Universidade Estadual de Londrina

VICTOR EZEQUIEL DE LA HOZ CORONELL

THE CASIMIR EFFECT IN THE KERR SPACETIME

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Dissertação de Mestrado apresentada ao Departamento de Física da Universidade Estadual de Londrina, como requisito parcial à obtenção do título de Mestre em Física.

Orientador: Prof. Dr. Andrey Bytsenko.
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ABSTRACT

we research the regularized vacuum enerry for a massless scalar field on Casimir cavity that is inside two parallel plates that are moving in a circular equatorial orbit in the neighborhood of Kerr spacetime. We find this energy and we see that in the ZAMO limit the energy return to the Casimir in flat space.

Keywords: Casimir effect. Kerr spacetime. Klein-Gordon equation.

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1 INTRODUCTION

The Casimir effect takes its name from the dutch physicist H. Casimir who in 1948 published a document *called attraction of two neutral metallic plates* at the Academy of Sciences of the Netherlands. The Casimir's work has been so important that it has attracted the attention of the scientific community for two or three decades. The impulse of this work was an article by Casimir and Polder in 1948 in a Physical Review termed *The influence of Retardation on the London-Van der Waals Forces*, of this interaction of Van der Walls is that the Casimir effect is born. Van der Waals forces arise from the electrical interaction between two or more atoms, this interaction is at molecular level, and arising in the neighborhood of one or more atoms or molecules when its are very close. The interaction of Van der Waals is the weakest of all interactions (at the molecular level there is strong dipole-dipole force), however when you have many molecules this interaction can be very strong. The distance action of the Van der Waals interaction is $0.3 - 0.6\,nm$. The London-Van der Waals force is the interaction between dipoles induced instantaneously, this is also known in the literature as dispersion of energies. This attraction or repulsion of the London-Van der Waals interactions have the same origin os the Casimir effect, that arise from the quantum interaction with the zero-point field. Many paper has been published on Casimir energy in the frame of quantum field theory due to the mysterious character of the vacuum. Another importance of the Casimir effect is that it arise by imposing boundary conditions, then it impose geometry of the boundary's problem.

From the onset of the pioneering work of H. B. G. Casimir [1], many papers has been published of which we only cite in the direction in our interest, for example [9, 10] they are consider differents boundary conditions, such as, flat, mixed boundary conditions, curves, conical singularities, spaces with negative and positive curvature. For experimental evidence of this quantum fluctuations we can see [11, 12].

In this thesis we compute the Casimir energy for a particular system; by considering two parallel plates separated a distance L that rotating with one gravitational rotating source. In order to do this we find an approximately solution of the Klein-Gordon equation for massless and massive scalar field confined in the Casimir cavity moving in a circular equatorial orbit on Kerr metric. The Riemann ζ-function is used as a mathematical technique in order to regularized vacuum energy.

The goal is to calculate the casimir energy for a massless and massive scalar field in the Kerr time space and generalized the paper of Sorge [19], however, to keep under control the subtleties of the method of quantization and regularization, we first approach in detail in the chapter 2 the Casimir effect in the flat space. Later is necessary to introduce some properties of the Kerr metric in order to understand the motion of the plates in this metric, and also is important to obtain the Klein-Gordon equation in this curved space, this is in the chapter 3. In this process appear some problems such that the mode normalization of the solution on Klein-Gordon equation, for solve it we introduce the Vierbein (Tetrad) formalism on details in the appendix (E).

2 THE CASIMIR ENERGY IN FLAT SPACETIME

First review the Casimir energy in flat space to find it later in curved space. The principal point here is to consider the Lagrangian of real scalar field and its respective motion equation that we have resolved by imposing Dirichlet boundary conditions on the field and then quantized it. After this we consider the energy-momentum tensor for this field and find its Hamiltonian density via definition of π momentum and field φ as operators. The Lagrangian for a real scalar field have the following form [16]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2). \tag{2.1}$$

The Euler-Lagrange equation of motion provide the Klein Gordon equation [1] for the $\varphi(x)$ field as

$$\Box \varphi(x) + m^2 \varphi(x) = 0, \tag{2.2}$$

and the corresponding energy momentum tensor is

$$\begin{aligned} \theta^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi - g^{\mu\nu} \mathcal{L} \\ &= \partial^\mu \varphi \partial^\nu \varphi - \frac{1}{2} g^{\mu\nu} (\partial^\alpha \varphi \partial_\alpha \varphi - m^2 \varphi^2), \end{aligned} \tag{2.3}$$

the energy density is given by

$$\begin{aligned} \theta^{00} = \mathcal{H} &= \pi^2 - \frac{1}{2}(\partial^\alpha \varphi \partial_\alpha \varphi - m^2 \varphi^2), \\ &= \frac{1}{2}\pi^2 + |\nabla \varphi|^2 + \frac{m^2}{2}\varphi^2, \end{aligned} \tag{2.4}$$

from which by integrating we obtain the Hamiltonian of the system

$$H = \int d^3x \;\; \theta^{00}. \tag{2.5}$$

The Klein-Gordon eq (2.2) can be rewritten as[2]

$$(\partial_0^2 - \nabla^2 + m^2)\varphi = 0, \tag{2.6}$$

1 $\Box = \partial_\mu \partial^\mu$

2 here we use signature -2 and $\hbar = c = 1$

and admits solution of the form

$$\varphi(x) = A\, e^{\alpha_\mu x^\mu}. \tag{2.7}$$

The Klein-Gordon equation (2.6) impose that

$$\alpha_0^2 - (\alpha_i)^2 + m^2 = 0, \Rightarrow$$
$$\alpha_0^2 = \pm\sqrt{(\alpha_i)^2 - m^2}.$$

and we can rewrite the field eq (2.7) as

$$\varphi(x) = A\, e^{\pm\sqrt{(\alpha_i)^2 - m^2}t} e^{-\alpha^i x^i}. \tag{2.8}$$

2.1 Dirichlet Boundary Conditions for the Field

Before discussing the case of boundary conditions for Dirichlet we will first introduce the periodic case and then extend the idea to Dirichlet boundary condition. Impose periodic condition in the variable x of the field we obtain

$$\begin{aligned}
\varphi(x, y, z, t) &= \varphi(x + L, y, z, t), \;\; \textit{periodicity condition}, \\
A\, e^{\pm\sqrt{(\alpha_i)^2 - m^2}t} e^{-\alpha^x x - \alpha^y y - \alpha^z z} &= A\, e^{\pm\sqrt{(\alpha_i)^2 - m^2}t} e^{-\alpha^x (x+L) - \alpha^y y - \alpha^z z}, \\
e^{-\alpha^x x} &= e^{-\alpha^x (x+L)} \Rightarrow \\
e^{-\alpha^x L} &= 1 \Rightarrow \alpha^x = \imath\, k_x = \frac{2n\pi}{L}\imath.
\end{aligned}$$

To get consistent solutions also define

$$\alpha^y \to \imath\, k^2 \quad , \quad \alpha^z \to \imath\, k^3,$$

this implies the following form for the scalar field (2.8)

$$\begin{aligned}
\varphi_n(x) &= e^{\pm\imath\sqrt{(\frac{2n\pi}{L})^2 + k_y^2 + k_z^2 + m^2}t} e^{\imath \boldsymbol{k}\cdot\boldsymbol{x}}, \\
&= e^{-\imath\omega_n t}\, e^{\imath\left(\frac{2n\pi}{L}\right)x}\, e^{\imath(k_y y + k_z z)},
\end{aligned} \tag{2.9}$$

where

$$\omega_n = \sqrt{\left(\frac{2n\pi}{L}\right)^2 + k_y^2 + k_z^2 + m^2}, \tag{2.10}$$

the eq (2.10) is the frequency of the field by consider periodicity condition. If we now consider Dirichlet boundary conditions (DBC) in (2.9). i.e.

$$\varphi(x=0,y,z,t)=\varphi(x=L,y,z,t)=0,\ \ \textit{Dirichlet condition}, \tag{2.11}$$

we obtain the following solution

$$\varphi_n(x)=\sin\left[\left(\frac{n\pi}{L}\right)x\right]e^{-\imath\omega_n t}\,e^{\imath(k_y y+k_z z)}, \tag{2.12}$$

now $k_x=n\pi/L$, then the frequency (2.10) is rewritten as

$$\omega_n=\sqrt{\left(\frac{n\pi}{L}\right)^2+k_y^2+k_z^2+m^2}. \tag{2.13}$$

This is the frequency of the field by consider Dirichlet conditions. We can consider the orthogonality condition in the functions [7]

$$\begin{aligned}&\frac{1}{L}\int_0^L e^{\imath\frac{2\pi}{L}(n-m)x}\,dx=\delta_{mn},\\ &\int_{\pi}^{\pi}\sin(nx)\,\sin(mx)\,dx=\int_{-\pi}^{\pi}\cos(nx)\,\cos(mx)\,dx=\pi\delta_{mn},\\ &\int_0^L\sin(nx)\,\cos(mx)\,dx=0,\quad\forall n,m,\in\mathbb{Z}.\end{aligned} \tag{2.14}$$

by performing the following change of variable

$$x\to x'=\frac{\pi}{L}x,\ \Rightarrow dx\to\frac{\pi}{L}dx, \tag{2.15}$$

one can rewrite the orthogonality eq (2.14) as

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\,\sin\left(\frac{m\pi}{L}x\right)\,dx=\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\,\cos\left(\frac{m\pi}{L}x\right)\,dx=L\delta_{mn}. \tag{2.16}$$

As for the continuous variables we use, the Dirac delta function representation

$$\delta(x)=\int_{-\infty}^{\infty}\frac{d^4p}{(2\pi)^4}e^{\imath p_\mu x^\mu}. \tag{2.17}$$

The solution of (2.12) in a discrete base is given by

$$\varphi_n(x,y,z,t) = \sum_{n=1}^{\infty} A_n \sin\left[\left(\frac{n\pi}{L}\right)x\right] e^{-\imath\omega_n t}\, e^{\imath(k_y y + k_z z)}, \tag{2.18}$$

and after we find that $A_n = \sqrt{2/L}$. Now we proceed to express the solution of the Klein-Gordon eq (2.18) as an expansion in the orthonormal base of the sin function and one integration in the momentum, the factors $1/2\pi$ are introduced by the definition of the delta function (2.17), see the quantization of the scalar field in the reference [16]

$$\begin{aligned}\varphi(x) = {} & \sqrt{\frac{2}{L}}\,\frac{1}{2\pi}\sum_{n=1}^{\infty}\int \frac{d^2k}{\sqrt{2\omega_{nk}}} \sin\left(\frac{n\pi}{L}x\right) \\ & \times \left[a_n(k)\, e^{\imath(k_y y + k_z z)}\, e^{-\imath\omega_{nk} t} + a_n^{\dagger}(k)\, e^{-\imath(k_y y + k_z z)}\, e^{\imath\omega_{nk} t}\right].\end{aligned} \tag{2.19}$$

A more suitable notation for the scalar field can be done defining the orthonormal bases

$$\begin{aligned} u_n(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right), \\ & \int_0^L u_n(x)\, u_m(x)\, dx = \delta_{nm}, \\ f_k(x) &= \frac{1}{2\pi}\,\frac{1}{\sqrt{2\omega_{nk}}}\, e^{\imath(k_y y + k_z z)}\, e^{-\imath\omega_n t} = \frac{1}{2\pi}\,\frac{1}{\sqrt{2\omega_{nk}}}\, e^{-\imath k_T x}, \\ k_T x &= k_0 x^0 - k_y y - k_z z, \end{aligned} \tag{2.20}$$

for this, the expression for the field in (2.19) became

$$\varphi(x) = \sum_{n=1}^{\infty}\int d^2k \left[a_n(k)\, f_k(x) + a_n^{\dagger}(k)\, f_k^{*}(x)\right] u_n(x). \tag{2.21}$$

In order to obtain the Hamiltonian using eq (2.5) we can express this using eq (2.4) in the following form

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\varphi|^2 + \frac{m^2}{2}\varphi^2, \\ &= \mathcal{H}_\pi + \mathcal{H}_{\nabla\varphi} + \mathcal{H}_\varphi, \\ H &= \int d^3x\ \mathcal{H}_\pi + \int d^3x\ \mathcal{H}_{\nabla\varphi} + \int d^3x\ \mathcal{H}_\varphi, \end{aligned} \tag{2.22}$$

where

$$\mathcal{H}_\pi = \frac{1}{2}\pi^2, \quad \mathcal{H}_{\nabla\varphi} = |\nabla\varphi|^2, \quad \mathcal{H}_\varphi = \frac{m^2}{2}\varphi^2, \tag{2.23}$$

then

$$H_\pi = \int d^3x\, \frac{1}{2}\pi^2, \quad H_{\nabla\varphi} = \int d^3x\, |\nabla\varphi|^2, \quad H_\varphi = \int d^3x\, \frac{m^2}{2}\varphi^2. \tag{2.24}$$

The detailed calculation of this Hamiltonian are presented in appendix (A). We obtain the following vacuum energy

$$\begin{aligned} \langle 0|\, H \,|0\rangle &= \frac{1}{2}\sum_n \int d^2k\; \delta^{(2)}(0)\omega_n(k), \\ \delta^{(2)}(0) &= \delta^{(2)}(k-k) \to \frac{L_y}{2\pi}\frac{L_z}{2\pi}, \end{aligned} \tag{2.25}$$

where L_y and L_z are the dimensions that corresponds of the transverse components of the moments. i.e. k_y, k_z. The expression (2.25) it is obviously divergent, in order to get a regular solution of the vacuum energy we proceed to use some mathematical techniques that convert this energy regular.

2.2 Regularization to Vacuum Energy

To compute the integral in the momentum space (wave numbers) that appear in the eq (2.25), we use the well-known formula of dimensional regularization to reduce the differential d^np in n-dimensions to one dimension by the formula

$$\int_0^\infty d^nr\, f(r) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty dr r^{n-1}\, f(r), \tag{2.26}$$

where $r = \sqrt{x^ix^i}$ $i = 1, ...n$, using this result in the vacuum energy eq (2.25),

$$\begin{aligned} \langle 0|\, H \,|0\rangle &= \frac{1}{2}\frac{L_y\, L_z}{(2\pi)^2}\sum_n \int d^2k\; \omega_n(k), \\ &= \frac{1}{2}\frac{L_y\, L_z}{(2\pi)^2}\sum_n \int d^2k\; \sqrt{\left(\frac{n\pi}{L}\right)^2 + k_y^2 + k_z^2 + m^2}, \\ &= \frac{1}{2}\frac{L_yL_z}{2\pi}\sum_n \int_0^\infty dk_T\, k_T\, \sqrt{\left(\frac{n\pi}{L}\right)^2 + k_T^2 + m^2}, \end{aligned} \tag{2.27}$$

Integration into the k_T variable can be done directly via change of variable, but the result provides an infinite value for energy. To regularized this divergence instead of compute the integral in eq (2.27) we compute the integral

$$\mathcal{I} = \int_0^\infty dk_T\, k_T \left[\left(\frac{n\pi}{L}\right)^2 + k_T^2 + m^2\right]^{-s/2}. \tag{2.28}$$

by using the following change of variable,

$$\begin{aligned} u &= \left(\frac{n\pi}{L}\right)^2 + k_T^2 + m^2, \\ du &= 2\, k_T\, dk, \end{aligned} \tag{2.29}$$

one can rewrite the eq (2.28) as

$$\begin{aligned} \mathcal{I} &= \frac{1}{2}\int_{u(0)}^\infty du\, u^{-s/2} = \frac{1}{2}\left[\lim_{u\to\infty} \frac{u^{-s/2+1}}{-s/2+1} - \frac{[u(0)]^{-s/2+1}}{-s/2+1}\right], \\ &= \frac{1}{2}\left[-\frac{[u(0)]^{-s/2+1}}{-s/2+1}\right], \quad if\ \Re\, s > 2, \\ &= \frac{1}{2}\frac{\left[\left(\frac{n\pi}{L}\right)^2 + m^2\right]^{1-s/2}}{s/2-1} = \frac{\left[\left(\frac{n\pi}{L}\right)^2 + m^2\right]^{1-s/2}}{s-2}. \end{aligned} \tag{2.30}$$

in the context of this calculation, the “vacuum energy” is

$$\langle 0|\, H\, |0\rangle \equiv \mathcal{E}(s) = \frac{1}{4\pi} L_y L_z \sum_n \frac{\left[\left(\frac{n\pi}{L}\right)^2 + m^2\right]^{1-s/2}}{s-2}, \tag{2.31}$$

consider a massless scalar field, in this case we have,

$$\begin{aligned} \mathcal{E}(s) &= \frac{1}{4\pi}\frac{L_y L_z}{s-2}\sum_n \left[\left(\frac{n\pi}{L}\right)^2\right]^{1-s/2}, \\ &= \frac{1}{4\pi}\frac{L_y L_z}{s-2}\sum_{n=1}^\infty \left(\frac{n\pi}{L}\right)^{2-s}, \\ &= \frac{1}{4\pi}\frac{L_y L_z}{s-2}\left(\frac{\pi}{L}\right)^{2-s}\sum_{n=1}^\infty n^{-(s-2)}, \\ &= \frac{1}{4\pi}\frac{L_y L_z}{s-2}\left(\frac{\pi}{L}\right)^{2-s}\zeta(s-2). \end{aligned} \tag{2.32}$$

This expression, Eq. (2.32) is not in fact the vacuum energy we try calculate. This is because to recover the vacuum energy of expression is necessary to make this equation

$s = -1$, but this expression is not defined for this value of s according to Eq. (2.30). For this we do analytic continuation of the zeta function [9] see (B)

$$\zeta(-1-2) = -\frac{B_4}{4} = -\frac{1}{4}\left(-\frac{1}{30}\right) = \frac{1}{120}, \tag{2.33}$$

using this result in (2.32),

$$\mathcal{E}(-1) \equiv E_C(L) = \frac{1}{4\pi}\left(\frac{L_y L_z}{-3}\right)\left(\frac{\pi}{L}\right)^3 \frac{1}{120} = -L_y L_z \frac{\pi^2}{1440 L^3} \tag{2.34}$$

that is the Casimir vacuum energy for a massless scalar field subject to Dirichlet boundary conditions. We can find the Casimir energy density by

$$\varepsilon_C(L) = \frac{1}{L_y L_z\, L} E_C(L) = -\frac{\pi^2}{1440 L^4}, \tag{2.35}$$

Which is the vacuum energy (Casimir energy) for a massless scalar field subject to Dirichlet boundary conditions. Note that this energy is negative which will imply an attractive force.

3 KERR SPACETIME AND CURVED KLEIN-GORDON EQUATION

In this chapter we will discuss the Kerr solution of the Einstein equation because our interest is study the Casimir energy in the frame of Kerr. The Kerr metric describe one rotating massive gravitational object, our purpose is now find the Casimir energy in the neighborhood on this rotating source, but firstly we see some properties of the Kerr solution that we have to construct the correct expression for the vacuum energy.

3.1 Kerr spacetime

We consider one specific curved space called Kerr spacetime that have the following metric in Boyer-Lindquist coordinates [2] (t, r, θ, φ) with signature -2 is given by

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mar}{\Sigma} \sin^2\theta dt d\varphi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \frac{A\sin^2\theta}{\Sigma} d\varphi^2, \tag{3.1}$$

where $a = J/M$ is the angular momentum by unit mass and

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 - 2Mr, \quad A = (r^2 + a^2)\Sigma + 2Mra^2\sin^2\theta. \tag{3.2}$$

The metric $g_{\mu\nu}$ is given by the eq (C.1) in the appendix (C). We assume that the plates are moving in a circular equatorial orbit with angular velocity $\Omega = d\varphi/dt$. For this case eq (3.2) take the form

$$\Sigma = r^2, \quad \Delta = r^2 + a^2 - 2Mr, \quad A = (r^2 + a^2)r^2 + 2Mra^2, \tag{3.3}$$

and the metric eq (3.1) is

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \frac{4Ma}{r} dt d\varphi - \frac{A}{r^2} d\varphi^2. \tag{3.4}$$

The four-velocity is defined as $u^\mu = dx^\mu/d\tau$ where τ is the proper time. i.e. the time measured in the local rest frame of the observer (also called coframe). If τ is the proper time then we can parametrize with the arc-length ds [4], in order to rewrite the four-velocity as

$$u^\mu = \frac{dx^\mu}{ds}. \tag{3.5}$$

Using the result obtained in eq (3.4) and the fact that the plates are moving in a circular equatorial orbit with angular velocity Ω one have

$$u^{\mu} = C(\Omega)(1, 0, 0, \Omega), \tag{3.6}$$

where $C(\Omega)$ is especified in eq (C.12)

3.2 Comoving Coordinates

Our objective is study the Casimir energy of the scalar field inside the cavity rotating around a body of mass M, in a circular equatorial orbit. In order to perform the calculations one introduce a coordinate system fixed in the cavity, i.e. a comoving system for this we define a new variables (coordinate system)

$$\begin{aligned} t' &= t, \\ r' &= r, \\ \theta' &= \theta, \\ \varphi' &= \varphi - \Omega t. \end{aligned} \tag{3.7}$$

The metric in this new system can be calculated by using

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}. \tag{3.8}$$

From which the nonvanishing elements are

$$\begin{aligned} g'_{tt} &= C^{-2}(\Omega), \\ g'_{t\varphi'} &= g_{\varphi\varphi}(\Omega - \omega_d), \\ g'_{rr} &= g_{rr}, \;\; g'_{\varphi\varphi} = g_{\varphi\varphi}, \;\; g'_{\theta\theta} = g_{\theta\theta}. \end{aligned} \tag{3.9}$$

That gives the metric in the comoving frame as

$$ds'^2 = C^{-2}(\Omega)dt'^2 + 2\frac{A}{r^2}(\omega_d - \Omega)dt'd\varphi' - \frac{r^2}{\Delta}dr'^2 - r^2 d\theta' - \frac{A}{r^2}d\varphi'^2. \tag{3.10}$$

Because the cavity is rectangular, we can define Cartesian coordinates (x, y, z) in the comoving frame eq (3.10) as $dx = r\, d\varphi'$, $dy = -r d\theta'$, $dz = dr'$, $dt = dt'$ and get a new metric namely $\hat{g}_{\mu\nu}$ such that

$$d\hat{s}^2 = C^{-2}(\Omega)dt^2 + 2\frac{A}{r^3}(\omega_d - \Omega)dtdx - \frac{A}{r^4}dx^2 - dy^2 - \frac{r^2}{\Delta}dz^2. \tag{3.11}$$

At this point is necessary specify that the dimension on this coordinates is very small due to it is a Casimir cavity. In this new coordinate system the metric $\hat{g}_{\mu\nu}$ has determinant a $det(\hat{g}_{\mu\nu}) = -1$. The inverse metric $\hat{g}^{\mu\nu}$ reads

$$\hat{g}^{\mu\nu} = \begin{pmatrix} \frac{A}{r^2\Delta} & (\omega_d - \Omega)\frac{A}{r\Delta} & 0 & 0 \\ (\omega_d - \Omega)\frac{A}{r\Delta} & -\frac{r^2}{\Delta}C^{-2}(\Omega) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{\Delta}{r^2} \end{pmatrix}. \tag{3.12}$$

3.3 Scalar Field on Curved Space

Consider a spacetime [17][18] of arbitrary dimension $D = d+1$ with metric $(+,-\cdots-)$. The action functional for scalar field in curved space is given by

$$S = \int d^D x \sqrt{|g|}\frac{1}{2}\left[g^{\mu\nu}\partial_\mu\varphi\partial^\nu\varphi - \left(m^2 + \xi R\right)\varphi^2\right], \tag{3.13}$$

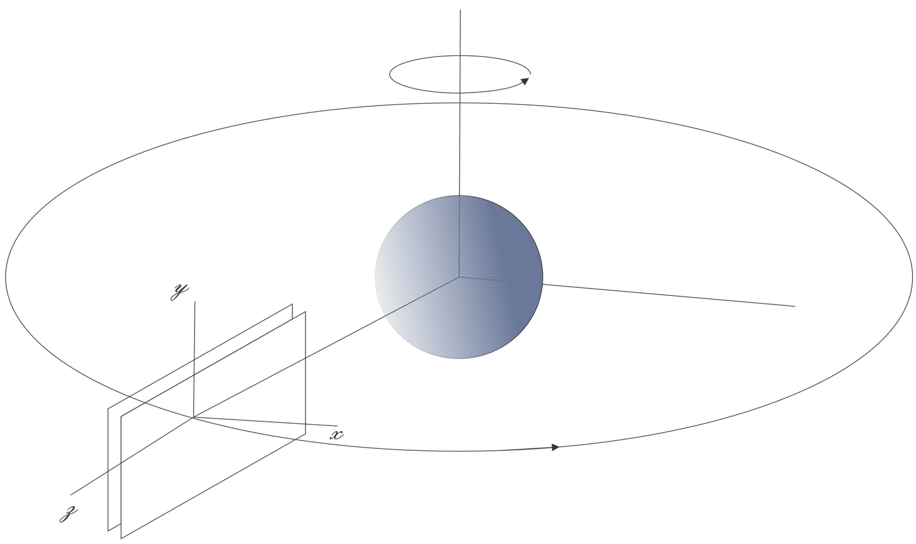
and the motion equation equation is

$$\left(\Box + m^2 + \xi R\right)\varphi = 0, \Box = \frac{1}{\sqrt{|g|}}\partial_\mu\sqrt{|g|}g^{\mu\nu}\partial_\nu. \tag{3.14}$$

Here ξ is a coupling constant between the scalar field and curvature. When $\xi = 0$ usually we called as minimal coupling, but for our case $R = 0$, therefore does not minimal coupling.

3.4 Normal Modes of the Scalar Field with $m = 0$

Figure 1 – Cavity with the scalar field orbiting a gravitational rotating source.



Source: Author.

The scalar field is confined that orbiting the massive object as show the figure (1), we take the Klein-Gordon equation in curved spacetime eq (3.14) to a massless scalar field, and also we take the approximation $\hat{g}_{\mu\nu} \simeq const$ inside the cavity. Remember that the dimensions of the cavity are very small, for this reason the metric tensor leaves the derivative in eq (3.14), with this approximation eq (3.14) became

$$\hat{g}^{\mu\nu}\partial_\mu\partial_\nu\psi = 0. \tag{3.15}$$

This equation can be solved by using the separation variable method presented in the appendix (D). The solution for the field inside the cavity is

$$\psi(x,y,z,t) = N_n\, e^{-\imath\omega_n t}\, e^{\imath(k_y y + k_z z)}\, e^{-\imath\beta_n x}\, \sin\left(\frac{n\pi}{L}x\right), \tag{3.16}$$

where N_n is a normalization constant that is compute in the section (3.4.1). The eigenfrequencies are given by

$$\begin{aligned} \omega_n &= \frac{r}{\sqrt{\Delta}C^2(\Omega)}\left[\left(\frac{n\pi}{L}\right)^2 + \frac{\Delta}{r^2}C^2(\Omega)\left(k_y^2 + \frac{\Delta}{r^2}k_z^2\right)\right]^{1/2}, \\ b_n &= \omega_n\, C^2(\Omega), \\ &= \frac{r}{\sqrt{\Delta}}\left[\left(\frac{n\pi}{L}\right)^2 + \frac{\Delta}{r^2}C^2(\Omega)\left(k_y^2 + \frac{\Delta}{r^2}k_z^2\right)\right]^{1/2}, \\ \beta_n &= \frac{b_n A}{r^3}(\omega_d - \Omega), \end{aligned} \tag{3.17}$$

where the detailed calculations is performed in Appendix eq (D.13) for more details. The frequencies (3.17) depends on the angular velocity of the massive object ω_d and the angular velocity of the cavity Ω and is according for admissible orbits ($r > r_+$), see [19] for more details.

3.4.1 Mode Normalization

For ψ_n and ψ_m solutions of the Klein-Gordon equation, we can take the scalar product defined by [18]

$$(\psi_n, \psi_m) = \imath \int_S \left[(\partial_\mu \psi_n)\psi_m^* - \psi_n(\partial_\mu \psi_m^*)\right] \sqrt{\hat{g}_S}\; n^\mu dS. \tag{3.18}$$

The problem is to define the surface S in order to avoid causality problems, for this we choose the surface S as a spacelike, then the induced metric in this surface is given by $\hat{g}_S = -\hat{g}/\hat{g}_{00}$, and n^μ is a timelike future-directed unit vector orthogonal to S [19]. Taking $dS = dxdydz$ (for more details about this computations see the vierbeins in the appendix (E) we find that

$$n^\mu = \left[\frac{1}{r}\sqrt{\frac{A}{\Delta}}, \sqrt{\frac{A}{\Delta}}(\omega_d - \Omega), 0, 0\right], \tag{3.19}$$

see (E.15). The condition for the orthogonality

$$(\psi_n, \psi_m) = \delta^{(2)}(\vec{k}_{\perp,n} - \vec{k}_{\perp,m})\delta_{mn}, \tag{3.20}$$

and the normalization constant

$$N_n^2 = \frac{1}{(2\pi)^2 L\omega_n}\,\frac{1}{r}\sqrt{\frac{A}{\Delta}}C^{-3}(\Omega). \tag{3.21}$$

4 COMPUTATION OF THE ENERGY-MOMENTUM TENSOR ON KERR SPACETIME

In analogy with (2.3) we find that the energy-momentum tensor for the massless scalar field is

$$\theta_{00} = \partial_0\psi_n\partial_0\psi_n^* - \frac{1}{2}\hat{g}_{00}\left(\hat{g}^{\mu\nu}\partial_\mu\psi_n\partial_\nu\psi_n^*\right), \tag{4.1}$$

where ψ_n and ψ_n^* are given by

$$\begin{aligned} \psi_n &= N_n e^{-\imath\omega_n t}\, e^{\imath(k_y y + k_z z)} e^{-\imath\beta_n x}\, \sin\left(\frac{n\pi}{L}x\right), \\ \psi_n^* &= N_n e^{\imath\omega_n t}\, e^{-\imath(k_y y + k_z z)} e^{\imath\beta_n x}\, \sin\left(\frac{n\pi}{L}x\right). \end{aligned} \tag{4.2}$$

The energy momentum-tensor (4.1) is construct for the tensor $\theta_{\mu\nu}$ by rewrite the fields $\psi(x)$ in forms of the fields modes ψ_k and using the result

$$\langle 0|\, a^\dagger_{\vec{k}} = 0, \tag{4.3}$$

together with

$$\langle 0|\, a_{\vec{k}} a^\dagger_{\vec{k}'}\, |0\rangle = \delta_{\vec{k}\vec{k}'}, \tag{4.4}$$

one obtain

$$\langle 0|\, \psi_{,\alpha}\psi_{,\beta}\, |0\rangle = \sum_k \psi_{\vec{k},\alpha}\psi^*_{\vec{k},\beta}, \tag{4.5}$$

to get in general

$$\langle 0|\, \theta_{\alpha\beta}\, |0\rangle = \sum_k \theta_{\alpha\beta}(\psi_k, \psi_k^*). \tag{4.6}$$

For detail see [18]. For this computations we need the following derivatives

$$
\begin{aligned}
&\partial_0\psi_n = -\imath\omega_n\psi_n,\\
&\partial_0\psi_n^* = \imath\omega_n\psi_n,\\
&\partial_1\psi_n = -\imath\beta_n\psi_n + N_n e^{-\imath\omega_n t} e^{\imath(k_y y + k_z z)} e^{-\imath\beta_n x}\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}x\right),\\
&\partial_1\psi_n^* = \imath\beta_n\psi_n^* + N_n e^{\imath\omega_n t} e^{-\imath(k_y y + k_z z)} e^{\imath\beta_n x}\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}x\right),\\
&\partial_2\psi_n = \imath k_y\psi_n,\\
&\partial_2\psi_n^* = -\imath k_y\psi_n^*,\\
&\partial_3\psi_n = \imath k_z\psi_n,\\
&\partial_3\psi_n^* = -\imath k_z\psi_n^*,
\end{aligned}
\tag{4.7}
$$

computing the second term of the right hand in (4.1)

$$
\hat{g}^{\mu\nu}\partial_\mu\psi_n\partial_\nu\psi_n^* = \overset{①}{\hat{g}^{0\nu}\partial_0\psi_n\partial_\nu\psi_n^*} + \overset{②}{\hat{g}^{1\nu}\partial_1\psi_n\partial_\nu\psi_n^*} + \overset{③}{\hat{g}^{2\nu}\partial_2\psi_n\partial_\nu\psi_n^*} + \overset{④}{\hat{g}^{3\nu}\partial_3\psi_n\partial_\nu\psi_n^*} \tag{4.8}
$$

①

$$\begin{aligned}\hat{g}^{0\nu}\partial_0\psi_n\partial_\nu\psi_n^* &= \hat{g}^{00}\partial_0\psi_n\partial_0\psi_n^* + \hat{g}^{01}\partial_0\psi_n\partial_1\psi_n^* + \cancelto{0}{\hat{g}^{02}}\partial_0\psi_n\partial_2\psi_n^* + \cancelto{0}{\hat{g}^{03}}\partial_0\psi_n\partial_3\psi_n^* \\ &= \frac{A}{r^2\Delta}\omega_n^2\psi_n\psi_n^* + \frac{A}{r\Delta}(\omega_d-\Omega)(-i\omega_n\psi_n)\left[\imath\beta_n\psi_n^* + N_n e^{\imath\omega_n t}e^{-\imath(k_y y+k_z z)}e^{\imath\beta_n x}\right. \\ &\left.\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}x\right)\right] \\ &= \frac{A}{r^2\Delta}\omega_n^2\psi_n\psi_n^* + \frac{A}{r\Delta}(\omega_d-\Omega)\left[\omega_n\beta_n\psi_n\psi_n^* - \imath\omega_n N_n^2\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\right].\end{aligned} \tag{4.9}$$

②

$$\begin{aligned}\hat{g}^{1\nu}\partial_1\psi_n\partial_\nu\psi_n^* &= \hat{g}^{10}\partial_1\psi_n\partial_0\psi_n^* + \hat{g}^{11}\partial_1\psi_n\partial_1\psi_n^* + \cancelto{0}{\hat{g}^{12}}\partial_1\psi_n\partial_2\psi_n^* + \cancelto{0}{\hat{g}^{13}}\partial_1\psi_n\partial_3\psi_n^* \\ &= \frac{A}{r\Delta}(\omega_d-\Omega)\left[-\imath\beta_n\psi_n + N_n e^{-\imath\omega_n t}e^{\imath(k_y y+k_z z)}e^{-\imath\beta_n x}\right. \\ &\left.\times\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}x\right)\right](\imath\omega_n\psi_n^*) - \frac{r^2}{\Delta}C^{-2}(\Omega)\left[-\imath\beta_n\psi_n\right. \\ &\left.+N_n e^{-\imath\omega_n t}e^{\imath(k_y y+k_z z)}e^{-\imath\beta_n x}\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}x\right)\right]\times \\ &\left[\imath\beta_n\psi_n^* + N_n e^{\imath\omega_n t}e^{-\imath(k_y y+k_z z)}e^{\imath\beta_n x}\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}x\right)\right] \\ &= \frac{A}{r\Delta}(\omega_d-\Omega)\left[\omega_n\beta_n\psi_n\psi_n^* + \imath\omega_n N_n^2\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\right] \\ &- \frac{r^2}{\Delta}C^{-2}(\Omega)\left[\beta_n^2\psi_n\psi_n^* + N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right)\right].\end{aligned} \tag{4.10}$$

③

$$\begin{aligned}\hat{g}^{2\nu}\partial_2\psi_n\partial_\nu\psi_n^* &= \cancelto{0}{\hat{g}^{20}}\partial_2\psi_n\partial_0\psi_n^* + \cancelto{0}{\hat{g}^{21}}\partial_2\psi_n\partial_1\psi_n^* + \hat{g}^{22}\partial_2\psi_n\partial_2\psi_n^* + \cancelto{0}{\hat{g}^{23}}\partial_2\psi_n\partial_3\psi_n^* \\ &= -(\imath k_y\psi_n)(-\imath k_y\psi_n^*) \\ &= -k_y^2\psi_n\psi_n^*.\end{aligned} \tag{4.11}$$

④

$$\begin{aligned}\hat{g}^{3\nu}\partial_3\psi_n\partial_\nu\psi_n^* &= \cancelto{0}{\hat{g}^{30}}\partial_3\psi_n\partial_0\psi_n^* + \cancelto{0}{\hat{g}^{31}}\partial_3\psi_n\partial_1\psi_n^* + \cancelto{0}{\hat{g}^{32}}\partial_2\psi_n\partial_3\psi_n^* + \hat{g}^{33}\partial_3\psi_n\partial_3\psi_n^* \\ &= -\frac{\Delta}{r^2}(\imath k_z\psi_n)(-\imath k_z\psi_n^*) \\ &= -\frac{\Delta}{r^2}k_z^2\psi_n\psi_n^*.\end{aligned} \tag{4.12}$$

Inserting eqs (4.9), (4.10), (4.11), (4.12) into the eq (4.8) yields

$$\begin{aligned}\hat{g}^{\mu\nu}\partial_\mu\psi_n\partial_\nu\psi_n^* &= \frac{A}{r^2\Delta}\omega_n^2\psi_n\psi_n^* + \frac{A}{r\Delta}(\omega_d-\Omega)\left[\omega_n\beta_n\psi_n\psi_n^* - \imath\omega_n N_n^2\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\right]\\ &+\frac{A}{r\Delta}(\omega_d-\Omega)\left[\omega_n\psi_n\psi_n^* + \imath\omega_n N_n^2\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\right]\\ &-\frac{r^2}{\Delta}C^{-2}(\Omega)\left[\beta_n^2\psi_n\psi_n^* + N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right)\right]\\ &-k_y^2\psi_n\psi_n^* - \frac{\Delta}{r^2}k_z^2\psi_n\psi_n^*.\end{aligned} \tag{4.13}$$

$$\begin{aligned}\hat{g}^{\mu\nu}\partial_\mu\psi_n\partial_\nu\psi_n^* &= \frac{A}{r^2\Delta}\omega_n^2\psi_n\psi_n^* + \frac{A}{r\Delta}(\omega_d-\Omega)\omega_n\beta_n\psi_n\psi_n^*\\ &-\frac{A}{r\Delta}(\omega_d-\Omega)\imath\omega_n N_n^2\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\\ &+\frac{A}{r\Delta}(\omega_d-\Omega)\omega_n\beta_n\psi_n\psi_n^* + \frac{A}{r\Delta}(\omega_d-\Omega)\imath\omega_n N_n^2\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\\ &-\frac{r^2}{\Delta}C^{-2}(\Omega)\left[\beta_n^2\psi_n\psi_n^* + N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right)\right]\\ &-k_y^2\psi_n\psi_n^* - \frac{\Delta}{r^2}k_z^2\psi_n\psi_n^*.\\ &=\left[\frac{A}{r^2\Delta}\omega_n^2 + 2\frac{A}{r\Delta}(\omega_d-\Omega)\omega_n\beta_n - \frac{r^2}{\Delta}C^{-2}(\Omega)\beta_n^2 - k_y^2 - \frac{\Delta}{r^2}k_z^2\right]\psi_n\psi_n^*\\ &-\frac{r^2}{\Delta}C^{-2}(\Omega)N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right).\end{aligned} \tag{4.14}$$

Introducing eq (4.14), eq (E.25) intoeq (4.1) we obtain

$$\begin{aligned}\theta_{00} &= \omega_n^2\psi_n\psi_n^* - \frac{1}{2}C^{-2}(\Omega)\left(\hat{g}^{\mu\nu}\partial_\mu\psi_n\partial_\nu\psi_n^*\right),\\ &= \omega_n^2\psi_n\psi_n^* - \frac{1}{2}C^{-2}\left(\left[\frac{A}{r^2\Delta}\omega_n^2 + 2\frac{A}{r\Delta}(\omega_d-\Omega)\omega_n\beta_n - \frac{r^2}{\Delta}C^{-2}(\Omega)\beta_n^2 - k_y^2 - \frac{\Delta}{r^2}k_z^2\right]\psi_n\psi_n^*\right.\\ &\left.-\frac{r^2}{\Delta}C^{-2}(\Omega)N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right)\right).\end{aligned} \tag{4.15}$$

Using the fact that

$$\begin{aligned}\beta_n &= \frac{b_n A}{r^3}(\omega_d-\Omega); \quad b_n = \omega_n C^2(\Omega)\\ \frac{\omega_n}{b_n} &= C^{-2}(\Omega),\end{aligned} \tag{4.16}$$

then

$$\begin{aligned}
2\frac{A}{r\Delta}(\omega_d-\Omega)\omega_n\beta_n &= 2\frac{A}{r\Delta}(\omega_d-\Omega)\omega_n\,\frac{b_nA}{r^3}(\omega_d-\Omega)\\
&= 2\frac{A}{r\Delta}(\omega_d-\Omega)\omega_n\,\frac{\omega_nC^2(\Omega)A}{r^3}(\omega_d-\Omega)\\
&= \frac{2A^2}{r^4\Delta}(\omega_d-\Omega)^2\omega_n^2C^2(\Omega),
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
\frac{r^2}{\Delta}C^{-2}(\Omega)\beta_n^2 &= \frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{b_nA(\omega_d-\Omega)}{r^3}\right)^2,\\
&= \frac{r^2}{\Delta}C^{-2}(\Omega)\,\frac{b_n^2A^2(\omega_d-\Omega)^2}{r^6},\\
&= \frac{C^{-2}(\Omega)}{\Delta}\,\frac{\omega_n^2C^4(\Omega)A^2(\omega_d-\Omega)^2}{r^4},\\
&= \frac{A^2}{r^4\Delta}(\omega_d-\Omega)^2\,\omega_n^2C^2(\Omega).
\end{aligned} \tag{4.18}$$

By using this results the Hamiltonian density in (4.15) become

$$\begin{aligned}
\theta_{00} &= \omega_n^2\psi_n\psi_n^* + \frac{1}{2}C^{-2}(\Omega)\left(\omega_n^2\left[-\frac{A}{r^2\Delta} - 2\cancel{\frac{A^2}{r^4\Delta}(\omega_d-\Omega)^2C^2(\Omega)} + \cancel{\frac{A^2}{r^4\Delta}(\omega_d-\Omega)^2C^2(\Omega)}\right]\psi_n\psi_n^*\right.\\
&\quad\left.+\left[k_y^2+\frac{\Delta}{r^2}k_z^2\right]\psi_n\psi_n* + \frac{r^2}{\Delta}C^{-2}(\Omega)N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right)\right),\\
&= \omega_n^2\psi_n\psi_n^* + \frac{1}{2}\left(\omega_n^2\psi_n\psi_n^*\left[-\frac{A}{r^2\Delta}C^{-2}(\Omega) - \frac{A^2}{r^4\Delta}(\omega_d-\Omega)^2\right]\right.\\
&\quad\left.+\left[k_y^2+\frac{\Delta}{r^2}k_z^2\right]\psi_n\psi_n^*C^{-2}(\Omega) + \frac{r^2}{\Delta}C^{-4}(\Omega)N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right)\right).
\end{aligned} \tag{4.19}$$

By substituting the explicity definition of $C(\Omega)$ in the eq (C.12), and using in the equation (4.19) one get

$$\begin{aligned}
\frac{A}{r^2\Delta}C^{-2}(\Omega) &= \frac{A}{r^2\Delta}\left[\frac{r^2\Delta}{A}\left(1-\frac{A^2}{r^4\Delta}(\Omega-\omega_d)^2\right)\right],\\
&= 1-\frac{A^2}{r^4\Delta}(\Omega-\omega_d)^2.
\end{aligned} \tag{4.20}$$

Replacing eq (4.20) into the eq (4.19) we obtain

$$\begin{aligned}
\theta_{00} &= \omega_n^2\psi_n\psi_n^* + \frac{1}{2}\left(\omega_n^2\psi_n\psi_n^*\left[-1+\cancel{\frac{A^2}{r^4\Delta}(\Omega-\omega_d)^2} - \cancel{\frac{A^2}{r^4\Delta}(\omega_d-\Omega)^2}\right]\right.\\
&\quad\left.+\left[k_y^2+\frac{\Delta}{r^2}k_z^2\right]\psi_n\psi_n^*C^{-2}(\Omega) + \frac{r^2}{\Delta}C^{-4}(\Omega)N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right)\right),
\end{aligned} \tag{4.21}$$

or

$$\theta_{00} = \omega_n^2\psi_n\psi_n^* - \frac{1}{2}\omega_n^2\psi_n\psi_n^* + \frac{1}{2}\left[k_y^2 + \frac{\Delta}{r^2}k_z^2\right]\psi_n\psi_n^* C^{-2}(\Omega) + \frac{1}{2}\frac{r^2}{\Delta}C^{-4}(\Omega)N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right),$$
$$= \frac{1}{2}\omega_n^2\psi_n\psi_n^* + \frac{1}{2}\left[k_y^2 + \frac{\Delta}{r^2}k_z^2\right]\psi_n\psi_n^* C^{-2}(\Omega) + \frac{1}{2}\frac{r^2}{\Delta}C^{-4}(\Omega)N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right). \tag{4.22}$$

Using eq (4.16) one can present the eq (4.22) in the form

$$\theta_{00} = \frac{1}{2}\left[\omega_n^2 + \frac{\omega_n}{b_n}\left(k_y^2 + \frac{\Delta}{r^2}k_z^2\right)\right]\psi_n\psi_n^* + \frac{1}{2}\frac{r^2}{\Delta}\frac{\omega_n^2}{b_n^2}N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right),$$
$$= \frac{1}{2}\left[\omega_n^2 + \frac{\omega_n}{b_n}\left(k_y^2 + \frac{\Delta}{r^2}k_z^2\right)\right]N_n^2\sin^2\left(\frac{n\pi}{L}x\right) + \frac{1}{2}\frac{r^2}{\Delta}\frac{\omega_n^2}{b_n^2}N_n^2\left(\frac{n\pi}{L}\right)^2\cos^2\left(\frac{n\pi}{L}x\right),$$
$$\theta_{00} = \frac{1}{2}N_n^2\left[\mathcal{F}_n\sin^2\left(\frac{n\pi}{L}x\right) + \mathcal{G}_n\cos^2\left(\frac{n\pi}{L}x\right)\right] \tag{4.23}$$

where

$$\mathcal{F}_n = \omega_n^2 + \frac{\omega_n}{b_n}\left(k_y^2 + \frac{\Delta}{r^2}k_z^2\right),$$
$$\mathcal{G}_n = \frac{\omega_n^2}{b_n^2}\frac{r^2}{\Delta}\left(\frac{n\pi}{L}\right)^2. \tag{4.24}$$

4.1 Computation of the Casimir Energy in Curved Spacetime

The vacuum energy density for the scalar field inside the cavity measured by the comoving observer $\hat{u}$ in eq (3.6), reads [19]

$$\langle\varepsilon_{vac}\rangle = \frac{1}{V_p}\int_V dxdydz\sqrt{g_S}\hat{u}^\mu\hat{u}^\nu\langle 0|T_{\mu\nu}|0\rangle, \tag{4.25}$$

where $V_p = \int_V dxdydz\sqrt{g_S}$ is the proper volume of the cavity measured by the comoving observer, and g_S is the induced metric in the surface.

$$\langle\varepsilon_{vac}\rangle = \frac{C^2(\Omega)}{L}\int_0^L dx\sum_n\int d^2k_T\,\theta_{00}(\psi_n,\psi_n^*), \tag{4.26}$$

substituting eq (4.23) into the eq (4.26) yields

$$\langle\varepsilon_{vac}\rangle = \frac{C^2(\Omega)}{L}\int_0^L dx\sum_n\int d^2k_T\,\frac{1}{2}N_n^2\left[\mathcal{F}_n\sin^2\left(\frac{n\pi}{L}x\right) + \mathcal{G}_n\cos^2\left(\frac{n\pi}{L}x\right)\right]$$
$$= \frac{C^2(\Omega)}{L}\sum_n\int d^2k_T\,\frac{1}{2}N_n^2\int_0^L dx\left[\mathcal{F}_n\sin^2\left(\frac{n\pi}{L}x\right) + \mathcal{G}_n\cos^2\left(\frac{n\pi}{L}x\right)\right], \tag{4.27}$$

$$\langle \varepsilon_{vac} \rangle = \frac{C^2(\Omega)}{\not{L}} \sum_n \int d^2k_T \, \frac{1}{2} N_n^2 \left[\mathcal{F}_n \frac{\not{L}}{2} + \mathcal{G}_n \frac{\not{L}}{2} \right], \\ \frac{C^2(\Omega)}{4} \sum_n \int d^2k_T \, N_n^2 \left[\mathcal{F}_n + \mathcal{G}_n \right], \tag{4.28}$$

Introducing the normalization constant given by eq (3.21) we obtain

$$\begin{aligned} \langle \varepsilon_{vac} \rangle &= \frac{\cancel{C^2(\Omega)}}{4} \sum_n \int d^2k_T \left[\frac{1}{(2\pi)^2 L \omega_n} \frac{1}{r} \sqrt{\frac{A}{\Delta}} \cancel{C^{-3}(\Omega)} \right] \left[\mathcal{F}_n + \mathcal{G}_n \right], \\ &= \frac{C^{-1}(\Omega)}{4} \frac{1}{(2\pi)^2 L} \frac{1}{r} \sqrt{\frac{A}{\Delta}} \sum_n \int d^2k_T \, \frac{1}{\omega_n} \left[\mathcal{F}_n + \mathcal{G}_n \right], \\ &= \frac{C^{-1}(\Omega)}{4} \frac{1}{(2\pi)^2 L} \frac{1}{r} \sqrt{\frac{A}{\Delta}} \sum_n \int d^2k_T \, \frac{1}{\omega_n} \left[\omega_n^2 + \frac{\omega_n}{b_n} \left(k_y^2 + \frac{\Delta}{r^2} k_z^2 \right) + \frac{\omega_n^2}{b_n^2} \frac{r^2}{\Delta} \left(\frac{n\pi}{L} \right)^2 \right]. \end{aligned} \tag{4.29}$$

Now one can use the eq (4.16) in order to simplify the eq (4.29)

$$\begin{aligned} \mathcal{F}_n + \mathcal{G}_n &= \omega_n^2 + \frac{\omega_n}{b_n} \left(k_y^2 + \frac{\Delta}{r^2} k_z^2 \right) + \frac{\omega_n^2}{b_n^2} \frac{r^2}{\Delta} \left(\frac{n\pi}{L} \right)^2, \\ &= \omega_n^2 + \frac{\cancel{\omega_n}}{\cancel{\omega_n} C^2(\Omega)} \left(k_y^2 + \frac{\Delta}{r^2} k_z^2 \right) + \frac{\cancel{\omega_n^2}}{\cancel{\omega_n^2} C^4(\Omega)} \frac{r^2}{\Delta} \left(\frac{n\pi}{L} \right)^2, \\ &= \omega_n^2 + C^{-2}(\Omega) \left(k_y^2 + \frac{\Delta}{r^2} k_z^2 \right) + C^{-4}(\Omega) \frac{r^2}{\Delta} \left(\frac{n\pi}{L} \right)^2, \\ &= \omega_n^2 + \underbrace{C^{-4}(\Omega) \frac{r^2}{\Delta} \left[\left(\frac{n\pi}{L} \right)^2 + C^2(\Omega) \frac{\Delta}{r^2} \left(k_y^2 + \frac{\Delta}{r^2} k_z^2 \right) \right]}_{\omega_n^2 \;\; see \;\; (D.13)}, \\ \mathcal{F}_n + \mathcal{G}_n &= 2\omega_n^2. \end{aligned} \tag{4.30}$$

and get for the mean value of the Casimir energy

$$\begin{aligned} \langle \varepsilon_{vac} \rangle &= \frac{C^{-1}(\Omega)}{\cancel{4}} \frac{1}{(2\pi)^2 L} \frac{1}{r} \sqrt{\frac{A}{\Delta}} \sum_n \int d^2k_T \, \frac{1}{\cancel{\omega_n}} \cancel{2} \omega_n^{\cancel{2}}, \\ \langle \varepsilon_{vac} \rangle &= \frac{C^{-1}(\Omega)}{2} \frac{1}{(2\pi)^2 L} \frac{1}{r} \sqrt{\frac{A}{\Delta}} \sum_n \int d^2k_T \;\; \omega_n(k). \end{aligned} \tag{4.31}$$

4.2 Regularization of the Energy Density

The density energy that appear in (4.31) is divergent in the sum over n and in the integral, in order to calculate a finite value for the vacuum energy one consider the extension of the integral for s dimensions, and the analytic continuation of the sum for this, let

$$\sum_n \int d^2k_T\omega_n = \sum_n \int d^2k_T\, C^{-1}(\Omega)\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{n\pi}{L}\right)^2 + k_y^2 + \frac{\Delta}{r^2}k_z^2\right]^{1/2}, \tag{4.32}$$

let

$$\begin{aligned} \tilde{k}_y &= k_y, \\ \tilde{k}_z &= \frac{\sqrt{\Delta}}{r}k_z,\ then; \\ \tilde{k}_T^2 &= \tilde{k}_y^2 + \tilde{k}_z^2,\ d^2\tilde{k}_T = d\tilde{k}_y d\tilde{k}_z,\ therefore, \\ d^2k_T = dk_y dk_z &= \frac{r}{\sqrt{\Delta}}d\tilde{k}_y d\tilde{k}_z = \frac{r}{\sqrt{\Delta}}\,d^2\tilde{k}_T, \end{aligned} \tag{4.33}$$

inserting this into (4.32)

$$\sum_n \int d^2k_T\omega_n = \frac{r}{\sqrt{\Delta}}\sum_n \int d^2\tilde{k}_T\, C^{-1}(\Omega)\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{n\pi}{L}\right)^2 + \tilde{k}_T^2\right]^{1/2}, \tag{4.34}$$

the first regularization is use the dimensional regularization, see (2.26) for more details, therefore $d^2\tilde{k}_T = 2\pi\tilde{k}_T d\tilde{k}_T$, taking only the positive wave number $d^2\tilde{k}_T = \frac{2\pi}{4}\tilde{k}_T d\tilde{k}_T = d^2\tilde{k}_T = \frac{\pi}{2}\tilde{k}_T d\tilde{k}_T$. Other point is that instead of to compute the integral in (4.34) we perform the following integral

$$\mathcal{I}_n(s) = \frac{r}{\sqrt{\Delta}}\frac{\pi}{2}\sum_n \int \tilde{k}_T\, d\tilde{k}_T\, C^{-1}(\Omega)\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{n\pi}{L}\right)^2 + \tilde{k}_T^2\right]^{-s/2}, \tag{4.35}$$

that take the form (4.34) for $s = -1$. Let

$$\begin{aligned} u &= \frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{n\pi}{L}\right)^2 + \tilde{k}_T^2, \\ du &= 2\tilde{k}_T\, d\tilde{k}_T, \\ \mathcal{I}_n(s) &= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\int_{u(0)}^{\infty} du\, u^{-s/2} = \pi\left[\lim_{u\to\infty}\frac{u^{-s/2+1}}{-s/2+1} - \frac{(u(0))^{-s/2+1}}{-s/2+1}\right], \\ &= -\frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\frac{(u(0))^{-s/2+1}}{-s/2+1},\ \ if\ \mathbb{R}s > 2, \\ &= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\frac{\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{n\pi}{L}\right)^2\right]^{1-s/2}}{s/2-1}, \end{aligned} \tag{4.36}$$

$$
\begin{aligned}
\sum_n \mathcal{I}_n(s) &= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\frac{\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\right]^{1-s/2}}{s/2-1}\sum_n\left(\frac{n\pi}{L}\right)^{2-s} \\
&= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\frac{\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\right]^{1-s/2}}{s/2-1}\left(\frac{\pi}{L}\right)^{2-s}\sum_{n=1}^{\infty} n^{(2-s)} \\
&= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\frac{\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\right]^{1-s/2}}{s/2-1}\left(\frac{\pi}{L}\right)^{2-s}\sum_{n=1}^{\infty} n^{-(s-2)} \\
&= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\frac{\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\right]^{1-s/2}}{s/2-1}\left(\frac{\pi}{L}\right)^{2-s}\zeta(s-2)|_{s=-1} \\
&= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\frac{\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\right]^{3/2}}{-3/2}\left(\frac{\pi}{L}\right)^{3}\underbrace{\zeta(-3)}_{\frac{1}{120}} \\
\sum_n \mathcal{I}_n(s) &= -\frac{\pi^4 r^4}{180\Delta^2 C^4(\Omega)L^3},
\end{aligned}
\tag{4.37}
$$

where $\zeta(s)$ is the ζ-Riemann function, see appendix (B) for more details. Now, replacing (4.37) into the (4.31) we obtain the regularized Casimir energy in the Kerr space given by

$$
\begin{aligned}
\langle\varepsilon_{vac}\rangle\,|_{reg} &= \left(\frac{C^{-1}(\Omega)}{2}\frac{1}{(2\pi)^2 L}\frac{1}{r}\sqrt{\frac{A}{\Delta}}\right)\left(-\frac{\pi^4 r^4}{180\Delta^2 C^4(\Omega)L^3}\right) \\
\langle\varepsilon_{vac}\rangle\,|_{reg} &= -\frac{\pi^2 r^3}{1440\Delta^2 L^4 C^5(\Omega)}\sqrt{\frac{A}{\Delta}}.
\end{aligned}
\tag{4.38}
$$

Here L is only the coordinate length, i.e. unphysical. In order to take a physical description to the energy we use the proper cavity lenght L_p [4], that is

$$
L_p = C(\Omega)\frac{\sqrt{\Delta}}{r}L,
\tag{4.39}
$$

replacing eq (4.39) into eq (4.40)

$$
\begin{aligned}
\langle\varepsilon_{vac}\rangle\,|_{reg} &= -\frac{\pi^2 r^3}{1440\Delta^2\left(\frac{rL_p}{C(\Omega\sqrt{\Delta})}\right)^4 C^5(\Omega)}\sqrt{\frac{A}{\Delta}}, \\
&= -\frac{\pi^2}{1440L_p^4}\sqrt{\frac{A}{\Delta r^2}}C^{-1}(\Omega), \\
&= -\frac{\pi^2}{1440L_p^4}\sqrt{\frac{A}{\Delta r^2}}\left[\frac{\Delta r^2}{A}\left(1-\frac{A^2}{\Delta r^4}(\Omega-\omega_d)^2\right)\right]^{1/2}, \\
\langle\varepsilon_{vac}\rangle\,|_{reg} &= -\frac{\pi^2}{1440L_p^4}\left[1-\frac{A^2}{\Delta r^4}(\Omega-\omega_d)^2\right]^{1/2},
\end{aligned}
\tag{4.40}
$$

this is the regularized vacumm Casimir energy in term of the proper lenght of the cavity. For the case of ZAMO (zero angular momentum observer) observer $\Omega = \omega_d$, the Casimir energy in the Kerr space take the form of the Casimir energy in flat space (2.35).

5 CASIMIR ENERGY FOR A MASSIVE SCALAR FIELD IN KERR SPACETIME

our objective in this chapter is to compute the Casimir energy on curved Kerr spacetime for a massive scalar field that is on Casimir cavity. Taking the Klein-Gordon equation eq (3.14) with the approximation $\hat{g}_{\mu\nu} \simeq const$ inside the cavity, then we get

$$(\hat{g}^{\mu\nu}\partial_\mu\partial_\nu + m^2)\psi = 0, \tag{5.1}$$

a similar computation was done on Appendix (D). If you proceed on the same way on this appendix, when we include the mass m, we find that the frequency is now

$$\omega_n(k) = \frac{r}{\sqrt{\Delta}}C^{-2}(\Omega)\sqrt{\left(\frac{n\pi}{L}\right)^2 + \frac{\Delta}{r^2}C^2(\Omega)\left(k_y^2 + \frac{\Delta}{r^2}k_z^2 + m^2\right)}. \tag{5.2}$$

Casimir energy can be calculated by considering the difference between vacuum energy with boundary conditions imposed on the system and the vacuum energy without boundary condition. i.e

$$\varepsilon_{cas} = \varepsilon_{vac}(\Gamma) - \varepsilon_{vac}(0), \tag{5.3}$$

where Γ represent the boundary conditions especified on the problem. For our case, the mean value of the Casimir energy with boundary conditions, analogously to (4.31) become

$$\langle\varepsilon_{vac}(\Gamma)\rangle = \frac{C^{-1}(\Omega)}{2}\frac{1}{(2\pi)^2 L}\frac{1}{r}\sqrt{\frac{A}{\Delta}}\sum_n\int d^2k_T\ \ \omega_n(k). \tag{5.4}$$

This energy is divergent in both the sum and the integral, so it is necessary to introduce some mathematical method in order to find a regular energy. We first compute the following

$$\sum_n\int d^2k_T\omega_n(k) = \sum_n\int d^2k_T\, C^{-1}(\Omega)\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{n\pi}{L}\right)^2 + k_y^2 + \frac{\Delta}{r^2}k_z^2 + m^2\right]^{1/2}, \tag{5.5}$$

let

$$\begin{aligned}
\tilde{k}_y &= k_y,\\
\tilde{k}_z &= \frac{\sqrt{\Delta}}{r}k_z,\ \ then;\\
\tilde{k}_T^2 &= \tilde{k}_y^2 + \tilde{k}_z^2,\ \ d^2\tilde{k}_T = d\tilde{k}_y d\tilde{k}_z,\ \ therefore,\\
d^2k_T = dk_y dk_z &= \frac{r}{\sqrt{\Delta}}d\tilde{k}_y d\tilde{k}_z = \frac{r}{\sqrt{\Delta}}\,d^2\tilde{k}_T,
\end{aligned} \tag{5.6}$$

replacing this results on eq (5.5) yields

$$\sum_n \int d^2k_T\omega_n(k) = \frac{r}{\sqrt{\Delta}}\sum_n \int d^2\tilde{k}_T\, C^{-1}(\Omega)\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{n\pi}{L}\right)^2 + \tilde{k}_T^2 + m^2\right]^{1/2}. \quad (5.7)$$

In order to regularize this quantity we proceeding in the following form: The first step is to perform dimensional regularization in the integral over the wave number $\tilde{k}_T$, see the previous section for more details, and as a second step we use the Riemann ζ-function in order to regularize the sum in eq (5.7). Therefore applying the first step, we obtain

$$\sum_n \int d^2k_T\omega_n(k) = \frac{r}{\sqrt{\Delta}}\frac{\pi}{2}\sum_n \int \tilde{k}_T\, d\tilde{k}_T\, C^{-1}(\Omega)\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{n\pi}{L}\right)^2 + \tilde{k}_T^2 + m^2\right]^{1/2}, \quad (5.8)$$

now we define the following function

$$\mathcal{I}_n(s) = \frac{r}{\sqrt{\Delta}}\frac{\pi}{2}\sum_n \int \tilde{k}_T\, d\tilde{k}_T\, C^{-1}(\Omega)\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{n\pi}{L}\right)^2 + \tilde{k}_T^2 + m^2\right]^{-s/2}, \quad (5.9)$$

that returns to (5.8) when $s = -1$. Now we use the following change of variable

$$\begin{aligned} u &= \frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{n\pi}{L}\right)^2 + \tilde{k}_T^2 + m^2, \\ du &= 2\tilde{k}_T\, d\tilde{k}_T, \\ \mathcal{I}_n(s) &= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\int_{u(0)}^{\infty} du\, u^{-s/2} = \pi\left[\lim_{u\to\infty}\frac{u^{-s/2+1}}{-s/2+1} - \frac{(u(0))^{-s/2+1}}{-s/2+1}\right], \\ &= -\frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\frac{(u(0))^{-s/2+1}}{-s/2+1}, \quad if\ \mathbb{R}s > 2, \\ &= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\frac{\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{n\pi}{L}\right)^2 + m^2\right]^{1-s/2}}{s/2-1}, \end{aligned} \quad (5.10)$$

$$\begin{aligned} \sum_n \mathcal{I}_n(s) &= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\sum_n \frac{\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{\pi}{L}\right)^2 n^2 + m^2\right]^{1-s/2}}{s/2-1}, \\ &= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\sum_n \frac{\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{\pi}{L}\right)^2\left(n^2 + \frac{\Delta}{r^2}C^2(\Omega)\left(\frac{L}{\pi}\right)^2 m^2\right)\right]^{1-s/2}}{s/2-1}, \quad (5.11) \\ \sum_n \mathcal{I}_n(s) &= \frac{r}{\sqrt{\Delta}}\pi\, C^{-1}(\Omega)\frac{\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{\pi}{L}\right)^2\right]^{1-s/2}}{s/2-1}\sum_{n=1}^{\infty}\left(n^2+\alpha^2\right)^{-(s/2-1)}, \end{aligned}$$

where

$$\alpha = \frac{\sqrt{\Delta}}{r}\frac{L}{\pi}\,C(\Omega)\,m. \tag{5.12}$$

To perform the sum eq (5.11) we use "Epstein-Hurwitz"function [21], that have the form

$$Z(l,\alpha) = \sum_{n=1}^{\infty}\left(n^2+\alpha^2\right)^{-l}, \mathbb{R}\mathrm{e}\; l > 1/2, \tag{5.13}$$

comparing with our result, and using the analytic continuation [21] in (5.13) become

$$Z(s/2-1,\alpha) = -\frac{1}{2}\alpha^{2-s} + \frac{\sqrt{\pi}}{2\alpha^{s-3}\Gamma(\frac{s}{2}-1)}\left[\Gamma\left(\frac{s}{2}-\frac{3}{2}\right) + 4\sum_{n=1}^{\infty}(n\pi\alpha)^{\frac{s}{2}-\frac{3}{2}}\,K_{\frac{s}{2}-\frac{3}{2}}(2n\pi\alpha)\right]. \tag{5.14}$$

where K_ν are the modified Bessel polynomials Therefore our energy in (5.4) yields

$$\begin{aligned}
\langle\varepsilon_{vac}(\Gamma)\rangle &= \frac{C^{-1}(\Omega)}{2}\frac{1}{(2\pi)^2 L}\frac{1}{r}\sqrt{\frac{A}{\Delta}}\sum_n \mathcal{I}_n(s)|_{s=-1}, \\
&= \frac{C^{-1}(\Omega)}{2}\frac{1}{(2\pi)^2 L}\frac{1}{r}\sqrt{\frac{A}{\Delta}}\frac{\pi r}{\Delta}C^{-1}(\Omega)\frac{\left[\frac{r^2}{\Delta}C^{-2}(\Omega)\right]^{3/2}}{-3/2}\left(\frac{\pi}{L}\right)^3 Z\left(-3/2,\alpha\right), \\
&= -\frac{\pi^2 r^3}{12\Delta^2 C^5(\Omega)L^4}\sqrt{\frac{A}{\Delta}}Z\left(-3/2,\alpha\right).
\end{aligned} \tag{5.15}$$

The explicit form to the mean vacuum energy with boundary conditions by using eq (5.14) is therefore

$$\begin{aligned}
\langle\varepsilon_{vac}(\Gamma)\rangle &= -\frac{\pi^2 r^3}{12\Delta^2 C^5(\Omega)L^4}\sqrt{\frac{A}{\Delta}} \\
&\times\left(-\frac{1}{2}\alpha^3 + \frac{\sqrt{\pi}}{2\alpha^{-4}\Gamma(-\frac{3}{2})}\left[\Gamma\left(-2\right) + 4\sum_{n=1}^{\infty}(n\pi\alpha)^{-2}\,K_{-2}(2n\pi\alpha)\right]\right)
\end{aligned} \tag{5.16}$$

using the Elizalde results [21] we find the mean value of Casimir density energy (5.3), yields

$$\begin{aligned}
\langle\varepsilon_{cas}\rangle &= \langle\varepsilon_{vac}(\Gamma)\rangle - \langle\varepsilon_{vac}(0)\rangle \\
&= -\frac{\pi^2 r^3}{12\Delta^2 L C^5(\Omega)}\sqrt{\frac{A}{\Delta}}\frac{1}{L^3}\left[-\frac{1}{2}\left(\frac{\sqrt{\Delta}LC(\Omega)}{\pi r}m\right)^3\right. \\
&+\frac{\sqrt{\pi}}{2\left(\frac{\sqrt{\Delta LC(\Omega)}}{r\pi}m\right)^{-4}\Gamma\left(-\frac{3}{2}\right)} \\
&\left.\times\left(\Gamma(-2) + 4\sum_{n=1}^{\infty}(n\pi\alpha)^{-2}K_{-2}(2n\pi\alpha)\right)\right] \\
&-\left(-\frac{\pi^2 r^3}{12\Delta^2 L C^5(\Omega)}\sqrt{\frac{A}{\Delta}}\frac{1}{L^3}\left[-\frac{1}{2}\left(\frac{\sqrt{\Delta}LC(\Omega)}{\pi r}m\right)^3 + \frac{\sqrt{\pi}\,\Gamma(-2)}{2\left(\frac{\sqrt{\Delta}C(\Omega)}{\pi r}m\right)^3\Gamma(-\frac{3}{2})}\right]\right).
\end{aligned} \tag{5.17}$$

Term independent of L have no physical significance due to belong to the zero modes over the momentum integral. Therefore simplifying the expression in (5.17) and using the fact that $\Gamma(-3/2) = 4\sqrt{\pi}/3$, we obtain the physically relevant energy

$$\langle \varepsilon_{cas} \rangle = -\frac{\pi^2 r^3}{8\Delta^2 L^4 C^5(\Omega)} \sqrt{\frac{A}{\Delta}} \sum_{n=1}^{\infty} (n\pi)^{-2} \alpha^2 K_2(2n\pi\alpha). \tag{5.18}$$

The energy that appear in eq (5.18) is the correct expression to a massive scalar field inside the Casimir cavity, we can see that this have a similar form of the massless case eq (4.38) multiplied by a corrective factor that depends on the mass. We can take the limit to $m \to 0$. We expand $K_2(x)$ for small argument, see [23] for more details, the energy in eq (5.18) became

$$\begin{aligned}
\lim_{m\to 0} \langle \varepsilon_{cas} \rangle &= -\frac{\pi^2 r^3}{8\Delta^2 L^4 C^5(\Omega)} \sqrt{\frac{A}{\Delta}} \sum_{n=1}^{\infty} (n\pi)^{-2} \alpha^2 \frac{1}{2} \Gamma(2) \left(\frac{2n\pi\alpha}{2} \right)^{-2}, \\
&= -\frac{\pi^2 r^3}{16\Delta^2 L^4 C^5(\Omega)} \sqrt{\frac{A}{\Delta}} \pi^{-4} \sum_{n=1}^{\infty} n^{-4}, \\
&= -\frac{r^3}{16\pi^2 \Delta^2 L^4 C^5(\Omega)} \sqrt{\frac{A}{\Delta}} \zeta(4), \\
&= -\frac{r^3}{16\pi^2 \Delta^2 L^4 C^5(\Omega)} \sqrt{\frac{A}{\Delta}} \frac{\pi^4}{90}, \\
&= -\frac{\pi^2 r^3}{1440\Delta^2 L^4 C^5(\Omega)} \sqrt{\frac{A}{\Delta}},
\end{aligned} \tag{5.19}$$

this is the same expression for a massless scalar field eq (4.38), if we put (5.19) in terms of the proper lenght (4.39) we get

$$\lim_{m\to 0} \langle \varepsilon_{cas} \rangle = -\frac{\pi^2}{1440 L_p^4} \left[1 - \frac{A^2}{\Delta r^4} (\Omega - \omega_d)^2 \right]^{1/2}, \tag{5.20}$$

this energy is the same for a massless scalar field confined on the Casimir cavity that is rotating around the gravity source. Now we proceed to take the large mass limit, that is, using the asymptotic behaviour for Bessel function to large arguments. i.e.

$$\begin{aligned}
\lim_{m\to \infty} \langle \varepsilon_{cas} \rangle &= -\frac{\pi^2 r^3}{8\Delta^2 L^4 C^5(\Omega)} \sqrt{\frac{A}{\Delta}} \sum_{n=1}^{\infty} (n\pi)^{-2} \lim_{m\to\infty} \alpha^2 K_2(2n\pi\alpha), \\
&= -\frac{\pi^2 r^3}{8\Delta^2 L^4 C^5(\Omega)} \sqrt{\frac{A}{\Delta}} \sum_{n=1}^{\infty} (n\pi)^{-2} \lim_{m\to\infty} \frac{\alpha^2}{\sqrt{4n\alpha}} e^{-2n\pi\alpha} \left(1 + \frac{15}{16n\pi\alpha} + \cdots \right).
\end{aligned} \tag{5.21}$$

As a result of this expression it is observed that the Casimir energy density tends to cancel strongly, due to energy dependence in the exponential function decreasing $e^{-2n\pi\alpha}$. the most important result for large massive field in the Casimir energy is that is impossible change the attractive to repulsive character of the force of consequence of asymptotic behaviour of the Bessel especial function of kind ν, given by

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4\nu^2 - 1}{8z} + \cdots\right), \quad z \to \infty. \tag{5.22}$$

In the Elizalde paper [21] ν is related with the dimension of the problem, If we take a general ν the unique possibility to change the force sign of attractive to repulsive is by $4\nu^2 - 1$, that have no sence for integer dimension of space. But if we assume that the inequality is valid, even so the factor $\frac{4\nu^2-1}{16n\pi\alpha}$ is very small compared with 1, hence does not reverse the sign of the energy maintaining the attractive character of the force.

6 CONCLUSIONS

We find the regularized Casimir energy for a massless scalar field in the Kerr metric via Riemann ζ-function, and the massive scalar field via Epstein-Hurwitz zeta function, we conclude that the inclusion of mass does not change the attractive character of the force, and we see that when we take the ZAMO limit, the energy return to Casimir energy in flat space.

The goal of this thesis was generalized the Sorge results [19] by considering massive scalar field. In this procedure also we perform the following

- We used some properties of the Kerr space and certain transformations in order to fix our solution on Casimir cavity.
- In the procedure to find the regularized Casimir energy was necessary to introduce the Vierbein formalism to take the normalization condition of the field via inner product of Klein-Gordon solution.
- For the massive case we take the limit $m \to 0$ and we saw that the results are the same as for the massless case.
- We consider the asymptotic limit $m \to \infty$ and we show that for large massive fields the Casimir energy does no change its sign due to the asymptotic behaviour of the Bessel especial function.

For later work we could compare our results with de Sitter or anti de Sitter space. Also we can change the boundary condition because that could change the character of the force [14]. Other important question is perform an analysis of the conical “ring”singularity that appear in the Kerr spacetime [22].

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Appendix

APPENDIX A – HAMILTONIAN FOR FLAT CASIMIR

The idea is to find the expressions on (2.24) from (2.23) to find the Hamiltonian H (2.5). We know that $\pi = \dot{\varphi}(x)$, then performing one derivative with respect to time in (2.19) we get

- Calculating H_π

$$\pi \;=\; \imath \sum_{n=1}^{\infty} \left[-a_n(k)\, f_k(x) + a_n^{\dagger}(k)\, f_k^*(x)\right] u_n(x)\;\omega_n(k), \tag{A.1}$$

$$\begin{aligned}
\mathcal{H}_\pi &= \frac{1}{2}\left[\imath \sum_{n=1}^{\infty} \int d^2k \left[-a_{nk}\, f_k(x) + a_{nk}^{\dagger}\, f_k^*(x)\right] u_n(x)\;\omega_{nk}\right] \\
&\times \left[\imath \sum_{m=1}^{\infty} \int d^2l \left[-a_{ml}\, f_l(x) + a_{ml}^{\dagger}\, f_l^*(x)\right] u_m(x)\;\omega_{ml}\right], \\
&= -\frac{1}{2}\sum_{m,n} \int d^2k\; d^2l\; \left[-a_{nk}\, f_k(x) + a_{nk}^{\dagger}\, f_k^*(x)\right] \\
&\times \left[-a_{ml}\, f_l(x) + a_{ml}^{\dagger}\, f_l^*(x)\right] u_n(x) u_m(x) \omega_{nk}\omega_{ml}. \\
H_\pi &= -\frac{1}{2}\sum_{m,n} \int d^3x \int d^2k\; d^2l\; \left[-a_{nk}\, f_k(x) + a_{nk}^{\dagger}\, f_k^*(x)\right] \\
&\times \left[-a_{ml}\, f_l(x) + a_{ml}^{\dagger}\, f_l^*(x)\right] u_n(x) u_m(x) \omega_{nk}\omega_{ml}, \\
&= -\frac{1}{2}\sum_{m,n} \int dx\, dy\, dz \int d^2k\; d^2l\; \left[-a_{nk}\, f_k(x) + a_{nk}^{\dagger}\, f_k^*(x)\right] \\
&\times \left[-a_{ml}\, f_l(x) + a_{ml}^{\dagger}\, f_l^*(x)\right] u_n(x) u_m(x) \omega_{nk}\omega_{ml},
\end{aligned} \tag{A.2}$$

the function $f_k(x)$ only depedent on variables (t, y, z), then the integral over x is in the $u_n(x)$ function, see (2.20), then we have

$$\begin{aligned}
H_\pi &= -\frac{1}{2}\sum_{m,n} \int dy\, dz \int d^2k\; d^2l\; \left[-a_{nk}\, f_k(x) + a_{nk}^{\dagger}\, f_k^*(x)\right] \\
&\times \left[-a_{ml}\, f_l(x) + a_{ml}^{\dagger}\, f_l^*(x)\right] \omega_{nk}\omega_{ml} \underbrace{\int_0^L dx\, u_n(x) u_m(x),}_{\delta_{mn}} \\
&= -\frac{1}{2}\sum_{n} \int dy\, dz \int d^2k\; d^2l\; \left[-a_{nk}\, f_k(x) + a_{nk}^{\dagger}\, f_k^*(x)\right] \\
&\times \left[-a_{nl}\, f_l(x) + a_{nl}^{\dagger}\, f_l^*(x)\right] \omega_{nk}\omega_{nl},
\end{aligned} \tag{A.3}$$

we proceeding to integrate over y, z, the functions f because ω only depend on k and l. We consider, for example, one of the expressions

$$\begin{aligned}\int d^2x\, f_k(x) f_l(x) &= \left(\frac{1}{2\pi\sqrt{2\omega_{nk}}}\right)\left(\frac{1}{2\pi\sqrt{2\omega_{nl}}}\right) e^{-\imath\omega_{nk}t} e^{-\imath\omega_{nl}t} \int dy\, e^{\imath(k_y+l_y)y} \int dz\, e^{\imath(k_z+l_z)z}, \\ &= \left(\frac{1}{2\pi\sqrt{2\omega_{nk}}}\right)\left(\frac{1}{2\pi\sqrt{2\omega_{nl}}}\right) e^{-\imath\omega_{nk}t} e^{-\imath\omega_{nl}t}\, 2\pi\,\delta(k_y+l_y)\, 2\pi\,\delta(k_z+l_z), \\ &= \left(\frac{1}{\sqrt{2\omega_{nk}}}\right)\left(\frac{1}{\sqrt{2\omega_{nl}}}\right) e^{-\imath\omega_{nk}t} e^{-\imath\omega_{nl}t}\, \delta(k_y+l_y)\, \delta(k_z+l_z),\end{aligned} \tag{A.4}$$

the integration over the momenta are null due to k_i, l_i is in espherical coordinates. i.e. is integrate over $0 < k_i < \infty$ then the delta function is valid when $k_y = -l_y$ and $k_z = -l_z$ then this this provides a negative value of k and l then the integral over momenta is null, so

$$\begin{aligned}\sum_n \int_0^\infty d^2k\, d^2l\, a_n(k) a_n(l) \int d^2x\, f_k(x) f_l(x) &= \sum_n \int_0^\infty dk_y\, dk_z \int_0^\infty dl_y\, dl_z a_n(k) a_n(l) \\ &\times \left(\frac{1}{\sqrt{2\omega_{nk}}}\right)\left(\frac{1}{\sqrt{2\omega_{nl}}}\right) e^{-\imath\omega_{nk}t} e^{-\imath\omega_{nl}t}\, \delta(k_y+l_y)\, \delta(k_z+l_z) = 0,\end{aligned} \tag{A.5}$$

the same situation happen with the other term that is proportional to $a^\dagger a^\dagger$. The non-vanishing terms are the crossed terms, providing

$$H_\pi = -\frac{1}{2}\sum_n \int dy\, dz \int d^2k\ d^2l\, \left[-a_{nk} f_k(x)\, a^\dagger_{nl}\, f^*_l(x) - a^\dagger_{nk}\, f^*_k(x) a_{nl}\, f_l(x)\right] \omega_{nk}\omega_{nl} \tag{A.6}$$

again computing

$$\begin{aligned}\int dy\, dz\, f_k(x) f^*_l(x) &= \left(\frac{1}{2\pi\sqrt{2\omega_{nk}}}\right)\left(\frac{1}{2\pi\sqrt{2\omega_{nl}}}\right) e^{-\imath\omega_{nk}t} e^{\imath\omega_{nl}t} \int dy\, e^{\imath(k_y-l_y)y} \int dz\, e^{\imath(k_z-l_z)z}, \\ &= \left(\frac{1}{2\pi\sqrt{2\omega_{nk}}}\right)\left(\frac{1}{2\pi\sqrt{2\omega_{nl}}}\right) e^{-\imath\omega_{nk}t} e^{\imath\omega_{nl}t}\, 2\pi\,\delta(k_y-l_y)\, 2\pi\,\delta(k_z-l_z), \\ &= \left(\frac{1}{\sqrt{2\omega_{nk}}}\right)\left(\frac{1}{\sqrt{2\omega_{nl}}}\right) e^{-\imath\omega_{nk}t} e^{\imath\omega_{nl}t}\, \delta(k_y-l_y)\, \delta(k_z-l_z),\end{aligned} \tag{A.7}$$

replacing (A.7) into the (A.6)

$$
\begin{aligned}
H_\pi &= \frac{1}{2}\sum_n \int_0^\infty dk_y dk_z \int_0^\infty dl_y dl_z \left[a_{nk}\, a_{nl}^\dagger + a_{nk}^\dagger a_{nl}\right] \omega_{nk}\omega_{nl} \\
&\times \left(\frac{1}{\sqrt{2\omega_{nk}}}\right)\left(\frac{1}{\sqrt{2\omega_{nl}}}\right) e^{-\imath\omega_{nk}t} e^{\imath\omega_{nl}t}\, \delta(k_y - l_y)\, \delta(k_z - l_z), \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \int d^2k \left[a_n(k)\, a_n^\dagger(k) + a_n^\dagger(k) a_n(k)\right] \frac{\omega_n(k)}{2}, \\
H_\pi &= \frac{1}{4}\sum_{n=1}^{\infty} \int d^2k \left[a_n(k)\, a_n^\dagger(k) + a_n^\dagger(k) a_n(k)\right] \omega_n(k),
\end{aligned}
\tag{A.8}
$$

- $H_{\nabla\varphi}$

Firstly we compute $\nabla\varphi$ in (2.21) where the operator ∇ denotes the usual gradient in 3-dim

$$
\begin{aligned}
|\nabla\varphi(x)| = &\sum_{n=1}^{\infty} \int d^2k \left[a_n(k)\, \nabla f_k(x) + a_n^\dagger(k)\, \nabla f_k^*(x)\right] u_n(x) \\
&+ \sum_{n=1}^{\infty} \int d^2k \left[a_n(k)\, f_k(x) + a_n^\dagger(k)\, f_k^*(x)\right] \nabla u_n(x).
\end{aligned}
\tag{A.9}
$$

performing the respective the derivatives in (2.20)

$$
\begin{aligned}
\nabla f_k(x) &= \imath(k_y + k_k) f_k(x), \\
\nabla f_k^*(x) &= -\imath(k_y + k_k) f_k^*(x), \\
\nabla u_n(x) &= \sqrt{\frac{2}{L}}\frac{n\pi}{L} \cos\left(\frac{n\pi}{L}x\right) \equiv \frac{n\pi}{L} g_n(x), \\
g_n(x) &= \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi}{L}x\right)
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
|\nabla\varphi(x)|^2 = &\left[\sum_{n=1}^{\infty} \int d^2k \left[a_{nk} f_k - a_{nk}^\dagger f_k^*\right] \imath(k_y + k_z) u_n(x) + \sum_{n=1}^{\infty} \int d^2k \left[a_{nk} f_k + a_{nk}^\dagger f_k^*\right] \frac{n\pi}{L} g_n(x)\right] \\
\times &\left[\sum_{m=1}^{\infty} \int d^2l \left[a_{ml} f_l - a_{ml}^\dagger f_l^*\right] \imath(l_y + l_z) u_m(x) + \sum_{m=1}^{\infty} \int d^2k \left[a_{ml} f_l + a_{ml}^\dagger f_l^*\right] \frac{m\pi}{L} g_m(x)\right]
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
|\nabla\varphi(x)|^2 = &\sum_{n,m=1}^{\infty} \left[\int d^2k \left[a_{nk} f_k - a_{nk}^\dagger f_k^*\right] \imath(k_y + k_z) u_n(x) + \int d^2k \left[a_{nk} f_k + a_{nk}^\dagger f_k^*\right] \frac{n\pi}{L} g_n(x)\right] \\
\times &\left[\int d^2l \left[a_{ml} f_l - a_{ml}^\dagger f_l^*\right] \imath(l_y + l_z) u_m(x) + \int d^2k \left[a_{ml} f_l + a_{ml}^\dagger f_l^*\right] \frac{m\pi}{L} g_m(x)\right]
\end{aligned}
\tag{A.12}
$$

using (2.22) $H_{\nabla\varphi} = \int d^3x\, |\nabla\varphi(x)|^2$ integrate over x variable and using the orthogonality conditions (2.14) and (2.15) we see that the crossed elements $u_n g_m$ in (A.12) are vanish, and $n = m$

$$\begin{aligned} H_{\nabla\varphi} &= \frac{1}{2}\int dydz|\nabla\varphi(x)|^2, \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\int dydz\int d^2k\int d^2l\left[a_{nk}f_k - a^\dagger_{nk}f^*_k\right]\left[a_{nl}f_l - a^\dagger_{nl}f^*_l\right][-(k_y+k_z)(l_y+l_z)] \\ &+ \frac{1}{2}\sum_{n=1}^{\infty}\int dydz\int d^2k\int d^2l\left[a_{nk}f_k + a^\dagger_{nk}f^*_k\right]\left[a_{nl}f_l + a^\dagger_{nl}f^*_l\right]\left(\frac{n\pi}{L}\right)^2 \end{aligned} \tag{A.13}$$

the terms that are proportional to $a\,a$ and $a^\dagger\,a^\dagger$ are vanish when we integrate in y, z variables (A.4) and then in k space (A.5). only the crossed terms survive in the integration on y, z see (A.7)

$$\begin{aligned} H_{\nabla\varphi} = &\frac{1}{2}\sum_{n=1}^{\infty}\int d^2k\int d^2l\left[-a_{nk}a^\dagger_{nl} - a^\dagger_{nk}a_{nl}\right][-(k_y+k_z)(l_y+l_z)] \\ &\times\left(\frac{1}{\sqrt{2\omega_{nk}}}\right)\left(\frac{1}{\sqrt{2\omega_{nl}}}\right)e^{-\imath\omega_{nk}t}e^{\imath\omega_{nl}t}\,\delta(k_y-l_y)\,\delta(k_z-l_z) \\ &+\frac{1}{2}\sum_{n=1}^{\infty}\int d^2k\int d^2l\left[a_{nk}a^\dagger_{nl} + a^\dagger_{nk}a_{nl}\right]\left(\frac{n\pi}{L}\right)^2 \\ &\times\left(\frac{1}{\sqrt{2\omega_{nk}}}\right)\left(\frac{1}{\sqrt{2\omega_{nl}}}\right)e^{-\imath\omega_{nk}t}e^{\imath\omega_{nl}t}\,\delta(k_y-l_y)\,\delta(k_z-l_z), \end{aligned} \tag{A.14}$$

applying the properties of delta function we eliminate the the l variable and simplifying we get

$$H_{\nabla\varphi} = \frac{1}{4}\int d^2k\left[a_n(k)a^\dagger_n(k) + a^\dagger_n(k)a_n(k)\right]\frac{1}{\omega_n(k)}\left[\left(\frac{n\pi}{L}\right)^2 + k_y^2 + k_z^2\right], \tag{A.15}$$

with a similar procedure we can find the term H_φ in (2.24) to get

$$H_\varphi = \frac{1}{4}\int d^2k\left[a_n(k)a^\dagger_n(k) + a^\dagger_n(k)a_n(k)\right]\frac{m^2}{\omega_n(k)}, \tag{A.16}$$

then replacing (A.15), (A.16) and (A.8) into the (2.22) yields

$$
\begin{aligned}
H &= \frac{1}{4}\int d^2k \left[a_n(k)a_n^\dagger(k) + a_n^\dagger(k)a_n(k)\right]\left[\omega_n(k) + \frac{m^2}{\omega_n(k)} + \frac{1}{\omega_n(k)}\left[\left(\frac{n\pi}{L}\right)^2 + k_y^2 + k_z^2\right]\right], \\
&= \frac{1}{4}\int d^2k \left[a_n(k)a_n^\dagger(k) + a_n^\dagger(k)a_n(k)\right]\left[\omega_n(k) + \frac{1}{\omega_n(k)}\underbrace{\left[\left(\frac{n\pi}{L}\right)^2 + k_y^2 + k_z^2 + m^2\right]}_{\omega_n^2(k)}\right], \\
&= \frac{1}{4}\int d^2k \left[a_n(k)a_n^\dagger(k) + a_n^\dagger(k)a_n(k)\right] 2\omega_n(k), \\
H &= \frac{1}{2}\int d^2k \left[a_n(k)a_n^\dagger(k) + a_n^\dagger(k)a_n(k)\right] \omega_n(k),
\end{aligned}
\tag{A.17}
$$

the Hamiltonian in (A.17) is due to the field imposing Dirichlet conditions so that the frequency in this equation is given by (2.13). The same Hamiltonian is find it only with periodic condition by the field (2.9), the difference is that frequency in (A.17) is given by (2.10).The commutation relations to the operators a and $a^\dagger$ following the commutation relations of the fields and conjugate momenta [16]

$$
\begin{aligned}
[\varphi(\boldsymbol{x},t), \pi(\boldsymbol{y},t)] &= \imath\delta^3(\boldsymbol{x}-\boldsymbol{y}), \\
[\varphi(\boldsymbol{x},t), \varphi(\boldsymbol{y},t)] &= [\pi(\boldsymbol{x},t), \pi(\boldsymbol{y},t)] = 0,
\end{aligned}
\tag{A.18}
$$

using the orthogonality of the base

$$
\begin{aligned}
\imath\int d^3x\, f_k^*(\boldsymbol{x},t)\overleftrightarrow{\partial_0} f_{k'}(\boldsymbol{x},t) &= \delta^3(\boldsymbol{k}-\boldsymbol{k'}), \\
\imath\int d^3x\, f_k(\boldsymbol{x},t)\overleftrightarrow{\partial_0} f_{k'}(\boldsymbol{x},t) &= 0,
\end{aligned}
\tag{A.19}
$$

with the definition

$$
a(t)\overleftrightarrow{\partial_0} b(t) = a(t)\frac{\partial b(t)}{\partial t} - \frac{\partial a(t)}{\partial t}b(t),
$$

providing

$$
a(t) = \imath\int d^3x\, f_k^*(\boldsymbol{x},t)\overleftrightarrow{\partial_0}\varphi(\boldsymbol{x},t), \tag{A.20}
$$

where the expression is obtained for the operator $a^\dagger$. The commutation relation for the creation and annihilation operators are by the equations (2.21) (2.22) (A.18)

$$
\begin{aligned}
\left[a(k), a^\dagger(k')\right] &= \imath\int d^3x\, f_k^*(\boldsymbol{x},t)\overleftrightarrow{\partial_0} f_{k'}(\boldsymbol{x},t) = \delta^3(\boldsymbol{k}-\boldsymbol{k'}), \\
[a(k), a(k')] &= \left[a^\dagger(k), a^\dagger(k')\right] = 0,
\end{aligned}
\tag{A.21}
$$

using the last expression into the (A.17) we obtain that the Hamiltonian is,

$$H = \frac{1}{2}\sum_n \int d^2k \left[2a_n^\dagger(k)a_n(k) + \delta^{(2)}(0)\right] \omega_n(k), \tag{A.22}$$

the vacuum state in our Hamiltonian

$$\begin{aligned} \langle 0|\, H\, |0\rangle &= \frac{1}{2}\sum_n \int d^2k\ \delta^{(2)}(0)\omega_n(k), \\ \delta^{(2)}(0) &= \delta^{(2)}(k-k) \to \frac{L_y}{2\pi}\frac{L_z}{2\pi}, \end{aligned} \tag{A.23}$$

where l_y and l_z are the dimensions that corresponds of the transverse components of the moments. i.e. k_y, k_z. This is the same expression as in (2.25).

APPENDIX B – ζ-RIEMANN FUNCTION

The ζ-Riemann function is defined by [9]

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s \in \mathbb{C}, \;\; \Re(s) > 1, \tag{B.1}$$

this serie converge only to $\Re(s) > 1$, On the other hand, the integral representation is

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt \, \frac{t^{s-1}}{e^t - 1}. \tag{B.2}$$

The reference [9] proof that the integral representation of the ζ-Riemann function is the analytic continuation of the sum (B.1) in all s except to $s = 1$ where the function has a pole with residues 1.
Some particular values of the ζ-Riemann functions are

$$\begin{aligned} \zeta(0) &= -\frac{1}{2}, \\ \zeta(-2n) &= 0, \\ \zeta(1-2n) &= -\frac{B_{2n}}{2n}, \quad n = 1, 2, 3, ... \end{aligned} \tag{B.3}$$

where B_{2n} are the Bernoulli numbers. Another important formula is the reflexion formula

$$\begin{aligned} \zeta(s) &= 2^2 \pi^{s-1} \sin\left(\frac{\pi\, s}{2}\right) \Gamma(1-s)\zeta(1-s), \\ \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \pi^{s-1/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s). \end{aligned} \tag{B.4}$$

APPENDIX C – SOME REMARKS ABOUT THE KERR SOLUTION

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2Mr}{\Sigma} & 0 & 0 & \frac{2Mar}{\Sigma}\sin^2\theta \\ 0 & \frac{-\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & -\Sigma & 0 \\ \frac{2Mar}{\Sigma}\sin^2\theta & 0 & 0 & \frac{-A\sin^2\theta}{\Sigma} \end{pmatrix}, \tag{C.1}$$

with determinant

$$g = -\Sigma^2\sin^2\theta, \tag{C.2}$$

$$\det(g^{\mu\nu}) = -\frac{1}{\Sigma^2\sin^2\theta}.$$

the inverse metric is

$$(g^{\mu\nu}) = \begin{bmatrix} g^{00} & 0 & 0 & g^{03} \\ 0 & g^{11} & 0 & 0 \\ 0 & 0 & g^{22} & 0 \\ g^{30} & 0 & 0 & g^{33} \end{bmatrix}, \tag{C.3}$$

where

$$\begin{aligned} g^{00} &= \frac{A}{\Delta\Sigma}, \quad g^{30} = g^{03} = \frac{2Mar}{\Delta\Sigma}, \\ g^{11} &= -\frac{\Delta}{\Sigma}, \quad g^{22} = -\frac{1}{\Sigma}, \\ g^{33} &= -\left(1-\frac{2Mr}{\Sigma}\right)\frac{1}{\Delta\sin^2\theta} = \frac{\Delta - a^2\sin^2\theta}{\Delta\Sigma\sin^2\theta}. \end{aligned} \tag{C.4}$$

The arc-length (3.4) is

$$ds = \sqrt{\left(1-\frac{2M}{r}\right) + \frac{4Ma}{r}\Omega - \frac{A}{r^2}\Omega^2}\; dt \tag{C.5}$$

where $t=\tau$ is the proper time, then

$$\begin{aligned} u^\mu = \frac{dx^\mu}{ds} &= \left[\left(1-\frac{2M}{r}\right) + \frac{4Ma}{r}\Omega - \frac{A}{r^2}\Omega^2\right]^{-1/2}\left(\frac{dx^0}{dt}, 0, 0, \frac{dx^3}{dt}\right), \\ &= \left[\left(1-\frac{2M}{r}\right) + \frac{4Ma}{r}\Omega - \frac{A}{r^2}\Omega^2\right]^{-1/2}(1,0,0,\Omega) \\ C(\Omega) &= \left[\left(1-\frac{2M}{r}\right) + \frac{4Ma}{r}\Omega - \frac{A}{r^2}\Omega^2\right]^{-1/2}, \\ &= \left[g_{33}\Omega^2 + 2g_{03}\Omega + g_{00}\right]^{-1/2}. \end{aligned} \tag{C.6}$$

Other form to obtain the same expression in (C.6) is only by considering the condition $u^\mu u_\mu = 1$, this mean that the 4-velocity is a timelike vector.

C.1 Conditions for the angular velocity Ω

in the equation (C.6) we note that this must be satisfied some conditions to be a real constant, that is

$$g_{33}\Omega^2 + 2g_{03}\Omega + g_{00} > 0, \Longrightarrow \;\; \Omega_\pm = -\frac{g_{03}}{g_{33}} \pm \sqrt{\left(\frac{g_{03}}{g_{33}}\right)^2 - \frac{g_{00}}{g_{33}}}, \tag{C.7}$$

let

$$\omega_d = -\frac{g_{03}}{g_{33}}, \tag{C.8}$$

is termed as angular dragging velocity, is the angular velocity of the rotating source (Black Hole)

$$\begin{aligned}
\left(\frac{g_{03}}{g_{33}}\right)^2 - \frac{g_{00}}{g_{33}} &= \omega_d^2 + \frac{r(r-2M)}{A} = \frac{4M^2a^2r^2}{A^2} + \frac{r(r-2M)}{A}, \quad \frac{g_{00}}{g_{33}} = -\frac{r(r-2M)}{A}, \\
&= \frac{4M^2a^2r^2}{A^2} + \frac{rA(r-2M)}{A^2}, \\
&= \frac{1}{A^2}\left[4M^2a^2r^2 - 2MAr + r^2A\right], \\
&\overset{(3.3)}{=} \frac{1}{A^2}\Big[4M^2a^2r^2 - 2M((r^2+a^2)r^2 + 2Mra^2)r \\
&+ r^2((r^2+a^2)r^2 + 2Mra^2)\Big], \\
&= \frac{1}{A^2}\Big[\cancel{4M^2a^2r^2} - \cancel{4M^2a^2r^2} - 2Mr^3(r^2+a^2) \\
&+ r^4(r^2+a^2) + 2Mr^3a^2\Big], \\
&= \frac{1}{A^2}\left[-2Mr^5 - \cancel{2Mr^3a^2} + r^4(r^2+a^2) + \cancel{2Mr^3a^2}\right], \\
&= \frac{1}{A^2}\left[r^4\left(r^2+a^2-2Mr\right)\right], \\
&\overset{(3.3)}{=} \frac{r^4\Delta}{A^2},
\end{aligned} \tag{C.9}$$

substituting this result in (C.7) yields

$$\Omega_\pm = \omega_d \pm \frac{r^2}{A}\sqrt{\Delta}, \tag{C.10}$$

returning to the computation of $C(\Omega)$ in (C.6)

$$
\begin{aligned}
\left(1-\frac{2M}{r}\right)+\frac{4Ma}{r}\Omega-\frac{A}{r^2}\Omega^2 &= -\frac{A}{r^2}\left[\Omega^2-\frac{r^2}{A}\left(1-\frac{2M}{r}\right)-\Omega\frac{r^2}{A}\left(\frac{4Ma}{r}\right)\right], \\
&= -\frac{A}{r^2}\left[\Omega^2-\frac{r^2}{A}+\frac{2Mr}{A}-\frac{4Mar}{A}\Omega\right], \\
&\overset{(C.8)}{=} -\frac{A}{r^2}\left[\underbrace{\Omega^2-2\omega_d\Omega}_{(\Omega-\omega_d)^2-\omega_d^2}-\frac{r^2}{A}+\frac{2Mr}{A}\right], \\
&= -\frac{A}{r^2}(\Omega-\omega_d)^2+\frac{A}{r^2}\left(\omega_d^2+\frac{r^2}{A}-\frac{2Mr}{A}\right), \\
&= -\frac{A}{r^2}(\Omega-\omega_d)^2+\frac{A}{r^2}\left(\omega_d^2+\frac{r(r-2M)}{A}\right), \\
&\overset{(C.8)}{=} -\frac{A}{r^2}(\Omega-\omega_d)^2+\frac{\cancel{A}}{r^{\cancel{2}}}\left(\frac{r^{\cancel{4}}\Delta}{A^{\cancel{2}}}\right), \\
&= -\frac{A}{r^2}(\Omega-\omega_d)^2+\frac{r^2\Delta}{A}, \\
&= \frac{r^2\Delta}{A}\left(1-\frac{A^2}{r^4\Delta}(\Omega-\omega_d)^2\right),
\end{aligned}
\tag{C.11}
$$

replacing into the (C.6) yields

$$
C(\Omega)=\left[\frac{r^2\Delta}{A}\left(1-\frac{A^2}{r^4\Delta}(\Omega-\omega_d)^2\right)\right]^{-1/2}. \tag{C.12}
$$

APPENDIX D – SOLUTION OF THE KLEIN-GORDON EQUATION

To solve (3.15) we need the inverse metric (3.12), then, inserting (3.12) into the (3.15) we obtain the following differential equation

$$\left[\frac{A}{r^2\Delta}\frac{\partial^2}{\partial t^2}+\frac{2A}{r\Delta}(\omega_d-\Omega)\frac{\partial}{\partial x}\frac{\partial}{\partial t}-\frac{r^2}{\Delta}C^{-2}(\Omega)\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial y^2}-\frac{\Delta}{r^2}\frac{\partial^2}{\partial z^2}\right]\psi=0, \tag{D.1}$$

We consider the following solution in the form of separation of variables

$$\psi(x,y,z,t)=\chi(x)Y(y)Z(z)T(t), \tag{D.2}$$

which give us

$$\frac{A}{r^2\Delta}\frac{1}{T}\frac{d^2T}{dt^2}+\frac{2A}{r\Delta}(\omega_d-\Omega)\frac{1}{T\chi}\frac{d\chi}{dx}\frac{dT}{dt}-\frac{r^2}{\Delta}C^{-2}(\Omega)\frac{1}{\chi}\frac{d^2\chi}{dx^2}-\frac{1}{Y}\frac{d^2Y}{dy^2}-\frac{\Delta}{r^2}\frac{1}{Z}\frac{d^2Z}{dz^2}=0, \tag{D.3}$$

then we can choose

$$\begin{aligned}\frac{1}{T}\frac{d^2T}{dt^2}&=-\omega^2,\longrightarrow T(t)=T_0e^{-\imath\omega t}\\ \frac{1}{Y}\frac{d^2Y}{dy^2}&=-k_y^2,\longrightarrow Y(y)=Y_0e^{\imath k_y y}\\ \frac{1}{Z}\frac{d^2Z}{dz^2}&=-k_z^2,\longrightarrow Z(z)=Z_0e^{\imath k_z z}\end{aligned} \tag{D.4}$$

substituting into (D.3) and simplifying yields

$$a\frac{d^2\chi}{dx^2}+b\frac{d\chi}{dx}+c\chi=0, \tag{D.5}$$

where

$$\begin{aligned}a&=\frac{A}{r^2\Delta},\\ b&=2\imath\omega\frac{A}{r\Delta}(\omega_d-\Omega),\\ c&=\left[\omega^2\frac{A}{r^2\Delta}-k_y^2-\frac{\Delta}{r^2}k_z^2\right],\end{aligned} \tag{D.6}$$

that admits solution of the form

$$\chi(x) = \chi_0\, e^{\alpha x}, \tag{D.7}$$

giving

$$\alpha = -\frac{\imath\omega A(\omega_d - \Omega)}{r^3 C^{-2}(\Omega)} \pm \frac{\imath\Delta C^2(\Omega)}{r^2}\sqrt{\left[\omega(\Omega - \omega_d)\frac{A}{r\Delta}\right]^2 + \frac{r^2}{\Delta}C^{-2}(\Omega)\left[\omega^2\frac{A}{r^2\Delta} - k_y^2 - \frac{\Delta}{r^2}k_z^2\right]}. \tag{D.8}$$

We can simplify this equation by introducing $C^{-2}(\Omega)$ in (C.12) to obtain

$$\begin{aligned}\alpha &= -\frac{\imath\omega A(\omega_d - \Omega)}{r^3 C^{-2}(\Omega)} \pm \frac{\imath\Delta C^2(\Omega)}{r^2}\left(\frac{r^2}{\Delta}\left[\omega^2 - C^{-2}(\Omega)\left(k_y^2 + \frac{\Delta}{r^2}k_z^2\right)\right]\right)^{1/2}, \\ &= -\frac{\imath\omega A(\omega_d - \Omega)}{r^3 C^{-2}(\Omega)} \pm \imath\left(\frac{\Delta C^4(\Omega)}{r^2}\left[\omega^2 - C^{-2}(\Omega)\left(k_y^2 + \frac{\Delta}{r^2}k_z^2\right)\right]\right)^{1/2},\end{aligned} \tag{D.9}$$

therefore, substituting this last equation into the (D.7) yields

$$\chi(x) = \chi_0 e^{-\frac{\imath\omega A(\omega_d - \Omega)}{r^3 C^{-2}(\Omega)}}\left[C_1\, e^{\imath\sqrt{\left(\frac{\Delta C^4(\Omega)}{r^2}\left[\omega^2 - C^{-2}(\Omega)\left(k_y^2 + \frac{\Delta}{r^2}k_z^2\right)\right]\right)}\,x} + C_2\, e^{-\imath\sqrt{\left(\frac{\Delta C^4(\Omega)}{r^2}\left[\omega^2 - C^{-2}(\Omega)\left(k_y^2 + \frac{\Delta}{r^2}k_z^2\right)\right]\right)}\,x}\right], \tag{D.10}$$

now, we can to perform Dirichlet condition (2.11) to get

$$\chi(0) = \chi(L) = 0, \;\longrightarrow C_1 = -C_2, \tag{D.11}$$

but, by the condition $\chi(L) = 0$, then

$$\sqrt{\left(\frac{\Delta C^4(\Omega)}{r^2}\left[\omega^2 - C^{-2}(\Omega)\left(k_y^2 + \frac{\Delta}{r^2}k_z^2\right)\right]\right)}L = n\pi, \tag{D.12}$$

of this equation we obtain the frequecies of the field confined in the cavity orbiting the black hole. by

$$\omega \equiv \omega_n = \frac{r}{\sqrt{\Delta}}C^{-2}(\Omega)\sqrt{\left(\frac{n\pi}{L}\right)^2 + \frac{\Delta}{r^2}C^2(\Omega)\left(k_y^2 + \frac{\Delta}{r^2}k_z^2\right)}. \tag{D.13}$$

Then the full solution of (D.1) is given by

$$\psi = N_n\, e^{-\imath\omega_n t}\, e^{\imath(k_y y + k_z z)}\, e^{-\imath\beta_n x}\, \sin\left(\frac{n\pi}{L}x\right), \tag{D.14}$$

where N_n is a normalization constant that is calculated in appendix (E)

$$\begin{aligned} \beta_n &= \frac{b_n A}{r^3}(\omega_d - \Omega), \\ b_n &= \omega_n \, C^2(\Omega) \end{aligned} \tag{D.15}$$

APPENDIX E – TETRAD FORMALISM

Tetrad coordinates also termed Vierbein (4-dimensional) of the metric (3.11)

$$d\hat{s}^2 = C^{-2}(\Omega)dt^2 - 2\frac{A}{r^3}(\Omega-\omega_d)dtdx - \frac{A}{r^4}dx^2 - dy^2 - \frac{r^2}{\Delta}dz^2. \tag{E.1}$$

The basis in the tetrad formalism are one-forms, but in the second term of this metric contain an crossed term, we should try to factor this term and find a quadratic form for this and try to eliminate this cross term, then completing the square in the second a third term yields

$$\begin{aligned}
-\beta dx^2 - \alpha dtdx &= -\beta\left(dx^2 + \frac{\alpha}{\beta}dtdx\right) = -\beta\left(dx^2+\gamma dx\right)\\
\alpha = 2\frac{A}{r^3}(\Omega-\omega_d),\ \ \beta &= \frac{A}{r^4},\ \ \gamma = \frac{\alpha}{\beta}dt = 2r(\Omega-\omega_d)dt\\
-\beta dx^2 - \alpha dtdx &= -\beta\left(dx^2+\gamma dx + \left(\frac{\gamma}{2}\right)^2 - \left(\frac{\gamma}{2}\right)^2\right)\\
&= -\beta\left[\left(dx+\frac{\gamma}{2}\right)^2 - \left(\frac{\gamma}{2}\right)^2\right]\\
&= -\frac{A}{r^4}\left[(dx + r(\Omega-\omega_d)dt)^2 - r^2(\Omega-\omega_d)^2dt^2\right]\\
&= -\frac{A}{r^4}\left(dx + r(\Omega-\omega_d)dt\right)^2 + \frac{A}{r^2}(\Omega-\omega_d)^2dt^2.
\end{aligned} \tag{E.2}$$

Replacing this in the metric then

$$\begin{aligned}
d\hat{s}^2 &= C^{-2}(\Omega)dt^2 - \frac{A}{r^4}\left(dx + r(\Omega-\omega_d)dt\right)^2 + \frac{A}{r^2}(\Omega-\omega_d)^2dt^2 - dy^2 - \frac{r^2}{\Delta}dz^2,\\
&= \left(C^{-2}(\Omega) + \frac{A}{r^2}(\Omega-\omega_d)^2\right)dt^2 - \frac{A}{r^4}\left(dx + r(\Omega-\omega_d)dt\right)^2 - dy^2 - \frac{r^2}{\Delta}dz^2.\\
d\hat{s}^2 &= r^2\frac{\Delta}{A}dt^2 - \frac{A}{r^4}\left(dx + r(\Omega-\omega_d)dt\right)^2 - dy^2 - \frac{r^2}{\Delta}dz^2.
\end{aligned} \tag{E.3}$$

For more details see the references [2] [20]. Introducing the coframe basis vectors

$$\begin{aligned}
ds^2 &= \eta_{ab}e^ae^b = (e^1)^2 - (e^2)^2 - (e^3)^2 - (e^4)^2,\\
&= (e^1_\mu dx^\mu)^2 - (e^2_\mu dx^\mu)^2 - (e^3_\mu dx^\mu)^2 - (e^4_\mu dx^\mu)^2,\\
&= (e^1_t dt + e^1_x dx + e^1_y dy + e^1_z dz)(e^1_t dt + e^1_x dx + e^1_y dy + e^1_z dz)\\
&- (e^2_t dt + e^2_x dx + e^2_y dy + e^2_z dz)(e^2_t dt + e^2_x dx + e^2_y dy + e^2_z dz)\\
&- (e^3_t dt + e^3_x dx + e^3_y dy + e^3_z dz)(e^3_t dt + e^3_x dx + e^3_y dy + e^3_z dz)\\
&- (e^4_t dt + e^4_x dx + e^4_y dy + e^4_z dz)(e^4_t dt + e^4_x dx + e^4_y dy + e^4_z dz)
\end{aligned} \tag{E.4}$$

comparing (E.4) with (E.3) we get

$$
\begin{aligned}
&e^1_t = r\sqrt{\frac{\Delta}{A}},\ e^1_x = e^1_y = e^1_z = 0,\\
&e^2_t = \frac{\sqrt{A}}{r}(\Omega - \omega_d),\ e^2_x = \frac{\sqrt{A}}{r^2},\ \ e^2_y = e^2_z = 0,\\
&e^3_t = 0,\ e^3_x = 0,\ e^3_y = 1,\ e^3_z = 0,\\
&e^4_t = 0,\ e^4_x = 0,\ e^4_y = 0,\ e^4_z = \frac{r}{\sqrt{\Delta}},
\end{aligned}
\tag{E.5}
$$

the vierbeins are

$$
(e^a_\mu) = \begin{pmatrix} r\sqrt{\frac{\Delta}{A}} & 0 & 0 & 0\\ \frac{\sqrt{A}}{r}(\Omega-\omega_d) & \frac{\sqrt{A}}{r^2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \frac{r}{\sqrt{\Delta}} \end{pmatrix},
\tag{E.6}
$$

and the basis coframe are $e^a = e^a_\mu dx^\mu$

$$
\begin{aligned}
e^1 &= r\sqrt{\frac{\Delta}{A}}dt,\\
e^2 &= \frac{\sqrt{A}}{r}(\Omega - \omega_d)dt + \frac{\sqrt{A}}{r^2}dx,\\
e^3 &= dy\\
e^4 &= \frac{r}{\sqrt{\Delta}}dz,
\end{aligned}
\tag{E.7}
$$

we use this basis to find the volume form

$$
dV = e^1 \wedge e^2 \wedge e^3 \wedge e^4
\tag{E.8}
$$

$$
\begin{aligned}
dV &= r\sqrt{\frac{\Delta}{A}}dt \wedge \frac{\sqrt{A}}{r}\left[(\Omega - \omega_d)dt + \frac{dx}{r}\right] \wedge dy \wedge \frac{r}{\sqrt{\Delta}}dz,\\
dV &= dt \wedge dx \wedge dy \wedge dz,
\end{aligned}
\tag{E.9}
$$

this results is according to the fact that

$$
det[e^a_\mu] = 1.
\tag{E.10}
$$

We now define the surface element by

$$
dS_a = \frac{1}{3!}\varepsilon_{abcd} e^b \wedge e^c \wedge e^d,
\tag{E.11}
$$

we take a spacelike surface by

$$\begin{aligned} dS^1 &= e^2 \wedge e^3 \wedge e^4, \\ &= \frac{\sqrt{A}}{r}\left[(\Omega-\omega_d)dt + \frac{dx}{r}\right] \wedge dy \wedge \frac{r}{\sqrt{\Delta}}dz, \\ &= \sqrt{\frac{A}{\Delta}}(\Omega-\omega_d)dt\wedge dy\wedge dz + \frac{1}{r}\sqrt{\frac{A}{\Delta}}dx\wedge dy\wedge dz, \end{aligned} \tag{E.12}$$

this define the following surfaces,

- Spacelike surface in the direction of the timelike vector

$$dS^1_t = dx\wedge dy\wedge dz, \rightarrow n^\mu = \left(\frac{1}{r}\sqrt{\frac{A}{\Delta}}, 0, 0, 0\right) \tag{E.13}$$

- Null surface in the direction of the null vector

$$dS^1_x = dy\wedge dt\wedge dz, \rightarrow n^\mu = \left(0, -\sqrt{\frac{A}{\Delta}}(\Omega-\omega_d), 0, 0\right) = \left(0, \sqrt{\frac{A}{\Delta}}(\omega_d-\Omega), 0, 0\right) \tag{E.14}$$

now we can consider the vector

$$n^\mu = \left(\frac{1}{r}\sqrt{\frac{A}{\Delta}}, \sqrt{\frac{A}{\Delta}}(\omega_d-\Omega), 0, 0\right) \tag{E.15}$$

$$\hat{g}_{\mu\nu} = \begin{pmatrix} C^{-2}(\Omega) & (\omega_d-\Omega)\frac{A}{r^3} & 0 & 0 \\ (\omega_d-\Omega)\frac{A}{r^3} & -\frac{A}{r^4} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{r^2}{\Delta} \end{pmatrix}. \tag{E.16}$$

$$n_\mu = \hat{g}_{\mu\nu}n^\nu \tag{E.17}$$

$$\begin{aligned} n_0 = \hat{g}_{0\nu}n^\nu &= \sqrt{\frac{A}{\Delta}}\frac{1}{r}\left[C^{-2}(\Omega) + \frac{A}{r^2}(\Omega-\omega_d)^2\right] = \sqrt{\frac{A}{\Delta}}\frac{1}{r}\left[r^2\frac{\Delta}{A}\right] \\ n_0 n^0 &= \sqrt{\frac{A}{\Delta}}\frac{1}{r}\left[r^2\frac{\Delta}{A}\right]\frac{1}{r}\sqrt{\frac{A}{\Delta}} = 1. \end{aligned} \tag{E.18}$$

$$n_1 = \hat{g}_{1\nu}n^\nu = -(\Omega-\omega_d)\frac{A}{r^4}\sqrt{\frac{A}{\Delta}} + (\Omega-\omega_d)\frac{A}{r^4}\sqrt{\frac{A}{\Delta}} = 0 \tag{E.19}$$

then instead of use (E.15) we can use only

$$n^{\mu} = \left(\frac{1}{r}\sqrt{\frac{A}{\Delta}}, 0, 0, 0 \right) \tag{E.20}$$

and

$$n_{\mu} n^{\mu} = 1 \tag{E.21}$$

then n^{μ} is a timelike vector. A hypesurface $\mathcal{N}$ is called spacelike, if the vector N_x is normal at each point $x \in \mathcal{N}$ and is timelike,i.e. $g(N_x, N_x) > 0$ in signature $(+, -, -, -)$.[1]

E.1 Computation of the integral Surface

The idea is to compute the surface integral in (3.18)

$$(\psi_n, \psi_m) = \imath \int_S \left[(\partial_{\mu}\psi_n)\psi_m^* - \psi_n(\partial_{\mu}\psi_m^*) \right] \sqrt{\hat{g}_S}\; n^{\mu} dS, \tag{E.22}$$

using the unit vector (E.15) and the following determinant of the induced metric

$$(\hat{g}_{\mu\nu}) = \begin{bmatrix} C^{-2}(\Omega) & -(\Omega - \omega_d)\frac{A}{r^3} & 0 & 0 \\ -(\Omega - \omega_d)\frac{A}{r^3} & -\frac{A}{r^4} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{r^2}{\Delta} \end{bmatrix}.$$

$$\hat{g} = 1 = g_{tt} g_S - g_{tx} g_{S1} + 0 g_{S2} + 0 g_{S3}$$

fixing $dxdt = 0$ we find that

$$\hat{g} = 1 = g_{tt}\hat{g}_S \Longrightarrow \hat{g}_S = \frac{1}{\hat{g}_{tt}} = C^2(\Omega)$$

and the mode solutions (3.16) in the form

$$\begin{aligned} \psi_n &= N_n e^{-\imath\omega_n t}\, e^{\imath(k_y y + k_z z)} e^{-\imath\beta_n x} \sin\left(\frac{n\pi}{L}x\right), \\ \psi_n^* &= N_n e^{\imath\omega_n t}\, e^{-\imath(k_y y + k_z z)} e^{\imath\beta_n x} \sin\left(\frac{n\pi}{L}x\right). \end{aligned} \tag{E.23}$$

By the unit vector (E.15)

$$n^{\mu} = \left(\frac{1}{r}\sqrt{\frac{A}{\Delta}}, \sqrt{\frac{A}{\Delta}}(\omega_d - \Omega), 0, 0 \right), \tag{E.24}$$

[1] if the signature is $(-, +, +, +)$ then one timelike vector is when $g(N_x, N_x) < 0$.

putting $\eta^\mu = w^\mu$ in (E.24) we see that the surface integral in the components $\eta^2 = \eta^3 = 0$, is vanish, then the inner product in (E.22) is only in η^0 and η^1, then we only perform the following derivatives

$$\begin{aligned}
\partial_0 \psi_n &= -\imath\omega_n \psi_n, \\
\partial_0 \psi_n^* &= \imath\omega_n \psi_n^*, \\
\partial_1 \psi_n &= -\imath\beta_n \psi_n + N_n e^{-\imath\omega_n t} e^{\imath(k_y y + k_z z)} e^{-\imath\beta_n x} \left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi}{L}x\right), \\
\partial_1 \psi_n^* &= \imath\beta_n \psi_n^* + N_n e^{\imath\omega_n t} e^{-\imath(k_y y + k_z z)} e^{\imath\beta_n x} \left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi}{L}x\right),
\end{aligned} \tag{E.25}$$

putting (E.22) in the following form

$$(\psi_n, \psi_m) = \imath(\mathcal{I}_0 + \mathcal{I}_1), \tag{E.26}$$

where

$$\begin{aligned}
\mathcal{I}_0 &= \int_S \left[(-\imath\omega_n)\psi_n\psi_m^* - (\imath\omega_m)\psi_n\psi_m^*\right] C(\Omega)\sqrt{\frac{A}{\Delta}}\frac{1}{r} dxdydz, \\
&= C(\Omega)\sqrt{\frac{A}{\Delta}}\frac{1}{r}\left[-\imath(\omega_n + \omega_m)\right] \int_S \psi_n\psi_m^* \, dxdydz,
\end{aligned} \tag{E.27}$$

using (E.23) and separating the integrals in x, y, z yields

$$\begin{aligned}
\mathcal{I}_0 = C(\Omega)\sqrt{\frac{A}{\Delta}}\frac{1}{r}&\left[-\imath(\omega_n + \omega_m)\right] N_n N_m e^{\imath(\omega_m - \omega_n)} \underbrace{\int_0^L dx\, e^{\imath(\beta_m - \beta_n)} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right)}_{\frac{L}{2}\delta_{mn}} \\
&\times \underbrace{\int_{-\infty}^{\infty} dy\, e^{\imath(k_y - k_y')y}}_{2\pi\delta(k_y - k_y')} \underbrace{\int_{-\infty}^{\infty} dz\, e^{\imath(k_z - k_z')z}}_{2\pi\delta(k_z - k_z')}, \\
&= -\imath(\omega_n + \omega_m) C(\Omega)\sqrt{\frac{A}{\Delta}}\frac{1}{r} N_n N_m e^{\imath(\omega_m - \omega_n)} \frac{L}{2}\delta_{mn} 2\pi\delta(k_y - k_y')\, 2\pi\delta(k_z - k_z'), \\
&= -\imath(\omega_n + \omega_m) C(\Omega)\sqrt{\frac{A}{\Delta}}\frac{1}{r}\frac{L}{2}(2\pi)^2 N_n N_m e^{\imath(\omega_m - \omega_n)}\, \delta_{mn}\delta(k_y - k_y')\, \delta(k_z - k_z'),
\end{aligned} \tag{E.28}$$

comparing with (3.20) is the condition of orthogonality of the modes, then

$$\mathcal{I}_0 = -\imath\omega_n\, C(\Omega)\sqrt{\frac{A}{\Delta}}\frac{1}{r} L(2\pi)^2\, N_n^2. \tag{E.29}$$

$$\mathcal{I}_1 = \int_S \left[\left[-\imath\beta_n\psi_n + N_n e^{-\imath\omega_n t} e^{\imath(k_y y + k_z z)} e^{-\imath\beta_n x} \left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}x\right)\right]\psi_m^* \right.$$
$$\left. -\psi_n \left[\imath\beta_m\psi_m^* + N_m e^{\imath\omega_m t} e^{-\imath(k'_y y + k'_z z)} e^{\imath\beta_m x} \left(\frac{m\pi}{L}\right)\cos\left(\frac{m\pi}{L}x\right)\right]\right] C(\Omega)\sqrt{\frac{A}{\Delta}}(\omega_d - \Omega) dxdydz, \tag{E.30}$$

in this integral the terms that contains the crossed terms $\sin(n\pi)\cos(m\pi)$ are zero by orthogonality condition, therefore this integral is reduced to

$$\begin{aligned} \mathcal{I}_1 &= \int_S [-\imath\beta_n\psi_n\psi_m^* - \imath\beta_m\psi_n\psi_m^*] C(\Omega)\sqrt{\frac{A}{\Delta}}(\omega_d - \Omega) dxdydz \\ &= C(\Omega)\sqrt{\frac{A}{\Delta}}(\omega_d - \Omega)(-\imath\beta_n - \imath\beta_m)\int \psi_n\psi_m^*\, dxdydz \end{aligned} \tag{E.31}$$

the integral that appear in (E.31) was compute above, give

$$\int \psi_n\psi_m^*\, dxdydz = N_n N_m e^{\imath(\omega_m - \omega_n)}\frac{L}{2}(2\pi)^2\,\delta_{mn}\,\delta(k_y - k'_y)\,\delta(k_z - k'_z), \tag{E.32}$$

replacing into the (E.31) yields

$$\mathcal{I}_1 = -\imath\beta_n C(\Omega)\sqrt{\frac{A}{\Delta}}(\omega_d - \Omega)(2\pi)^2 L\, N_n^2, \tag{E.33}$$

inserting (E.33) and (E.29) into the (E.26) yields

$$\begin{aligned} (\psi_n, \psi_m) &= \imath\left[-\imath\omega_n\, C(\Omega)\sqrt{\frac{A}{\Delta}}\frac{1}{r}L(2\pi)^2\, N_n^2 - \imath\beta_n C(\Omega)\sqrt{\frac{A}{\Delta}}(\omega_d - \Omega)(2\pi)^2 L\, N_n^2\right], \\ &= C(\Omega)\sqrt{\frac{A}{\Delta}}L(2\pi)^2\, N_n^2\left[\frac{\omega_n}{r} + \beta_n(\omega_d - \Omega)\right] = C(\Omega)\sqrt{\frac{A}{\Delta}}\frac{1}{r}L(2\pi)^2\, N_n^2\left[\omega_n + r\beta_n(\omega_d - \Omega)\right]. \end{aligned} \tag{E.34}$$

β_n is given in (3.17) by

$$\begin{aligned}
\beta_n &= \frac{b_n A}{r^3}(\omega_d - \Omega), \\
b_n &= \omega_n\, C^2(\Omega), \\
\beta_n &= \omega_n\, C^2(\Omega)\frac{A}{r^3}(\omega_d - \Omega), \\
[\omega_n + r\beta_n(\omega_d - \Omega)] &= \omega_n + \omega_n\, C^2(\Omega)\frac{A}{r^2}(\omega_d - \Omega)^2, \\
&= \omega_n\, C^2(\Omega)\left[C^{-2}(\Omega) + \frac{A}{r^2}(\omega_d - \Omega)^2\right], \\
&= \omega_n\, C^2(\Omega)\left[\frac{\Delta\, r^2}{A} - \cancel{\frac{A}{r^2}(\omega_d - \Omega)^2} + \cancel{\frac{A}{r^2}(\omega_d - \Omega)^2}\right], \\
&= \omega_n\, C^2(\Omega)\frac{\Delta\, r^2}{A}
\end{aligned} \tag{E.35}$$

replacing into (E.36) yields

$$(\psi_n, \psi_m) = C(\Omega)\sqrt{\frac{A}{\Delta}}\frac{1}{r}L(2\pi)^2\, N_n^2 \omega_n\, C^2(\Omega)\frac{\Delta\, r^2}{A}, \tag{E.36}$$

therefore the normalization constant is given by

$$N_n^2 = \frac{1}{(2\pi)^2\, L\, \omega_n}\frac{1}{r}\sqrt{\frac{A}{\Delta}}C^{-3}(\Omega), \tag{E.37}$$

that is according to (3.21)

APPENDIX F – DIMENSION OF THE FIELD IN 4-D

in the natural units $[\hbar] = [c] = 1$, then

$$[E] = M, \quad L = T \tag{F.1}$$

but we know that the action is defined as

$$S = \int dt\ L_{lagrangian}, \quad [S] = [E][T], \tag{F.2}$$

in order to get the action dimensionless $[S] = 1$ then

$$[E] = T^{-1} = L^{-1}. \tag{F.3}$$

For the Lagrangian density we have that

$$\begin{aligned} S = \int dt \int d^3x\mathcal{L} = \int d^4x\mathcal{L}, \\ [\mathcal{L}] = \frac{[L_{lagrangian}]}{L^3} = \frac{[E]}{L^3} = \frac{1}{L^4} = [E^4] \end{aligned} \tag{F.4}$$

where $\mathcal{L}$ is the Lagrandian density. Then this mean that

$$\begin{aligned} \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi\right] =& \frac{1}{L^2}[\phi^2] = [E^4], \\ [\phi^2] =& [E^4]L^2 = [E^4][((E)^{-1})^2] = [E^2],\ [\phi] = [E] = M. \end{aligned} \tag{F.5}$$

The dimension of the orthogonal functions and the field are

$$\begin{aligned} [\phi] &= [E] = M = L^{-1}, \\ [u] &= \frac{1}{L^{1/2}} = [E^{1/2}], \\ [f] &= L^{1/2} = [E^{-1/2}], \\ [a^\dagger] &= L = [E^{-1}]. \end{aligned} \tag{F.6}$$

The full dimensions of $\hbar$ are the same that the dimension of the action (F.2)

$$\begin{aligned} E &= hf, \\ [E] &= [h][f], \\ [h] &= [E][T], \end{aligned} \tag{F.7}$$

remenber that the units of the frequency are T^{-1}.