^2}{L^2}} - \varepsilon_0(R). \tag{3.24}$$

And the total energy (3.18) will be,

$$E = -R\sqrt{1 - \frac{2m}{R} + \frac{R^2}{L^2}} - 4\pi R^2\varepsilon_0(R). \tag{3.25}$$

In the thermodynamic picture, the entropy obeys the same relation that we had before, $S = \pi r_+^2$, where r_+ is the horizon radius, which obeys,

$$N(r_+) = 1 - \frac{2m}{r_+} + \frac{r_+^2}{L^2} = 0. \tag{3.26}$$

Now, if we identify the energy (3.26) with the internal energy for the S-AdS black hole spacetime within boundary R, then E will be a function of entropy since $m = m(r_+)$ and one can find the following temperature,

$$T \equiv \left(\frac{\partial E}{\partial S}\right) = \frac{1}{\sqrt{1 - \frac{2m}{R} + \frac{R^2}{L^2}}}\left(\frac{1 + 3r_+^2/L^2}{4\pi r_+}\right). \tag{3.27}$$

The first term in the above expression is just the Tolman redshift [21] factor for temperature in a stationary gravitational field. The second factor in (3.27) is exactly the contribution that arises from the surface gravity κ times $1/2\pi$, thus, the temperature measured at R is proportional to the product of the surface gravity κ times the Tolman redshift factor,

$$T_B = \frac{1}{2\pi}\frac{\kappa}{N(R)}. \tag{3.28}$$

We are going to refer to the temperature (3.28) as "Brown's temperature". One can also define a thermodynamic surface pressure as,

$$\mathcal{P} \equiv -\left(\frac{\partial E}{\partial (4\pi R^2)}\right) = \frac{1}{8\pi R N(R)}\left(1 - \frac{m}{R} + 2\frac{R^2}{L^2}\right) + \frac{\partial \left(R^2 \varepsilon_0\right)}{\partial (R^2)}, \tag{3.29}$$

$$= \frac{1}{8\pi R}\left(\frac{1 - \frac{m}{R} - \frac{2}{3}\Lambda R^2}{1 - \frac{2m}{R} - \frac{1}{3}\Lambda R^2}\right) + \frac{\partial \left(R^2 \varepsilon_0\right)}{\partial (R^2)}. \tag{3.30}$$

And one can observe that the temperature (3.27) does not depends on the ε_0 function, but the pressure (3.29) does depends on ε_0. In the same way, we are going to refer to $\mathcal{P}$ in (3.29) as "Brown's pressure". In Brown's thermodynamic description, the first law of thermodynamics assumes the form,

$$dE = TdS - \mathcal{P}d\left(4\pi R^2\right). \tag{3.31}$$

With this, one can investigate the behavior of the heat capacity at constant surface pressure $\mathcal{P}$,

$$C_R \equiv \left(\frac{\partial E}{\partial T}\right)_R = \left(\frac{\partial E}{\partial r_+}\right)_R \left(\frac{\partial T}{\partial r_+}\right)_R^{-1} \tag{3.32}$$

One can easily see that the temperature has a minimum, $T = T_0$, at $r_+ = \sqrt{-1/\Lambda} = L/\sqrt{3}$, and, therefore, there are two black hole solutions for $T > T_0$. Since $\partial E/\partial r_+$ is strictly positive, one of the solutions will be an smaller and unstable ($C_R < 0$) black hole and the other solution will be a bigger and stable ($C_R > 0$) black hole. For $T < T_0$ there will be no black hole solutions. Thus, there are no Schwarzchild Anti-de-Sitter black holes with temperature at $r = R$ less than T_0. If the temperature at $r = R$ is less than T_0 the system will only have radiation, and if the temperature is greater than T_0 there will be the two black hole solutions.

Now consider the heat capacity in the limit $R \to \infty$ maintaining the black hole size r_+ fixed. If the cosmological constant is negative, the temperature will redshift to zero in this limit, if the cosmological constant is zero, the temperature will go to $1/4\pi r_+$. In general, the heat capacity assumes the form,

$$\lim_{R\to\infty} C_R = -2\pi r_+ \left(\frac{1 + 3r_+^2/L^2}{1 - 3r_+^2/L^2}\right). \tag{3.33}$$

Thus, one can see from (3.33) that in the large R limit, the minimum T_0 coincides with the divergence point of the heat capacity and thus, this is saying to us that in the R limit the small-large black hole phase transition also occurs, but this requires a negative cosmological constant. If the cosmological constant vanishes, then the minimum T_0 becomes infinite, which indicates that the black hole size must also diverges (since the minimum point is $r_+ = \sqrt{-1/\Lambda}$). Thus, it is impossible to keep the size of the black hole fixed in the large R limit. These results indicate to us, that a black hole can be in thermal equilibrium in a infinite space, only if the cosmological constant is negative, and not if the cosmological constant is zero. This is the same result that we have discussed in section 5.1, and the same result reached by Hawking and Page [13].

4 Extended thermodynamic theory with finite boundary

In the last chapter we have discussed the approach due to Brown, Creighton and Mann [6] to investigate the thermodynamical properties of black holes in spacetimes that are not asymptotically flat, in particular spacetimes that are asymptotically Anti de-Sitter, this was done considering a gravitational theory with finite boundary conditions. With this approach they were able to correct the Hawking temperature, which should be red shift to zero in the infinity AdS [21], this approach also gives a definition to the total energy E of the gravitational field within a region of space with a boundary B. This energy, in general, is not equal to the ADM mass, as one can see from equations (3.18) and (3.20).

Although this approach corrects the temperature with the red shift factor, it still fails when one analyzes the homogeneity of the thermodynamical quantities, since the cosmological constant is not considered as a thermodynamical variable. In this chapter we want to generalize the thermodynamics described in the chapter 3 using the Hamiltonian formalism described in chapter 2. This can be done by extending the space of the thermodynamical variables by inserting the AdS radius, L^2, as a thermodynamical parameter. This will define a new equation of state ξ associated with L^2, and thus will give rise to a new thermodynamic degree of freedom.

In this chapter we are going to use D to represent the dimension of spacetime, to avoid confusion with the lapse function N.

4.1 Thermodynamics with three parameters

The AdS radius can be inserted as thermodynamical variable if one demands that the lapse function N is a equation of state,

$$N^2\left(R, L^2\right) = 1 - \frac{\tilde{m}}{R^{D-3}} - \tilde{\Lambda} R^2, \tag{4.1}$$

$$\tilde{m} = \frac{16\pi}{(D-2)\,B_{D-2}} M, \quad \tilde{\Lambda} = \frac{2\Lambda}{(D-1)\,(D-2)}. \tag{4.2}$$

Where we are considering that the cosmological constant is fixed by the geometry as,

$$2\Lambda = -\frac{(D-1)\,(D-2)}{L^2}.$$

The mass of the black hole can be written as,

$$M = \frac{(D-2)\, B_{D-2}}{16\pi} \left(\frac{4S}{B_{D-2}} \right)^{\frac{D-3}{D-2}} \left(1 + \frac{1}{L^2} \left(\frac{4S}{B_{D-2}} \right)^{\frac{2}{D-2}} \right). \tag{4.3}$$

With (4.3), the equation (4.1) can be written as,

$$N^2 \left(R, L^2 \right) = 1 - \left(\frac{1}{R^{D-2}} \frac{4S}{B_{D-2}} \right)^{\frac{D-3}{D-2}} + \frac{R^2}{L^2} \left[1 - \left(\frac{1}{R^{D-2}} \frac{4S}{B_{D-2}} \right)^{\frac{D-1}{D-2}} \right]. \tag{4.4}$$

Let us introduce new variables in order to make the connection with the Hamiltonian formalism discussed in chapter 2,

$$\tau^{-\frac{1}{c}\frac{2}{D-2}} = \frac{1}{L^2}, \quad X = \frac{1}{R^{D-2}} \frac{S}{B_{D-2}}. \tag{4.5}$$

With this identification, we can use (4.4) to write the AdS radius as,

$$\tau^{-\frac{1}{c}\frac{2}{D-2}} = \frac{1}{R^2} \frac{\left(N^2 - 1 + X^{\frac{D-3}{D-2}} \right)}{\left(1 - X^{\frac{D-1}{D-2}} \right)}. \tag{4.6}$$

One can think of (4.6) in terms of Λ, and is easily to see that it has exactly the form of equation (2.106), with a given form of the $f(X)$ function. Since N is homogeneous of order 0 one can see that X is also an homogeneous function of order 0, thus, R must be homogeneous of order $1/(D-2)$, and in the same sense, L^2 must be homogeneous of order $1/(D-2)$. Because of the homogeneity of τ, the constant c remains undefined, it can assume any value that respects the condition in (2.109), and for simplicity we will set $c = 1$, in order that τ will have the same homogeneity of S and R^2.

For an arbitrary point in the AdS spacetime the temperature will be given by Brown's temperature, with the cosmological constant being a variable in the phase space of the theory. Thus, we have an additional term in equation (3.27). The energy (3.25) can be written in D dimensions as,

$$E = -\frac{B_{D-2}}{4\pi} \left(\frac{r}{B_{D-2}} \right)^{\frac{D-3}{D-2}} \left(1 - \frac{16\pi M}{(D-2)\, B_{D-2}} \left(\frac{r}{B_{D-2}} \right)^{-\frac{D-3}{D-2}} + \tau^{-\frac{2}{D-2}} \left(\frac{r}{B_{D-2}} \right)^{\frac{2}{D-2}} \right)^{1/2} \tag{4.7}$$

$$- r\varepsilon_0 (r, \tau) .$$

Where we have introduced $r = B_{D-2} R^{D-2}$. Taking the differential of (4.7), one can find

$$dE = \frac{\partial E}{\partial S} dS + \frac{\partial E}{\partial r} dr + \frac{\partial E}{\partial \tau} d\tau. \tag{4.8}$$

Where,

$$\frac{\partial E}{\partial S} = \frac{1}{N}\frac{2}{(D-2)}\frac{\partial M}{\partial S}, \tag{4.9}$$

$$\frac{\partial E}{\partial r} = -\frac{\left(\frac{r}{B_{D-2}}\right)^{-\frac{1}{D-2}}\left((D-3) - \frac{8\pi M}{B_{D-2}}\left(\frac{D-3}{D-2}\right)\left(\frac{r}{B_{D-2}}\right)^{-\frac{D-3}{D-2}} + (D-2)\,\tau^{-\frac{2}{D-2}}\left(\frac{r}{B_{D-2}}\right)^{\frac{2}{D-2}}\right)}{4\pi(D-2)N} - \frac{\partial(r\varepsilon_0)}{\partial r}, \tag{4.10}$$

$$\frac{\partial E}{\partial \tau} = \frac{1}{N}\frac{1}{(D-2)}\left(2\frac{\partial M}{\partial \tau} - \frac{B_{D-2}}{4\pi}\tau^{-\frac{D}{D-2}}\left(\frac{r}{B_{D-2}}\right)^{\frac{D-1}{D-2}}\right) - \frac{\partial}{\partial \tau}(r\varepsilon_0). \tag{4.11}$$

The mass derivatives,

$$\frac{\partial M}{\partial S} = \frac{1}{4\pi}\left((D-3)\left(\frac{4S}{B_{D-2}}\right)^{-\frac{1}{D-2}} + (D-1)\,\tau^{-\frac{2}{D-2}}\left(\frac{4S}{B_{D-2}}\right)^{\frac{1}{D-2}}\right), \tag{4.12}$$

$$\frac{\partial M}{\partial \tau} = -\frac{B_{D-2}}{8\pi}\tau^{-\frac{D}{D-2}}\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}}. \tag{4.13}$$

One can identify (4.10) with the negative of a superficial pressure $\mathcal{P}$ that is associated with the boundary defined by the position of the observer, and (4.11) in the same sense, can be identified as a superficial pressure ξ associated with the boundary of the AdS spacetime. Thus,

$$\begin{aligned}\mathcal{P} &= -\frac{\partial E}{\partial r}, \\ &= \frac{\left(\frac{r}{B_{D-2}}\right)^{-\frac{1}{D-2}}\left((D-3) - \frac{8\pi M}{B_{D-2}}\left(\frac{D-3}{D-2}\right)\left(\frac{r}{B_{D-2}}\right)^{-\frac{D-3}{D-2}} + (D-2)\,\tau^{-\frac{2}{D-2}}\left(\frac{r}{B_{D-2}}\right)^{\frac{2}{D-2}}\right)}{4\pi(D-2)N} + \frac{\partial(r\varepsilon_0)}{\partial r},\end{aligned} \tag{4.14}$$

$$\xi = \frac{\partial E}{\partial \tau} = -\frac{1}{N}\frac{\tau^{-\frac{D}{D-2}}}{(D-2)}\frac{B_{D-2}}{4\pi}\left(\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}} + \left(\frac{r}{B_{D-2}}\right)^{\frac{D-1}{D-2}}\right) - \frac{\partial}{\partial \tau}(r\varepsilon_0). \tag{4.15}$$

Equation (4.15) defines to us the new constraint in the theory,

$$\chi = 0, \tag{4.16}$$

$$\chi = \xi + \frac{1}{N}\frac{\tau^{-\frac{D}{D-2}}}{(D-2)}\frac{B_{D-2}}{4\pi}\left(\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}} + \left(\frac{r}{B_{D-2}}\right)^{\frac{D-1}{D-2}}\right) + \frac{\partial}{\partial \tau}(r\varepsilon_0). \tag{4.17}$$

The only new information here is the addition of $\tau^{\frac{2}{D-2}} = L^2$ as thermodynamic variable, everything else is a generalization of the previously treatment for D dimensional spacetime.

4.2 Homogeneity relations for the extended theory

The thermodynamic theory that we have found, is described by the following equations of state,

$$\xi = -\frac{1}{N}\frac{\tau^{-\frac{D}{D-2}}}{(D-2)}\frac{B_{D-2}}{4\pi}\left(\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}} + \left(\frac{r}{B_{D-2}}\right)^{\frac{D-1}{D-2}}\right) - \frac{\partial}{\partial\tau}(r\varepsilon_0), \tag{4.18}$$

$$\mathcal{P} = -\frac{\partial E}{\partial r},$$
$$= \frac{\left(\frac{r}{B_{D-2}}\right)^{-\frac{1}{D-2}}}{4\pi(D-2)N}\left(D-3-\frac{8\pi M}{B_{D-2}}\left(\frac{D-3}{D-2}\right)\left(\frac{r}{B_{D-2}}\right)^{-\frac{D-3}{D-2}} + (D-2)\tau^{-\frac{2}{D-2}}\left(\frac{r}{B_{D-2}}\right)^{\frac{2}{D-2}}\right) \tag{4.19}$$

$$+\frac{\partial(r\varepsilon_0)}{\partial r}, \tag{4.20}$$

$$T = \frac{1}{N}\frac{1}{2\pi(D-2)}\left((D-3)\left(\frac{4S}{B_{D-2}}\right)^{-\frac{1}{D-2}} + (D-1)\tau^{-\frac{2}{D-2}}\left(\frac{4S}{B_{D-2}}\right)^{\frac{1}{D-2}}\right). \tag{4.21}$$

Thus, we are able to find the homogeneity of all quantities by imposing the extensivity of the entropy, i.e., $S \to \lambda S$,

$$\begin{aligned} &R \to \lambda^{\frac{1}{D-2}}R, \quad r \to \lambda r, \quad E \to \lambda^{\frac{D-3}{D-2}}E,\\ &\mathcal{P} \to \lambda^{-\frac{1}{D-2}}\mathcal{P}, \quad \tau \to \lambda\tau,\\ &\xi \to \lambda^{\frac{D-5}{D-2}}\xi, \quad M \to \lambda^{\frac{D-3}{D-2}}M, \quad T \to \lambda^{-\frac{1}{D-2}}T. \end{aligned} \tag{4.22}$$

Using the Euler theorem for homogeneous functions, we are able to find,

$$\begin{aligned} \left(\frac{D-3}{D-2}\right)E &= \left(\frac{\partial E}{\partial S}\right)S + \left(\frac{\partial E}{\partial r}\right)r + \left(\frac{\partial E}{\partial \tau}\right)\tau,\\ (D-3)E &= (D-2)(TS - \mathcal{P}r + \xi\tau). \end{aligned} \tag{4.23}$$

Equation (4.23) can be thought as the Smarr formula for the black hole in the extended thermodynamic theory.

4.3 Heat capacities and stability

One can compute the heat capacity for this theory. If one considers the heat capacity for R and τ fixed, i.e., the observer does not change his position and the AdS radius does not vary (the geometry remains fixed). Thus the heat capacity $C_{R,\tau}$ can be found as,

$$C_{R,\tau} = \left(\frac{\partial E}{\partial T}\right)_{R,\tau} = T\left(\frac{\partial S}{\partial T}\right)_{R,\tau}. \tag{4.24}$$

And the derivative of the entropy can be found as,

$$\left(\frac{\partial S}{\partial T}\right)_{R,\tau} = -\frac{\pi}{2}\left(D-2\right)^2 B_{D-2}N\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}}\frac{1}{v\left(S,r,\tau\right)}, \tag{4.25}$$

$$v\left(S,r,\tau\right) = D-3-\left(D-1\right)\tau^{-\frac{2}{D-2}}\left(\frac{4S}{B_{D-2}}\right)^{\frac{2}{D-2}}. \tag{4.26}$$
$$-2\pi^2T^2\left(D-2\right)^2\left(\frac{4S}{B_{D-2}}\right)^{\frac{D-1}{D-2}}\left(\frac{r}{B_{D-2}}\right)^{-\frac{D-3}{D-2}}.$$

Thus, using equations (4.25) and (4.21), the heat capacity can be found as,

$$C_{R,\tau} = -\left(D-2\right)S\frac{\left(\left(D-3\right)+\left(D-1\right)\tau^{-\frac{2}{D-2}}\left(\frac{4S}{B_{D-2}}\right)^{\frac{2}{D-2}}\right)}{v\left(S,r,\tau\right)}. \tag{4.27}$$

If we take the limit of a infinite boundary $r \to \infty$, with a non zero cosmological constant, the heat capacity assumes the form,

$$\lim_{r\to\infty} C_{R,\tau} = -\left(D-2\right)S\frac{\left(\left(D-3\right)+\left(D-1\right)\tau^{-\frac{2}{D-2}}\left(\frac{4S}{B_{D-2}}\right)^{\frac{2}{D-2}}\right)}{\left(\left(D-3\right)-\left(D-1\right)\tau^{-\frac{2}{D-2}}\left(\frac{4S}{B_{D-2}}\right)^{\frac{2}{D-2}}\right)}. \tag{4.28}$$

It is easy to see that equation (4.28) coincides exactly with equation (3.33) for the case where $D = 4$, and all the conclusions we have taken before are also valid here.

5 Conclusions

We have considered the thermodynamics of Schwarzchild black holes in anti de Sitter spacetimes. First attempts to describe the thermodynamical behavior of black holes in anti de Sitter are not entirely consistent. The minimal description lacks homogeneity, which is required for extensivity [15] and for the existence of an integrating factor for the reversible heat exchange and, as a consequence, the well definition of entropy [5]. Besides that, the temperature in the minimal description is not well defined, since in stationary anti de Sitter spacetime (or in a general asymptotic curved spacetime), the Killing vector that generates the horizon diverges asymptotically and thus, the surface gravity (and consequently the temperature), are not unique in such spacetimes.

One way to deal with this problem is by constructing quasi-local charges [6, 7] defined in some $(D-2)$-Surface of a D dimensional spacetime. In this formalism one does not solve the arbitrariness in the temperature associated with the non uniqueness of the surface gravity. But instead, one can obtain a thermodynamical description (in a natural way) in which the temperature of the black hole have a dependency in the position R of the observer in the spacetime. This dependency appears as the Tolman factor [21], which tell us that in a asymptotically curved spacetime, the temperature of the black hole should be red shifted to zero.

This formalism of the quasi-local charges, that give us a position dependent temperature, also have the same issues that appears in the minimal description. Thus we make use of the Hamiltonian formalism [3, 2], in which the thermodynamic equations of state are realized as constraints on phase space. The Hamiltonian formalism allow us to extend the theory by introducing a new variable that is related to the cosmological constant Λ, more specifically, the AdS radius L^2. The Hamiltonian formalism tell us exactly how the cosmological constant should appear in the theory, if we want that the theory have a well defined homogeneity.

Thus, we are able to find a new thermodynamic, that is described by three equations of state, and is able to predict the small-large black hole phase transition. The theory described by the entropy S, the position of the observer R and the AdS radius L, has all the properties of the previously descriptions and has the advantage that is also homogeneous.

Concerning the physical interpretation of these extras quantities, it was shown in [6] that the conjugate momenta $\mathcal{P}$ can be seen as a surface pressure, and also, can be related to the trace of the spatial stress of the spacetime. In the other hand, the physical interpretation of the conjugate momenta ξ is a bit mysterious, and for now, there is no much clues of the meaning of this quantity.

One natural route to continue what was discussed in this work, is analyzing the consequences of this new thermodynamics, such as, phase transitions, critical exponents and so on. The analysis of the heat capacities in a procedure where we hold $\mathcal{P}$ and ξ fixed could give some insight on the theory. Although the expressions for these quantities are extremely complex, in some limit (as in the $r \to \infty$), they may be simplified and give some useful information.

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Apêndices

APÊNDICE A – Brief review of general relativity

The main references for this discussion will be [17, 8, 16, 18].

A.1 General ideas

As the name suggests, general relativity is a generalization of special relativity. From special relativity we know that the speed light is a constant in all reference frames, and hence, one can fix a unity system where $c = 1$. Which gives the following line element

$$ds^2 = -dt^2 + d\vec{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{A.1}$$

Where $\eta_{\mu\nu}$ is the so called Minkowski metric with signature $(- + ++)$. The symmetry group of special relativity is the one that leaves the line element (A.1) invariant, called $\mathcal{SO}\left(1, D-1\right)$, or *Lorentz group.* This physically corresponds to transformations between inertial frames, and includes a particular case of rotations, the boosts. The Lorentz transformation is then a generalized rotation, given by

$$x'^\mu = \Lambda^\mu_\nu x^\nu; \quad \Lambda^\mu_\nu \in \mathcal{SO}\left(1, 3\right). \tag{A.2}$$

Therefore the statement of special relativity is that physics is Lorentz invariant, i.e., physics can be written in the same way in terms of transformed coordinates as in terms of the original coordinates.

In general relativity we keep this idea, but one considers a more general space-time, specifically a curved space-time, where the line element can be written as

$$ds^2 = g_{\mu\nu}\left(x\right) dx^\mu dx^\nu. \tag{A.3}$$

Where $g_{\mu\nu}\left(x\right)$ are arbitrary functions collectively called *the metric*, and x^μ are arbitrary parameterizations of the space-time, i.e., coordinates on the manifold.

But what the curvature has to do with gravity? Einstein formulated his theory in order to modify Newton's theory of gravity at strong gravitational fields and high velocities. Einstein then proceeded to construct the gravitational theory by making two assumptions (i) gravity is geometry, i.e., matter follows geodesics in a curved space-time, and the resulting motion appears to us as the effect of gravity; (ii) matter sources gravity, in other words, matter curves space, i.e., the curvature of space-time is generated by a matter distribution.

In a more formal way, these two assumptions can be stated as:

- Physics is invariant under general coordinate transformations (diffeomorphisms)[1],

$$x'^{\mu} = x'^{\mu}\left(x^{\nu}\right) \Rightarrow ds^2 = g_{\mu\nu}\left(x\right) dx^{\mu} dx^{\nu} = ds'^2 = g'_{\mu\nu}\left(x'\right) dx'^{\mu} dx'_{\nu}. \tag{A.4}$$

  So, the invariance of physics under a Lorentz transformation that we had in special relativity is now generalized to a invariance under general coordinate transformations, i.e., all physics equations take the same form in terms of x^{μ} or x'^{μ}.

- *The equivalence principle*, which comes in a variety of forms, the first one is the WEP (Weak equivalence principle), which basically states that "the motion of freely-falling particles are the same in a gravitational field and a uniformly accelerated frame in small enough regions of space-time. The second version is the EEP (Einstein equivalence principle), which is a extrapolation from the WEP and it is stated as, "In small enough regions of space-time, the laws of physics reduce to those of special relativity; it is impossible to detect the presence of a gravitational field by means of local experiments".

In other words, physics is general coordinate (diffeomorphisms) invariant, and both gravity and acceleration are manifestation of the curvature of space-time.

A.2 Kinematics of general relativity

As we saw, the metric can change under a coordinate transformation, and it obeys

$$g'_{\mu\nu}\left(x'\right) = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}\left(x\right). \tag{A.5}$$

Where x'^{μ} stands for the new coordinate system, and x^{μ} for the old one. Since the metric is symmetric, it has $D\left(D+1\right)/2$ components, but there are D coordinates transformations $x'^{\mu}\left(x^{\nu}\right)$ one can make in order to leave the physics invariant, which leave us with only $D\left(D-1\right)/2$ degrees of freedom that describes the curvature of space-time. Also, one can always find a coordinate transformation in order to arrange $g_{\mu\nu} = \eta_{\mu\nu}$ around an arbitrary point, that suggests that $g_{\mu\nu}$ is not a good measure for telling us whether there is curvature or not around some point.

[1] A map $f : M \to N$ is called diffeomorphism if it is a bijection and the inverse f^{-1} is also differentiable.

Thus, one need to introduce a new object, but before that let us recall the notion of relativistic tensors. A contravariant tensor A^{μ} is defined as an object that transforms as

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}. \tag{A.6}$$

And a covariant tensor B_{μ} is defined as the object that transforms as

$$B'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} B_{\nu}. \tag{A.7}$$

By means of (A.6) and (A.7), one can obtain that the transformation for $\partial_{\rho} g_{\mu\nu}$ is

$$\partial'_{\rho} g'_{\mu\nu}\left(x'\right) = \frac{\partial x^{\sigma}}{\partial x'^{\rho}} \frac{\partial}{\partial x'^{\sigma}} \left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \right) g_{\alpha\beta} + \frac{\partial x^{\sigma}}{\partial x'^{\rho}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \partial_{\sigma} g_{\alpha\beta}. \tag{A.8}$$

Then one can easily see that the derivative of the metric is not a tensor, because it does not obey the correct transformation law. The answer to remedy this problem is to look to gauge theory. We need to introduce a new object that play the role of a gauge field of gravity, and then construct a covariant derivative by introducing this new object to the regular derivative. We define the covariant derivative as

$$D_{\mu} T^{\nu} = \partial_{\mu} T^{\nu} + \Gamma^{\nu}_{\sigma\mu} T^{\sigma}. \tag{A.9}$$

Where the $\Gamma^{\nu}_{\sigma\mu}$ is called *Christoffel symbol*. We can generalize (A.9) to act on 2-rank tensor, as

$$D_{\mu} T^{\rho}_{\nu} \equiv \partial_{\mu} T^{\rho}_{\nu} + \Gamma^{\rho}_{\sigma\mu} T^{\sigma}_{\nu} - \Gamma^{\sigma}_{\mu\nu} T^{\rho}_{\sigma}. \tag{A.10}$$

Now, by means of the *equivalence principle* we know that the space-time should be locally flat[2], which tell us that the first derivative of the metric should be zero. If one require that the covariant derivative should vanishes, one can obtain:

$$\nabla_{\mu} g_{\nu\rho} = \partial_{\mu} g_{\nu\rho} - \Gamma^{\sigma}_{\mu\nu} g_{\sigma\rho} - \Gamma^{\sigma}_{\rho\mu} g_{\sigma\nu} = 0. \tag{A.11}$$

One can take all the cyclic permutations of (A.11) sum each one of them and obtain a explicit form for the Γ

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha} \left(\partial_{\mu} g_{\alpha\nu} + \partial_{\nu} g_{\alpha\mu} - \partial_{\alpha} g_{\mu\nu} \right). \tag{A.12}$$

Where $g^{\sigma\alpha} = \left(g^{-1}\right)_{\sigma\alpha}$ is the inverse metric. One can easily verify that locally the Christoffel symbols vanishes, which agree with our assumption that the space-time is locally flat. Particles in GR follows world lines that are called *geodesics*, this world lines can be obtained from a extremization principle. Let us consider the invariant distance

[2] The spacetime metric $g_{\mu\nu}$ must be compatible with the Minkowski metric $\eta_{\mu\nu}$ locally.

in a curved space-time with a metric tensor $g_{\mu\nu}(x)$ that is given by equation (A.3), one can write down the action associated with this line element as follows

$$S = \int \left(-g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda. \tag{A.13}$$

In equation (A.13) we are assuming that the world lines are parametrized by some affine parameter λ, such that $x^\mu = x^\mu(\lambda)$. One can ask, what is the world line that extremizes the action (A.13)?, the answer for that question is given by making the variation with respect to $x^\alpha(\lambda)$, and one can obtain the following expression

$$\delta S = -\int \left(g_{\mu\sigma} \frac{d^2 x^\mu}{d\lambda^2} + \frac{1}{2} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right) \delta x^\sigma d\lambda. \tag{A.14}$$

By imposing that the variation (A.14) should be an extrema, one can get the following equation

$$\frac{d^2 x^\alpha}{d\lambda^2} + \frac{1}{2} g^{\sigma\alpha} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \tag{A.15}$$

In (A.15) one can identify the term that accompanies the two derivatives as the Christoffel symbol, and thus, (A.15) can be written as,

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \tag{A.16}$$

Equation (A.16) is know as the *geodesic equation.* Geodesics can be thought as the generalization of a "straight line" for a curved space-time, the geodesics are curves that parallel transport their tangent vector with respect to the Christoffel connection associated with the metric.

Now, we are able to introduce the curvature tensor. Imagine that we are given a loop defined by two vectors, A^μ and B^ν. We want to make the parallel transport of a vector V^σ around this loop. After the parallel transport we will obtain another vector say, $V^{\sigma'}$, and then we have options. If $V^{\sigma'} = V^\sigma$, the spacetime does not have any curvature, but, if $V^\sigma \neq V^{\sigma'}$ then, the spacetime have an intrinsic curvature and the difference between the two vectors after the parallel transportation is proportional to a tensor $R^\mu_{\nu\rho\sigma}$, which is commonly called *Riemann tensor,* and is given by,

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\lambda\rho} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma} \Gamma^\lambda_{\nu\rho}. \tag{A.17}$$

The Riemann tensor satisfies various symmetry properties

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} = R_{\rho\sigma\mu\nu}. \tag{A.18}$$

One can also get two more objects from (A.17), just by contractions with the metric

$$R_{\nu\sigma} = R^{\mu}_{\nu\mu\sigma}, \tag{A.19}$$

$$R = g^{\nu\sigma} R_{\nu\sigma}. \tag{A.20}$$

The first one is the *Ricci tensor* and the second one is the *Ricci scalar* or *curvature scalar.*

A.3 Einstein's equations

Our purpose now is to find the dynamical equations of gravity, or by its famous name, the Einstein's field equations. To start our derivation, the first thing we have to note is that the volume element that is invariant is not $d^d x$ anymore, but rather $\sqrt{-g} d^d x$. To see that, consider a transformation $x^\mu \to x'^\mu$, the volume element becomes

$$d^d x = \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right) d^d x'. \tag{A.21}$$

On the other hand, if we take the determinant on both sides of the metric transformation (A.5), we obtain

$$g' = \left[\det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)\right]^2 g. \tag{A.22}$$

Where g denotes the determinant of the metric. It is easy to see that the element volume that is invariant under a general coordinate transformation, is the quantity $\sqrt{-g} d^d x$, since

$$\sqrt{-g} d^d x = \sqrt{-g'} d^d x'. \tag{A.23}$$

The minus sign comes from the signature of the metric, which means that $\det\left(g_{\mu\nu}\right) < 0$.

Now, we must write down an action for gravity. The Lagrangian that defines the action, obviously, should be invariant under a general coordinate transformation, thus it must be a scalar. There are several possibilities, but the simplest will suffice, and the simplest one is the Ricci scalar. This gives rise to the *Einstein-Hilbert action for gravity*

$$S_{EH} = \frac{1}{16\pi G} \int d^d x \sqrt{-g} R. \tag{A.24}$$

One can add a constant to the action (A.24), because that would not change the equations of motion. In fact, for our purposes, the most general action can be written as

$$S = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left(R - 2\Lambda\right) + S_{matter}. \tag{A.25}$$

Where Λ is called *Cosmological constant* and S_{matter} is a source of matter in the space-time, then, the vacuum is achieved when $S_{matter} = 0$. In order to obtain the equations of motion associated with the action (A.25) we must remember that now the space-time itself is our dynamical quantity. So, the equations of motion are obtained by looking for some extrema of S with respect to the metric $g_{\mu\nu}$. Thus, we must find the equations that satisfies the following relation

$$\frac{\delta S}{\delta g_{\mu\nu}\left(x^{\alpha}\right)} = 0. \tag{A.26}$$

Which gives

$$\frac{1}{16\pi G}\int d^{d}x\left[\frac{\delta}{\delta g_{\mu\nu}\left(x^{\alpha}\right)}\left(\sqrt{-g}\right)\left(R-2\Lambda\right)+\sqrt{-g}\frac{\delta R}{\delta g_{\mu\nu}\left(x^{\alpha}\right)}\right]+\frac{\delta S_{matter}}{\delta g_{\mu\nu}} = 0. \tag{A.27}$$

The last term in (A.27) can be identified with the stress-energy tensor by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}}. \tag{A.28}$$

And what is left for us is to compute, is the functional derivative of the first term. To do that, lets recall

$$\det\left(M\right) = e^{Tr\left(\ln M\right)} \Rightarrow \delta\det\left(M\right) = \det\left(M\right)Tr\left(\delta M\, M^{-1}\right) = -\det\left(M\right)Tr\left(\left(\delta M^{-1}\right)M\right). \tag{A.29}$$

If we apply that for the metric, one can obtain

$$\frac{\delta g}{\delta g^{\mu\nu}} = -g g_{\rho\sigma}\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}\delta^{(d)}\left(x^{\alpha}-y^{\alpha}\right), \tag{A.30}$$

Next, we have the variation of the Ricci scalar, which is

$$\frac{\delta R}{\delta g_{\mu\nu}} = R_{\alpha\beta}\frac{\delta g^{\alpha\beta}}{\delta g_{\mu\nu}}+g^{\alpha\beta}\frac{\delta R_{\alpha\beta}}{\delta g_{\mu\nu}}. \tag{A.31}$$

Now, we must show that the second term in (A.31) contributes in the form of a total derivative. For simplicity, we work in a local coordinate system, where the first derivative of the metric and the Christoffel symbols are zero. In this coordinate system, one can write

$$\frac{\delta R_{\mu\nu}}{\delta g_{\alpha\beta}} = \frac{\delta}{\delta g_{\alpha\beta}}\left(\partial_{\rho}\Gamma^{\rho}_{\mu\nu}\right)-\frac{\delta}{\delta g_{\alpha\beta}}\left(\partial_{\nu}\Gamma^{\rho}_{\mu\rho}\right). \tag{A.32}$$

Even that the Christoffel symbols are zero in this coordinate system, the first derivatives of them are not necessarily zero, in fact is straightforward to show that the variation of the Christoffel symbols are

$$\frac{\delta\Gamma^{\mu}_{\nu\rho}}{\delta g_{\alpha\beta}} = \frac{1}{2}g^{\mu\lambda}\left(D_\rho\frac{\delta g_{\lambda\mu}}{\delta g_{\alpha\beta}} + D_\nu\frac{\delta g_{\rho\lambda}}{\delta g_{\alpha\beta}} - D_\lambda\frac{\delta g_{\nu\rho}}{\delta g_{\alpha\beta}}\right). \tag{A.33}$$

Where we have used the freedom that the coordinate system give us to interchange between ∂s and Ds. Things that are worth noting: $\delta\Gamma^{\mu}_{\nu\rho}$ is a tensor, because it transforms correctly, and also, is a covariant expression, thus, even if we used a specific coordinate system, the equation (A.33) is valid for any coordinate system and not just for our special one. Given that $\delta\Gamma^{\mu}_{\nu\rho}$ is a tensor, we can change the normal derivatives in (A.32) for the covariant derivatives for free in our coordinate system where Γs are zero, and again, the expression we will obtain is a covariant one, so, is true for any coordinate system. Plugging (A.33) in (A.32) and multiplying by the metric $g^{\mu\nu}$, and recalling that $D_\mu g_{\nu\rho} = 0$, one obtain

$$g^{\mu\nu}\frac{\delta R_{\mu\nu}}{\delta g_{\alpha\beta}} = D_\mu\left(g^{\nu\rho}\frac{\delta\Gamma^{\mu}_{\nu\rho}}{\delta g_{\alpha\beta}}\right) - D_\rho\left(g^{\nu\rho}\frac{\delta\Gamma^{\mu}_{\nu\mu}}{\delta g_{\alpha\beta}}\right). \tag{A.34}$$

When we plug (A.34) inside the action, it will contributes as a boundary, which integrates to zero. Putting all the pieces together, (A.27) takes the form

$$\frac{1}{8\pi G}\int d^dx\left[\frac{1}{\sqrt{-g}}\frac{\delta g}{\delta g^{\mu\nu}}\left(-\frac{1}{2}R+\Lambda\right) + \sqrt{-g}R_{\alpha\beta}\frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}}\right] - T_{\mu\nu} = 0, \tag{A.35}$$

$$\frac{1}{8\pi G}\int d^dx\left[\sqrt{-g}\frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}}\left(-\frac{1}{2}g_{\alpha\beta}R+g_{\alpha\beta}\Lambda\right) + \sqrt{-g}R_{\alpha\beta}\frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}}\right] - T_{\mu\nu} = 0, \tag{A.36}$$

$$\frac{1}{8\pi G}\int d^dx\left[\sqrt{-g}\delta^d\left(y^\rho - x^\rho\right)\left(-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda\right) + \sqrt{-g}R_{\mu\nu}\delta^d\left(y^\rho - x^\rho\right)\right] - T_{\mu\nu} = 0. \tag{A.37}$$

Which gives the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu}. \tag{A.38}$$

APÊNDICE B – Anti de-Sitter Space-time and Penrose diagrams

This references for this discussion are [8, 16, 12, 17].

B.1 General definitions

Anti de-Sitter space, is a space with Lorentzian signature $(-++\ldots+)$ and constant negative curvature. In D dimensions, the AdS space-time is defined by the embedding in $\mathbb{H}^{D+1}$,

$$ds^2 = -dX_0^2 + \sum_{i=1}^{D-1} dX_i^2 - dX_{D+1}^2, \tag{B.1}$$

$$-L^2 = -X_0^2 + \sum_{i=1}^{D-1} X_i^2 - X_{D+1}^2. \tag{B.2}$$

Where $(X_0, X_{D+1}, X_1, \ldots, X_{D-1})$ are the coordinates of the space, and L is the *AdS radius.* We can see that is explicitly invariant under transformations of the group $\mathbb{SO}(2, D-1)$ that rotates coordinates $X_\mu = (X_0, X_{D+1}, X_1, \ldots, X_{D-1})$ by $X'_\mu = \Lambda^\mu_\nu X^\nu$. The metric of this space can be written in many different forms, each one of them corresponding to a different coordinate system that solves the embedding equation (B.2). For example, consider the following solution

$$\begin{aligned} X_0 &= \frac{1}{2u}\left(1 + u^2\left(L^2 + \vec{x}^2 - t^2\right)\right), \\ X^{D+1} &= Lut, \\ X^i &= Lux^i, \quad i = 1, \ldots, D-2. \\ X^{D-1} &= \frac{1}{2u}\left(1 - u^2\left(L^2 - \vec{x}^2 + t^2\right)\right). \end{aligned} \tag{B.3}$$

Inserting this into (B.3), one can obtain

$$ds^2 = L^2\left(u^2\left(-dt^2 + \sum_{i=1}^{D-2} dx_i^2\right) + \frac{du^2}{u^2}\right). \tag{B.4}$$

Where $0 < u < +\infty$. This coordinate system is known as *Poincaré coordinates.* This form is explicitly invariant under the Poincaré group of rotations and translations on $(t, \vec{x})$, $\mathbb{ISO}(1, D-2)$, and under the group $\mathbb{SO}(1, 1)$, a scaling symmetry acting by

$(t, \vec{x}, u) \to (\lambda t, \lambda \vec{x}, \lambda^{-1} u)$. We can also change variables to $u = 1/x_0$, and obtain another form of the Poincaré coordinates,

$$ds^2 = \frac{L^2}{x_0^2}\left(-dt^2 + \sum_{i=1}^{D-2} dx_i^2 + dx_0^2\right). \tag{B.5}$$

Where, $\infty < t,\, x^i < \infty$ and $0 < x_0 < \infty$. Therefore, from (B.5), we can see that this is just the D dimensional Minkowski space up to a conformal factor, do not be misled at this point, this does not mean that the Minkowski space and the AdS space are the same up to a conformal factor, in fact, the Poincaré coordinates only cover a piece of the AdS space, what is often called *the Poincaré patch*, hopefully, this will be better understood later, when we discuss the construction of the Penrose diagram associated for these spaces. The Poincaré coordinates does not cover the entire space by the following calculation. First, consider the change of coordinates $e^{-y} = x_0/L$, with this (B.5) becomes,

$$ds^2 = e^{2y}\left(-dt^2 + \sum_{i=1}^{D-2} dx_i^2\right) + L^2 dy^2. \tag{B.6}$$

However, even though the coordinates are infinite in extent, they do not cover the entire space! Imagine that we send a light ray to infinity in y coordinates, which corresponds to a boundary of the space $(x_0 = 0)$, and also consider that we do this at constant x_i, one can obtain

$$t = L\int_0^\infty e^{-y} dy < \infty. \tag{B.7}$$

Thus, it takes a finite amount of time for the light ray to reach the boundary, but since t is not finite, in principle, it can go further. It can, for example, reflect from the boundary and travel back to another region of the AdS space, so, even though the Poincaré coordinates are very useful in many applications, they do not cover the entire space, in fact we can find the coordinate system that does, and we discuss that in the following.

Poincaré coordinates, are just one of many coordinates systems that we can use to investigate the geometry of the AdS space. We just argued in the previous paragraph that the Poincaré coordinates does not cover the entire space, and in fact the coordinate system that does that is what we call *global coordinate system* which is given by,

$$\begin{aligned} X_0 &= L\cosh\rho\cos\tau, \\ X_i &= L\sinh\rho\,\Omega_i, \quad i = 1, \ldots, D-1. \\ X_{D+1} &= L\cosh\rho\sin\tau. \end{aligned} \tag{B.8}$$

It is straightforward to check that the coordinates in (B.8) solves the constraint of the AdS space (B.2). Ω_i are the coordinates for a unit sphere, defined by $\Omega_i\Omega_i = 1$, and we also have that $0 < \rho < \infty$ and $0 < \tau < 2\pi$. The line element in global coordinates take the following form,

$$ds^2 = L^2 \left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega^2_{D-2}\right). \tag{B.9}$$

Where $d\Omega^2_{D-2}$ is the line element associated with a unit $(D-2)$ dimensional sphere. Near $\rho = 0$, the metric can be written as

$$ds^2 = L^2 \left(-d\tau^2 + d\rho^2 + \rho^2 d\Omega^2_{D-2}\right). \tag{B.10}$$

From (B.10) we can see that the topology of the AdS space is given by $\mathbb{S}^1 \times \mathbb{R}^{D-1}$, with $\mathbb{S}^1$ being the periodic time. But, this is very uncomfortable! The periodicity in the time direction, means that our time-like curves are closed, which means that our space does not have causal properties, one can in principle imagine that we can go back in time by traveling in one of those curves, despite how tempting this can be, we should avoid this kind of things, and to do that is very simple, just "unwrap" the circle $\mathbb{S}^1$ and let $-\infty < \tau < \infty$ with no identifications. With this we obtain a causal space-time which is known as *universal cover of AdS space.* There is some others useful coordinate systems that are worth mentioning. Static coordinates are defined by making the transformation $r = \sinh\rho$ in the metric defined by (B.9), with this, one can obtain

$$\frac{ds^2}{L^2} = -\left(r^2 + 1\right) d\tau^2 + \frac{dr^2}{r^2 + 1} + r^2 d\Omega^2_{D-2}. \tag{B.11}$$

A *static metric* can be understood as stationary metric (no dependency on time) and time reversal invariant, this last invariance implies that there are no cross terms of time-space, $dxdt$. Another useful coordinate system is called *Conformal coordinates*, and are defined by the transformation $\tan\theta = \sinh\rho$ in the metric defined by (B.9), with this, the line element becomes:

$$\frac{ds^2}{L^2} = \frac{1}{\cos^2\theta} \left(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2_{D-2}\right), \tag{B.12}$$

Where,

$$-\pi/2 < \theta < \pi/2, \quad D = 2.$$
$$0 < \theta < \pi/2, \quad D > 2.$$

This coordinates will be useful in the next section, when in the discussion about the Penrose diagrams for the AdS space.

B.2 Penrose diagrams

Curved space-times can be a real challenge to understand, they are infinity in extent, have complicated topological and causal structure and a bunch of other properties to make our life a bit harder. Fortunately, we are able to draw standardized representations of space-time diagrams that capture the global properties and causal structure of sufficiently symmetric space-times, and such diagrams are called *Conformal diagrams* (or Carter-Penrose diagrams, or just Penrose diagrams).

A Conformal diagram is simply an ordinary space-time diagram for a metric which we have made a particularly coordinate transformation. Our goal is to portray the causal structure of a space-time and to bring the coordinates that are infinity in extent to a finite distance. First thing we have to note, the causal structure is given by the light cone, on which the line element is given by $ds^2 = 0$ so, in order for the new metric preserves the causal structure of the space-time we just have to find a new metric that is conformally related to the old metric (the two metrics can be related up to an overall factor that is called Conformal factor). Basically we want to leave a coordinate system described by $(t, r)$where null rays satisfies $(dt/dr) = \pm 1$ and go to a coordinate system that is described by a new set (T, R) and the null rays also satisfies $(dR/dT) = \pm 1$.

Let us draw the Penrose diagram for the Minkowski space to exemplify what we are saying. The Minkowski metric in polar coordinates can be written as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{D-2}^2. \tag{B.13}$$

Where $d\Omega_{D-2}^2$ is the line element along a unit $(D-2)$ Sphere, and $-\infty < t < \infty$, $0 \leq r < \infty$. It's obvious that we can draw light cones at 45° everywhere, but the coordinates still have infinite ranges and we want to bring those infinities to a finite range. Nothing unusual will happen with the line element $d\Omega_{D-2}^2$, and in principle one can just ignore this term and proceed in the exactly same manner as follows. Let us make a change of coordinates, from (t, r) go to null coordinates $(u\, v)$,

$$u = t - r, \quad v = t + r. \tag{B.14}$$

Where, $-\infty < u, v < \infty$ and $u \leq v$. These coordinates are illustrated in Fig.(1), on which each point represents a $(D-2)$-Sphere of radius $r = \frac{1}{2}(v-u)$. The Minkowski metric (B.13) in null coordinates take the form

$$ds^2 = -\frac{1}{2}(dudv + dvdu) + \frac{1}{4}(v-u)^2 d\Omega_{D-2}^2. \tag{B.15}$$

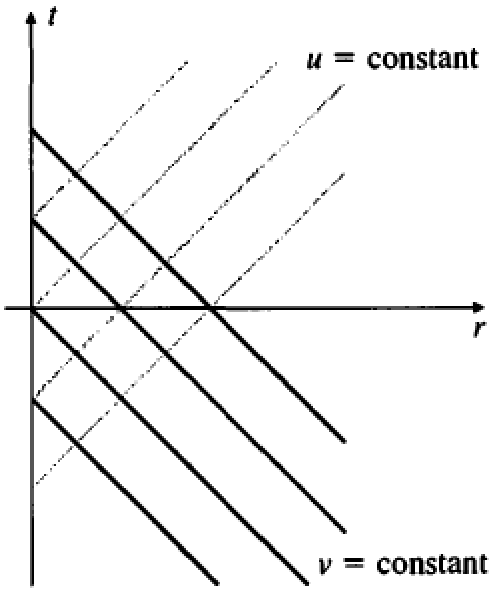

Figura 1 – Null radial coordinates in Minkowski space

Now one can make a new change of coordinates to bring the infinity to a finite value, consider the following transformation

$$U = \arctan u, \quad V = \arctan v. \tag{B.16}$$

Where $-\pi/2 < U, V < \pi/2$ and $U \leq V$. With this transformation, is straightforward to show that the metric (B.15) becomes:

$$ds^2 = \frac{1}{4\cos^2 U \cos^2 V}\left(-2\left(dUdV + dVdU\right) + \sin^2\left(V - U\right) d\Omega^2_{D-2}\right). \tag{B.17}$$

Now, let us get back to time and radial coordinates, via

$$T = V + U, \quad R = V - U. \tag{B.18}$$

Where $0 \leq R < \pi$, $|T| + R < \pi$. The metric (B.17) then becomes

$$ds^2 = \frac{1}{\omega^2\left(T, R\right)}\left(-dT^2 + dR^2 + \sin^2 R \, d\Omega^2_{D-2}\right). \tag{B.19}$$

Where,

$$\begin{aligned} \omega\left(T, R\right) &= 2\cos U \cos V, \\ &= 2\cos\left(\frac{1}{2}\left[T - R\right]\right)\cos\left(\frac{1}{2}\left[T + R\right]\right), \\ &= \cos T + \cos R. \end{aligned} \tag{B.20}$$

As we have said before, we are just interested in maintaining the causal structure invariant, so the line element $d\Omega^2_{D-2}$ is not really important and we could have dropped him in the beginning and still get the same results, with that said, one can think that the original metric and the "unphysical" metric are related by the conformal factor $\omega\,(T,\,R)$ by:

$$\begin{aligned} d\tilde{s}^2 &= \omega^2\,(T,\,R)\,ds^2, \\ &= -dT^2 + dR^2 + \sin^2 R\, d\Omega^2_{D-2}. \end{aligned} \tag{B.21}$$

The metric in (B.21) describes the manifold $\mathbb{R} \times \mathbb{S}^{D-1}$, where the $(D-1)$ sphere is purely space like, perfectly round and unchanging in time. In fact, equation (B.21) represents the *Einstein static universe*, a static solution of the Einstein equations with a positive cosmological constant.

Of course, the full range of the coordinates in $\mathbb{R} \times \mathbb{S}^{D-1}$ would usually be $-\infty < T < \infty$ and $0 \leq R \leq \pi$, while the Minkowski space is mapped into the subspace of $\mathbb{R} \times \mathbb{S}^{D-1}$ with $|T| + R < \pi$ and $0 \leq R < \pi$. We can represent $\mathbb{R} \times \mathbb{S}^{D-1}$ as a cylinder where each circle of constant T represents a $(D-1)$ sphere, and the Minkowski space is represented by the shaded region in Fig(2).

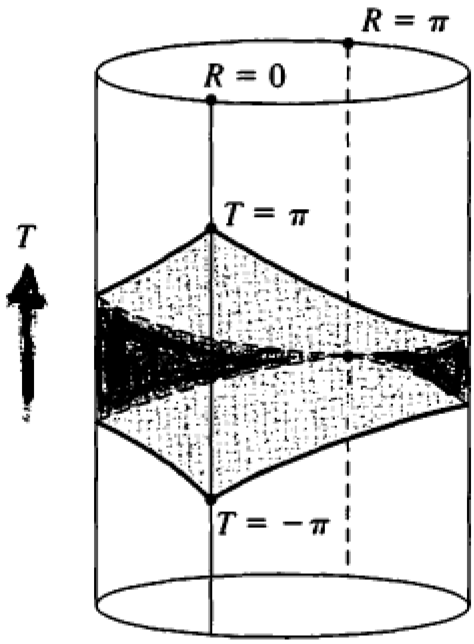

Figura 2 – Einstein static universe, $\mathbb{R} \times \mathbb{S}^3$ portrayed as a cylinder. The shaded region is conformally related to the Minkowski space.

In fact, Minkowski space is only the interior of the diagram of the Fig(2), including $R = 0$, the boundaries are not part of the original space-time. The boundaries are referred to as *conformal infinity*, and the union of the original space-time with the conformal infinity is the *conformal compactification.* The conformal diagram for the Minkowski space is portrayed in Fig(3).

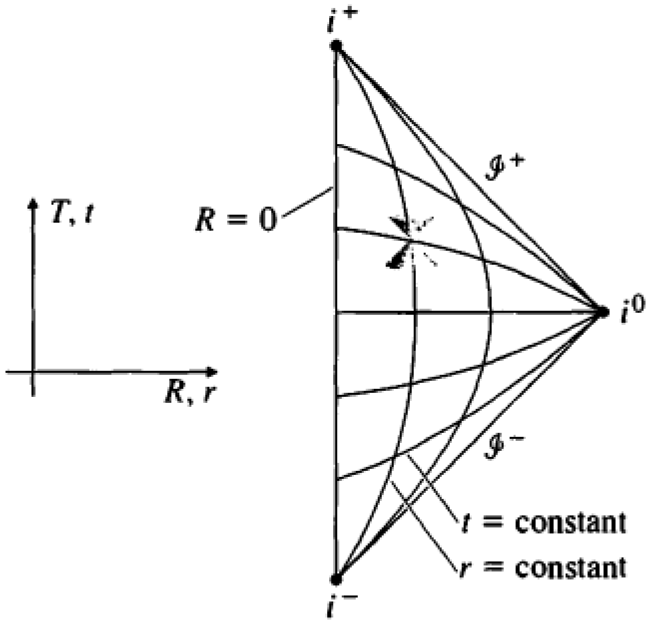

Figura 3 – Conformal diagram for the Minkowski space.

The points i^+, i^- and i^0 are the future and past time-like infinity and the spatial infinity, respectively. $\mathscr{I}^+$ and $\mathscr{I}^-$ are the future and the past null infinity, with the topology $\mathbb{R} \times \mathbb{S}^{D-2}$. Now, we turn our attention to the AdS spacetime. For this case is simpler because we already built the line element necessary to draw the conformal diagram. In conformal coordinates the line element becomes:

$$\frac{ds^2}{R^2} = \frac{1}{\cos^2\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2_{D-2}\right). \tag{B.22}$$

We have written the line element again just to make things more clear. Let us consider the case where the dimension $D > 2$ and forget about the line element for the sphere $d\Omega^2$ it will not be important to draw the diagram, of course, neither the conformal factor $R^2/\cos^2\theta$ will be important.

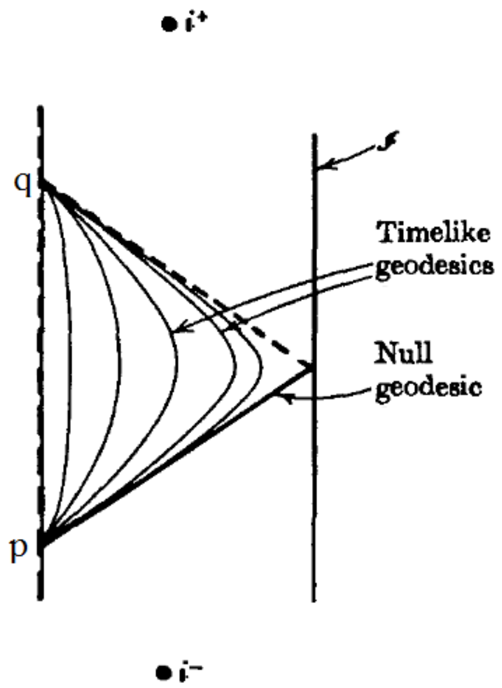

Figura 4 – Conformal diagram for the AdS spacetime.

With those considerations, the diagram can be seen in Fig.(4), the triangle region is called *Poincaré patch* of the AdS spacetime. The points i^{+}, i^{-} represent, in the same way as before, the future and past time like infinity, the line $\mathscr{I}$ represents the real boundary of the space (corresponds to the value of $\rho = \infty$) and the other line that we did not specified is a fake boundary of the AdS spacetime.

As we already stated before, the AdS spacetime has this interesting property where a particle can reach the boundary in a finite time, as we can see from the Fig.(4). But, the space has two further properties. First, the Cauchy problem is not well defined in AdS spacetime, for example, if we want to specify the behavior at some poiont $\mathscr{O}$ one need to consider all the causal past for this point, and for this we have to specify the initial conditions on a Cauchy surface at the past time like infinity, in any other regular space there is no problem with that, but in AdS spacetime, as a particle can reach the boundary at a finite time, just specifying the conditions on the Cauchy surface is not enough to specify the behavior at $\mathscr{O}$, one also need to specify boundary conditions for the spacetime, this feature of the AdS spacetime is sometimes called *absence of hyperbolicity*, also, once one specify the conditions on such surface, one cannot predict beyond the Cauchy development of the surface, any attempt to predict beyond this region is prevented by fresh information coming in from the time like infinity.

Secondly, one can also observe from the diagram shown in Fig.(4) that given some point p in the diagram, a time like geodesic leaving from p will converge at some point q in the future of p. In fact, all time like geodesics from any point in this space (past or future) re-converge to an image point, diverging again from this image point to refocus at a second image point and so on. The future time like geodesics from p therefore never reach $\mathscr{I}$, in contrast to the future null geodesics which go to $\mathscr{I}$ from p and form the boundary of the future of p.

B.3 Cosmological constant

As we have mentioned before, the Anti de-Sitter spacetime is the maximally symmetric solution of the Einstein's equations which comes from the action defined in (A.25). From the Einstein's equations, one can write[1]:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0, \tag{B.23}$$

Contracting (B.23) with the metric $g^{\mu\nu}$, one can obtain

$$R = \left(\frac{2D}{D-2}\right)\Lambda, \tag{B.24}$$

The Ricci scalar, R, for the N dimensional AdS spacetime is

$$R = -\frac{D(D-1)}{L^2}. \tag{B.25}$$

Combining the equations (B.25) and (B.24), one can find the general expression for the cosmological constant in the D dimensional AdS spacetime, which is given by

$$2\Lambda = -\frac{(D-1)(D-2)}{L^2}. \tag{B.26}$$

Another important feature, the introduction of a cosmological constant in the Einstein's equations is equivalent to introducing a vacuum energy density. Let us consider the energy momentum tensor for a perfect fluid $T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$[2], comparing to the energy momentum tensor for the vacuum $T^{(vac)}_{\mu\nu} = -\rho_{vac} g_{\mu\nu}$, one sees that the two are exactly the same, once we have $p_{vac} = -\rho_{vac}$. The vacuum behaves like a perfect fluid with an isotropic pressure opposite in sign to the energy density. This energy density should be constant throughout the entire spacetime, since a gradient would not be Lorentz invariant.

Let us write down the Einstein's equations with a decomposed energy momentum tensor

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T^{(M)}_{\mu\nu} - \rho_{vac} g_{\mu\nu}. \tag{B.27}$$

[1] We are using units were $8\pi G_N = 1$.

[2] U_μ is the four velocity.

Where the $T_{\mu\nu}^{(M)}$ is the energy momentum tensor that takes into account only matter contributions. Now, compare (B.27) with (B.23), one can see that the cosmological constant is proportional to the vacuum energy density by the relation

$$\rho_{vac} = \frac{\Lambda}{8\pi G_D}. \tag{B.28}$$

Where in (B.28) we have returned the constants just to be more explicit. The terms, *vacuum energy* and *cosmological constant* are used, commonly, interchangeably.

APÊNDICE C – Stability and phase transitions for the S-AdS black holes

C.1 Hawking-page and Small-Large black hole phase transitions

The following discussion is based on [26]. The partition function for the gravitational theory can be written as,

$$Z = \int D\left[g,\, \phi\right] e^{-I[g,\phi]}. \tag{C.1}$$

Where now we have put the fields ϕ explicitly. The action $I\left[g,\phi\right]$ is the full action for the gravitational theory, which involves the Einstein-Hilbert term, the Hawking-Gibbons boundary term and, of course, the counter term action to regularize the result. For the following discussion, things can be simplified and we do not have to take into account the Hawking-Gibbons action nor do the counter term, the regularization in what follows will be made by inserting a cutoff.

Let us consider the Einstein-Hilbert action $I = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2\Lambda\right)$, from the Einstein's equations we can see that for the AdS vacuum solutions satisfies $R = 4\Lambda$, and with this the Einstein-Hilbert action becomes,

$$I = \frac{\Lambda}{8\pi}\int d^4x\sqrt{-g}. \tag{C.2}$$

We will not need to take into account boundary terms, because we want to take the difference at the asymptotic infinity of the action for the AdS spacetime with the action for the S-AdS spacetime, and then, any boundary term will cancel. The integral in (C.2) is infinite if we take over all the space, to regularize this action we will insert a cutoff in the radial coordinate say, $r = K$. Then, for the AdS metric, (C.2) becomes,

$$I_1 = \frac{\Lambda}{8\pi}\int_0^{\beta_1} dt \int_0^K r^2 dr \int_{S^2} d\Omega_2 = \frac{\Lambda}{6}\beta_1 K^3, \tag{C.3}$$

And for the S-AdS metric,

$$I_2 = \frac{\Lambda}{8\pi}\int_0^{\beta_0} dt \int_{r_+}^K r^2 dr \int_{S^2} d\Omega_2 = \frac{\Lambda}{6}\beta_0\left(K^3 - r_+^3\right), \tag{C.4}$$

Where β_0 is the inverse temperature for the S-AdS geometry, which can be found as,

$$T_{BH} = \beta_0^{-1} = \frac{L^2 + 3r_+^2}{4\pi L^2 r_+}. \tag{C.5}$$

While β_0 is well defined, β_1 can in principle assume any value, since the AdS geometry does not have a horizon, and then, does not need to be periodic in the imaginary time. β_1 can be fixed by requiring that both geometries match for the $r = K$ hyper surface, in particular the interesting equality appears by matching the time coordinate of both metrics, which tell us that they must have the same period when we take the imaginary time,

$$\beta_1\sqrt{1+\frac{K^2}{L^2}} = \beta_0\sqrt{1-\frac{2M}{K}+\frac{K^2}{L^2}}. \tag{C.6}$$

Then, the difference between the action for the S-AdS geometry and for the AdS geometry, can be written as,

$$\begin{aligned} I = I_2 - I_1 &= \frac{\Lambda}{6}\beta_0\left(K^3 - r_+^3\right) - \frac{\Lambda}{6}\beta_1 K^3, \\ &= \frac{\Lambda}{6}\beta_0\left(K^3 - r_+^3 - \frac{\beta_1}{\beta_0}K^3\right). \end{aligned} \tag{C.7}$$

And using (C.6) and taking the expansion for large K, one can rewrite (C.7) as,

$$I \approx \frac{L^2 r_+^2\left(L^2 - r_+^2\right)}{L^2 + 3r_+^2}. \tag{C.8}$$

The plot of the temperature (C.5) is showed in Fig.(5)

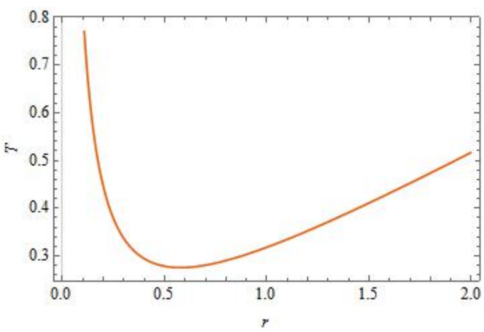

Figura 5 – Temperature vs horizon radius for the S-AdS black hole.

One can see that below some critical temperature, $T_c = \sqrt{3}/\left(2\pi L\right)$, there is no black hole solution, there is only radiation. Above this critical temperature, one can have two solutions, a small black hole and a larger black hole. The smaller black hole is represented by the branch, $r < r_0$, below the minimum radius $r_0 = L/\sqrt{3}$ and the larger black hole, is represented by the branch, $r > r_0$. If we compute the heat capacity for this black hole, one will find,

$$C = \frac{4\pi^2 T r_+^2 L^2}{3r_+^2 - L^2}. \tag{C.9}$$

Thus, it is easily seen from (C.9), that the heat capacity diverges at $r = r_0$ which indicates a first order phase transition, for the small black hole, $r < r_0$, the heat capacity is negative, thus, the small black hole is unstable and will eventually evaporates in pure radiation, on the other hand, the larger black hole, $r > r_0$, have positive heat capacity and is stable.

Back to (C.8), we see that the free energy $F = TI$ will be proportional to $(L^2 - r_+^2)$, thus, when $r_+ = L$ the free energy vanishes, and the temperature at this point is $T_{HP} = T_1 = 1/\pi L$, and of course, for $r_+ < L$ the free energy will be positive and will have a temperature T where $T < T_1$, thus, thermal radiation will dominate the partition function. On the other limit, $r_+ > L$ the temperature will be higher than T_1, and black holes will dominate the partition function.

At the temperature[1] $T = T_1$ a thermal phase transition occurs, where the preferred state becomes the larger black hole, instead of radiation, below the temperature T_1, large black hole solutions are allowed, but is preferred to the system that these large black holes switches to the smaller one and then evaporates becoming pure radiation. This transition is called *Hawking-Page phase transition* and it was first presented in a paper by Hawking and Page in the 80's[13]. There is another critical temperature $T_2 > T_1$ for which radiation can no longer sustain itself, and eventually collapses into a black hole, all of what was discussed here, can be better understand in the diagram presented at Fig.6

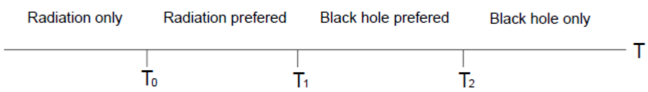

Figura 6 – Phase diagram for the Hawking-Page phase transition.

Thus, in summary, for $T < T_0$, black holes can not exist and all the space is dominated by pure radiation. For $T_0 < T < T_1$, black holes can exist but they are not the preferred state, the large black hole can reduce its free energy by tunneling into a smaller black hole and evaporating into pure radiation. For $T_1 < T < T_2$ black holes are the preferred state, at this temperature, pure radiation can reduce its free energy by collapsing into a black hole. Finally, for $T > T_2$, radiation can no longer exist, and every portion of radiation must collapse! The universe is now dominated only by black holes.

[1] Observe that $T_1 > T_0$.