Universidade Estadual de Londrina

RODRIGO CORSO BAPTISTA DOS SANTOS

DUALITIES IN $(2+1)$ DIMENSIONAL QUANTUM FIELD THEORY

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A thesis submitted in fulfillment of the requirements for the degree of Masters in the Physics Department .

Advisor: Prof. Dr. Pedro Rogerio Sergi Gomes

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RESUMO

Neste trabalho analisamos a dualidade entre o limite de baixas energias do modelo de Thirring e a teoria de Maxwell-Chern-Simons. Isto é feito introduzindo um campo vetorial que elimina a interação quártica de Thirring e integrando os graus de liberdade fermiônicos. Também investigamos uma série de dualidades, que podem ser obtidas a partir de operações simples na dualidade de bosonização mestre. Com este método obtemos outra dualidade bóson-férmion e duas dualidades partícula-vórtice, uma para bósons e outra para férmions. Também discutimos uma maneira de obter dualidades massivas a partir da dualidade mestre.

Palavras-chave: Bosonização. Teoria Quântica de Campos. Dualidades.

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ABSTRACT

In this work we analyze the duality between the low energy limit of the Thirring model and the Maxwell-Chern-Simons theory. This is done by introducing a gauge field to eliminate the quartic Thirring interaction and integrating out the fermionic degrees of freedom. We also investigate a series of dualities, which can be obtained by simple operations from a master boson-fermion duality. We obtain another boson-fermion duality and two particle-vortex duality, one for fermions and one for bosons. We also discuss a way to give mass to the master duality and derive a series of massive dualities.

Keywords: Bosonization. Quantum Field Theory. Dualities.

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1 INTRODUCTION

Many quantum field theories present the remarkable property to be dual to another one. This has proven to be specially useful when one of the theories can be exactly solved, allowing us to draw conclusions from a theory that we previously could not solve. One of the earliest known duality is between the Sine-Gordon and the Thirring theories in (1+1) space-time dimensions [1, 2, 3]. This is also an exact duality, thus we have an exact relation between the fermionic and bosonic fields. As this duality relates two theories with different statistics it is called a bosonization duality.

In (2+1) space-time dimensions things are more subtle as bosonization dualities involve distinct mechanisms. This can be checked by counting the number of degrees of freedom in each theory. In (2+1) dimensions the spinor has two complex components and the scalar field has one complex component, thus we should not expect to obtain an exact duality without the addition of another degree of freedom. We usually find a duality in the low-energy limit, where one of the degrees of freedom of the spinor is suppressed, or a duality that involves a scalar, a fermion and a gauge field.

The first suggestions of the existence of boson-fermion dualities came to light in the study of the motion of quantum particles in (2+1) dimensions. The conclusion of such studies was that the particles could have any statistics, not only bosons and fermions. Such particles were called anyons. The way to implement the transmutation of quantum numbers in a quantum field theory is through a Chern-Simons term. As we will see, this term will be of great relevance to our work. One place where this term is of seminal importance is in the study of the quantum Hall effect, where an effective theory in terms of the Chern-Simons allowed the explanation of fractional filling fraction, statistics and a series of other interesting effects.

Beyond that we also find that an odd number of fluxes transmutes the statistics of a particle from fermion to boson and vice-versa [4, 5]. Further support to the bosonization dualities comes from [6], where we find that a monopole operator also changes the spin of a particle in a way that is compatible with the statistics transmutation. This motivated the search of explicit dualities between two different theories [7, 8, 9]. These dualities relate a free fermion to an interacting boson, or vice-versa. Thus the duality states that the interacting theories behaves like a free theory, but with the statistics changed. Again the Chern-Simons term is responsible for the transmutation from bosons to fermions.

One of the dualities we find is the original one by Peskin, between the XY model and the Abelian Higgs model, but with a field theory approach. We also show a particle-vortex duality between the free massless fermion and the (2+1) dimensional QED. This duality has been especially useful in the understanding of condensed matter systems, as the quantum Hall effect [10], superconductors [11] and topological insulators [12, 13] among many other applications.

This work is mostly of review character, where we discuss some well known facts about 3D quantum field theory, papers of great recent interest and also fill some gaps with discussions not present in the literature. It is organized as follows. In the second chapter we study the transmutation of spin and statistics in particles in $(2+1)$ space-time dimensions. In the third chapter we study the duality between the Thirring model and the Maxwell-Chern-Simons theory. These two chapters provide the basic elements to, in the fourth chapter, investigate a bosonization duality as well as a series of relations that follow from it. This group of relations is called web of dualities. In the fifth chapter we extend the dualities discussed in the third chapter to include the massive case. We conclude the work with some final remarks. Subsidiary calculations are carried out in the final remarks.

2 SPIN-STATISTICS TRANSMUTATION

In this we chapter discuss how a monopole operator or the Chern-Simons term can promote the change of quantum numbers, like spin, statistics and Lorentz spin. To do so we also study some general aspects of monopole operators, the Chern-Simons action and the state-operator map in conformal field theory. This will be a motivation to, in the future, introduce a duality between a boson and a fermion partition functions, which is the main focus of this work.

2.1 STATISTICS

In $(2+1)$ dimensional field theory the spin is associated with the $SO(2)$ group, which has no non-trivial commutation relation, as it is an Abelian group. This can easily checked if we think of the $SO(2)$ group as a representation of the group of rotations around a fixed axis. Thus we do not expect to obtain any restriction on the spin of particles. By the spin-statistics theorem we also do not expect to find any limitation on the statistics of such particles, which are called anyons. We shall investigate these aspects from now on.

Let us assume that we have the wave function of a pair of non-interacting particles, except for a hard-core repulsion. If we rotate one particle around the other we do not expect the probability density to change as they are identical particles and we assume the system to be rotational invariant. Thus the wave function of these particles can change, at the most, by a phase factor. It is natural to assume that the phase depends on the angle of rotation, thus we write

$$\psi'(1,2) = e^{i\nu\Delta\theta}\psi(1,2), \tag{2.1}$$

where ν is an unknown constant, which we call the statistics parameter.

As we know from quantum mechanics, to compute the particle's statistics we must exchange the particles and check the phase that the wave function picks. This can be accomplished by rotating one particle around the other by an angle of $\Delta\theta = \pi$ or $\Delta\theta = -\pi$

$$\psi(1,2) \to \psi(2,1) = e^{i\nu\pi}\psi(1,2) \quad \text{or} \quad \psi(2,1) = e^{-i\nu\pi}\psi(1,2). \tag{2.2}$$

In three or more spacial dimensions we can deform one path into the other, even with hard-core interaction, as we have an extra dimension to go around the particle. This is equivalent to saying that both paths in the figure 2.1 are equivalent, and that the negative and positive phases in the equation (2.2) must be the same i.e., $e^{i\nu\pi} = e^{-i\nu\pi}$. This restricts the statistics parameter in such dimensions to even ν, corresponding to bosons with symmetric wave function and to odd ν for fermions with anti-symmetric wave function.

In two spacial dimensions these paths are not equivalent, as we can not deform one path into the other because of the hard-core repulsion. By this argument the two ways we wrote $\psi(2,1)$ in the equation (2.2) are not equivalent. Thus, in two spacial dimensions there are no limitations to the statistics parameter of a particle ν.

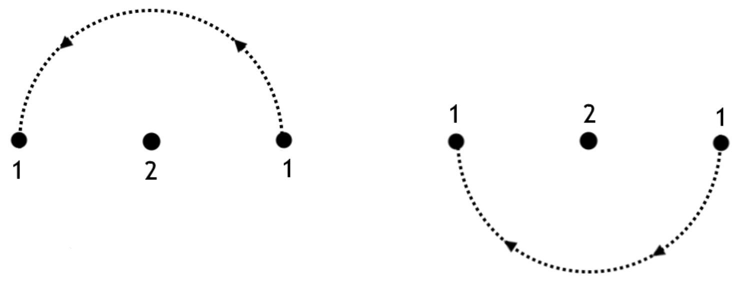

Figura 2.1: Transport of one particle around the other

Now we would like to incorporate this effect in a local quantum field theory, because this will be important for our purposes in the remaining of the work . The way to implement statistics transmutation into a particle is through a Chern-Simons term

$$S_{CS}[a] = \frac{k}{4\pi} \int \mathrm{d}^3x \, \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma, \tag{2.3}$$

where k is called the Chern-Simons level. The Chern-Simons term is of great relevance to many areas of physics. One of these fields of study is the Quantum Hall Effect, where the Chern-Simons action is considered the effective action emerging from the interaction of the large number of strongly coupled electrons. A good review on the subject can be found in [14]. Another noteworthy property is that the Chern-Simons term is topological, that is, its form does not vary with the metric of the manifold in which it is inserted.

One important aspect of the Chern-Simons action is the quantization of the level k. We can see this by placing the theory in a manifold $\mathbb{S}^2 \times \mathbb{R}$, in the presence of a magnetic monopole with a Dirac quantization condition

$$\frac{1}{2\pi} \int \mathrm{d}^2x \, f_{12} = \mathbb{Z}, \tag{2.4}$$

where d^2x is the integration over spacial $\mathbb{S}^2$ and f_{12} is the component of the field strength corresponding to the magnetic field. For convenience we work on a compact Euclidean time τ, such that $\tau \in [0, \beta]$, where we consider a large gauge transformations

$$\Lambda = 2\pi \frac{\tau}{\beta}. \tag{2.5}$$

This can be thought as a finite temperature system. Under these gauge transformation only the temporal component of the gauge field undergoes a transformation, $a_0' = a_0 + \frac{2\pi}{\beta}$.

Now we rewrite the Chern-Simons action as

$$S_{CS}[a] = \frac{k}{4\pi}\int \mathrm{d}^3x \, (a_0 f_{12} + a_2 f_{01} + a_1 f_{20}) \, . \tag{2.6}$$

We leave the first term as it is and rewrite the second and third terms in terms of a

$$a_2(\partial_0 a_1 - \partial_1 a_0) + a_1(\partial_2 a_0 - \partial_0 a_2). \tag{2.7}$$

We integrate by parts the terms that contain a_0 and leave the others. The upshot is that when we consider gauge transformations of the form (2.5) the first term in the equation (2.6) can be multiplied by a factor of two and the others can be ignored. After this procedure it is easy to see that the change of the Chern-Simons action under large gauge transformations reads

$$\begin{aligned} \delta S_{CS} &= \frac{k}{2\pi}\int \mathrm{d}^3x \, \frac{2\pi}{\beta} f_{12} \\ &= \frac{k}{2\pi} 2\pi \int \mathrm{d}^2x \, f_{12} \\ &= 2\pi k \, \mathbb{Z}. \end{aligned} \tag{2.8}$$

Here we integrated x_0 and used the Dirac quantization condition, equation (2.4). Although the action is not gauge invariant we can still define a quantum field theory, given in terms of $e^{iS_{CS}[a]}$, which is invariant if k is an integer. Thus the Chern-Simons partition function is gauge invariant if the Chern-Simons level is an integer.

To show how a particle transmutes its statistics we add to the Lagrangian a kinetic term of a set of particles and couple them to the Chern-Simons field through the minimal coupling. The complete action reads

$$S = \int \mathrm{d}^3x \left[\frac{k}{4\pi}\epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma - j_\mu a^\mu\right] + \int \mathrm{d}t \sum_I^N \frac{1}{2} m \vec{v}_I^2 \tag{2.9}$$

$$= \frac{k}{4\pi}\int \mathrm{d}^3x \, \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma + \int \mathrm{d}t \sum_I^N \left[\frac{1}{2} m \vec{v}_I^2 + e\vec{v}_I \cdot \vec{a} - e a_0\right], \tag{2.10}$$

where $j^\mu = (\rho, \vec{j})$ is the Minkowsky current given by

$$\begin{aligned} \rho(x) &= \sum_I^N e\delta^{(2)}\left(\vec{x} - \vec{r}_I(t)\right) \\ \vec{j}(x) &= \sum_I^N e\vec{v}_I(t)\delta^{(2)}\left(\vec{x} - \vec{r}_I(t)\right). \end{aligned} \tag{2.11}$$

We take the equations of motion of the vector potential a and of the particle.

The equation of motion for the particle reads

$$m\dot{v}_I^i = e\left(E^i\left(r_I\right) + \epsilon^{ij} v_I^j B\left(r_I\right)\right), \tag{2.12}$$

while the equation of motion for a_μ is

$$k\epsilon^{\mu\nu\sigma}\partial_\nu a_\sigma = 2\pi j^\mu. \tag{2.13}$$

In terms of the components,

$$B = -\frac{2\pi}{k}\rho \qquad \text{and} \qquad E^i = \frac{2\pi}{k}\epsilon^{ij} j^j, \tag{2.14}$$

where E^i and B are the electrical and magnetic fields that can be obtained from $E^i = -\partial_0 a^i - \partial_i a_0$ and $B = \epsilon^{ij}\partial_i a_j$.

The equation of motion for the magnetic field allows us to calculate the flux over a small region around a single particle I

$$\Phi_I = \int_I \mathrm{d}^2x\, B = -\frac{2\pi e}{k}\int_I \mathrm{d}^2x \sum_{J=1}^{N} \delta^{(2)}\left(x - r_J\right) = -\frac{2\pi e}{k}. \tag{2.15}$$

We see that the coupling to a Chern-Simons term attaches a magnetic flux to all charged particles.

In conjunction with the explicit form of the currents, equation (2.11), the equations of motion for the vector potential, equation (2.14), allows us to find an explicit form for the vector potential

$$a_I^i(\vec{r}_1, \cdots, \vec{r}_N) = \frac{e}{k}\sum_{I\neq J} \epsilon^{ij}\frac{\left(r_I^j - r_J^j\right)}{\left|\vec{r}_I - \vec{r}_J\right|^2}, \tag{2.16}$$

in the gauge where $\partial_i a^i$. The subindex I designates that this is the potential of felt by the particle I. This vector potential produces the magnetic field

$$B_I(\vec{r}_1, \cdots, \vec{r}_N) = -\frac{2\pi e}{k}\sum_{I\neq J}\delta^{(2)}\left(\vec{r}_I - \vec{r}_J\right). \tag{2.17}$$

Now that we have an explicit form the magnetic field we can find the statistics parameter. We do this by transporting adiabatically one particle around the other and checking the phase that the wave function picks under this operation.The Aharonov-Bohm pase is

$$\exp -iq\oint \mathrm{d}\vec{r}\cdot\vec{a} = \exp -iq\int \mathrm{d}^2x\, B, \tag{2.18}$$

where q is the Noether charge under $U(1)$ gauge transformation. Calculating the Noether conserved charge under $U(1)$ for the action (2.9) we find $q = \frac{e}{2}$. The discrepancy with the expected result, $q = e$, is due to the coupling to the Chern-Simons term. If we were to develop the same theory with the Maxwell term in place of the Chern-Simons, we would find the expected result, $q = e$. A detailed calculation can be found in [15].

Now we are able to calculate the Aharonov-Bohm phase using the magnetic field given by the equation (2.17)

$$\exp -iq \int \mathrm{d}^2x\, B = \exp \frac{\pi i e^2}{k}. \tag{2.19}$$

By comparing with equation (2.2), we identify the statistics parameter as

$$\nu = \frac{e^2}{k}. \tag{2.20}$$

With this we learn that the addition of a Chern-Simons term to the action can make the statistics of the system fractional, as the Chern-Simons level is quantized. This result is similar to what can happen in strongly coupled system where fractional statistics can be experimentally observed. One of these systems is the quantum Hall effect [16].

2.2 Lorentz Spin

In this section we wish to work in the Euclidean $(2+1)$ space, or $\mathbb{R}^3$. The rotation group is the $SO(3)$. As we know this group is non-Abelian, thus we expect to find restrictions on the Lorentz spin of particles. Here we will study how a monopole operator can transmute the Lorentz spin of a boson to a fermion and vice-versa. It is not easy to study how a monopole operator does this in a quantum field theory. In contrast, in a conformal field theory it is possible to circumvent this difficulty by using a state-operator map, which allows us to study monopole operators in a quantum mechanics setting. This correspondence relates fields (operators) in $\mathbb{R}^3$ to wave functions (states) in $\mathbb{S}^2 \times \mathbb{R}$, where $\mathbb{R}$ is the time direction. For completeness we give a hint at how this formalism works before we start with our study.

The maps between the two spaces can be achieved by taking the metric for $\mathbb{R}^3$,

$$\mathrm{d}s^2_{\mathbb{R}^3} = \mathrm{d}r^2 + r^2\, \mathrm{d}s^2_{\mathbb{S}^2}\,, \tag{2.21}$$

and making the conformal transformation $r = e^\tau$. It is easy to check that the result is the metric for $\mathbb{S}^2 \times \mathbb{R}$ up to a conformal factor $e^{2\tau}$, i.e.

$$\mathrm{d}s^2_{\mathbb{R}^3} = e^{2\tau}\, \mathrm{d}s^2_{S^2\times\mathbb{R}}\,. \tag{2.22}$$

With this it is easy to see that the infinite past, $\tau \to -\infty$, corresponds to the the point $r = 0$.

Thus a monopole operator inserted at the origin in a conformal field theory can be studied by considering asymptotic states in $\mathbb{S}^2 \times \mathbb{R}$

Now we show precisely how to define the map between states and operators. It is a well known fact that in quantum mechanics the wave function is defined by projecting a state, $|\psi, t\rangle$, in a coordinate basis, $|\vec{x}\rangle$, i.e.

$$\psi(\vec{x}, t) = \langle \vec{x} | \psi, t \rangle \, . \tag{2.23}$$

We want to build an analogous object in quantum field theory. We introduce a basis that diagonalize the field operator $\hat{\phi}(\vec{x}) \, |\phi(\vec{x})\rangle = \phi(\vec{x}) \, |\phi(\vec{x})\rangle$. We define a state as

$$\Psi \left[\phi(\vec{x}), t \right] \equiv \langle \phi(\vec{x}) | \psi, t \rangle \, . \tag{2.24}$$

Note that the state we defined is an object fixed in time but defined in the entire space, in contrast to as operator, $\mathcal{O} \left[\phi(\vec{x}) \right]$, that is defined in a fixed position in space-time.

The connection can be achieved when we project the equation for time evolution of the state,

$$|\psi, t\rangle = U(t, 0) \, |\psi, 0\rangle \, , \tag{2.25}$$

into the basis that diagonalize the field operator. By introducing a completeness relation we obtain

$$\Psi \left[\phi_f(\vec{x}), \tau_f \right] = \int \mathcal{D}\phi_i \int_{\phi(\tau_i)=\phi_i}^{\phi(\tau_f)=\phi_f} \mathcal{D}\phi \; \Psi \left[\phi_i(\vec{x}), \tau_i \right] e^{-S[\phi]}, \tag{2.26}$$

written in the Euclidean. Under the conformal transformation $r = e^{\tau}$, this equation becomes

$$\Psi \left[\phi_f(\vec{x}), rf \right] = \int \mathcal{D}\phi_i \int_{\phi(r_i)=\phi_i}^{\phi(r_f)=\phi_f} \mathcal{D}\phi \; \Psi \left[\phi_i(\vec{x}), r_i \right] e^{-S[\phi]}, \tag{2.27}$$

where the integration is carried over the region between two concentric spheres of radius r_i and r_f. The path integrals are calculated first with a fixed starting and ending points, and then summing over all starting points. This is equivalent to taking all the path integrals with a fixed ending point. Then we consider $r = 0$ and obtain

$$\Psi \left[\phi_f, r_f \right] = \int^{\phi(r_f)=\phi_f} \mathcal{D}\phi \Psi \left[\phi(0), 0 \right] e^{-S[\phi]}. \tag{2.28}$$

Now the map is manifest. We interpret the functional $\Psi[\phi(0), 0]$ as a local operator in the path integral language and $\Psi \left[\phi_f, r_f \right]$ as the state.

2.2.1 Monopole Operators in $\mathbb{R}^3$

In this section we discuss some basic aspects of monopole operators. Before we begin our discussion we must introduce the concept of a section and indicate why it is useful. In the absence of a monopole the equation $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$ is true and states that the total flux of the magnetic field vanishes. But in the presence of a monopole this equality is no longer true at one point on every closed surface surrounding the monopole, making the magnetic flux over a closed surface not zero. This can be achieved by letting $\vec{A}$ be divergent in one point in every closed surface around the monopole, the connection of all of these points is called a Dirac string, see figure 2.2 [17, 18].

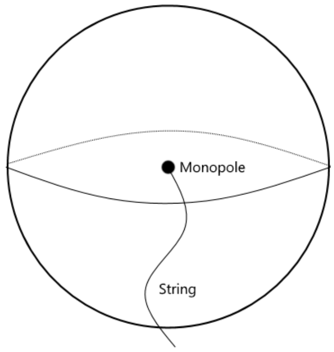

Figura 2.2: Dirac string emanating from the monopole

In order to avoid the string we define two overlapping vector potentials, in a way that the strings are never in the domain of the potential. One possible choice is [19]

$$A_r^{\pm} = A_\theta^{\pm} = 0, \quad A_\phi^{\pm} = \frac{g}{r \sin\theta}(\pm 1 - \cos\theta), \tag{2.29}$$

where g is the monopole strength. Here the plus and minus sign represent the north and south caps, respectively. This vector potential produces the magnetic field

$$\vec{B} = \frac{g}{r^2}\hat{r} \quad \text{for} \quad r \neq 0. \tag{2.30}$$

This has the same behaviour as the electric field for a charged particle. We can verify that this magnetic field is generated by a monopole using the equivalent of the Gauss law for the magnetic field in the presence of magnetic particles

$$\vec{\nabla} \cdot \vec{B} = \rho_m = 4\pi g \delta^{(3)}(r). \tag{2.31}$$

The right-hand-side clearly describes a magnetic monopole at the origin.

As the two potentials are connected by a gauge transformation, they describe the same magnetic field.

$$\vec{A}^{+} = \vec{A}^{-} + \frac{1}{eZ}\vec{\nabla}\alpha(x), \tag{2.32}$$

where $\alpha(x) = 2q\phi$ and eZ is the charge of the particle. The number q characterizes the section. It is connected to the monopole strength in Dirac's unit, $D \equiv 2eg$, by the relation $q = DZ/2$. As D and Z are integer, so is $2q$.The gauge transformation can be easily checked substituting the equation (2.29) into the one above

$$\begin{aligned}
\vec{A}^{-} + \frac{1}{eZ}\vec{\nabla}\alpha(x) &= \frac{g}{r\sin\theta}\left(-1-\cos\theta\right)\hat{\phi} + \frac{1}{eZ}\frac{1}{r\sin\theta}\partial_\phi(2q\phi)\hat{\phi} \\
&= \frac{g}{r\sin\theta}\left(-1-\cos\theta\right)\hat{\phi} + \frac{2q}{eZr\sin\theta}\hat{\phi} = \frac{g}{r\sin\theta}\left(-1-\cos\theta\right)\hat{\phi} + \frac{2g}{r\sin\theta}\hat{\phi} \\
\vec{A}^{-} + \frac{1}{eZ}\vec{\nabla}\alpha(x) &= \vec{A}^{+} \qquad (2.33)
\end{aligned}$$

Under gauge transformations the wave function undergoes the usual $U(1)$ transformation

$$\psi^{+} = e^{i\alpha(x)}\psi^{-}. \tag{2.34}$$

The way the wave function transforms is what defines a section. A section is a function that has different definitions on different domains. In our case we have ψ^{+} and ψ^{-} for the north and south caps of the sphere, respectively.

2.2.2 Scalar Particles

In this subsection and the next we will study how particles behave in the presence of a monopole. Here the particles will be described by wave functions and the monopole operator by a magnetic flux. It is important to remember that it is possible to return to a field theory description using the state-operator map we discussed in the beginning of this section. Using this mechanism, we are able to map the wave functions back into fields (operators) and the magnetic flux into a monopole operator.

Now that we understand how to work with a magnetic monopole we are able to understand how the monopole changes the spin of a scalar field. The presence of the monopole induces the non-trivial commutation relation [20]

$$[\pi_i, \pi_j] = -i\epsilon^{ijk}\frac{e}{c}B_k(\mathbf{x}), \tag{2.35}$$

where $\vec{\pi} = \vec{p} - e\vec{A}$. The angular momentum which is conserved and satisfies the usual commutation relation, $[L_i, L_j] = i\epsilon^{ijk}L_k$, involves an additional term

$$\vec{L} = \vec{r}\times\vec{\pi} - q\frac{\vec{r}}{r}. \tag{2.36}$$

Alternatively, we could find this angular momentum starting from the action of a particle coupled to the gauge field plus the Maxwell term and build the angular momentum using the Noether theorem. The only non-trivial part is that we must choose the field configuration corresponding to a monopole.

As usual, the operators $\vec{L}^2$ and L_z can be diagonalized simultaneously. Here it is important to note that we have an additional quantum number q in the spherical harmonics to differentiate the section.

$$\vec{L}^2 Y_{q,l,m} = l(l+1)Y_{q,l,m}; \qquad L_z Y_{q,l,m} = mY_{q,l,m}. \tag{2.37}$$

One can solve the second equation to find the ϕ dependence and then impose $L_- Y_{q,l,-l} = 0$ to obtain

$$Y^{\pm}_{q,l,-l} = \left[\frac{(2l+1)!}{4\pi 2^{2l}(l-q)!(l+q)!}\right]^{\frac{1}{2}} \sqrt{1-\cos\theta}^{\,l-q}\sqrt{1+\cos\theta}^{\,l+q} e^{i\phi(m\pm q)}. \tag{2.38}$$

The other $Y^{\pm}_{q,l,m}(\theta)$ can be found applying L_+ to $Y^{\pm}_{q,l,-l}$, where $L_\pm = L_x \pm iL_y$.

We impose that the sections $Y^{\pm}_{q,l,-l}$ must be single valued and obtain $m \pm q \in \mathbb{Z}$. This means that q and m must be integer or half integer. The wave function is ill defined for $l < |q|$, thus we restrict the wave function to $l \geq |q|$. Therefore the possible eigenvalues are

$$l = |q|, |q|+1, |q|+2\cdots \qquad m = -l, -l+1\cdots, l \tag{2.39}$$

The only angular momentum we can measure is the orbital angular momentum, thus it must be the total angular momentum. As the minimum eigenvalue of the angular momentum is $|q|$, it also behaves as the spin of the system composed by monopole plus the scalar particle. Because q is integer or half integer, this is equivalent to saying that the monopole transmutes the spin of the particle. Furthermore, we also know that the scalar wave function has no spin content, thus q must be the monopole's spin.

2.2.3 Fermions

Now we wish to repeat the same argument for fermions with spin $\frac{1}{2}$. In order to do so, we must diagonalize $\vec{J}^2$, $\vec{L}^2$ and J_z. For spin $\frac{1}{2}$ the total angular momentum is given by

$$\vec{J} = \vec{L} + \frac{\vec{\sigma}}{2}, \tag{2.40}$$

such that $\vec{\sigma}$ are the Pauli matrices and $\vec{L}$ is given by the equation (2.37). The eigenfunctions of $\vec{J}^2$, $\vec{L}^2$ and J_z operators are [6]

$$\phi^{\pm}_{j,l,m_j} = \begin{pmatrix} \sqrt{\frac{l+m+1}{2l+1}} Y^{\pm}_{q,l,m} \\ \sqrt{\frac{l-m}{2l+1}} Y^{\pm}_{q,l,m+1} \end{pmatrix} \qquad \text{for } j = l + \frac{1}{2}, \;\; m_j = m + \frac{1}{2}; \tag{2.41}$$

$$\phi^{\pm}_{j,l,m_j} = \begin{pmatrix} -\sqrt{\frac{l-m}{2l+1}} Y^{\pm}_{q,l,m} \\ \sqrt{\frac{l+m+1}{2l+1}} Y^{\pm}_{q,l,m+1} \end{pmatrix} \qquad \text{for } j = l - \frac{1}{2}, \;\; m_j = m + \frac{1}{2}. \tag{2.42}$$

In this way

$$\vec{J}^2 \phi_{j,l,m_j} = j(j+1)\phi_{j,l,m_j}, \quad \vec{L}^2 \phi_{j,l,m_j} = l(l+1)\phi_{j,l,m_j} \quad \text{and} \quad J_z \phi_{j,l,m_j} = m_j \phi_{j,l,m_j}. \tag{2.43}$$

The quantum numbers l and m obey the same relations as before

$$l = |q|, \; |q|+1 \cdots \quad ; \quad m = -l, \cdots, l-1, \; l, \tag{2.44}$$

and, to avoid divergences on $Y_{q,l,m}$, the total angular momentum eigenvalue j obeys

$$j = |q| - \frac{1}{2}, \; |q| + \frac{1}{2}, \; |q| + \frac{3}{2}, \; \cdots . \tag{2.45}$$

It is important to note that for $q = l = 0$ the second spinor is not allowed because it would lead to $j = -\frac{1}{2}$

The presence of a monopole leads to the same consequences on spinors as on scalars. For half-integer q, j assumes integer values, thus the system monopole plus spinor behaves like a boson. For integer q the system behaves like a fermion. Just like in the scalar case it follows that the monopole transmutes the spin of the spinor. Furthermore, we know that the spinor has $\frac{1}{2}$ spin, hence the monopoles's spin is q.

For completeness we will find the energy spectrum in the quantum theory. Under the variable change the Dirac Lagrangian reads

$$\mathcal{L}_{\mathbf{S}^2\times\mathbb{R}} = i\bar{\psi}\sigma_r \left(\frac{\partial}{\partial\tau} - \left(\vec{J}^2 - \vec{L}^2 + \frac{1}{4} \right) - q\sigma_r \right) \psi. \tag{2.46}$$

We write the wave function as $\psi = \sum_{l,jm_j} R_{l,j,m_j}(\tau)\phi_{l,j,m_j}(\theta,\phi)$ and find the radial part equation. It is important to remember that, as we are in a radial quantization scheme, $R(\tau)$ is the time dependence of the wave function

$$\frac{dR_{ljm_j}(\tau)}{d\tau} - \left(j(j+1) - l(l+1) + \frac{1}{4} \right) R_{ljm_j}(\tau) - \sum_{l'j'm'_j} qR_{l'j'm'_j}(\tau) \left\langle ljm_j \middle| \sigma_r \middle| l'j'm'_j \right\rangle = 0. \tag{2.47}$$

A more detailed calculation about how to solve the differential equation can be found in [6]. We write $R^{\pm}(\tau)$ instead of $R^{j=l\pm\frac{1}{2}}(\tau)$. As τ is the time in the radial quantization the energy of the state can be read by the factor $e^{-E\tau}$ in the radial part.

For $q = 0$, we find

$$R^{\pm}(\tau) = C^{\pm} e^{\pm(j+\frac{1}{2})\tau} \tag{2.48}$$

and the energy is $E = \pm(j + \frac{1}{2})$, thus there are no zero energy solutions.

For $q \neq 0$ and $j = |q| - \frac{1}{2}$, we have no solution for $l = j - \frac{1}{2}$, thus we find

$$R^{-} = C, \tag{2.49}$$

thus the energy is zero.

For $q \neq 0$ and $j = |q| - \frac{1}{2} + p$, with $p = 1, 2, 3, \cdots$, we find

$$R^{+}(\tau) = qC_1^{+} e^{\tau\sqrt{\left(j+\frac{1}{2}\right)^2 - q^2}} + qC_2^{+} e^{-\tau\sqrt{\left(j+\frac{1}{2}\right)^2 - q^2}} \tag{2.50}$$

and

$$\begin{aligned} R^{-}(\tau) =& C_1^{-} \left[\sqrt{\left(j + \frac{1}{2}\right)^2 - q^2} - \left(j + \frac{1}{2}\right) \right] e^{\tau\sqrt{\left(j+\frac{1}{2}\right)^2 - q^2}} \\ &+ C_2^{+} \left[\sqrt{\left(j + \frac{1}{2}\right)^2 - q^2} + \left(j + \frac{1}{2}\right) \right] e^{-\tau\sqrt{\left(j+\frac{1}{2}\right)^2 - q^2}}. \end{aligned} \tag{2.51}$$

These solutions have energy $E = \pm\sqrt{\left(j + \frac{1}{2}\right)^2 - q^2} = \pm\sqrt{2|q|p + p^2}$. Since the second equality the energy does not depend on j, we find that the degeneracy of the p-th state is $j = |q| - \frac{1}{2} + p$.

3 (2+1) DIMENSIONAL THIRRING MODEL

We already know that in $(2+1)$ dimensions the line that separates fermions and bosons is tenuous, but before we start with our main objective, we can demonstrate an explicit duality between bosons and fermions. This is the case of the Thirring model in the large mass limit and the Maxwell-Chern-Simons theory. This chapter is based on the studies of Frandkin and Schaposnik [21].

3.1 The Free Fermion

Before we start with the Thirring model we will give some general properties of fermions in $(2+1)$ dimensions. We start with the Lagrangian

$$\mathcal{L}_0 = \bar{\psi}\left(i\partial\!\!\!/ + m\right)\psi \tag{3.1}$$

where $\partial\!\!\!/ = \gamma^\mu\partial_\mu$, $\bar{\psi} = \psi^\dagger\gamma^0$. The Dirac matrices obey $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_{2\times 2}$, where the metric tensor has the signature $\eta^{\mu\nu} = \text{diag}(+,-,-)$. We choose the following representation for the Dirac matrices

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1 \quad \text{and} \quad \gamma^2 = i\sigma^2, \tag{3.2}$$

where σ^i are the Pauli matrices. With this representation it is easy to show that $(\gamma^0)^2 = \mathbb{I}$ and $(\gamma^i)^2 = -\mathbb{I}$.

Now we want to analyze the discreet symmetries of parity and time-reversal. In $(2+1)$ dimensions the parity transformation inverts only one axis, because if we inverted both axis it would be equivalent to a rotation. If we choose to invert x^1, the coordinates and the vector potential transforms as $x^1 \to -x^1$, $A^1 \to -A^1$, $x^2 \to x^2$ and $A^2 \to A^2$. To find how this transformation acts on the spinors we consider the massless case and assume that the kinetic term is invariant

$$i\psi^\dagger\mathcal{P}^\dagger\gamma^0\left(\gamma^0\partial_0 - \gamma^1\partial_1 + \gamma^2\partial_2\right)\mathcal{P}\psi = i\psi^\dagger\gamma^0\left(\gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2\right)\psi. \tag{3.3}$$

Comparing each term and using that $\gamma^\mu\gamma^\nu \propto \epsilon^{\mu\nu\sigma}\gamma_\sigma$ for $\mu \neq \nu$, we find that

$$\mathcal{P}^\dagger\mathcal{P} = \mathbb{I}, \qquad \left\{\mathcal{P}, \gamma^2\right\} = 0 \quad \text{and} \quad \left[\mathcal{P}, \gamma^1\right] = 0. \tag{3.4}$$

The second relation is only satisfied if $\mathcal{P}$ is proportional to γ^0 or γ^1. The third relation allows γ^1 and the identity, thus the only possibility is $\mathcal{P} \sim \gamma^1$. We also know that $\mathcal{P}^2 = \mathbb{I}$, thus we find

$\mathcal{P} = -i\gamma^1 = \sigma^1$. It is easy to check that the mass term breaks parity invariance

$$\bar{\psi}'\psi' = -\psi^\dagger\gamma^1\gamma^0\gamma^1\psi = \psi^\dagger\gamma^0(\gamma^1)^2\psi = -\bar{\psi}\psi. \tag{3.5}$$

Now we want to repeat the process for time reversal, $\mathcal{T}$. We know that $\mathcal{T}$ is anti-unitary, so we write $\mathcal{T} = K\tilde{\mathcal{T}}$, where K performs the complex conjugation. As we did before we impose that the kinetic term is invariant under time reversal and we will find the form of $\tilde{\mathcal{T}}$

$$-i\psi^\dagger\tilde{\mathcal{T}}^\dagger\gamma^0\left(-\gamma^0\partial_0 - \gamma^1\partial_1 + \gamma_2\partial^2\right)\tilde{\mathcal{T}}\psi = i\psi^\dagger\gamma^0\left(\gamma^0\partial_0 + \gamma_1\partial^1 + \gamma^2\partial^2\right)\psi. \tag{3.6}$$

Comparing the terms we find

$$\tilde{\mathcal{T}}^\dagger\tilde{\mathcal{T}} = \mathbb{I}, \qquad \tilde{\mathcal{T}}^\dagger\gamma^0\gamma^1\tilde{\mathcal{T}} = \gamma^0\gamma^1 \quad \text{and} \quad \tilde{\mathcal{T}}^\dagger\gamma^0\gamma^2\tilde{\mathcal{T}} = -\gamma^0\gamma^2. \tag{3.7}$$

From the second equation we find that $\left[\tilde{\mathcal{T}}, \gamma^2\right] = 0$, this leaves us with γ^2 and the identity. From the third we find $\left\{\tilde{\mathcal{T}}, \gamma^1\right\} = 0$. This leaves us with $\mathcal{T} = K\gamma^2$. The mass term also breaks time reversal

$$\bar{\psi}'\psi' = \psi^\dagger(\gamma^2)^\dagger\gamma^0\gamma^2\psi = -\psi^\dagger\gamma^2\gamma^0\gamma^2\psi = \psi^\dagger\gamma^0(\gamma^2)^2\psi = -\bar{\psi}\psi. \tag{3.8}$$

Thus the mass term breaks both parity and time reversal in $(2+1)$ dimensions.

One could also check whether the fermion Lagrangian is invariant under chiral transformations, defined as $\psi' = e^{i\theta\gamma^5}\psi$, where the γ^5 matrix is usually defined as being proportional to the product of all the other γ^μ. It has the property to anticommute with all the other γ^μ, and we adjust the constant of proportionality such that γ^5 is hermitian and $(\gamma^5)^2 = \mathbb{I}$. The problem is that in the minimal representation of fermions in $(2+1)$ space-time dimensions there is no γ^5 matrix because $i\gamma^0\gamma^1\gamma^2 = \mathbb{I}$, thus there is no way to define a chiral transformation. This fact is not exclusive to $(2+1)$ dimensions, it happens to all fermionic theories in odd space-time dimensions. If we were to work in a non-minimal representation to the spinors, for example we could work with four component instead of two component spinors, we would be able to build a chiral transformations as we would have the four dimensional Dirac matrices.

3.2 The Thirring Model

Now we can start to work with the Thirring model by defining its partition function

$$Z_{Th} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\ i\int \mathrm{d}^3x \left[\bar{\psi}(i\partial\!\!\!/ + m)\psi - \frac{g^2}{2}j_\mu j^\mu\right], \tag{3.9}$$

where g is the coupling constant and j^μ is the $U(1)$ Noether current

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{3.10}$$

Let us perform a dimensional analysis of the Thirring partition function in units of mass, $[x^\mu] = -1$. We find $[\psi] = 1$, $[j^\mu j_\mu] = 4$ and $[\bar{\psi}\psi] = 2$, making the Thirring interaction perturbatively irrelevant and the mass term relevant in the low-energy limit. Thus we expect that in low enough energies the Thirring interaction will not give any meaningful contribution to the effective action.

We wish to investigate the low-energy effective action of the Thirring model. As we shall discuss, this will provide us with a bosonic theory. To do this we integrate out the fermionic fields from the partition function (3.9). To do so, it is convenient to we introduce a new auxiliary vector field a_μ in a way that when we eliminate the quartic Thirring interaction

$$\exp\left[-i\frac{g^2}{2}\int \mathrm{d}^3x\, j_\mu j^\mu\right] = \int \mathcal{D}a \exp\, i\int \mathrm{d}^3x \left[\frac{1}{2}a_\mu a^\mu + g a_\mu j^\mu\right]. \tag{3.11}$$

After this we integrate out the fermionic fields and obtain an effective action for the vector field

$$\begin{aligned} Z_{Th} &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}a \exp\left[i\int \mathrm{d}^3x\, \bar{\psi}\left(i\not{\partial} + m + g\not{a}\right)\psi + \frac{1}{2}a_\mu a^\mu\right] \\ &= \int \mathcal{D}a \exp\left[\mathrm{tr}\ln\left(i\not{\partial} + m + g\not{a}\right) + i\int \mathrm{d}^3x\, \frac{1}{2}a_\mu a^\mu\right] \\ &\equiv \int \mathcal{D}a\; e^{S_{eff}[a]}. \end{aligned} \tag{3.12}$$

The calculation of this effective action can be represented by the following Feynman diagrams.

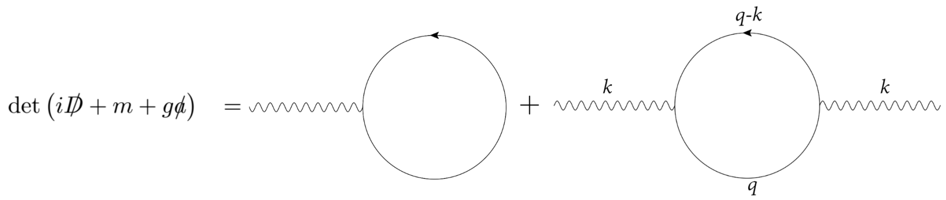

Figura 3.1: Diagramatic expansion for fermionic determinant

The relevant terms in the IR limit are the quadratic ones, coming from the second diagram

$$S_{eff} = \int \mathrm{d}^3x \left[\frac{1}{2}a_\mu a^\mu \mp \frac{g^2}{8\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \frac{g^2}{24m\pi}f_{\mu\nu}f^{\mu\nu}\right] + \cdots, \tag{3.13}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, $\epsilon^{\mu\nu\rho}$ is the Levi-Civita symbol and $\mp = -\mathrm{sign}(m)$. A detailed calculation of this effective action and its propagator can be found in Appendix A. In the large mass limit, $m \to \infty$, we recognize the first two terms of the effective action, equation (3.13), as the Self-Dual action [22]. Thus, up to the order $1/m$, we have demonstrated the equivalence between the Thirring partition function and the Self-Dual partition function

$$Z_{Th} \approx Z_{SD}. \tag{3.14}$$

Here $\approx$ means that the duality is valid up to the order of $1/m$.

With the propagator corresponding to the equation (3.13)

$$G^{\mu\nu}(k) = -\frac{1}{9g^4k^2m^2 - 4\left(g^2k^2 - 6\pi m\right)^2}\left[24\pi m\left(6\pi m - g^2k^2\right)\eta^{\mu\nu} + 36i\pi g^2 m\,|m|\,\epsilon^{\mu\nu\sigma}k_\sigma \right.$$
$$\left. + \left(g^4\left(4k^2 - 9m^2\right) - 24\pi g^2 m\right)k^\mu k^\nu\right], \tag{3.15}$$

we can extract some information on the physical content of the model. There will be only bound states excitations if we find real poles with $k_\pm^2 < 4m^2$. Considering $m > 0$ and the poles of the propagator,

$$k_\pm^2 = \frac{9g^4m^2 + 48\pi g^2 m \pm 3m\sqrt{9g^8m^2 + 96\pi g^6 m}}{8g^4}, \tag{3.16}$$

we find that the bound state condition is satisfied when $g^2 > \frac{6\pi}{m}$ for k_+^2 and $g^2 > \frac{6\pi}{7m}$ for k_-^2. This means that for any given mass, there is a strong enough coupling g^2 that, up to order $1/m^2$, the Thirring action describes only bound states. This can be easily seen from the graph bellow.

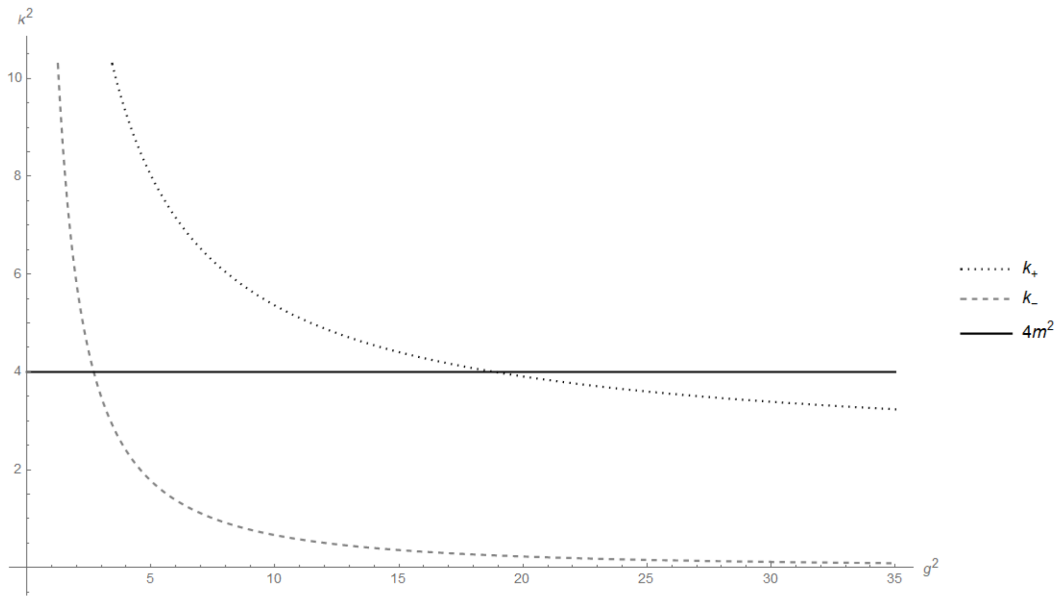

Figura 3.2: Plot for $k_\pm^2$ versus g^2 for $m = 1$

Now, by considering the low-energy limit, $m \to \infty$, we see that the Self-Dual action only describes bound states with mass $\frac{4\pi}{g^2}$. In the low-energy limit, or g^2 large enough, any remaining dynamics is due to bound states. This is the same result we get in the large-N limit [23].

It is interesting to note what happens when we consider the partition function (3.9) for a set of fermions, which is renormalizable in the $1/N$ expansion [23]. In this case the action arising from this theory is also the equation (3.13), thus in the low-energy it limit continues to be consisted by bound states. This happens because in the low-energy limit of the N fermions Thirring theory there will also be no free particles, even if there are multiple flavors, thus there should be no free particles in its effective action, the Self-Dual theory.

There also seems to be an inconsistency in the duality (3.14), if we take the limit $m \to 0$ in the Thirring action, equation (3.9). In this case we eliminate the term that breaks parity and time reversal, but this does not eliminate the Chern-Simons term in the Thirring effective action, which breaks both parity and time reversal, this is an anomaly. To further support the claim of the Chern-Simons term we note that the mass term acts as a Pauli-Villars regulator in the partition function (3.9), a detailed calculation of how this happens can be found in [24]. Even with a different regulator, like a ζ function regulator, we will find the same anomalous Chern-Simons term [25].

Now we prove the equivalence between the Self-Dual theory and the Maxwell-Chern-Simons theory. This equivalence was first studied by Deser and Jackiw [22], but we follow the reference [21] that uses the path integral approach. This duality will allow us to prove the equivalence between the Thirring and the Maxwell-Chern-Simons theories. To this end we introduce an interpolating partition function

$$Z_I = \int \mathcal{D}\tilde{a}\mathcal{D}a \exp i \int \mathrm{d}^3x \left[\frac{1}{2} a_\mu a^\mu - \epsilon^{\mu\nu\rho} a_\mu \partial_\nu \tilde{a}_\rho \mp \frac{2\pi}{g^2} \epsilon^{\mu\nu\rho} \tilde{a}_\mu \partial_\nu \tilde{a}_\rho \right]. \tag{3.17}$$

This partition function received this name because integrating out different vector fields yields two different theories. It is also important to note that this partition function is gauge invariant under transformations of $\tilde{a}$ but not of a.

We begin integrating over $\tilde{a}$,

$$\begin{aligned} Z_I &= \int \mathcal{D}a \; \bar{Z}_I[a] \exp\left(\frac{i}{2} \int \mathrm{d}^3x \, a_\mu a^\mu \right) \\ &= \int \mathcal{D}a \exp i \int \mathrm{d}^3x \left[\frac{1}{2} a_\mu a^\mu \mp \frac{g^2}{8\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \right] = Z_{SD}. \end{aligned} \tag{3.18}$$

The detailed calculation of $\bar{Z}_I[a]$ can be found in Appendix B. We recognize this as the Self-Dual partition function, thus we have proved the equivalence

$$Z_{SD} = Z_I. \tag{3.19}$$

Integrating the interpolating partition function over a instead of $\tilde{a}$, we get

$$Z_I = \int \mathcal{D}\tilde{a} \exp i \int \mathrm{d}^3x \left[-\frac{1}{4}\tilde{f}_{\mu\nu}\tilde{f}^{\mu\nu} \mp \frac{2\pi}{g^2}\epsilon^{\mu\nu\rho}\tilde{a}_\mu \partial_\nu \tilde{a}_\rho \right] = Z_{MCS}, \tag{3.20}$$

which is the Maxwell-Chern-Simons theory. Thus, using the duality (3.19) we have established that

$$Z_{SD} = Z_{MCS}, \tag{3.21}$$

and using the duality (3.14), we get a new duality

$$Z_{Th} \approx Z_{MCS}. \tag{3.22}$$

As before, $\approx$ means that the duality is valid up to the order $1/m$.

Something quite interesting happened here. The left hand side is not exactly solvable, but the right hand side is quadratic in the field $\tilde{a}$. Thus this duality should be suitable to exactly solve the Thirring model up to the order $1/m$. Another noteworthy property is that the coupling appears as g^2 in the Thirring action and as $\frac{1}{g^2}$ in the MCS theory.

The duality between the Self-Dual theory and the Maxwell-Chern-Simons theory is exact. Thus we would expect the propagator to have the same pole structure. The propagator of both theories can be calculated the same way we calculated the propagator of the Thirring effective action in Appendix B. It is important to remember to add a gauge fixing term in the MCS theory. This can be done with the term $\frac{\lambda}{2}(\partial_\mu a_\mu)^2$, but it is not needed in the Self-Dual theory because the term $a_\mu a^\mu$ breaks gauge invariance. We find that the Self-Dual theory propagator is

$$S^{SD}_{\mu\nu}(k) = -\frac{1}{g^4k^2 - 16\pi^2}\left[16\pi^2\eta_{\mu\nu} - g^4 k_\mu k_\nu \pm 4i\pi g^2 \epsilon_{\mu\sigma\nu}k^\sigma\right] \tag{3.23}$$

and the MCS theory propagator is given by

$$S^{MCS}_{\mu\nu}(k) = \frac{1}{g^4k^2 - 16\pi^2}\left[g^4\eta_{\mu\nu} + \frac{16\pi^2 - g^4k^2(\lambda+1)}{k^4\lambda}k_\mu k_\nu \pm \frac{4i\pi g^2}{k^2}\epsilon_{\mu\sigma\nu}k^\sigma\right]. \tag{3.24}$$

In the MCS propagator we must disregard poles involving the gauge fixing parameter λ, as it is not a physical constant. This makes the second term become $-g^4 k_\mu k_\nu / k^2$. In addition, what carries physical meaning is the S matrix. To calculate it we would need to take the product $k^\mu S^{MCS}_{\mu\nu}$, this is equivalent to selecting ony the transversal part of the propagator. This would eliminate both the remaining $1/k^2$ poles in the second and third terms, in such a way that the only pole is $k = 4\pi/g^2$. Thus both propagators have the same physical pole structure, as we expected for an exact duality.

We are also interested in the bosonization rule for this duality. This rule gives us a direct mapping between one object from the bosonic theory and one from the fermionic theory. To this end we introduce a coupling between the $U(1)$ current and a new external gauge field b_μ in the Thirring action. This can be achieved by making the shift $a_\mu \to a_\mu - \frac{1}{g}b_\mu$, and then we perform the path integral the same way in did in the Appendix A to obtain the duality between the Thirring and the Self-Dual theories. The Thirring partition function coupled to the new field b_μ reads

$$Z_{Th}[b] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}a \exp i \int \mathrm{d}^3x \left[\bar{\psi}\left(i\partial\!\!\!/ + m + g a\!\!\!/\right) + \frac{1}{2}a_\mu a^\mu + b_\mu j^\mu\right] \tag{3.25}$$

To connect the Thirring theory to the Maxwell-Chern-Simons theory we introduce a coupling in the interpolating partition function, equation (3.17) and repeat the calculations we did to prove duality between the Self-Dual and the Maxwell-Chern-Simons theory. We obtain that the Maxwell-Chern-Simons action coupled to b_μ is

$$S_{MCS} = \int \mathrm{d}^3x \left(-\frac{1}{4}\tilde{f}_{\mu\nu}\tilde{f}^{\mu\nu} + \frac{1}{g}\epsilon^{\mu\nu\sigma}b_\mu\partial_\nu\tilde{a}_\sigma \mp \frac{1}{g^2}\epsilon^{\mu\nu\sigma}\tilde{a}_\mu\partial_\nu\tilde{a}_\sigma\right). \tag{3.26}$$

We observe that b_μ is coupled to j^μ in the Thirring action and to $\frac{1}{g}\epsilon^{\mu\nu\sigma}\partial_\nu\tilde{a_\sigma}$. As we already know the duality between these two theories we conclude that the bosonization rule is

$$\bar{\psi}\gamma^\mu\psi \approx \frac{1}{g}\epsilon^{\mu\nu\sigma}\partial_\nu a_\sigma. \tag{3.27}$$

As the duality is valid up to orders of $1/m$ so is this bosonization rule. A detailed calculation of how the bosonization rule is be derived can be found in [21].

4 A WEB OF DUALITIES

In the previous chapter we have seen the equivalence between the Thirring model and the Maxwell-Chern-Simons theory in the low-energy limit, which can be thought as a bosonization duality. However, it is not a true bosonization, as it relates only the bound state sector of the Thirring to the Maxwell-Chern-Simons. Nevertheless, it states that there is an equivalence between two theories of different characters. An important lesson from the previous chapters is that the Chern-Simons term was present in all the discussion relating bosonic and fermionic theories. Therefore, it is expected that it will play a central role in the bosonization relation examined in this chapter. Our discussion is based on [8, 9].

4.1 Bosonization Dualities

Now we want to discuss two more bosonization dualities that arise when we attach flux to a theory. To proceed we first define

$$Z_{scalar+flux}[A] = \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}a \exp i\left(S_{scalar}[\phi, a] + S_{CS}[a] + S_{BF}[a; A]\right), \tag{4.1}$$

where

$$S_{BF}[a, A] = \frac{1}{2\pi}\int \mathrm{d}^3x\, \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu A_\rho, \tag{4.2}$$

$$S_{CS}[a] = \frac{1}{4\pi}\int \mathrm{d}^3x\, \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma, \tag{4.3}$$

$$S_{scalar}[\phi, a] = \int \mathrm{d}^3x\, |(\partial_\mu - iea_\mu)\phi|^2 + V(\phi). \tag{4.4}$$

These actions are gauge invariant, but only the scalar action is invariant under parity and time reversal. The potential must be of kind $|\phi|^4$ for the dualities to work, as it allows us to define the theory in a Wilson-Fischer fixed point, where the theory is scale invariant [7]. For convenience we will often omit this term from the action. We also define

$$Z_{fermion}[A] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{iS_{fermion}[\psi, A]}, \tag{4.5}$$

where the fermion action is the Dirac massless action

$$S_{fermion}[A] = \int \mathrm{d}^3x\, i\bar{\psi}(\not{\partial} - ie\not{A})\psi. \tag{4.6}$$

This action is invariant under gauge transformations. Parity and time reversal invariance can also be immediately inferred due to the absence of a mass term. As we discussed before, it is the mass term that breaks both $\mathcal{P}$ and $\mathcal{T}$.

As we learned in the previous chapter the result of attachment of a magnetic flux to a particle is the transmutation between bosons and fermions. Following [9], we will assume the duality

$$Z_{fermion}[A]e^{-\frac{i}{2}S_{CS}[A]} = Z_{scalar+flux}[A] \tag{4.7}$$

to be true and prove a series of other dualities. Because of this we call it the master duality. For the duality to be true we consider the scalar side to be in the Wilson-Fischer fixed point. To do this, we must consider that the potential has a $|\phi|^4$ form, as we will discuss shortly. We also consider that both gauge fields that appear in the duality (4.7) obey the Dirac quantization condition

$$\frac{1}{2\pi}\int_{S^2} \mathrm{d}^2x\, F_{12} = \mathbb{Z}. \tag{4.8}$$

Upon a first look there might seem to be a problem with the duality (4.7) since only the right hand side appears to be gauge invariant. We already encountered this kind of inconsistency before when we calculated the Thirring effective action. What is happening is that the fermion partition function is also anomalous, and the Chern-Simons term in the left hand side is what cancels this anomaly. The same thing happens with both parity and time reversal.

An argument to support this duality can be found if we take the equation of motion for a_0 from the partition function (4.1) in the absence of a background source $A = 0$,

$$\rho_{scalar} = -\frac{da}{2\pi}, \tag{4.9}$$

where $\rho_{scalar} = 2a^0\phi\phi^* + i\left(\phi\partial^0\phi^* - \phi^*\partial^0\phi\right)$. Integrating both sides, and using the quantization condition we see that the presence of the Chern-Simons term attaches flux to the scalar particles. From our previous discussion we expect a scalar attached to flux to behave like a fermion, thus the right hand side of the duality (4.7) behaves like a fermion.

We can also check that the Hall conductivity, $\sigma_{xy} = \frac{k}{2\pi}$, is matched by both sides of the duality [9, 8]. The Chern-Simons level k is the constant of the background field A in the partition function. To do this we must gap the theory by adding a mass term to the duality. In the fermion side we must integrate out the fermion fields. We already performed this calculation in the last chapter and obtained $\mp\frac{i}{2}S_{CS}[A]$. With the term that we already have in the duality (4.7), this will produce a Chern-Simons with level $k = 0$ for the plus sign and $k = -1$ for the minus sign. To find the Hall conductivity we take the functional derivative in respect to A_0. We obtain $\sigma_{xy} = 0$ for the plus sign and $\sigma_{xy} = \frac{1}{2\pi}$ for the minus sign.

In the bosonic side we gap the theory by modifying the potential with $V(\phi) = \lambda(|\phi|^2 \pm m^2)^2$. A precise way to add the mass term will be discussed in the Chapter 5. This potential in known as the Mexican Hat potential because of the form of its graph. For the minus sign the field acquires an expected value $|\phi| = m$. This motivates us to write the field as $\phi(x) = \rho(x)e^{i\theta(x)}$ with $\rho(x) = m + \chi(x)$. We work only with the terms that are important to our calculation

$$\mathcal{L} = \mathcal{L}_{CS}[a] + \mathcal{L}_{BF}[a, A] + \rho^2 \left(\partial_\mu \theta - e a_\mu\right)^2 + \partial_\mu \rho \partial^\mu \rho - \lambda \left(\rho^2 - m^2\right)^2$$
$$\mathcal{L} = \mathcal{L}_{CS}[a] + \mathcal{L}_{BF}[a, A] + \rho^2 \partial_\mu \theta \partial^\mu \theta + 2\rho^2 \theta a_\mu \partial^\mu \theta + e^2 \chi^2 a_\mu a^\mu + 2e^2 m \chi a_\mu a^\mu + e^2 m^2 a_\mu a^\mu$$
$$- \lambda \left(\rho^2 - m^2\right)^2 . \tag{4.10}$$

Now the dominant term is the mass term, thus the Hall conductivity is zero, matching one of the results of the fermion. For the plus sign in the potential the field do not acquire an expected value. Thus the Hall conductivity is given by the Chern-Simons term, we obtain $\sigma_{xy} = \frac{1}{2\pi}$.

From the duality (4.7) we can derive another bosonization. To do this, we promote the background gauge field to a dynamical one (every time we do this we will change the field from upper-case letters to lower-case letter $A \rightarrow a$) and couple it to a new background gauge field through BF coupling. The duality becomes

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}a \exp\ i \left(S_{fermion}[\psi, a] - \frac{1}{2} S_{CS}[a] - S_{BF}[a; A] \right)$$
$$= \int \mathcal{D}\phi \mathcal{D}\phi^* \mathcal{D}a \mathcal{D}\tilde{a} \exp\ i \left(S_{scalar}[\phi, a] + S_{CS}[a] + S_{BF}[a; \tilde{a}] - S_{BF}[\tilde{a}; A] \right)$$
$$= \int \mathcal{D}\phi \mathcal{D}\phi^* \exp\ i \left(S_{scalar}[\phi, A] + S_{CS}[A] \right) . \tag{4.11}$$

In the last step we used the equation of motion for $\tilde{a}$, $dA = da$, and integrated out $\tilde{a}$. We recognize the left hand side as the fermion attached to flux partition function. Doing this process we obtained a new duality between a fermionic theory and a bosonic theory

$$Z_{fermion+flux}[A] e^{-iS_{CS}[A]} = Z_{scalar}[A]. \tag{4.12}$$

Throughout this work we will assume that the process of promoting the background gauge field to a dynamical one and adding a BF coupling or a Chern-Simons term will not change the validity of the master duality, equation (4.7).

As a last discussion in this section we would like to talk about the time reversed version of these dualities because we will use them to derive the particle-vortex dualities. We know that the free theories are invariant. Under time reversal x_0 and A_i are odd, x_i and A_0 are even. With this in mind, it is easy to show that the Chern-Simons and the BF action are both odd under time reversal. With this we can find the time reversed version of the dualities. The duality (4.12) becomes

$$Z_{scalar}[A] = \bar{Z}_{fermion+flux}[A] e^{iS_{CS}[A]}, \tag{4.13}$$

where

$$\bar{Z}_{fermion+flux}[A] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}a \exp\ i\left(S_{fermion}[\psi, a] + \frac{1}{2}S_{CS}[a] + S_{BF}[a, A]\right). \quad (4.14)$$

And the duality (4.7) becomes

$$Z_{fermion}[A] = \bar{Z}_{scalar+flux}[A]e^{-\frac{i}{2}S_{CS}[A]}, \quad (4.15)$$

where

$$\bar{Z}_{scalar+flux}[A] = \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}a \exp i\left(S_{scalar}[\phi, a] - S_{CS}[a] - S_{BF}[a, A]\right). \quad (4.16)$$

4.2 Bosonic Particle-Vortex Duality

Now we obtain two particle-vortex dualities, one for bosons and one for fermions. The results will also serve as consistency check for our master duality, equation (4.7), as both particle-vortex dualities are present in the literature and the bosonic one is quite well known [26, 27].

We start with the duality (4.12). We promote the background gauge field A to a dynamical gauge field a, and introduce a new background gauge field A through BF coupling. The right hand side becomes the scalar QED partition function

$$Z_{scalar\ QED}[A] = \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}a \exp i\left(S_{scalar}[\phi, a] + S_{BF}[a, A]\right). \quad (4.17)$$

The left hand side reads

$$\begin{aligned}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}a\mathcal{D}\tilde{a} \exp\ i\left(S_{fermion}[\psi, \tilde{a}] - \frac{1}{2}S_{CS}[\tilde{a}] - S_{BF}[a; \tilde{a}] - S_{CS}[a] + S_{BF}[a, A]\right)\\ = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\tilde{a} \exp\ i\left(S_{fermion}[\psi, \tilde{a}] + \frac{1}{2}S_{CS}[\tilde{a}] - S_{BF}[\tilde{a}, A] + S_{CS}[A]\right).\end{aligned} \quad (4.18)$$

In the last step we have used the equation of motion for a, $a_\mu = A_\mu - \tilde{a}_\mu$, and integrated it out.

We recognize the first thee terms as the time reversed partition function $\bar{Z}_{fermion+flux}[-A]$, defined by the equation (4.14), thus we can replace the left hand side using the duality (4.13). The Chern-Simons actions cancel each other and we are left with

$$Z_{scalar}[-A] = Z_{scalar\ QED}[A]. \quad (4.19)$$

This is the original particle-vortex duality between the XY model and the Abelian Higgs model [26].

To see how the particle-vortex duality is manifested in the equation (4.19) we must understand what is a vortex state. Following [28, 29] we introduce the potential $V = \lambda(|\phi|^2 - \phi_0^2)^2$ we used in the last section. This induces a finite energy solution with a non-trivial boundary condition at spatial infinity, namely $\phi \to \phi_0 e^{in\theta}$ as $r \to \infty$, with ϕ_0 constant. To make $\phi(\theta = 0) = \phi(\theta = 2\pi)$ we must restrict n to integers.

Now we look at the energy for the scalar field

$$H = \int \mathrm{d}^2x \left(\pi\pi^* + (\vec{\nabla} - i\vec{a})\phi \cdot (\vec{\nabla} + i\vec{a})\phi^* + V(\phi) \right), \tag{4.20}$$

where $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* + ia_0\phi^*$ and $\pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} - ia_0\phi$. If we consider the static case and the absence of the gauge field, the spacial part becomes

$$\left|\vec{\nabla}\phi\right|^2 = \left|\frac{in}{r}\phi\, \hat{\theta}\right|^2 \tag{4.21}$$

and the energy is logarithmically divergent

$$H = n^2\phi_0^2 \int \frac{1}{r^2} r\, \mathrm{d}r\, \mathrm{d}\theta\,. \tag{4.22}$$

Thus we must choose a gauge field that eliminates this divergence. The simplest choice is

$$\vec{a} = \frac{n}{r}\,\hat{\theta} = \vec{\nabla}(n\theta), \quad \text{for} \quad r \to \infty. \tag{4.23}$$

With this $(\vec{\nabla} - i\vec{a})\phi = 0$ in the limit $r \to \infty$ and the energy of the vortex state is finite.

So what we learned is that the vortex configuration is not only composed by the scalar field, but also by a matching of the gauge field. Now we take the functional derivative on both sides of the duality (4.19) with respect to A_0. From the left hand side we obtain the current for $U(1)$ transformations. This is the current corresponding to charge conservation

$$j^0 = i\left(\phi^*\dot{\phi} - \phi\dot{\phi}^*\right) + 2A^0|\phi|^2. \tag{4.24}$$

On the right hand side we obtain the field strength for the monopole state

$$\frac{1}{2\pi} f_{12} = \frac{1}{2\pi} B. \tag{4.25}$$

In the end this duality can be viewed between the expected value of two objects of the theories, that is, between j^0 from the free theory and the magnetic field from the scalar QED.

We can also explicitly check the Dirac quantization condition for the gauge field that we choose, equation (4.23)

$$\frac{1}{2\pi}\int \mathrm{d}^2x\, B = \frac{1}{2\pi}\int \mathrm{d}\vec{r}\cdot A = \frac{n}{2\pi}\int \frac{1}{r} r\, \mathrm{d}\theta = n. \tag{4.26}$$

4.3 FERMIONIC PARTICLE-VORTEX DUALITY

The fermionic particle-vortex duality can be obtained through a similar process. We start rewriting the duality (4.7) as

$$Z_{fermion}[A] = Z_{scalar+flux}[A] e^{\frac{i}{2}S_{CS}[A]}. \tag{4.27}$$

Again we promote the background gauge A to a dynamical gauge field a and couple the partition functions to a new background gauge field A. The left hand side becomes the three dimensional QED partition function

$$Z_{QED}[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\mathcal{D}a \exp\ i\left(S_{fermion}[\psi, a] + S_{BF}[a, A]\right). \tag{4.28}$$

The right hand side reads

$$\begin{aligned} &\int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}a\mathcal{D}\tilde{a} \exp\ i\left(S_{scalar}[\phi, a] + S_{CS}[a] + S_{BF}[\tilde{a}, a] + \frac{1}{2}S_{BF}[\tilde{a}, A] + \frac{1}{2}S_{CS}[a]\right) \\ &\qquad = \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}a \exp\ i\left(S_{scalar}[\phi, a] - S_{CS}[a] - S_{BF}[a, A] - \frac{1}{2}S_{CS}[A]\right). \end{aligned} \tag{4.29}$$

On the last line we took the equation of motion for $\tilde{a}$, $\tilde{a}_\mu = -A_\mu - 2a_\mu$, and integrated out $\tilde{a}$.

We recognize the first three terms as the time reversed scalar attached to flux partition function, given by the equation (4.16). Now we use the duality (4.15) and obtain the new duality

$$Z_{fermion}[A] = Z_{QED}[A]. \tag{4.30}$$

Here the particle-vortex duality is manifested the same way as in the scalar case, that is between the gauge fields of the QED theory and the fermion field of the free theory. This duality was firstly proposed by Son in the study of the Quantum Hall Effect at filling fraction $\nu = \frac{1}{2}$ [10].

When we rewrote the duality (4.7) as the equation (4.27), we introduced the problem that both sides of the duality are not gauge invariant. The left hand side because we can perform the fermion path integral and obtain $\frac{i}{2}S_{CS}[A]$. The right hand side is not gauge invariant because of the same type of term. We can fix this if we choose the fields not to obey the standard Dirac quantization condition (4.8), but instead

$$\frac{1}{2\pi}\int_{S^2} \mathrm{d}A = 2\mathbb{Z}. \tag{4.31}$$

This makes sense for a background field, but we promoted A to a dynamical field. The way to fix this is to imagine that the Chern-Simons term with a fractional level is coming from

$$e^{\frac{i}{2}S_{CS}[a]} \to \int \mathcal{D}b \exp\ i\left(\gamma S_{CS}[b] + \alpha S_{BF}[b,a]\right) = e^{-i\frac{\alpha^2}{\gamma}S_{CS}[a]}. \tag{4.32}$$

Thus choosing α and γ suitably, $\gamma = -2\alpha^2$ in our case, we can produce a a term that is not gauge invariant from a gauge invariant partition function.

But even with this trick we are still left with another problem. To see what it is, let us take the equation of motion from the action (4.32)

$$\mathrm{d}b = -\frac{\alpha}{\gamma}\,\mathrm{d}a\,. \tag{4.33}$$

If we take the integral over both sides in the equation above it is easy to see that the Dirac quantization condition cannot be simultaneously satisfied on both sides of the equation. This is not a problem for us as we work with local physical quantities, like the Hall conductivity, thus we will not need to take a integral over the space.

5 MASSIVE DUALITIES

We already have some evidences on the importance of a mass term in the dualities from our discussion on the Hall conductivity, where we had to insert a mass term to check that the Hall conductivity is consistent between both sides of the duality. In this chapter we show precisely how to introduce a mass term in the dualities from the previous chapter and obtain a new method to derive the duality between the Thirring model and the Maxwell-Chern-Simons theory . We follow the method presented in [8].

5.1 MASSIVE BOSONIZATION DUALITIES

We start with the duality (4.7) with a symmetry breaking potential in the scalar partition function. The starting duality reads

$$Z_{fermion}[A]e^{-\frac{i}{2}S_{CS}[A]} = \int \mathcal{D}a \; e^{iS_{scalar}[a]+iS_{CS}[a]+iS_{BF}[a,A]}, \tag{5.1}$$

where

$$\begin{aligned} e^{iS_{scalar}[a]} &= \exp i \int \mathrm{d}^3x \left[|(\partial_\mu - ia_\mu)\,\phi|^2 - \frac{\lambda}{4}|\phi|^4 \right] \\ &= \int \mathcal{D}\sigma \; \exp i \int \mathrm{d}^3x \left[|(\partial_\mu - ia_\mu)\,\phi|^2 - \sigma|\phi|^2 + \frac{1}{\lambda}\sigma^2 \right] \end{aligned} \tag{5.2}$$

Using dimensional analysis we find that $[\lambda] = 1$. In the low-energy limit we must take $\lambda \to \infty$, whith this we loose the term σ^2 and this field can be seen as a Lagrange multiplier. If we think of σ a dynamical field we can integrate it out and return to the Wilson-Fischer, equation (5.2). On the other hand, if we think of σ a background field it then becomes a source for $|\phi|^2$.

By following [8] we will assume the following map between operators to be true

$$\bar{\psi}\psi \Longleftrightarrow -\sigma. \tag{5.3}$$

A similar version of this map is known to be true in the large-N limit [30, 31]. Following [8] we will assume this to be true outside the large-N limit. Naively, a real scalar field is considered to be even under both parity and time reversal. On the other hand, the map $\bar{\psi}\psi \Longleftrightarrow -\sigma$ implies that σ is odd, since $\bar{\psi}\psi$. However, as the boson theory is an interacting one, involving Chern-Simons, it is not easy to attribute quantum numbers to σ.

This motivates us to write the massive version of the duality (5.1) as

$$Z_{fermion}[A,m]e^{-\frac{i}{2}S_{CS[A]}} = \int \mathcal{D}a\ Z_{scalar}[a,m]e^{iS_{CS}[a]+iS_{BF}[a,A]}$$
$$Z_{fermion}[A,m]e^{-\frac{i}{2}S_{CS[A]}} = Z_{scalar+flux}[A,m], \tag{5.4}$$

where

$$Z_{fermion}[A,m] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp i \int \mathrm{d}^3x \left[\bar{\psi}(i\partial\!\!\!/ + A\!\!\!/)\psi - m\bar{\psi}\psi\right] \tag{5.5}$$

and

$$\begin{aligned} Z_{scalar}[a,m] &= \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}\sigma\ \exp i \int \mathrm{d}^3x \left[|(\partial_\mu - ia_\mu)\,\phi|^2 - \sigma(|\phi|^2 - m) + \frac{1}{\lambda}\sigma^2\right] \\ &= \int \mathcal{D}\phi\mathcal{D}\phi^* \exp i \int \mathrm{d}^3x \left[|(\partial_\mu - ia_\mu)\,\phi|^2 - \frac{\lambda}{4}\left(|\phi|^2 - m\right)^2\right] \\ &= \int \mathcal{D}\phi\mathcal{D}\phi^* \exp i \int \mathrm{d}^3x \left[|(\partial_\mu - ia_\mu)\,\phi|^2 - \frac{\lambda}{4}|\phi|^4 + \frac{m\lambda}{2}|\phi|^2\right]. \end{aligned} \tag{5.6}$$

Note that the sign of the mass term is inverted in the two dualities. This may seem odd, but it is this difference of sign that ensures the matching Hall conductivity between both sides of the duality. The way to calculate σ_{xy} here is quite similar to what we did in Chapter 4. In the fermion side σ_{xy} is zero or $\frac{1}{2\pi}$ for positive and negative mass respectively. In the bosonic side the Higgs mechanism happens for positive m leaving us with $\sigma_{xy} = 0$ and for negative mass we obtain the Hall conductivity as usual $\sigma_{xy} = \frac{1}{2\pi}$.

Now it is easy to see that the partition function (5.6) describes a massive scalar with mass $\frac{m\lambda}{2}$. The duality (5.4), that we just obtained, is the massive version of the duality (4.7). We want to use this duality to obtain the massive version of (4.12). As we did before, we promote the background field A to a dynamical field $\tilde{a}$ and couple it to a new background field. After this operation the duality reads

$$\int \mathcal{D}a\ Z_{fermion}[a,m]e^{-\frac{i}{2}S_{CS}[a]-iS_{BF}[a,A]} = \int \mathcal{D}\tilde{a}\mathcal{D}a\ Z_{scalar}[\tilde{a},m]e^{iS_{CS}[\tilde{a}]+iS_{BF}[\tilde{a},a]-iS_{BF}[a,A]}$$
$$Z_{fermion+flux}[A,m] = Z_{scalar}[A,m]e^{iS_{CS}[A]}, \tag{5.7}$$

where the fermion attached to flux partition function is the left hand side of the equation above.

As a comment we promote m to a dynamical field μ in the equation (5.4) and couple it to a new background field m. We obtain

$$\begin{aligned} \int \mathcal{D}\mu\ Z_{fermion}[A,\mu]e^{-\frac{i}{2}S_{CS}[A]-i\mu m} &= \int \mathcal{D}a\mathcal{D}\sigma\ \mathcal{D}\mu Z_{\text{scalar}}\,[a,\sigma]e^{i(S_{CS}[a]+S_{BF}[a;A]+\mu(\sigma-m))} \\ &= \int \mathcal{D}a\ Z_{\text{scalar}}\,[a,m]e^{i(S_{CS}[a]+S_{BF}[a;A])}. \end{aligned} \tag{5.8}$$

The left hand side can be viewed as the infrared limit of a Yukawa-type theory. For a large number of flavors it is known to describe a Gross-Neveu fixed-point [23]. However for a small number of flavors it is unknown if there is a fixed point. This hints that the map (5.3) may imply a second bosonization duality, in which the Gross-Neveu fermion is related to the free scalar with flux.

We see that both massive dualities, equation (5.7) and equation (5.4), are very similar to their non-massive versions, equations (4.7) and (4.12). As we did before we would like to obtain the time reversed versions of the massive dualities. This will allow us to obtain the massive particle-vortex dualities. We follow our previous discussion on time reversal, and remember that the massive fermionic term acquires a sign under time reversal.

Under time reversal, the duality (5.7) becomes

$$\bar{Z}_{fermion+flux}[A,m] = Z_{scalar}[A,m]e^{-iS_{CS}[A]}, \tag{5.9}$$

where

$$\bar{Z}_{fermion+flux}[A,m] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}a\ Z_{fermion}[a,-m]\exp i\left[\frac{1}{2}S_{CS}[a] + S_{BF}[a,A]\right]. \tag{5.10}$$

Under time reversal, the duality (5.4) becomes

$$Z_{fermion}[A,-m]e^{\frac{i}{2}S_{CS}[A]} = \bar{Z}_{scalar+flux}[A,m], \tag{5.11}$$

such that

$$\bar{Z}_{scalar+flux}[A,m] = \int \mathcal{D}a\ Z_{scalar}[a,m]e^{-iS_{CS}[a]-iS_{BF}[a,A]}. \tag{5.12}$$

5.2 Massive Particle-Vortex Dualities

Here we wish to examine the calculations from Chapter 4 in the massive case to obtain a massive version for the particle-vortex dualities. As we did before, we will manipulate one of the massive bosonization dualities and use a time reversed duality. We start by rewriting the equation (5.4) as

$$Z_{fermion}[A,m] = Z_{scalar+flux}[A,m]e^{\frac{i}{2}S_{CS}[A]}. \tag{5.13}$$

We promote the background gauge field A to a dynamical gauge field and couple it to a new background gauge field. The left hand side reads

$$Z_{QED}[A,m] \equiv \int \mathcal{D}a\ Z_{fermion}[a,m]e^{\frac{i}{2}S_{BF}[a,A]}. \tag{5.14}$$

The right hand side reads

$$\begin{aligned}
&\int \mathcal{D}\tilde{a}\mathcal{D}a\ Z_{scalar}[a,m] \exp i \left[\frac{1}{2}S_{CS}[\tilde{a}] + S_{CS}[a] + S_{BF}[a,\tilde{a}] + \frac{1}{2}S_{BF}[\tilde{a},A]\right] \\
&= \int \mathcal{D}a\ Z_{scalar}[a,m] \exp i \left[-\frac{1}{2}S_{CS}[A] - S_{CS}[a] - S_{BF}[a,A]\right] \\
&= Z_{fermion}[A,-m]. \qquad (5.15)
\end{aligned}$$

Here we integrated out the field $\tilde{a}$ using its equation of motion $\tilde{a}_\mu = -A_\mu - 2a_\mu$. In the last step we have used the time reversed duality (5.11). With this we have the massive fermion particle-vortex duality

$$Z_{QED}[A,m] = Z_{fermion}[A,-m]. \qquad (5.16)$$

Now we proceed similarly, but starting with the duality (5.7). As we did before we promote the background gauge field to a dynamical one and couple it to a new background field, the right hand side reads

$$Z_{scalar\ QED}[A,m] \equiv \int \mathcal{D}a\ Z_{scalar}[a,m] e^{iS_{BF}[a,A]}. \qquad (5.17)$$

And the right hand side reads

$$\begin{aligned}
&\int \mathcal{D}\tilde{a}\mathcal{D}a\ Z_{fermion}[a,m] \exp i \left[-\frac{1}{2}S_{CS}[a] - S_{BF}[a,\tilde{a}] - S_{CS}[\tilde{a}] + S_{BF}[\tilde{a},A]\right] \\
&= \int \mathcal{D}a Z_{fermion}[a,m] \exp i \left[\frac{1}{2}S_{CS}[a] - S_{BF}[a,A] + S_{CS}[A]\right] \\
&= Z_{scalar}[-A,-m]. \qquad (5.18)
\end{aligned}$$

In the second line we used the equation of motion $\tilde{a}_\mu = A_\mu - a_\mu$ and integrated out the field $\tilde{a}$. And in the last line we used the duality (5.11). Finally we write the massive fermion particle-vortex duality as

$$Z_{scalar}[-A,-m] = Z_{scalar\ QED}[A,m]. \qquad (5.19)$$

5.3 Thirring Duality from Particle-Vortex Duality

One could ask if it is possible to derive the duality between the Thirring model and the Maxwell-Chern-Simons theory through the same method we used to derive the other dualities. To do this we start with the duality (5.16) with the sign of the mass changed in both sides. We choose this one because we can easily make one side become the Thirring partition function and integrate the fermions to generate a Chern-Simons term in the other side. To generate the Thirring partition function we add a term $\frac{1}{2g^2}A_\mu A^\mu$ and integrate over A. Now the right hand side reads

$$Z_{Th} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}a \exp i \int \mathrm{d}^3x \left[\bar{\psi}(i\partial\!\!\!/ + m + a\!\!\!/)\psi + \frac{1}{2g^2}a_\mu a^\mu\right]. \tag{5.20}$$

The left hand side reads

$$\int \mathcal{D}a\mathcal{D}\tilde{a}\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp i \int \mathrm{d}^3x \left[\bar{\psi}\left(i\partial\!\!\!/ + a\!\!\!/ - m\right)\psi + \frac{1}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu\tilde{a}_\rho + \frac{1}{2g^2}\tilde{a}_\mu a^\mu\right]. \tag{5.21}$$

Integrating out the fermionic fields we obtain the same results we found in the Appendix A

$$\int \mathcal{D}a\mathcal{D}\tilde{a} \exp i \int \mathrm{d}^3x \left[\pm\frac{1}{8\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \frac{1}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu\tilde{a}_\rho + \frac{1}{2g^2}\tilde{a}_\mu\tilde{a}^\mu\right]. \tag{5.22}$$

We make the field rescaling $a \to -\frac{4\pi}{g}a$ and $\tilde{a} \to g\tilde{a}$ and are left with a partition function very similar to the interpolating partition function (3.17)

$$\int \mathcal{D}a\mathcal{D}\tilde{a} \exp i \int \mathrm{d}^3x \left[\pm\frac{2\pi}{g^2}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma - \epsilon^{\mu\nu\sigma}a_\mu\partial_\nu\tilde{a}_\sigma + \frac{1}{2}\tilde{a}_\mu\tilde{a}^\mu\right]. \tag{5.23}$$

Integrating over $\tilde{a}_\mu$, we get

$$Z_{Th} = \int \mathcal{D}a \exp i \int \mathrm{d}^3x \left[\pm\frac{2\pi}{g^2}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma - \frac{1}{4}f_{\mu\nu}f^{\mu\nu}\right] = Z_{MCS}. \tag{5.24}$$

This is the same duality found in the equation (3.20), except that here we have sign(m) with the Chern-Simons term and in the previous chapter we obtained $-$sign(m). This result is of great importance as it gives further support to the master duality, equation (4.7).

It is important to note that when we integrated out the fermion fields we left out terms of order $\mathcal{O}(\frac{1}{m})$. This is compatible with our discussion on the Thirring model effective action because there also we left out terms of order $\mathcal{O}(\frac{1}{m})$ to obtain the Self-Dual action, equation (3.18). As we expected both methods we used to obtain the Thirring duality are only valid in the low-energy limit.

6 FINAL REMARKS

In this work we have studied aspects of dualities in $(2+1)$ dimensional quantum field theory. As we saw, this dimensionality is special because the quantum numbers, spin and statistics, are not so rigid. In fact, there is no limitation to the statistics. The underlying mechanism is the conversion of quantum numbers like spin and statistics. This can be incorporated in a local quantum field theory using a Chern-Simons term, that can be easily incorporated only in a $(2+1)$ dimensional field theory.

After discussing the basic elements on the importance of the Chern-Simons term to the transmutation of spin-statistics, we analyzed dualities in specific models. We started with the one between the Thirring model and the Maxwell-Chern-Simons theory. The existence of this duality and our knowledge on the effects of the Chern-Simons term on the transmutation of spin-statistics motivated us to study a master bosonization duality, which passes several consistency checks, like the Hall conductivity and the attachment of flux. From this one we were able to derive another bosonization duality and two more particle-vortex dualities.

To calculate the Hall conductivity as a consistency check to our master duality we had to insert a mass term in our dualities. This motivates us to study the massive versions the two bosonization and particle-vortex dualities. Using these results we extended our discussion to show another way to obtain the duality between the Thirring model and the Maxwell-Chern-Simons theory.

Another method to obtain bosonization dualities, that we did not discuss here, is to work in a lattice. This method has had great success and some of the dualities we discussed in this work can also be obtained from this approach [26, 32, 33]. A newer approach to $(2+1)$ bosonization is to define a series of theories in $(1+1)$ dimensions and add an interaction between the theories. This is called the quantum wires approach [34]. The process we used to derive the dualities in the Chapter 4 could be extended using more external fields and the addition of Chern-Simons terms with different levels. This leads to a series of new dualities that we did not discuss here [9, 8].

Further studies could be focused in finding a way to derive the massive dualities from the non-massive ones and check whether the dynamically generated mass is compatible in the two theories. This kind of effect is known to happen through quantum corrections in the $(3+1)$ dimensional ϕ^4 theory as a consequence of the Coleman-Weinber mechanism [35]. On the fermion side the same is known to happen with the three dimensional Thirring model [23].

A APPENDIX A: THIRRING EFFECTIVE ACTION

In this appendix we want to perform the fermionic path integral in the partition function (3.9) and obtain the effective action (3.13). We eliminate the quartic Thirring interaction using a new auxiliary gauge field. We substitute

$$\exp\left[-i\frac{g^2}{2}j_\mu j^\mu\right] = \int \mathcal{D}a \exp\ i\int \mathrm{d}^3x \left[\frac{1}{2}a_\mu a^\mu + g a_\mu j^\mu\right], \tag{A.1}$$

in the partition function (3.9), and obtain

$$\begin{aligned} Z_{Th} &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}a \exp\left[i\int \mathrm{d}^3x\, \bar{\psi}\left(i\partial\!\!\!/ + m + g a\!\!\!/\right)\psi + \frac{1}{2}a_\mu a^\mu\right] \\ &= \int \mathcal{D}a \exp\left[\mathrm{tr}\ln\!\left(i\partial\!\!\!/ + m + g a\!\!\!/\right) + i\int \mathrm{d}^3x\, \frac{1}{2}a_\mu a^\mu\right]. \end{aligned} \tag{A.2}$$

In the last step we performed the fermionic path integral.

Now we focus our attention on the most relevant terms

$$\mathrm{tr}\ln\!\left(i\partial\!\!\!/ + m + g a\!\!\!/\right) = \mathrm{tr}\log\left(i\partial\!\!\!/ + m\right) + g\,\mathrm{tr}\left(\frac{1}{i\partial\!\!\!/ + m}a\!\!\!/\right) + \frac{g^2}{2}\,\mathrm{tr}\left(\frac{1}{i\partial\!\!\!/ + m}a\!\!\!/\frac{1}{i\partial\!\!\!/ + m}a\!\!\!/\right) + \cdots. \tag{A.3}$$

The first term is a c-number, thus it will not contribute to the effective action, so we will just ignore it. The second term is the tadpole diagram on the left hand side of FIG. [3.1], as it has no dependence on the external momentum, its value is zero. But this result can be explicitly demonstrated quite easily using that the trace of γ^μ. The third term will produce the effective action that we are looking for

$$\begin{aligned} MCS &= \frac{1}{2}\frac{g^2}{(2\pi)^6}\,\mathrm{tr}\int \mathrm{d}^3p\mathrm{d}^3q\mathrm{d}^3x\mathrm{d}^3y\ a_\mu(x)a_\nu(y)\left[\left(\frac{p\!\!\!/ - m}{p^2 - m^2}\right)\gamma^\mu\left(\frac{q\!\!\!/ - m}{q^2 - m^2}\right)\gamma^\nu\right] e^{ix(p-q)}e^{iy(q-p)}. \\ &= \frac{1}{2}\frac{g^2}{(2\pi)^6}\int \mathrm{d}^3x\mathrm{d}^3y\ a_\mu(x)a_\nu(y)\int \mathrm{d}^3k e^{ik(x-y)}\int \mathrm{d}^3q\,\mathrm{tr}\left[\left(\frac{k\!\!\!/ + q\!\!\!/ - m}{(k+q)^2 - m^2}\right)\gamma^\mu\left(\frac{q\!\!\!/ - m}{q^2 - m^2}\right)\gamma^\nu\right] \\ &= \frac{1}{2}\frac{g^2}{(2\pi)^6}\int \mathrm{d}^3x\mathrm{d}^3y a_\mu(x)a_\nu(y)\int \mathrm{d}^3k e^{ik(x-y)}\int \mathrm{d}^3q\frac{1}{(k+q)^2 - m^2}\frac{1}{q^2 - m^2} \\ &\times\left[2(k^\mu q^\nu - k^\sigma q_\sigma \eta^{\mu\nu} + k^\nu q^\mu + 2q^\mu q^\nu - q^\sigma q_\sigma \eta^{\mu\nu}) + 2imk_\sigma\epsilon^{\sigma\mu\nu} + 2m^2\eta^{\mu\nu}\right]. \end{aligned} \tag{A.4}$$

where we defined $k \equiv p - q$.

In order to obtain the Chern-Simons action we look at the $k_\sigma\epsilon^{\sigma\mu\nu}$ term in the low-energy limit, $k \approx 0$

$$\begin{aligned}
CS &= \frac{1}{2}\frac{g^2}{(2\pi)^6}\int \mathrm{d}^3x\mathrm{d}^3y\; a_\mu(x)a_\nu(y)\int\; \mathrm{d}^3k e^{ik(x-y)}2imk_\sigma\epsilon^{\sigma\mu\nu}\int \mathrm{d}^3q\left(\frac{1}{q^2-m^2}\right)^2\\
&= \frac{-g^2m}{64\pi^4|m|}\int \mathrm{d}^3x\mathrm{d}^3y\; a_\mu(x)a_\nu(y)\int\; \mathrm{d}^3k e^{ik(x-y)}k_\sigma\epsilon^{\sigma\mu\nu}\\
&= -i\mathrm{sign}(m)\frac{g^2}{8\pi}\int \mathrm{d}^3x\;\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\rho. \qquad \text{(A.5)}
\end{aligned}$$

Now we wish to obtain the Maxwell term from the equation (A.4). As the Maxwell term has two derivatives we expand the first fraction up to order $k_\alpha k_\beta$ and collect the appropriate terms.

$$\begin{aligned}
Maxwell &= \frac{1}{2}\frac{g^2}{(2\pi)^6}\int \mathrm{d}^3x\,\mathrm{d}^3y\;\; a_\mu(x)a_\nu(y)\int \mathrm{d}^3k\,k e^{ik(x-y)}\int \mathrm{d}^3q\left(-4k_\alpha q^\alpha\left[\frac{k^\mu q^\nu - k_\sigma q^\sigma\eta^{\mu\nu}+k^\nu q^\mu}{(q^2-m^2)^3}\right]\right.\\
&+ 2k_\alpha k_\beta\left[\frac{2\eta^{\alpha\beta}q^\mu q^\nu}{(q^2-m^2)^3} - 8\frac{q^\mu q^\nu q^\alpha q^\beta}{(q^2-m^2)^4} - \frac{q^2\eta^{\alpha\beta}\eta^{\mu\nu}}{(q^2-m^2)^3} + 4\frac{q^2\eta^{\mu\nu}q^\alpha q^\beta}{(q^2-m^2)^4}\right]\\
&\left.-2m^2\eta^{\mu\nu}k_\alpha k_\beta\left[\frac{\eta^{\alpha\beta}}{(q^2-m^2)^3} - 4\frac{q^\alpha q^\beta}{(q^2-m^2)^4}\right]\right). \qquad \text{(A.6)}
\end{aligned}$$

Due to the size of the equation above we will solve the q integral line by line

$$\begin{aligned}
I \equiv &-4k_\alpha\int \mathrm{d}^3q\,q^\alpha\left[\frac{k^\mu q^\nu - k_\sigma q^\sigma\eta^{\mu\nu}+k^\nu q^\mu}{(q^2-m^2)^3}\right]\\
= &-\frac{4}{3}\left[2k^\mu k^\nu - k^2\eta^{\mu\nu}\right]\int \mathrm{d}^3q\,\frac{q^2}{(q^2-m^2)^3} = \frac{i\pi^2}{m}\left(k^2\eta^{\mu\nu}-k^\mu k^\nu\right); \qquad \text{(A.7)}
\end{aligned}$$

$$\begin{aligned}
II \equiv &\int \mathrm{d}^3q\,2k_\alpha k_\beta\left[\frac{2\eta^{\alpha\beta}q^\mu q^\nu}{(q^2-m^2)^3} - 8\frac{q^\mu q^\nu q^\alpha q^\beta}{(q^2-m^2)^4} - \frac{q^2\eta^{\alpha\beta}\eta^{\mu\nu}}{(q^2-m^2)^3} + 4\frac{q^2\eta^{\mu\nu}q^\alpha q^\beta}{(q^2-m^2)^4}\right]\\
=&\frac{2}{3}k^2\eta^{\mu\nu}\int \mathrm{d}^3q\,\frac{q^2}{(q^2-m^2)^3} + \frac{8}{5}\left[\frac{4}{3}k^\mu k^\nu - k^2\eta^{\mu\nu}\right]\int \mathrm{d}^3q\,\frac{q^4}{(q^2-m^2)^4}\\
=&\frac{i\pi^2}{6m}\left[8k^\mu k^\nu - 3k^2\eta^{\mu\nu}\right]; \qquad \text{(A.8)}
\end{aligned}$$

$$\begin{aligned}
III \equiv &-2m^2\eta^{\mu\nu}k_\alpha k_\beta\int \mathrm{d}^3q\left[\frac{\eta^{\alpha\beta}}{(q^2-m^2)^3} - 4\frac{q^\alpha q^\beta}{(q^2-m^2)^4}\right]\\
=&\frac{i\pi^2}{6m}k^2\eta^{\mu\nu}. \qquad \text{(A.9)}
\end{aligned}$$

To simplify the calculations we used that the q integrals must be rotational invariant, in other words, under the integration sign $q^\mu q^\nu = \frac{1}{3}q^2\eta^{\mu\nu}$, $q^\mu q^\nu q^\sigma = 0$ and $q^\mu q^\nu q^\sigma q^\rho = \frac{1}{15}q^4(\eta^{\mu\nu}\eta^{\sigma\rho}+\eta^{\mu\sigma}\eta^{\nu\rho}+\eta^{\mu\rho}\eta^{\nu\sigma})$. Substituting the results (A.7), (A.8) and (A.9) in equation (A.6), we obtain the Maxwell term

$$\begin{aligned} Maxwell =& \frac{ig^2}{96m\pi^4}\int \mathrm{d}^3x\mathrm{d}^3y\; a_\mu(x)a_\nu(y)\int \mathrm{d}^3k e^{ik(x-y)}\left[k^2\eta^{\mu\nu}-k^\mu k^\nu\right] \\ =& \frac{ig^2}{12m\pi}\int \mathrm{d}^3x\left(\partial_\mu a_\nu\partial^\mu a^\nu-\partial_\mu a^\mu\partial_\nu a^\nu\right) \\ =& \frac{ig^2}{24m\pi}\int \mathrm{d}^3x\, f_{\mu\nu}f^{\mu\nu}, \end{aligned} \tag{A.10}$$

where $f_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu$ is the field strength.

The calculation of the Maxwell and the Chern-Simons terms can be summarized by the Feynman diagram on right hand side of FIG. [3.1]. If we wished, we could proceed with our calculations and obtain terms with higher derivatives in the gauge field. To do this we should have considered higher orders when we expanded $\frac{1}{(k+q)^2-m^2}$. We also could have considered terms with higher order in the gauge field in the series (A.3). One of the possible terms is $\epsilon^{\mu\nu\rho}a_\mu a_\nu a_\rho$, this term is identically zero in our case, but in non-abelian theories this term must be taken into account.

Putting all these results together, the Thirring partition function reads

$$Z_{Th}[a]=\int \mathcal{D}a\exp i\int \mathrm{d}^3x\left[\frac{1}{2}a_\mu a^\mu-\mathrm{sign}(m)\frac{g^2}{8\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho+\frac{g^2}{24m\pi}f_{\mu\nu}f^{\mu\nu}\right]. \tag{A.11}$$

We can extract some useful information calculating the propagator of the Thirring effective action, but first we rewrite it as

$$S_{eff}=\int \mathrm{d}^3x\,\frac{1}{2}a_\mu a^\mu+\frac{\alpha}{2}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma+\frac{\beta}{4}f_{\mu\nu}f^{\mu\nu}. \tag{A.12}$$

We removed the constants that appear in the partition function in favour of $\alpha=-\dfrac{\mathrm{sign}(m)}{4\pi}$ and $\beta=\frac{g^2}{6\pi m}$ for brevity. We take the equation of motion from the effective action

$$a^\mu+\alpha\epsilon^{\mu\nu\sigma}\partial_\nu a_\sigma+\beta\partial_\nu F^{\nu\mu}=0, \tag{A.13}$$

that can be rewritten as

$$\left[\left(\beta\Box+1\right)\eta_{\mu\nu}+\alpha\epsilon_{\mu\sigma\nu}\partial^\sigma-\beta\partial_\mu\partial_\nu\right]a^\nu=0. \tag{A.14}$$

The propagator is the Green function of the equation of motion

$$S_{\mu\nu}(x)G^{\nu\rho}(x-y)=\eta^\rho_\mu\delta^{(3)}(x-y), \tag{A.15}$$

where

$$S_{\mu\nu}(x) = (\beta\Box + 1)\,\eta_{\mu\nu} + \alpha\epsilon_{\mu\sigma\nu}\partial^{\sigma} - \beta\partial_{\mu}\partial_{\nu} \tag{A.16}$$

We can perform a Fourier transformation of the equation above and obtain its counterpart in the momentum space

$$S_{\mu\nu}(k)G^{\nu\rho}(k) = \eta_{\mu}^{\rho}, \tag{A.17}$$

where the equation of motion operator in the momentum space reads

$$S_{\mu\nu}(k) = \left[\beta k_{\mu}k_{\nu} - (\beta k^2 - 1)\eta_{\mu\nu} + i\alpha\epsilon_{\mu\sigma\nu}k^{\sigma}\right]. \tag{A.18}$$

Here we traded the differential equation for a simpler one. The catch is that to obtain $G^{\nu\rho}(x-y)$ we will need to calculate the Fourier transformation of $G^{\nu\rho}(k)$. As we are interested in the poles of the propagator we will not need to do a Fourier transformation.

Looking at the terms that appear in $S_{\mu\nu}(k)$ we decompose the propagator all in the possible second order tensors

$$G^{\nu\rho}(k) = A\eta^{\nu\rho} + B\epsilon^{\nu\lambda\rho}k_{\lambda} + Ck^{\nu}k^{\rho}. \tag{A.19}$$

Where A, B and C are functions of k^2. We propose this form imitating the kinds of therms that appear in the equation of motion. Imposing the equation (A.17) we are able to find the constants $A,\ B$ and C that make the propagator we proposed be the correct one.

$$\begin{aligned}\eta_{\mu}^{\rho} &= A\beta k_{\mu}k^{\rho} + \beta Ck^2k_{\mu}k^{\rho} - A(\beta k^2 - 1)\eta_{\mu}^{\rho} - B(\beta k^2 - 1)\epsilon_{\mu}{}^{\sigma\rho}k_{\sigma} \\ &- C(\beta k^2 - 1)k_{\mu}k^{\rho} + i\alpha A\epsilon_{\mu}{}^{\sigma\rho}k_{\sigma} + i\alpha B(k^{\rho}k_{\mu} - k^2\eta_{\mu}^{\rho}) \\ &= \big(A\beta + \beta Ck^2 + i\alpha B - C(\beta k^2 - 1)\big)\,k_{\mu}k^{\rho} \\ &+ \big(i\alpha A - B(\beta k^2 - 1)\big)\,\epsilon_{\mu}{}^{\sigma\rho}k_{\rho} - \big(i\alpha Bk^2 + A(\beta k^2 - 1)\big)\,\eta_{\mu}^{\rho}.\end{aligned} \tag{A.20}$$

Comparing the left and right hand side of the equation above the following system of equations must be satisfied

$$\begin{aligned} -\big(i\alpha Bk^2 + A(\beta k^2 - 1)\big) &= 1, \\ \big(i\alpha A - B(\beta k^2 - 1)\big) &= 0, \\ \big(A\beta + \beta Ck^2 + i\alpha B - C(\beta k^2 - 1)\big) &= 0. \end{aligned} \tag{A.21}$$

One can easily solve this system and find

$$
\begin{aligned}
A &= \frac{1-\beta k^2}{\beta^2 k^4 - k^2\left(\alpha^2+2\beta\right)+1}, \\
B &= \frac{-i\alpha}{\beta^2 k^4 - k^2\left(\alpha^2+2\beta\right)+1}, \\
C &= \frac{\beta\left(\beta k^2-1\right)-\alpha^2}{\beta^2 k^4 - k^2\left(\alpha^2+2\beta\right)+1}.
\end{aligned} \tag{A.22}
$$

Substituting $A,\ B$ and C in the equation (A.19) we write a closed form for the propagator

$$
\begin{aligned}
G^{\nu\rho}(k) = -\frac{1}{9g^4k^2m^2 - 4\left(g^2k^2-6\pi m\right)^2}\Big[24\pi m\left(6\pi m - g^2k^2\right)\eta^{\nu\rho} + 36 i\pi g^2 m\,|m|\,\epsilon^{\nu\lambda\rho}k_\lambda \\
+\left(g^4\left(4k^2-9m^2\right)-24\pi g^2 m\right)k^\nu k^\rho\Big].
\end{aligned} \tag{A.23}
$$

With this we find the poles

$$
\begin{aligned}
k_\pm^2 &= \frac{\alpha^2+2\beta\pm\sqrt{\alpha^4+4\alpha^2\beta}}{2\beta^2} \\
k_\pm^2 &= \frac{9g^4m^2+48\pi g^2 m \pm 3m\sqrt{9g^8m^2+96\pi g^6 m}}{8g^4}.
\end{aligned} \tag{A.24}
$$

If we plot $k_\pm^2$ versus g^2 for $m=1$, see FIG. [A.1], we see that there is a value g^2 for which $4m^2 \geq k_\pm^2$. The intersections between k_+^2 and k_-^2 with the $4m^2$ line happen at $g^2 = \frac{6\pi}{m}$ and $g^2 = \frac{6\pi}{7m}$ respectively.

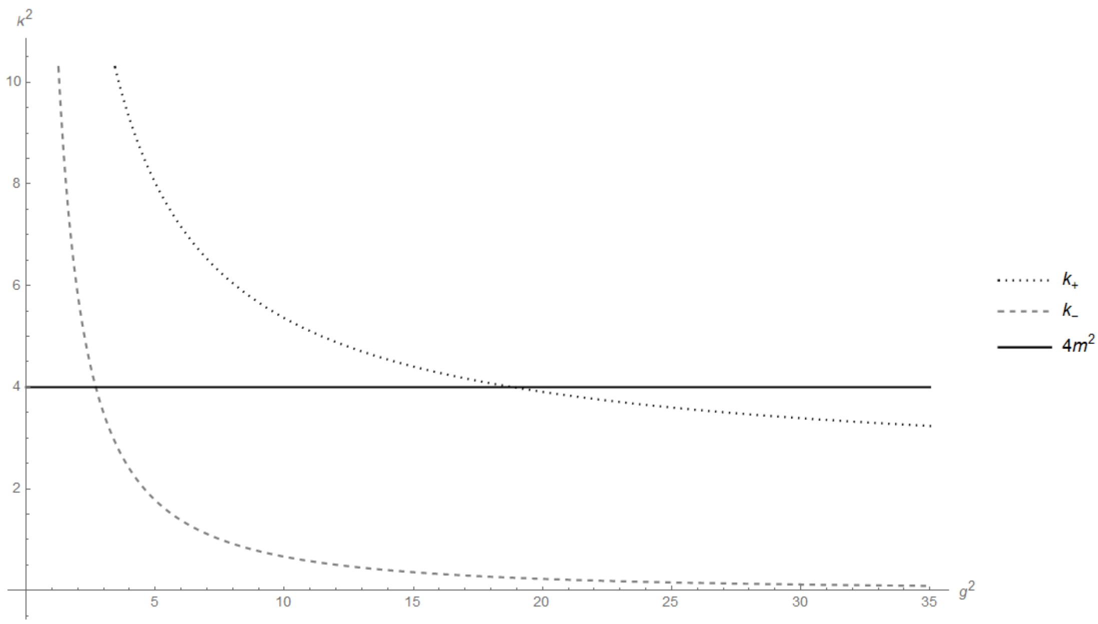

Figura A.1: Plot for $k_\pm^2$ versus g^2 for $m=1$

B APPENDIX B: INTERPOLATING PARTITION FUNCTION PATH INTEGRAL

Here we want to perform the path integral and find an effective action for a

$$
\begin{aligned}
\bar{Z}_I[a] &\equiv \int \mathcal{D}\tilde{a}\exp i\int \mathrm{d}^3x\left[\mp\frac{2\pi}{g^2}\epsilon^{\mu\nu\rho}\tilde{a}_\mu\partial_\nu\tilde{a}_\rho - \epsilon^{\mu\nu\rho}a_\mu\partial_\nu\tilde{a}_\rho\right] \\
&= \int \mathcal{D}\tilde{a}\exp i\int \mathrm{d}^3x\left[\mp\frac{1}{2}\epsilon^{\mu\nu\rho}\tilde{a}_\mu\partial_\nu\tilde{a}_\rho - \frac{g}{\sqrt{4\pi}}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu\tilde{a}_\rho\right] \\
&= \int \mathcal{D}\tilde{a}\exp i\int \mathrm{d}^3x\left[\frac{1}{2}\tilde{a}_\mu\tilde{S}^{\mu\nu}\tilde{a}_\nu - J^\mu\tilde{a}_\mu\right], \\
&= \exp\left(-\frac{i}{2}\int \mathrm{d}^3x\,\mathrm{d}^3y\,J^\mu(x)S^{-1}_{\mu\nu}(x,y)J^\nu(y)\right)
\end{aligned}
\tag{B.1}
$$

where $\tilde{S}^{\mu\nu} \equiv \mp\epsilon^{\mu\rho\nu}\partial_\rho$ and $J^\mu \equiv \frac{g}{\sqrt{4\pi}}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho$ and to get to the second line we made the shift $\tilde{a}_\mu \to \tilde{a}'_\mu = \frac{g}{\sqrt{4\pi}}\tilde{a}_\mu$. Now the problem resides on finding $\tilde{S}^{-1}_{\mu\nu}(x,y)$. Since $\tilde{S}_{\mu\nu}$ is not invertible we shall introduce a new regulated operator that returns to the old operator in the limit $\Lambda \to \infty$

$$
S^{\mu\nu} \equiv \tilde{S}^{\mu\nu} + \frac{1}{\Lambda}\partial^\mu\partial^\nu. \tag{B.2}
$$

The operator $S^{-1}_{\nu\sigma}$ obeys

$$
S^{\mu\nu}(x)S^{-1}_{\nu\sigma}(x-y) = \eta^\mu_\sigma\delta^{(3)}(x-y), \tag{B.3}
$$

or in the momentum space

$$
S^{\mu\nu}(k)S^{-1}_{\nu\sigma}(k) = \eta^\mu_\sigma, \tag{B.4}
$$

where $S^{\mu\nu}(k)$ can be obtained by performing a Fourier transformation on $S^{\mu\nu}(x)$

$$
S^{\mu\nu}(k) = \mp i\epsilon^{\mu\rho\nu}k_\rho - \frac{1}{\Lambda}k^\mu k^\nu. \tag{B.5}
$$

We propose the inverse operator

$$
S^{-1}_{\nu\sigma} = \alpha k_\nu k_\sigma + \beta\eta_{\nu\sigma} + \gamma\epsilon_{\nu\rho\sigma}k^\rho, \tag{B.6}
$$

where α, β and γ are functions of k^2. Imposing (B.4) we find

$$
S^{-1}_{\nu\sigma}(k) = \mp\frac{i}{k^2}\epsilon_{\nu\rho\sigma}k^\rho - \frac{\Lambda}{(k^2)^2}k_\nu k_\sigma, \tag{B.7}
$$

or in the coordinate space

$$S^{-1}_{\nu\sigma}(x-y) = \pm\frac{1}{4\pi}\epsilon_{\nu\rho\sigma}\partial^{\rho}\frac{1}{|x-y|} - \frac{\Lambda}{8\pi}\partial_{\nu}\partial_{\sigma}|x-y|. \tag{B.8}$$

With this result we can easily obtain

$$\bar{Z}_I[a] = \exp\left(i\int \mathrm{d}^3x \mp \frac{g^2}{8\pi}\epsilon^{\mu\nu\rho}a_{\mu}\partial_{\nu}a_{\rho}\right). \tag{B.9}$$

The term $\frac{1}{\Lambda}\partial^{\mu}\partial^{\nu}$ that we added in the operator (B.2) can be interpreted as originating from a gauge breaking term in the action (3.17) of the form $\dfrac{1}{2\Lambda}(\partial_{\mu}\tilde{a}^{\mu})^2$. We notice that the final partition function has no dependence in Λ, so the limit $\Lambda \to \infty$ is trivial.

REFERENCES

[1] S. Mandelstam, "Soliton operators for the quantized sine-gordon equation," *Phys. Rev. D*, vol. 11, pp. 3026–3030, May 1975.

[2] S. Coleman, "Quantum sine-gordon equation as the massive thirring model," *prd*, vol. 11, pp. 2088–2097, Apr. 1975.

[3] E. Abdalla, M. Abdalla, and K. Rothe, *Non-perturbative Methods in 2 Dimensional Quantum Field Theory*. World Scientific, 2001.

[4] F. Wilczek, "Magnetic flux, angular momentum, and statistics," *Physical Review Letters*, vol. 48, no. 17, p. 1144, 1982.

[5] T. H. Hansson, M. Rocek, I. Zahed, and S. C. Zhang, "Spin and Statistics in Massive (2+1)-dimensions QED," *Phys. Lett.*, vol. B214, pp. 475–479, 1988.

[6] V. Borokhov, A. Kapustin, and X. K. Wu, "Topological disorder operators in three-dimensional conformal field theory," *JHEP*, vol. 11, p. 049, 2002.

[7] N. Seiberg, T. Senthil, C. Wang, and E. Witten, "A duality web in 2+ 1 dimensions and condensed matter physics," *Annals of Physics*, vol. 374, pp. 395–433, 2016.

[8] A. Karch, B. Robinson, and D. Tong, "More Abelian Dualities in 2+1 Dimensions," *JHEP*, vol. 01, p. 017, 2017.

[9] A. Karch and D. Tong, "Particle-Vortex Duality from 3d Bosonization," *Phys. Rev.*, vol. X6, no. 3, p. 031043, 2016.

[10] D. T. Son, "Is the composite fermion a dirac particle?," *Physical Review X*, vol. 5, no. 3, p. 031027, 2015.

[11] C. P. Burgess and B. P. Dolan, "Particle vortex duality and the modular group: Applications to the quantum Hall effect and other 2-D systems," *Phys. Rev.*, vol. B63, p. 155309, 2001.

[12] C. Wang and T. Senthil, "Dual dirac liquid on the surface of the electron topological insulator,"

[13] M. A. Metlitski and A. Vishwanath, "Particle-vortex duality of 2d dirac fermion from electric-magnetic duality of 3d topological insulators,"

[14] D. Tong, "Lectures on the Quantum Hall Effect," 2016.

[15] A. Lerda, *Anyons: Quantum Mechanics of Particles with Fractional Statistics*. Stony Brook, NY: Springer Berlin Heidelberg, 1992.

[16] J. K. Jain, *Composite Fermions*. Cambridge University Press, 2007.

[17] T. T. Wu and C. N. Yang, “Concept of nonintegrable phase factors and global formulation of gauge fields,” *Phys. Rev. D*, vol. 12, pp. 3845–3857, Dec 1975.

[18] P. A. M. Dirac, “The theory of magnetic poles,” *Phys. Rev.*, vol. 74, pp. 817–830, Oct 1948.

[19] T. T. Wu and C. N. Yang, “Dirac Monopole Without Strings: Monopole Harmonics,” *Nucl. Phys.*, vol. B107, p. 365, 1976.

[20] H. J. Lipkin, W. I. Weisberger, and M. Peshkin, “Magnetic charge quantization and angular momentum,” *Annals Phys.*, vol. 53, pp. 203–214, 1969.

[21] E. H. Fradkin and F. A. Schaposnik, “The Fermion - boson mapping in three-dimensional quantum field theory,” *Phys. Lett.*, vol. B338, pp. 253–258, 1994.

[22] S. Deser and R. Jackiw, “'Selfduality' of Topologically Massive Gauge Theories,” *Phys. Lett.*, vol. 139B, pp. 371–373, 1984.

[23] M. Gomes, R. S. Mendes, R. F. Ribeiro, and A. J. da Silva, “Gauge structure, anomalies, and mass generation in a three-dimensional thirring model,” *Phys. Rev. D*, vol. 43, pp. 3516–3523, May 1991.

[24] A. N. Redlich, “Parity Violation and Gauge Noninvariance of the Effective Gauge Field Action in Three-Dimensions,” *Phys. Rev.*, vol. D29, pp. 2366–2374, 1984. [,2366(1983)].

[25] R. E. Gamboa Saravi, G. L. Rossini, and F. A. Schaposnik, “The Zeta function answer to parity violation in three-dimensional gauge theories,” *Int. J. Mod. Phys.*, vol. A11, pp. 2643–2660, 1996.

[26] M. E. Peskin, “Mandelstam-'t hooft duality in abelian lattice models,” *Annals of Physics*, vol. 113, no. 1, pp. 122–152, 1978.

[27] C. Dasgupta and B. I. Halperin, “Phase transition in a lattice model of superconductivity,” *Phys. Rev. Lett.*, vol. 47, pp. 1556–1560, Nov 1981.

[28] A. Zee, *Quantum Field Theory in a Nutshell*. Nutshell handbook, Princeton, NJ: Princeton Univ. Press, 2003.

[29] L. Ryder, *Quantum Field Theory*. Cambridge, UK: Cambridge University Press, 1996.

[30] G. Gur-Ari and R. Yacoby, "Three dimensional bosonization from supersymmetry," *Journal of High Energy Physics*, vol. 2015, p. 13, Nov 2015.

[31] S. Jain, S. Minwalla, T. Sharma, T. Takimi, S. R. Wadia, and S. Yokoyama, "Phases of large n vector chern-simons theories on $s2xs1$," *Journal of High Energy Physics*, vol. 2013, p. 9, Sep 2013.

[32] D. F. Mross, J. Alicea, and O. I. Motrunich, "Explicit derivation of duality between a free dirac cone and quantum electrodynamics in (2+1) dimensions,"

[33] J.-Y. Chen, J. H. Son, C. Wang, and S. Raghu, "Exact boson-fermion duality on a 3d euclidean lattice," *Phys. Rev. Lett.*, vol. 120, p. 016602, Jan 2018.

[34] C. A. Hernaski and P. R. S. Gomes, "Duality between 3d massive thirring and maxwell-chern-simons models from 2d bosonization," *Phys. Rev. Lett.*, vol. 121, p. 041601, Jul 2018.

[35] K. Huang, *Quarks, Leptons & Gauge Fields*. Singapore: World Scientific, 1992.