Universidade Estadual de Londrina

KESLEY RAIMUNDO

ON THE CLASSICAL THEORY FOR PSEUDO-HERMITIAN TWO-LEVEL SYSTEMS

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Londrina, 5 de março de 2020

Ao meu Pai, Adilson da Silva Raimundo.
A pessoa que, sem medir esforços, tornou tudo isso possível.

Agradecimentos

Não é a todo instante da vida que estamos no ápice da produção, em todos os seus sentidos. Por esse motivo, a necessidade de produzir quando não conseguimos sempre gera ansiedade. Embora esta seja natural para qualquer ser humano, não é todo mundo que consegue lidar com ela de uma forma saudável. Do ponto de vista de um estudante, com muitos prazos a cumprir e notas mínimas à serem atingidas, a má administração desta ansiedade pode ser catastrófica para a vida acadêmica. Em geral, como a relação entre professor e aluno se dá de uma forma muito profissional, é natural que a ansiedade em questão não seja problema do orientador, que por sua vez, continua cobrando produtividade do estudante, como deveria ser. Por este motivo, muitos alunos se cobram ainda mais quando não estão no ápice da produção, gerando assim, uma bola de neve extremamente problemática. Claro, escrevo isso baseado em minha experiência, seja passando por isso, ou vendo acontecer com pessoas próximas. Entretanto, este texto até aqui tem como objetivo (ou pelo menos eu tive à intenção de) exaltar a imensa compreensão que meu orientador, Prof. Dr. Mario Cesar Baldiotti, teve comigo durante todo esse tempo, desde a graduação até o presente momento no final do mestrado. Em outras palavras, existe uma chance imensa de que, se ele não tivesse essa compreensão, eu poderia ter estragado minha vida acadêmica na física de alguma forma, como quase estraguei. Por esse motivo, começo agradecendo meu orientador, Prof. Dr. Mario Cesar Baldiotti, primeiramente, pela paciência e compreensão em relação à este assunto. Além disso, agradeço também por todo incentivo e pelas discussões produtivas que sempre me recordavam como a física é maravilhosa. Também ressalto a escolha do projeto que gerou este trabalho. Como o assunto em questão é um problema além de uma revisão bibliográfica, fui capaz de finalmente começar a entender o que é ser um pesquisador, de sorte que obtive então a certeza de que escolhi o caminho certo. Por último, mas não menos importante, agradeço também a todos os convites para churrascos e rodas de violões, compostas por diversos outros alunos e, eventualmente, outros professores, que geraram bastante historia para contar, além de, ao menos da minha parte, um sentimento de amizade. Obrigado.

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Wasser ist die beste Erfindung
der Menschheit

Resumo

Neste trabalho, analisamos o limite clássico para sistemas pseudo-hermitianos com um número finito de níveis de energia. Estudando sistemas com campos complexos, descobrimos que uma transformação canônica na teoria clássica pode ser dada por uma transformação linear $R \in SO(3, \mathbb{C})$. Como um caso particular, podemos transformar um campo real em um campo complexo através desta rotação. Mostramos então que a condição que garante que R é uma transformação canônica na teoria clássica é uma das condições necessarias para que a teoria quântica seja pseudo-hermitiana. Propomos então um limite clássico correto para a teoria pseudo-hermitiana. Além disso, quando o sistema não é pseudo-hermitiano, o limite clássico produz a equação de Landau-Lifshitz-Gilbert como equação de movimento. Essa identificação nos permite interpretar a forma algébrica do campo externo complexo, que quebra a hermiticidade do problema, como um campo efetivo para sistemas de dois níveis abertos. Neste sentido, afirmamos que o Hamiltoniano proposto aqui descreve um amortecimento em sistemas de dois níveis. Como exemplo, aplicamos esse formalismo a um análogo do Problema Rabi e mostramos possíveis efeitos mensuráveis.

Palavras-chave: 1. Mecânica Quântica 2. Pseudo-Clássico 3. Pseudo-Hermitiano 4. Transformações Canonicas

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Abstract

In the present work, we analyze the classical limit of pseudo-hermitian systems with finite energy levels. By exploring systems coupled with complex external fields, we find that a linear transformation $SO(3, \mathbb{C})$ provides a canonical transformation in the classical theory. As a special case, we can rotate a real field into a complex field. On the quantum side, we show that the condition that ensures the classical transformation is canonical is a necessary condition so that the quantum theory is pseudo-hermitian. We then propose a classical limit for the pseudo-hermitian theory that yields the right classical equations of motion. Furthermore, when the system is not pseudo-hermitian, the classical limit yields the Landau-Lifshitz-Gilbert equation as the equations of motion. This identification allows us to interpret the algebraic form of the complex external field (which breaks the hermiticity of the problem) as an effective field for open two-level systems. We argue that the Hamiltonian proposed here describes damped system. As an example, we apply this formalism to an analog of the well-known Rabi Problem and calculate possible measurable effects.

Keywords: 1. Quantum Mechanics 2. Pseudo-Classical 3. Pseudo-Hermitian 4. Canonical Transformations

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Sumário

Parte I

Introduction

1 Introduction

The simplest system with non-trivial dynamics that we can build in quantum mechanics is the two-level system. Despite the adjective “simplest” one cannot underestimate the power of such set ups. In general, they are the best-understood quantum systems and adequately describe several physically relevant phenomena. Furthermore, they play an important role in the understanding of more intricate configurations.

In general, we can treat a quantum two-level system as a spin 1/2 particle interacting with an external magnetic field if the spatial dynamic is not taken into account. Thus, a two-level system is governed by the Pauli equation in $(0+1)$-dimension

$$i\frac{\partial v}{\partial t} = \hat{H}v, \text{ with } \hat{H} = \frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{F} \text{ and } v = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}. \tag{1.1}$$

Here, $v = (v_1(t), v_2(t))^T$ is a two-component spinor, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and $\boldsymbol{F} = (F_1(t), F_2(t), F_3(t))$ is an external field.[1] Therefore, solving a two-level system is equivalent to solving equation (1.1), which will be referred as the *Spin Equation* (SE).

Since the Pauli matrices are hermitian, the hermicity of $\hat{H}$ in SE depends strictly on whether $\boldsymbol{F}$ is real or not. In this case, due to our first notions of unitarity, one may be tempting to associate the unitarity of the theory described by $\hat{H}$ with the reality of $\boldsymbol{F}$. However, as we will see throughout this work, hermicity and unitarity are not fundamentally related. Of course, it is well known that a Hermitian Hamiltonian yields a unitary theory, however, a unitary theory does not need to be described exclusively by a hermitian operator [1]. In other words, there are circumstances where a non-hermitian operator, and consequently, complex external fields, yield a well defined quantum unitary theory.

The latter statement opens a range of physical theories that we can achieve exclusively with the requirement of unitarity, that is, theories described by non-hermitian Hamiltonians. As an application, we can explore unitarity conditions in quantum open-systems, since non-hermitian Hamiltonians are often found in this context [2]. These kinds of systems are intended to be quantum systems that interact with the environment in which they are embedded. Although this interaction is well-formulated in classical physics, it is not yet fully comprehended at the quantum level. Therefore, we may get some insight about the true nature of these problems from the relation of unitarity and non-hermicity.

[1] We are setting $\gamma = -1$, where $\gamma = \frac{gq}{2m}$ with q , m and g being, respectively, the charge, mass and the g-factor of the spin 1/2 particle. Also, in this description, $\boldsymbol{F}$ has dimension of energy.

Despite the difficulties in non-hermitian systems, non-unitary theories have been acquiring some attention in the physics community through the study of a certain class of non-hermitian operators called *pseudo-hermitian operators* (PHOs). PHOs define the so-called *pseudo-hermitian quantum mechanics* (PHQM). In PHQM, the freedom in defining an inner product in the associated Hilbert spaces is explored to recover the unitarity of a theory. The choice of inner product in some Hilbert spaces is indeed a freedom, since we cannot identify it by any measurement. In this context, we may think that the notion of non-unitarity arises because we are using the "wrong" inner product.

The liberty in choosing the inner product has already been studied long ago by Dirac and Pauli, among others [3–8]. These early developments were attempts in recovering unitarity using what they called *indefinite-metric quantum theories.* Here, the terminology "indefinite-metric" stands for non-positive-definite inner products. More recently, non-hermitian Hamiltonians with real eigenvalues were considered (see [9], for instance). Later on, a series of papers [1, 10–13] exploring whether a Hamiltonian must be hermitian or not was proposed. The authors argued that a weaker and physically transparent condition for the reality of the spectrum of the Hamiltonian operator $\hat{H}$ is $\mathcal{PT}$-symmetry, where $\mathcal{P}$ stands for the parity operator and $\mathcal{T}$ stands tor the time-reversal operator[2]. Also, they showed that if $\hat{H}$ has an unbroken $\mathcal{PT}$-symmetry, there is an operator $\mathcal{C}$, commuting with $\hat{H}$, that allows one to define a positive-definite inner product, with a metric operator given by $\eta = \mathcal{CPT}$. Further on, the question of what are the necessary and sufficient conditions for the reality of the spectrum of a linear operator were explored in [14–18]. It turns out that the answer to this problem initiated the research in the pseudo-hermitian quantum mechanics, in which the $\mathcal{PT}$-symmetry and the $\mathcal{C}$ operator are included, but do not play a fundamental role. Indeed, it can be shown from PHQM that $\eta = \mathcal{CPT}$ is just an example of a positive-definite metric operator [19]. There are several contexts where pseudo-hermitian operators appear [19]. In special, recent treatments on topological aspects of non-hermitian systems use the framework of PHQM [20–28].

A point explored by the PHQM framework is that a non-unitary similarity transformations between operators that act in different Hilbert spaces with possible two different inner products can be established. On the other hand, it is well-known that quantum canonical transformations are generated by similarity transformations, even if they are not implemented by unitary operators [29]. In addition, for systems with infinite energy levels, a physical meaning for the canonical transformations can be established by examining the classical limit of the theory [17, 19, 30]. This procedure is called *η-pseudo-hermitian canonical quantization.*

In the present work, the important observation is that, when dealing with systems with finite energy levels, there is no classical correspondence a priori. Nevertheless,

[2] $\langle x| \mathcal{P} |\psi, t\rangle = \psi(-x, t)$ and $\langle x| \mathcal{T} |\psi, t\rangle = \overline{\psi}(x, -t)$, where for any complex number $Z \in \mathbb{C}$, $\overline{Z}$ denotes the complex conjugate of Z. Also, throughout this text, $\mathbb{C}$ denotes the set of complex numbers.

quantum theories for systems with finite energy levels can still have a $\hbar \to 0$ limit . This limit yields what is called *pseudo-classical mechanics* [31–33], which consists of using Grassmann variables as phase-space coordinates for fermionic degrees of freedom. In this picture, the algebra of Grassmann variables is quantized to an anti-commutator, as is usually done for quantization of the algebra of fermionic degrees of freedom, such as spin.

In this paper, the pseudo-hermitian treatment will be extended to the pseudo-classical framework. Despite the existence of pseudo-classical mechanics, its relation with pseudo-hermitian theories has not yet been fully analyzed. Indeed, the aim of this work is to exploit this relation at the level of canonical transformations, considering both the pseudo-hermitian quantum theory and its pseudo-classical limit. For this purpose, complex external fields will be considered, which turn out to define, in general, non-unitary systems. We then study the pseudo-classical-quantum correspondence for the theory in order to assign a physical meaning for the complex fields. In exploring the consequences of this correspondence, we find that there is an interesting case where, through a $SO(3,\mathbb{C})$ canonical transformation, a real field can be promoted to a complex field. We interpret the results and provide possible experimental tests for the theory.

This work is organized as follows: in Chapter 2 we present the precession equation for a magnetization vector, as well as the Landau-Lifshitz-Gilbert equation, which describes a damped precession movement for a magnetization vector. In chapter 3 we present the pseudo-classical theory which gives the quantum theory for a spin 1/2 particle after canonical quantization. We then present a classical limit for this quantum theory in order to assign a physical meaning for complex fields. Also in this chapter, we present a canonical transformation between two general pseudo-classical theories with complex fields, which gives, among other things, a transformation of real field into a complex one. In Chapter 4 we present the pseudo-hermitian framework, which we use to interpret the results from Chapter 3. Also, we present a method to find the metric operator for a specific class of systems, which are the ones we are interested in. In Chapter 5 we discuss the η-canonical quantization scheme in order to motivate the correct classical limit for pseudo-hermitian systems, that is, the one that properly renders the notion of physical equivalence. In Chapter 6 we present a particular choice of for the generic external field, which in turn, defines a specific problem. In chapter 7 we explore possible experimental tests for the given theory, through possible measurable effects. Finally, in chapter 8 we give some final remarks and future perspectives for this work.

Parte II

Pseudo-Classical Framework

2 Classical Magnetization

2.1 Precession Equation

It is well-known that quantum descriptions of particles with half-integer spin, such as the electron, do not have a classical correspondent. However, we also know that there are macroscopic consequences of the concept of half-integer spin, such as those in a Stern-Gerlach experiment, for instance. The latter comes from the fact that a spin $1/2$ particle has an intrinsic magnetic moment. For this reason, it is interesting for us to look at how a magnetic moment behaves when immersed in an external magnetic field.

Classically, the magnetic moment of a particle $\boldsymbol{\mu}$ is a measure of how much it feels a torque when immersed in a magnetic field $\boldsymbol{B}$. The resulting torque $\boldsymbol{\tau}$ in this system is given by

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B} \ . \tag{2.1}$$

Since the resulting torque will always be perpendicular to the plane formed by $\boldsymbol{\mu}$ and $\boldsymbol{B}$, when the magnetic moment is not in the same direction of the magnetic field, $\boldsymbol{\mu}$ will move towards $\boldsymbol{\tau}$ at each instant changing its direction. When $\boldsymbol{B}$ is constant, Eq. (2.1) describes what is called precession movement. We then say that $\boldsymbol{\mu}$ describes a precession movement around $\boldsymbol{B}$.

For a charged particle with angular orbital moment $\boldsymbol{L}$, there is an associated magnetic momentum

$$\boldsymbol{\mu} = \gamma \boldsymbol{L} = \frac{gq}{2m} \boldsymbol{L} \ . \tag{2.2}$$

If we take the electron as an example, then $g \approx 2$ and $\gamma < 0$. This allows one to write Eq. (2.1) as

$$\dot{\boldsymbol{L}} = \gamma \boldsymbol{L} \times \boldsymbol{B} \ , \tag{2.3}$$

with the dot over $\boldsymbol{L}$ denoting the time derivative of $\boldsymbol{L}$. For the sake of simplicity, as we did in Eq. (1.1), we will set $\gamma = -1$ throughout the text. Therefore, we write Eq. (2.3) as

$$\dot{\boldsymbol{L}} = -\boldsymbol{L} \times \boldsymbol{B} \ . \tag{2.4}$$

Again, note that $\boldsymbol{B}$ has dimension of energy. Furthermore, despite the fact that the precession movement of $\boldsymbol{L}$ occurs only when $\boldsymbol{B}$ is constant, Eq. (2.4) will be referred as a *precession equation* even when $\boldsymbol{B}$ is time-dependent.

The interaction between the magnetic moment and the magnetic field is described by the Hamiltonian

$$H = -\boldsymbol{\mu} \cdot \boldsymbol{B} \ . \tag{2.5}$$

Hence, denoting the norm of $\boldsymbol{B}$ as B and the norm of $\boldsymbol{\mu}$ as μ, the total energy of the system can assume values from $H = -\mu B$, that is when $\boldsymbol{\mu}$ is parallel to $\boldsymbol{B}$, to $H = \mu B$, that is when $\boldsymbol{\mu}$ is anti-parallel to $\boldsymbol{B}$. From the latter we also infer that, when the total energy is not conserved, we expect that the magnetization tends to align parallel to the external field, which is the minimum-energy configuration.

Just for completeness, we can also derive the precession equation from (2.5) through Hamilton's equations. To achieve the latter, let us consider a charged particle such that all the magnetic moment comes from its angular momentum $\boldsymbol{L}$. We then can write explicitly

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \ , \tag{2.6}$$

in some frame of reference such that $\boldsymbol{r}$ is the position of this particle and $\boldsymbol{p}$ is its linear momentum. In this case, Hamilton's equation

$$\dot{p}_i = -\frac{\partial H}{\partial r_i} = -\left[\boldsymbol{p} \times \boldsymbol{B}\right]_l \quad \text{and} \quad \dot{r}_l = \frac{\partial H}{\partial p_i} = -\left[\boldsymbol{r} \times \boldsymbol{B}\right]_i \tag{2.7}$$

leads to

$$\dot{L}_i = -\left(r_k p_i - r_i p_k\right) B_k = -\varepsilon_{ijk} L_j B_k \ , \tag{2.8}$$

which in a vector notation is just the precession equation[1] (2.4), as it should be.

The precession equation is also often found in the literature as

$$\dot{\boldsymbol{L}} = \boldsymbol{L} \times \boldsymbol{\omega}_{\mathrm{L}} \ , \quad \text{where } \boldsymbol{\omega}_{\mathrm{L}} = \gamma \boldsymbol{B} \tag{2.9}$$

is called the Larmor frequency.

2.2 Landau-Lifshitz-Gilbert Equation

It is an experimental fact that the phenomenon of magnetic saturation occurs when a ferromagnetic material is subject to a very intense external field. In a simple way, with the increase of the external field, there is a critical point where the system can no longer absorb energy in the magnetization. Instead, it will dissipate the acquired energy.

As we see from (2.5), sticking with the particular case of a magnetization which precesses around a magnetic field, the energy loss means that the magnetization will eventually align with the external field. This describes a characteristic movement that will be referred to as *damped precession.*

In general, a realistic description of a mechanism of this kind is very complex. Indeed, in the bulk of some material, the energy dissipation may be described by a highly non-linear differential equation that can include chaotic motion [35]. However, it is possible to study this phenomenon including an *ad hoc* term in the equation of motion

[1] As in Eq. (2.8), throughout this work a sum over repeated indices is assumed

that corresponds to the damping. In 1935, Landau and Lifshitz suggested a damping term which gave rise to the *Landau-Lifshitz equation*

$$\dot{\boldsymbol{u}} = -\boldsymbol{u} \times \boldsymbol{B} - \alpha \boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{B}) \ . \tag{2.10}$$

Here, α is a real dimensionless parameter that controls the damping intensity and $\boldsymbol{B}$ has to be taken as the resulting effective field of all possible internal and external fields. Also, $\boldsymbol{u}$ is taken to be the unitary magnetization

$$\boldsymbol{u} = \frac{\boldsymbol{\mu}}{|\boldsymbol{\mu}|} \ . \tag{2.11}$$

Although the Landau-Lifshitz (LL) equation has done well in fitting a certain amount of available data, in 1956, Gilbert [36] suggested another version of the damping term that experimentally best describes the magnetization when the energy loss is large [37]. The main difference from the LL damping is that the one provided by Gilbert depends explicitly on the total derivative of $\boldsymbol{u}$, which means that the damping is smaller when the magnetization change is slower. Explicitly, the Gilbert damping is

$$\dot{\boldsymbol{u}} = -\boldsymbol{u} \times \boldsymbol{B} + \alpha \boldsymbol{u} \times \dot{\boldsymbol{u}} \ . \tag{2.12}$$

Iterating the latter equation, we obtain

$$\dot{\boldsymbol{u}} = -\frac{1}{(1+\alpha^2)} \boldsymbol{u} \times \boldsymbol{B} - \frac{\alpha}{(1+\alpha^2)} \boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{B}) \ . \tag{2.13}$$

The resulting equation is called the *Landau-Lifshitz-Gilbert* (LLG) equation. The form of Eq. (2.13) when $\alpha \ll 1$ is clearly the LL equation. In this sense, the LLG equation is a general form of the LL equation.

Later on, in 1996 Slonczewski [38] changes the equation in order to account for the spin-transfer torque effect. The resulting equation is called *Landau-Lifshitz-Gilbert-Slonczewski* equation. However, in the present work we are mainly interested in the simplest dissipative system described kinematically by a damped precession movement, which is the one-particle system. This means that the spin-transfer torque effect, which is associated with spin waves and lattices, is neglected. For this purpose, we will consider with the LLG equation and refer to α as the Gilbert damping.

An important concept to take into account here is that Eq. (2.13) is a classical phenomenological equation. Moreover, we stress that it is an equation for dissipative systems. We want to write the LLG equation as a classical limit of some quantum theory. However, it is not that simple to just guess which quantum theory would give us the desired result. So, we can try the other way around, that is, we can perform a quantization of the LLG equation, so that we have our quantum-dissipative theory. However, we must recall that we are discussing magnetization from a classical point of view. This is relevant due to the fact that we cannot obtain a quantum theory that describes a spin system from a classical theory in the usual sense. Despite that, we can perform the latter from the perspective of pseudo-classical mechanics.

3 Pseudo-Classical Theory

3.1 Grassmann Numbers

Although there is no classical correspondent for the spin, there is a theory in the limit $\hbar \to 0$ for systems with fermionic degrees of freedom. This theory is the so called pseudo-classical mechanics [31, 32] and it is based on an extension of the classical phase-space to incorporate fermionic degrees of freedom (besides the bosonic ones), described by anti-commuting variables. In this case, the "pseudo-classical" term means that, although it is not a classical theory, there is a quantization procedure such that a quantum theory representing spins is obtained, unlike the classical magnetization in the previous chapter. For this purpose, we aim in this section to present what are the anti-commuting variables, which are also referred as *Grassmann numbers*.

We start by considering the quantities ξ_i, for $i = 1, 2, ..., n$, forming a set of generators an n-dimensional Grassmann algebra. The latter says that the ξ_i's fulfill

$$\xi_i \xi_j + \xi_j \xi_i = 0 \ , \tag{3.1}$$

which defines the product in the *Grassmann algebra*. As a special case, $\xi_i^2 = 0$. As one can check, in an n-dimensional Grassmann algebra there are 2^n possible independent products between the Grassmann generators which can be built. Namely, they are all the possibles independent products between the ξ_i's, together with the unit, which is denoted by 1. As an example, since it will be interesting for us later, the $n = 3$ case has the following $2^3 = 8$ independent products:

$$1, \xi_1, \xi_2, \xi_3, \xi_1\xi_2, \xi_2\xi_3, \xi_3\xi_1, \xi_1\xi_2\xi_3 \ . \tag{3.2}$$

Since the the ξ_i's are taking to be elements of an n-dimensional algebra, they form a vector space whose dimension is 2^n. In this sense, we can expand any function in terms of these independents products. Namely for the $n = 3$ case,

$$f_a = a_0 + a_i \xi_i + a_{ij} \xi_i \xi_j + a \xi_1 \xi_2 \xi_3 \ , \tag{3.3}$$

where we have summed over repeated indices. When $\{a_0, a_i, a_{ij}, a\}$ are just complex numbers, which turns out to be our case, f_a is called in literature as a *supernumber*.

Since we will construct classical mechanics from a variational principle, it is important to specify how to construct real supernumbers, so that we can write actions properly. For this purpose, in the following we shall use an anti-involution such that the generators are real under its action. That is,

$$(f_a + f_b)^* = f_a^* + f_b^* \ , \quad (f_a f_b)^* = f_b^* f_a^* \text{ and } \xi_i = \xi_i^* \ . \tag{3.4}$$

Next we define left and right derivatives of general supernumbers

$$\frac{\partial_L}{\partial \xi_i}\left(\xi_j \xi_k\right) = \delta_{ij}\xi_k - \delta_{ki}\xi_k = -\frac{\partial_R}{\partial \xi_i}\left(\xi_j \xi_k\right) \ , \tag{3.5}$$

where ∂_L means that we take derivatives from the left (first in j and after in k). On the other hand, ∂_R means that we take derivatives from the right (first in k and after in j). However, by means to simplify the notation, we will always express the derivatives as taken from the left, that is,

$$\partial \equiv \partial_L \tag{3.6}$$

and when we have to take ∂_R derivatives we will compensate the sign, as in Eq. (3.5)

3.2 Pseudo-Classical Mechanics

In order to incorporate supernumbers in classical mechanics, one extends phase-space such that the generators ξ_i can be regarded as actual phase-space coordinates transforming as vectors under the $O(n)$ group [31–33]. In this case, since we are not interested in bosonic degrees of freedom, that is, we will only consider a single electron fixed in space, the present pseudo-classical theory will only depend on Grassmann variables.

A simple $O(n)$-invariant pseudo-classical theory is given by the action [32]

$$S = \int_{t_i}^{t_f} dt \left(\frac{i}{2}\xi_i \dot{\xi}_i - H\left(\{\xi_i\}\right)\right) \ . \tag{3.7}$$

whose equation of motion is

$$\dot{\xi}_i = -i\frac{\partial H}{\partial \xi_i} \ . \tag{3.8}$$

For the special case $n = 3$, ξ_i transforms as a vector under $O\left(3\right)$. In this case, by requiring a rotational and parity invariant theory, H must be of the form

$$H = -\frac{i}{2}\varepsilon_{ijk}\xi_i\xi_j F_k \ , \tag{3.9}$$

where F_k transforms as a pseudo-vector (like the magnetic field). In this case, the equations of motion are given by

$$\dot{\xi}_i = -\varepsilon_{ijk}\xi_j F_k \ , \tag{3.10}$$

which is the analog of the classical precession equation (2.4), that is, in analogy to a magnetic moment immersed in a magnetic field $\boldsymbol{F} = (F_1, F_2, F_3)$. If $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, Eq. (3.10) can be written in vector notation as

$$\dot{\boldsymbol{\xi}} = -\boldsymbol{\xi} \times \boldsymbol{F} \ . \tag{3.11}$$

Since we want to perform canonical quantization of this theory, we must evaluate the canonical conjugate momentum

$$\pi_i = \frac{\partial L}{\partial \dot{\xi}_i} = -\frac{i}{2}\xi_i \ . \tag{3.12}$$

It is immediate that this is a constrained system, whose constraints are given by

$$\chi_i = \pi_i + \frac{i}{2}\xi_i \ . \tag{3.13}$$

The Poisson brackets for any variables f and g is given by

$$\{f, g\} = \frac{\partial_R f}{\partial \xi_i}\frac{\partial_L g}{\partial \pi_i} - (-1)^{P_f P_q}\frac{\partial_R g}{\partial \xi_i}\frac{\partial_L f}{\partial \pi_i} \ , \tag{3.14}$$

where P_f and P_q stand for the parity of f and g, respectively, which assume the value 0 for odd functions and 1 for even ones. Also, as one can check, the constraints do not evolve on time since $\{H, \chi_i\} = 0$. Therefore, for the canonical quantization scheme, we need to replace the Poisson brackets with the Dirac brackets [33, 34]

$$\{f, g\}_D \equiv \{f, g\} - \{f, \chi_i\}|\{\chi_i, \chi_j\}|^{-1}\{\chi_j, g\} \ . \tag{3.15}$$

Hence, since

$$\{\chi_i, \chi_j\} = -i\delta_{ij} \ , \tag{3.16}$$

the fundamental Dirac brackets are given by

$$\{\xi_i, \xi_j\}_D = -i\delta_{ij} \ . \tag{3.17}$$

The quantization scheme for anti-commuting variables consists in promoting the ξ_i's to operators and using the Dirac rule

$$\left[\hat{\xi}_i, \hat{\xi}_j\right]_+ = i\,\{\xi_i, \xi_j\}_D = \delta_{ij} \ , \tag{3.18}$$

Hence, under canonical quantization, the Grassmann algebra is promoted to a Clifford algebra. In the present work, we consider a straightforward representation for the Clifford algebra, given by the Pauli matrices:

$$\hat{\xi}_i \to \frac{\hat{\sigma}_i}{\sqrt{2}} \, , \, \text{for}\, i = 1, 2, 3 \ . \tag{3.19}$$

In what follows, we will always choose this representation, unless otherwise specified.

We then perform the canonical quantization of (3.9) using (3.18), which results in the quantized Hamiltonian

$$\hat{H} = -i\left(F_1\hat{\xi}_2\hat{\xi}_3 + F_2\hat{\xi}_3\hat{\xi}_1 + F_3\hat{\xi}_1\hat{\xi}_2\right), \text{with } \left[\hat{\xi}_i, \hat{\xi}_j\right]_+ = \delta_{ij} \ . \tag{3.20}$$

So, in terms of the chosen representation (3.19), we can write (3.20) as

$$\hat{H} = \frac{1}{2}\left(F_1\hat{\sigma}_1 + F_2\hat{\sigma}_2 + F_3\hat{\sigma}_3\right) \ . \tag{3.21}$$

From the SE, equation (1.1), we see that $\hat{H}$ is exactly the quantum problem that defines a two-level system interacting with an external magnetic field $\boldsymbol{F} = (F_1, F_2, F_3)$. In this case, since we have obtained (3.21) from a quantization of a classical theory with Grassmann variables ξ_i's, we can say indeed that (3.7) can be regarded somewhat as a classical theory for a two-level system. Also, this realization with the Pauli matrices ensures that $\hat{H}$ is Hermitian, $\hat{H} = \hat{H}^\dagger$, when F is real.

3.3 Classical Limit

At this point, it is worth recalling that we aim to write a quantum theory that has the LLG equation as its classical limit. First of all, we start by letting the external field $\boldsymbol{F}$ to be complex. In this sense, $\hat{H}$ is non-hermitian, that is $H \neq H^{\dagger}$, and we expect a non-unitary theory. Then, we define the classical limit as the mean value of the $\hat{\sigma}_i$'s operators, that is,

$$\sigma_i(t) = \langle\psi|\, \hat{\sigma}_i \,|\psi\rangle \ , \tag{3.22}$$

where $|\psi\rangle$ is some time-dependent state. In this case, using the Schrödinger equation, we obtain

$$\dot{\sigma}_i(t) = i \langle\psi| \left(\hat{H}^{\dagger}\hat{\sigma}_i - \hat{\sigma}_i\hat{H}\right) |\psi\rangle \ . \tag{3.23}$$

Also, it is convenient to explicitly write

$$\hat{H} = \frac{1}{2}F_i\hat{\sigma}_i = \frac{1}{2}\mathrm{Re}\,(F_i)\,\hat{\sigma}_i + \frac{i}{2}\mathrm{Im}\,(F_i)\,\hat{\sigma}_i \ , \tag{3.24}$$

which in this case, from equation (3.23) we obtain

$$\begin{aligned} \dot{\sigma}_i(t) &= \frac{i}{2}\langle\psi|\, \mathrm{Re}\,(F_j)\,[\hat{\sigma}_j, \hat{\sigma}_i] - i\mathrm{Im}\,(F_j)\,\{\hat{\sigma}_j, \hat{\sigma}_i\}\,|\psi\rangle \\ &= -\varepsilon_{ijk}\sigma_j(t)\,\mathrm{Re}\,(F_k) + \mathrm{Im}\,(F_i)\,\langle\psi|\psi\rangle \ . \end{aligned} \tag{3.25}$$

Due the non-hermicity of $\hat{H}$, the quantity $\langle\psi|\psi\rangle$ may be not constant for every t. Because of that, it is convenient to define the quantity

$$n_i \equiv \frac{\sigma_i(t)}{\langle\psi|\psi\rangle} = \frac{\langle\psi|\, \hat{\sigma}_i \,|\psi\rangle}{\langle\psi|\psi\rangle} \ , \tag{3.26}$$

whose time-evolution is given by

$$\dot{n}_i = \frac{1}{\langle\psi|\psi\rangle}\left[\dot{\sigma}_i(t) - n_i\frac{d}{dt}\left(\langle\psi|\psi\rangle\right)\right] = \frac{\dot{\sigma}_i(t)}{\langle\psi|\psi\rangle} - n_i\mathrm{Im}\,(F_j)\,n_j \ . \tag{3.27}$$

Therefore, using the expression (3.25) for $\dot{\sigma}_i(t)$, we obtain after some algebra,

$$\dot{\boldsymbol{n}} = -\boldsymbol{n}\times\mathrm{Re}\,(\boldsymbol{F}) - \boldsymbol{n}\times\left(\boldsymbol{n}\times\mathrm{Im}\,(\boldsymbol{F})\right) \ . \tag{3.28}$$

It follows that when the external field is real, that is, when $\mathrm{Im}\,(\boldsymbol{F}) = 0$, Eq. (3.28) coincides with Feynman's results in [40] and the equation of motion reproduces the classical precession equation (3.11). However, for $\mathrm{Im}\,(\boldsymbol{F}) \neq 0$, Eq. (3.28) has an additional term that leads to damping in the dynamic of $\boldsymbol{n}$ that could not be obtained from (3.7) just by taking $\boldsymbol{F}$ to be complex from the very beginning. In other words, if we take $\boldsymbol{F}$ to be complex in Eq. (3.9) we still have the result (3.11) without the second term present in (3.28). Indeed, the damping term on (3.28) can only be obtained from a variational principle if the Lagrangian has a term $\xi_i\xi_j\xi_k$, so that when we take the derivative in ξ_l, only quadratic monomials would remain in the equations of motion.

However, to construct a scalar with cubic terms, we can only contract the indices ijk with ε_{ijk}, δ_{ij} and another field F_i. Then, if we want parity invariance, we are only left with the term $\xi_i\xi_j\xi_k F_i \delta_{jk}$. In this case, the equations of motion for this contribution are

$$\dot{\xi}_i = F_i\xi_j\xi_j + F_j\left(\xi_i\xi_j + \xi_j\xi_i\right) \ , \tag{3.29}$$

which also vanishes. In other words, we would have no damping in the pseudo-classical theory since the anti-commutator of the Grassmann variables vanishes.

As expected, damping in Eq. (3.28) arises exactly from the imaginary part of $\boldsymbol{F}$, which is what breaks the hermiticity of $\hat{H}$ in Eq. (3.21). Also, considering that $\boldsymbol{F}$ is time-independent, the Lagrangian in Eq. (3.7) is also time-independent. In this case, the energy of the system is conserved and we would not expect damping, that is, the equations of motion are (3.11). However, when the energy is not conserved, we cannot write a pseudo-classical theory from a variational principle. Instead, we can only start from a non-unitary quantum theory and take the classical limit to obtain the result (3.28).

There is a similarity between (2.13) and (3.28). If we choose the external field to be

$$\boldsymbol{F} = \frac{1 + i\alpha}{1 + \alpha^2}\boldsymbol{G} \ , \tag{3.30}$$

then the classical limit for $\hat{H}$ yields the LLG equation for every real external field $\boldsymbol{G}$. Also, another feature that we must emphasize is that the LLG equation is a phenomenological equation, while (3.28) was obtained from a classical limit of a quantum theory.

At this point, all these ideas strongly suggest that a complex external field leads to damping. In this sense, we can interpret a complex field as an effective field that describes the interaction of the system with the environment. Furthermore, the specific form of (3.30), which yields the LLG equation, also says that we can interpret the Gilbert damping α as a coupling constant for this interaction. Therefore, when we turn it off, that is, set $\alpha = 0$, then the external field is real and there is no damping at all.

3.4 Canonical Transformations

In order not to loose focus, so far we have presented a classical theory for a (non-relativistic) spin 1/2 particle; showed that, if the external field has a non zero imaginary part, the theory is non-unitary and the classical limit yields a damped precession equation. We have then concluded that complex fields describe damping, while real fields do not. From now on, we aim in this section to perform a canonical transformation in the pseudo-classical theory with a complex field and explore the consequences. However, let us first briefly introduce what is a canonical transformation in the usual (bosonic) sense.

A canonical transformation is a change of the canonical coordinates $\rho_i(t) \to \rho_i'(\rho_1, ..., \rho_{2n}, t)^1$ that preserves the symplectic structure $\boldsymbol{\Omega} = \sum_{i=1}^{n} d\rho_i \wedge d\rho_{i+n}$ of the $2n$-dimensional phase-space. Equivalently, a canonical transformations is leaves the Poisson brackets $\{\rho_i, \rho_j\} = \boldsymbol{\Omega}_{ij}$ invariant, where $\boldsymbol{\Omega}_{ij}$ are the components of the symplectic form $\boldsymbol{\Omega}$.

The pseudo-classical mechanics was obtained by extending the phase-space to incorporate Grassmann degrees of freedom. In this case, we expect that the notion of a canonical transformation is rather straightforward: a linear transformation R that takes the set of Grassmann numbers $\boldsymbol{\xi}$ into another set of Grassmann numbers $\boldsymbol{\zeta}(\xi)$ is canonical if

$$\{\xi_i, \xi_j\} = \{\zeta_i, \zeta_j\} \ . \tag{3.31}$$

Furthermore, there is the important statement that two systems that differ by a time-independent canonical transformation are said to be physically equivalent to each other.

As a preamble to this construction, in order to simplify the presentation, we will set F_2 in H, Eq. (3.9), to zero. Note that, even when $\boldsymbol{F}$ is complex, the quantum theory that describes the interaction between the system with $\boldsymbol{F}$ allows us to set some component $F_i = 0$ for both its real and imaginary part, without losing information or generality. This is what is called a *reduction of the external field* and it is possible due to constraints on the SE that allow us to construct one solution from another [41].

We are then left with the following pseudo-classical theory

$$H = -i(F_1\xi_2\xi_3 + F_3\xi_1\xi_2) \ , \tag{3.32}$$

whose canonical quantization yields

$$\hat{H} = -i(F_1\hat{\xi}_2\hat{\xi}_3 + F_3\hat{\xi}_1\hat{\xi}_2) \ , \text{with } \left[\hat{\xi}_i, \hat{\xi}_j\right]_+ = \delta_{ij} \ . \tag{3.33}$$

Or choosing the representation (3.19),

$$\hat{H} = \frac{1}{2}(F_1\hat{\sigma}_1 + F_3\hat{\sigma}_3) \ . \tag{3.34}$$

One should note here that the Dirac brackets (3.17) are invariant under the action of $O(n, \mathbb{R})$, recalling that we imposed rotational and parity symmetry in the action (3.7). We expect that rotations are canonical transformations in this sense. However, we claim that the relations in Eq. (3.17) are also invariant under linear transformations $R \in O(n, \mathbb{C})$, where we stress that $\mathbb{C}$ stands for the field of complex numbers. To see what this implies and discuss possible consequences, we present the following linear transformation

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \frac{1}{B_1^2 + B_3^2} \begin{pmatrix} F_1B_1 - B_3F_3 & 0 & F_1B_3 + B_1F_3 \\ 0 & -B_1^2 - B_3^2 & 0 \\ F_1B_3 + B_1F_3 & 0 & -(F_1B_1 - B_3F_3) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} . \tag{3.35}$$

1 For compactness, the n canonical coordinates are $q_i(t) = \rho_i(t)$ and their respective canonical conjugate moments are $p_i(t) = \rho_{i+n}(t)$.

Under the condition

$$F_1^2 + F_3^2 = B_1^2 + B_3^2 \ , \tag{3.36}$$

It can be checked that the transformation (3.35) is canonical even when $\boldsymbol{F}$ and $\boldsymbol{B}$ are complex entries.[2] In other words, it can be shown that the Poisson brackets for the ζ_i's variables are invariant even for $\{F_1, F_3, B_1, B_3\} \in \mathbb{C}$.

Moreover, if we write (3.35) as $\zeta_i = (R)_{ij}\xi_j$, we see that $\det(R) = 1$ and $R \in SO(3, \mathbb{C})$. In other words, the matrix $R \in SO(3, \mathbb{C})$ is orthogonal and it is defined over the field of the complex numbers $\mathbb{C}$. In addition, R is idempotent, $R^2 = I$.

The resulting Hamiltonian that we obtain when we perform the transformation R, under (3.36), is given by

$$H_I = -i\left(B_1\zeta_2\zeta_3 + B_3\zeta_1\zeta_2\right) \ , \tag{3.37}$$

whose canonical quantization leads us to

$$\hat{H}_I = -i\left(B_1\hat{\zeta}_2\hat{\zeta}_3 + B_3\hat{\zeta}_1\hat{\zeta}_2\right), \text{ with } \left[\hat{\zeta}_i, \hat{\zeta}_j\right]_+ = \delta_{ij} \ . \tag{3.38}$$

Therefore, by reproducing the quantization procedure previously discussed, the Clifford algebra can be realized with Pauli matrices so that

$$\hat{H}_I = \frac{1}{2}(B_1\hat{\sigma}_1 + B_3\hat{\sigma}_3) \ . \tag{3.39}$$

The latter result is also simplified in the sense that $B_2 = 0$, although it still is a general field, according to [41].

From (3.28) we showed that, when $\mathrm{Im}(\boldsymbol{B}) \neq 0$, there is damping in the theory. Also, we have just argued that we can connect two theories with complex external fields through a canonical transformation. Furthermore, canonical transformations also define a physical equivalence between theories. In this case we see that, when $\mathrm{Im}(\boldsymbol{F}) \neq 0$ and $\mathrm{Im}(\boldsymbol{B}) \neq 0$, both Hamiltonians $\hat{H}$ and $\hat{H}_I$ generate damping, which means that the associated theories are non-unitary. On the other hand, when $\mathrm{Im}(\boldsymbol{F}) = 0$ and $\mathrm{Im}(\boldsymbol{B}) = 0$, both the Hamiltonians $\hat{H}$ and $\hat{H}_I$ generate unitary theories, without damping.

Let us now turn our attention to the particular case in which $\mathrm{Im}(\boldsymbol{B}) = 0$ and $\mathrm{Im}(\boldsymbol{F}) \neq 0$. This scenario is interesting due the fact that we can, through a canonical transformation, take a real field $\boldsymbol{B}$ into a complex field $\boldsymbol{F}$. This possibility raises the question of how can a unitary theory be physically equivalent to a non-unitary theory. Naturally, the two descriptions are not equivalent. As we will show in next section, when $\boldsymbol{B}$ is real, $\boldsymbol{F}$ fulfills the pseudo-hermicity condition that is associated to a unitary theory, even if $\boldsymbol{F}$ is complex. However, if a complex field $\boldsymbol{F}$ defines a unitary theory, it cannot produce the damping precession equation as its classical limit, although that is what one would expect for a complex field. Even though things may seem inconsistent, we hope that the next chapter clears the confusion.

[2] Note that there is no reason, a priori, to assume that F_i and B_i are complex, apart from the fact that, if they are, (3.35) still is a canonical transformation.

For that purpose, from now on, we will only examine the particular case where $\mathrm{Im}\left(\boldsymbol{B}\right) = 0$ and $\mathrm{Im}\left(\boldsymbol{F}\right) \neq 0$. This particular restriction captures the essential points to be studied in the present work.

Parte III

Pseudo-Hermitian Framework

4 Pseudo-Hermitian Operators

4.1 The Inner-Product Problem

In order to explain how a complex field defines a unitary theory, we will first look at the main reason why unitarity can be broken. This reason is intrinsically related with the notion of orthogonality. In other words, the loss of unitarity lies on the fact that the eigenvectors with distinct eigenvalues of a non-hermitian Hamiltonian are not, in general, orthogonal to each other. Beyond that, the notion of orthogonality between two vectors $v, u \in \mathcal{H}$, where $\mathcal{H}$ is some vector space, only arises when we define an inner product[1] $\langle , \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ in $\mathcal{H}$. Moreover, since in our specific case $\mathcal{H} = \mathbb{C}^2$, we start with the natural choice

$$\langle v, u \rangle =: \begin{pmatrix} v_1^* & v_2^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} . \tag{4.1}$$

On the other hand, are other choices physically relevant?

The answer to the latter question is straightforward: the true nature of the inner product in quantum physics cannot be measured directly. In other words, the final and only test of a theory is the experiment. In this sense, since in quantum physics only statistics of quantities arising from inner products are measurable, namely, probabilities, apparently we are free to define any inner product structure in $\mathcal{H}$, provided we get the same result from experiments. Of course, we always think that the fewer and the simpler additional structures, the better. However, in dealing with non-unitary systems, we do not yet have a consistent notion for probability and measurement. The latter suggests that we are probably dealing with an incorrect Hilbert space realization, and, perhaps, we may have a non-canonical inner product that recovers the notion of probability and measurement.

Let us illustrate the latter statement with the following brief example [19]: Let $\mathcal{H}$ be a 2-dimensional Hilbert space and $|v\rangle = (v_1, v_2)^T \in \mathcal{H}$ be a generic state in $\mathcal{H}$. Also, consider an operator $\hat{H} : \mathcal{H} \to \mathcal{H}$ such that

$$\hat{H} |v\rangle = \begin{pmatrix} v_1 \\ v_1 - v_2 \end{pmatrix} . \tag{4.2}$$

If one uses the canonical basis[2] as a basis for $\mathcal{H}$, then the matrix representation of $\hat{H}$ is given by

$$\hat{H} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} . \tag{4.3}$$

[1] If the notion of "inner product" is not natural, see appendix A.

[2] The canonical basis is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

In this case, $\hat{H}$ is non-hermitian with respect to the canonical inner product (4.1). However, if we write the same operator in the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} , \tag{4.4}$$

then $\hat{H}$ is represented by the matrix

$$\hat{H} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} , \tag{4.5}$$

which is hermitian with respect to (4.1), that is,

$$\left\langle v, \hat{H}u \right\rangle = \left\langle \hat{H}v, u \right\rangle . \tag{4.6}$$

However, if we define another inner product, denoted by $\langle , \rangle_\eta$, given by

$$\langle v, u \rangle_\eta =: \langle v, \eta u \rangle = \begin{pmatrix} v_1^* & v_2^* \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} , \tag{4.7}$$

the basis $\mathcal{B}$ is now orthogonal and $\hat{H}$ is Hermitian in that basis. In other words,

$$\left\langle v, \hat{H}u \right\rangle_\eta = \left\langle \hat{H}v, u \right\rangle_\eta . \tag{4.8}$$

The main point is already clear. The notion of hermicity of an operator $\hat{H}$ depends on whether the basis of the Hilbert where $\hat{H}$ acts is orthogonal or not. Consequently, the notion of hermicity depends on the inner product in $\mathcal{H}$. In addition, since a change of orthogonal basis is implemented by a unitary transformation $\hat{T}$, a hermitian operator $\hat{H}$ transforms like[3] $\hat{H}' = \hat{T}\hat{H}\hat{T}^\dagger$ and its hermicity condition $\hat{H}^\dagger = \hat{H}$, namely Eq. (4.6), holds with the same inner product. That is $\hat{H}'^\dagger = (\hat{T}\hat{H}\hat{T}^\dagger)^\dagger = \hat{T}\hat{H}\hat{T}^\dagger = \hat{H}'$. However, when the new basis is non-orthogonal, we can change the inner product in $\mathcal{H}$ in order to find a representation such that a non-hermitian operator turns into a hermitian operator.

The question that naturally arises is the following: given a non-hermitian operator $\hat{H} : \mathcal{H} \to \mathcal{H}$, is it always possible to find an inner product $\langle , \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ in $\mathcal{H}$ that renders $\hat{H}$ hermitian? In general, the answer is no. However there is a class of operators in which the answer is yes, namely, the class of *pseudo-hermitian operators*. These operators define a generalization of the standard quantum mechanics called *Pseudo-Hermitian Quantum Mechanics* (PHQM). By generalization we mean that the standard QM is a special case of the PHQM, that is, when the inner product is (4.1).

Any vector space with an inner product has an induced notion of norm and, consequently, a induced notion of metric.[4] In this sense, we interpret (4.7) as $\langle v, u \rangle_\eta =:$

[3] Provided $\hat{T}$ is time-independent. Otherwise, $H' = i\frac{\partial \hat{T}}{\partial t}\hat{T}^\dagger + \hat{T}H\hat{T}^\dagger$.
[4] See appendix A.

$\langle u | \eta | v \rangle$ where $\eta : \mathcal{H} \to \mathcal{H}$ is often called the *metric operator*. However, one should be very careful with names since an authentic metric in a vector space is always a positive-definite structure and, in general, η can be non-positive-definite. Furthermore, η has not even the domain and range to be an authentic metric. The cases where η is non-positive-definite will be commented through the text when necessary in order to avoid confusions. Nevertheless, from now on, we will refer to an inner product $\langle , \rangle_\eta$ as given by a metric operator η. Furthermore, with this interpretation, it may be clearer that QM with $\eta = 1$ is somewhat a special case of PHQM.

4.2 Definitions

Let us now be a little more consistent by writing some definitions and names properly. We define a pseudo-hermitian as an operator acting on a finite-dimensional Hilbert space $\mathcal{H}$ with inner product $\langle , \rangle$, for which there is a hermitian operator η such that $\mathcal{H}$ is hermitian according to the inner product $\langle , \rangle_\eta$ defined by

$$\langle \phi, \psi \rangle_\eta \equiv \langle \phi, \eta\psi \rangle \ . \tag{4.9}$$

In this section, we will start by motivating the definition of pseudo-hermicity often found in the literature by showing some useful consequences of pseudo-hermicity.

Consider a non-hermitian linear operator $\hat{H} : \mathcal{H} \to \mathcal{H}$ according to the canonical inner product $\langle , \rangle$, and two generic elements of $\mathcal{H}$ given by $|\psi\rangle$ and $|\phi\rangle$. Then the following relation holds:

$$\left\langle \phi, \hat{H}\psi \right\rangle = \left\langle \hat{H}^\dagger \phi, \psi \right\rangle = \overline{\left\langle \psi, \hat{H}^\dagger \phi \right\rangle} \ , \tag{4.10}$$

where the over-line stands for complex conjugation. Suppose now that we define another inner product $\langle , \rangle_\eta$ in $\mathcal{H}$, defined by a hermitian metric operator $\eta : \mathcal{H} \to \mathcal{H}$, given by (4.9). In this case, Eq. (4.10) becomes

$$\left\langle \phi, \hat{H}\psi \right\rangle_\eta = \left\langle \hat{H}^\dagger \phi, \psi \right\rangle_\eta = \left\langle \phi, \eta\hat{H}\psi \right\rangle = \left\langle (\eta\hat{H})^\dagger \phi, \psi \right\rangle \ . \tag{4.11}$$

Thus, if η is the metric that renders $\hat{H}$ to be hermitian according to $\langle , \rangle_\eta$, the operator $\eta\hat{H} : \mathcal{H} \to \mathcal{H}$ must be hermitian with respect to $\langle , \rangle$. The latter naturally says that

$$\eta\hat{H} = \hat{H}^\dagger \eta \tag{4.12}$$

It is clear from (4.12) that, if η is the identity, then $\hat{H}$ is hermitian. In other words, equation (4.12) says that the pseudo-Hermitian operators form a set, in which the Hermitian operators form a subset.

Although the relation (4.12) comes out in a straightforward manner, it indeed defines what is a pseudo-hermitian operator and we can see this as follows. Suppose $\hat{H}$

fulfills $\hat{H} = \eta^{-1}\hat{H}^{\dagger}\eta$. In this case,

$$\begin{aligned} \left\langle \phi, \hat{H}\psi \right\rangle_{\eta} &= \left\langle \phi, \eta\hat{H}\psi \right\rangle = \left\langle \phi, \eta\left(\eta^{-1}\hat{H}^{\dagger}\eta\right)\psi \right\rangle \\ &= \left\langle \phi, \hat{H}^{\dagger}\eta\psi \right\rangle = \left\langle \hat{H}\phi, \eta\psi \right\rangle = \left\langle \hat{H}\phi, \psi \right\rangle_{\eta} \end{aligned} \tag{4.13}$$

for any $\psi, \phi \in \mathcal{H}$. That is, $\hat{H}$ is hermitian according to $\langle , \rangle_{\eta}$. On the other hand, suppose $\hat{H}$ is hermitian according to $\langle , \rangle_{\eta}$. In this case,

$$\begin{aligned} \left\langle \phi, \hat{H}\psi \right\rangle_{\eta} &= \left\langle \phi, \eta\hat{H}\psi \right\rangle = \phi_c^{\dagger}\left(\eta_c\hat{H}_c\right)\psi_c \\ &= \left\langle \hat{H}\phi, \psi \right\rangle_{\eta} = \left\langle \hat{H}\phi, \eta\psi \right\rangle = \left(\hat{H}_c\phi_c\right)^{\dagger}\eta_c\psi_c = \phi_c^{\dagger}\hat{H}_c^{\dagger}\eta_c\psi_c \ , \end{aligned} \tag{4.14}$$

where $\phi_c, \psi_c, \hat{H}_c$ and η_c are the representations for $\phi, \psi\hat{H}$ and η in the canonical basis, respectively. Since it is true for any $\phi, \psi \in \mathcal{H}$, then $\eta\hat{H} = \hat{H}^{\dagger}\eta$ in the canonical basis.

Probabilities in QM are real numbers that arise from inner products. In addition, it is natural to say that the probability of finding the state $|\psi\rangle$, given the initial condition $|\phi\rangle$, is the same of finding $|\phi\rangle$, in the initial condition $|\psi\rangle$. Therefore, we write probabilities as

$$P_{\phi\to\psi} = \langle \phi, \psi \rangle \overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle \overline{\langle \psi, \phi \rangle} = P_{\psi\to\phi}. \tag{4.15}$$

With this in mind, we expect that by endowing a Hilbert space with a metric operator, the probabilities will be written, using (4.11), as

$$P_{\phi\to\psi} = \langle \phi, \eta\psi \rangle \overline{\langle \phi, \eta\psi \rangle} = \langle \psi, \eta\phi \rangle \overline{\langle \psi, \eta\phi \rangle} = P_{\psi\to\phi} \, , \tag{4.16}$$

which is only true if

$$\langle \phi, \eta\psi \rangle = \overline{\langle \psi, \eta\phi \rangle} \ . \tag{4.17}$$

Hence, if $P_{\phi\to\psi} = P_{\psi\to\phi}$, which turns out to be a natural requirement, then η must be hermitian. Indeed, we impose the latter in the very beginning, that is, Eq. (4.9). However, the inner product $\langle \phi, \psi \rangle_{\eta}$ must by definition be skew symmetric (See appendix). In this case, we have

$$\begin{aligned} \langle \phi, \psi \rangle_{\eta} &= \langle \phi, \eta\psi \rangle = \overline{\langle \eta\psi, \phi \rangle} \\ &= \overline{\langle \psi, \phi \rangle_{\eta}} = \overline{\langle \psi, \eta\phi \rangle}, \end{aligned} \tag{4.18}$$

which says that η must be indeed hermitian.

Finally, the definition often found in the literature is straightforward from (4.12): *A linear operator $\hat{H} : \mathcal{H} \to \mathcal{H}$ is said to be pseudo-hermitian if there is a hermitian metric operator $\eta : \mathcal{H} \to \mathcal{H}$ such that*

$$\hat{H}^{\dagger} = \eta\hat{H}\eta^{-1}. \tag{4.19}$$

The definition (4.19) does not guarantee that η exists for a given $\hat{H}$. Nevertheless, if we are able to find a such metric, then $\hat{H}$ is pseudo-hermitian. In addition, (4.19) also does not say how to find η even if it exists. Furthermore, even if one finds metric operator, there is no guarantee that it will be unique or positive-definite. Indeed, there are, in general, an infinite number of metric operators that render the same operator hermitian.[5]

Apparently, this is not very practical. In other words, we would like to look at a particular operator and say whether it is pseudo-hermitian or not, so we would not be looking for something that does not exist. Luckily, as one may have noted so far, there is a way of dealing with this problem by looking at the spectrum of the operator. The latter is the subject of the next section.

4.3 Spectrum and Symmetry

Let $\mathcal{H}$ be a D-dimensional Hilbert space with an inner product $\langle,\rangle$ and $\hat{H}:\mathcal{H}\to\mathcal{H}$ a pseudo-hermitian operator. By definition, the spectrum of $\hat{H}$ is the set of complex numbers $\{E_n\}$, for $1\leq n\leq D$, such that the operator[6] $\hat{H}-E_n$ is non-invertible, that is, $\det(\hat{H}-E_n)=0$. Since η is invertible by definition, using Eq. (4.19) we can write

$$\det(\hat{H}-E_n)=\det(\eta(\hat{H}-E_n)\eta^{-1})=\det(\hat{H}^\dagger-E_n)=0\ . \tag{4.20}$$

Therefore, we see that there must be a one-to-one map between the spectrum of $\hat{H}^\dagger$ and $\hat{H}$. In addition, since the spectrum of $\hat{H}^\dagger$ is the complex conjugate of $\{E_n\}$, the spectrum of a pseudo-hermitian operator must be either real or come in complex conjugate pairs.

It turns out that the converse is also true. That is, if the spectrum of a given operator $\hat{H}$ is either real or comes in complex conjugate pairs, there is a metric operator η that renders $\hat{H}$ hermitian. Equivalently, there is an operator η such that equation (4.19) holds. The latter can also be seen from (4.20). The only way that $\hat{H}$ would have the same spectrum as $\hat{H}^\dagger$ is through the existence of an invertible operator η which fulfills (4.19).

If $\hat{H}:\mathcal{H}\to\mathcal{H}$ is a diagonalizable operator, then the condition on the spectrum of $\hat{H}$ implies that $\det(\hat{H})\in\mathbb{R}$. One can see this by writing $\hat{H}$ in its diagonal form. In this case, since the complex entries are complex conjugate pairs, their products will always produce a real number. One should note, however, that the converse is not necessarily true. A real determinant is not a sufficient condition for a given operator to be pseudo-hermitian. Nevertheless, this is a sufficient condition for traceless 2×2 matrices, which turns out to be an useful information for two-level systems. One can also see this by writing $\hat{H}$ in its diagonal form. In this case, when $\hat{H}$ is traceless, the diagonal form is proportional to $\hat{\sigma}_3$ and, consequently, even if there are complex eigenvectors, the determinant is real since they will come in pairs.

[5] This is easily seen by re-scaling the metric, although this is not the only way of seeing this.

[6] There is always an implicit identity multiplying complex numbers when they are in a sum with operators.

Another interesting feature is that, if η_1 and η_2 are two metric operators that render $\hat{H}$ hermitian, that is, $\eta_1 \hat{H} \eta_1^{-1} = \hat{H}^\dagger = \eta_2 \hat{H} \eta_2^{-1}$, then there is an operator $\mathcal{C} = \eta_2^{-1}\eta_1$, commuting with $\hat{H}$, that is, there is a linear symmetry of $\hat{H}$. This feature is actually what shows that the $\mathcal{CPT}$ inner product is an example of a positive-definite metric operator η [19]. Also, if there is a positive-definite operator η_+, it will be not unique, however, any two η_+ and $\eta_+^{'}$ are related according to $\eta'_+ = \hat{A}^\dagger \eta_+ \hat{A}$, where $\hat{A}$ is some invertible linear operator commuting with $\hat{H}$.

4.4 Soft-Limit method for the determination of a metric operator

As concluded from the last section, we can say whether an operator is pseudo-hermitian or not by looking at its spectrum. In this case, if we have a Hamiltonian $\hat{H}$ whose spectrum is real, then there is a metric operator η that renders $\hat{H}$ hermitian, that is, such that (4.19) is fulfilled. However, we still do not know how to find η. Indeed, the problem of finding a metric operator depends strongly on the system with which we are dealing. Because of that, we aim in this section to first define what kind of system we are interested in, and then present a schematic way of finding a metric operator for them.

First of all, recall that the result we are trying to explore here is how an operator defined in terms of a complex field can describe a unitary theory, so that we may further explore the situation described in section 3.4. An important point to be noticed is that we have two physically equivalent theories, one for a real field $\boldsymbol{B} \in \mathbb{R}$ and another for a complex field $\boldsymbol{F} \in \mathbb{C}$. We rewrite both Hamiltonians here for convenience.

$$\hat{H}_I = \frac{1}{2}\left(B_1\hat{\sigma}_1 + B_3\hat{\sigma}_3\right) \text{ , with } \boldsymbol{B} = (B_1, 0, B_3) \in \mathbb{R} \tag{4.21}$$

$$\hat{H} = \frac{1}{2}\left(F_1\hat{\sigma}_1 + F_3\hat{\sigma}_3\right) \text{ , with } \boldsymbol{F} = (F_1, 0, F_3) \in \mathbb{C} \tag{4.22}$$

Since $\hat{H}$ and $\hat{H}_I$ are related through a canonical transformation and $\hat{H}_I$ is hermitian according to the canonical inner product while $\hat{H}$ is not, if there is a physical equivalence between these two theories, it must be implemented by an operator which is non unitary.[7] This follows from the well-known result that quantum canonical transformations need not be unitary transformations [29].

A quantum physical equivalence is also an isometry, that is, a linear norm-preserving isomorphism, which consequently relies on the notion of the inner product. In this case, since the Hilbert spaces where $\hat{H}$ and $\hat{H}_I$ are defined will have different inner products, we denote $\hat{H} : \mathcal{H} \to \mathcal{H}$ and $\hat{H}_I : \mathcal{H}_I \to \mathcal{H}_I$, with $\mathcal{H}$ and $\mathcal{H}_I$ being the Hilbert spaces spanned by the eigenvectors of $\hat{H}$ and $\hat{H}_I$, respectively. Also, the definition of the η-inner

[7] The word unitary here means that, if $\mathcal{M}$ is the operator which implements the physical equivalence, then $\mathcal{M} \neq \mathcal{M}^\dagger$, where the $\dagger$ operation is according to the canonical inner product.

product in $\mathcal{H}$ is Eq. (4.9) where $\langle , \rangle_\eta : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, $\langle , \rangle : \mathcal{H}_I \times \mathcal{H}_I \to \mathbb{C}$, $|\psi\rangle \in \mathcal{H}_I$, and η is such that (4.19) holds. We stress here that the $\dagger$ operation in Eq. (4.19) is according to the canonical inner product $\langle , \rangle$.

We write the isometry $\mathcal{M} : \mathcal{H}_I \to \mathcal{H}$ in the following way:

$$\left|\psi_\pm^\eta\right\rangle = \mathcal{M}\left|\psi_\pm\right\rangle \;,\; \left|\psi_\pm^\eta\right\rangle \in \mathcal{H} \;,\; \left|\psi_\pm\right\rangle \in \mathcal{H}_I \;, \tag{4.23}$$

where $|\psi_\pm\rangle$ are eigenvectors of $\hat{H}_I$, while $\left|\psi_\pm^\eta\right\rangle$ are the eigenvectors of $\hat{H} \in \mathcal{H}$. Consequently, for systems where we can define this isometry, the operator η is given by

$$\left\langle \psi_\pm^\eta, \psi_\pm^\eta \right\rangle_\eta = \langle \mathcal{M}\psi_\pm, \eta\mathcal{M}\psi_\pm\rangle = \langle \psi_\pm, \psi_\pm\rangle \Rightarrow \eta = \left(\mathcal{M}\mathcal{M}^\dagger\right)^{-1} . \tag{4.24}$$

There are few things we should emphasize here. First of all, it follows from (4.24) that η is a positive-definite operator, being interpreted as the metric induced by the inner product (4.9). Secondly, also from Eq. (4.24), if we choose any pair of vectors in $\mathcal{H}_I$ that differ from $|\psi_\pm\rangle$ by a unitary transformation, the isometry $\mathcal{M}$ would change, albeit leaving η invariant. Hence, the metric does not depend on a special choice of $|\psi_\pm\rangle$ or $\left|\psi_\pm^\eta\right\rangle$. However, we are interested in systems where the non-hermicity of $\hat{H}$ is broken continuously, namely by a real-valued parameter α. For instance, we are interested in fields of the form (3.30). We will consider systems where there is a well-defined limit $\alpha \to 0$, where F_1 and F_3 are real fields. In this case, $\hat{H}$ becomes hermitian and both theories will only differ by a unitary transformation.

An important observation is that although the remaining unitary transformation is indeed arbitrary, for a consistent physical interpretation we impose the condition

$$\lim_{\alpha\to 0} \boldsymbol{F} = \lim_{\alpha \to 0} \boldsymbol{B} \in \mathbb{R} \; . \tag{4.25}$$

By imposing Eq. (4.25), we ensure that when $\alpha \to 0$, $\hat{H} \to \hat{H}_I$, which consequently implies

$$\left|\psi_\pm^\eta\right\rangle \to |\psi_\pm\rangle \Rightarrow \mathcal{M} \to 1 \Rightarrow \eta \to 1 \; . \tag{4.26}$$

The prescription (4.25), together with the choice of the eigenvectors of both $\hat{H}$ and $\hat{H}_I$ to construct η, is denoted as *soft-limit* in the present work.

Let us then see how to use this prescription to construct η. First of all, we have

$$\left|\psi_\pm^\eta\right\rangle = \frac{1}{F_1}\begin{pmatrix} F_3 \pm E \\ F_1 \end{pmatrix} \text{ with } E_\pm = \pm\frac{E}{2} = \pm\frac{1}{2}\sqrt{F_1^2 + F_3^2} \;, \tag{4.27}$$

from $\hat{H}\left|\psi_\pm^\eta\right\rangle = E_\pm \left|\psi_\pm^\eta\right\rangle$, and also,

$$|\psi_\pm\rangle = \frac{1}{B_1}\begin{pmatrix} B_3 \pm E_I \\ B_1 \end{pmatrix} \text{ with } E_{I\pm} = \pm\frac{E_I}{2} = \pm\frac{1}{2}\sqrt{B_1^2 + B_3^2} \;, \tag{4.28}$$

from $\hat{H}_I \left|\psi_\pm\right\rangle = E_{I\pm} \left|\psi_\pm\right\rangle$. Following the soft-limit prescription, the isometry $\mathcal{M}$ can be explicitly implemented by the operator

$$\mathcal{M} = \frac{1}{F_1 E_I} \begin{pmatrix} B_1 E & F_3 E_I - B_3 E \\ 0 & F_1 E_I \end{pmatrix} , \tag{4.29}$$

and the metric operator in $\mathcal{H}$ will be given by

$$\eta = \frac{1}{B_1^2 \left|E\right|^2} \begin{pmatrix} \left|F_1\right|^2 E_I^2 & F_1^* E_I \left(B_3 E - F_3 E_I\right) \\ F_1 E_I \left(B_3 E^* - F_3^* E_I\right) & B_1^2 \left|E\right|^2 + \left|B_3 E - F_3 E_I\right|^2 \end{pmatrix} . \tag{4.30}$$

As one can check, the operator η in Eq. (4.30) renders $\hat{H}$ hermitian since

$$\hat{H}^\dagger = \frac{E^2}{\left|E\right|^2} \eta \hat{H} \eta^{-1}, \text{ and } \left\langle \psi_\pm^\eta, \psi_\pm^\eta \right\rangle_\eta = \delta_{\pm\pm} . \tag{4.31}$$

We can compare this procedure with the example in section 4.1. In this case, although $\hat{H}$ is non-hermitian according to the canonical inner product, it is hermitian according to $\langle,\rangle_\eta$. Consequently, in this picture the eigenvectors $\left|\psi_\pm\right\rangle_\eta$ are orthogonal with respect to $\langle,\rangle_\eta$ and the evolution will be unitary. However, it should be mentioned that the quantum theory defined by $\hat{H}$ itself may include the case where E is not real. In other words, we could from the very beginning start from (4.22), without the necessity of (3.35). In this case, F_1 and F_3 need not satisfy the constraint (3.36) and the only condition for η to exist is

$$\mathrm{Re}\left(F_1\right) \mathrm{Im}\left(F_1\right) + \mathrm{Re}\left(F_3\right) \mathrm{Im}\left(F_3\right) = 0 , \tag{4.32}$$

which allows complex conjugate pairs of eigenvalues. In the latter case, E is purely imaginary and η still is a positive-definite operator. However, (4.31) says that $\hat{H}$ should be an anti-pseudo-hermitian, rather than a pseudo-hermitian operator.

Summarizing, for E imaginary, the condition (3.36) is not satisfied with $\boldsymbol{B} \in \mathbb{R}$ and there will be no connection between the quantum theory and the pseudo-classical theory with a real external field. Therefore, in this present work, the conditions (3.36) and $E = E_I \in \mathbb{R}$ are assumed. Furthermore, as one can check from Eq. (4.25), the soft-limit condition guarantees that $\mathcal{M} \to 1$, $\eta \to 1$, $\hat{H}_I \to \hat{H}$ and $\left|\psi_\pm^\eta\right\rangle \to \left|\psi_\pm\right\rangle$ (or simply, $\mathcal{H} \to \mathcal{H}_I$), in the limit $\alpha \to 0$.

Assuming that the operator $\hat{H}$ is time-independent, the dynamics of the associated problem is obtained by simply exponentiating $\hat{H}$. In this case, if we are interested, for instance, in evaluating a transition amplitude between the eigenvectors of $\hat{\sigma}_3$, that is, the states of "spin-up" and "spin-down" in $\mathcal{H}_I$, denoted by $\left|\pm\right\rangle$, we can construct these states in $\mathcal{H}$ using the isometry $\mathcal{M}$. This transition amplitude can be written as

$$\left\langle +^\eta, -_t^\eta \right\rangle_\eta = \left\langle +^\eta, \eta \exp\left(-i\hat{H}t\right) -^\eta \right\rangle = -i \frac{B_1}{E_I} \sin\left(\frac{E}{2} t\right) , \tag{4.33}$$

where

$$\left|\pm^\eta\right\rangle = \mathcal{M} \left|\pm\right\rangle . \tag{4.34}$$

As it should be, the transition amplitude is unitary since $\hat{H}$ is hermitian.

5 Physical Equivalence

5.1 Canonical Transformations Again

Based on the previous example, where we recover unitarity, we further discuss the physical equivalence underlying the isometry between two Hilbert spaces with different inner products.

In general, we see from Eq. (4.24) that an operator $\hat{A} : \mathcal{H} \to \mathcal{H}$ has the same matrix elements as an operator $\hat{A}_I : \mathcal{H}_I \to \mathcal{H}_I$ given by

$$\hat{A}_I = \mathcal{M}^\dagger \eta \hat{A} \mathcal{M} \ , \tag{5.1}$$

that is,

$$(A_I)_{ij} = \left\langle e_i, \eta \hat{A} e_j \right\rangle = \left\langle e_i, \mathcal{M}^\dagger \eta \hat{A} \mathcal{M} e_j \right\rangle \ . \tag{5.2}$$

Therefore, we can write

$$\hat{A}_I = \mathcal{M}^{-1} \hat{A} \mathcal{M} \ . \tag{5.3}$$

It turns out that Eq. (5.3) says that there is a similarity relation between operators that act in $\mathcal{H}_I$ and $\mathcal{H}$. This is good in general since similarity transformations do not change the algebra of the operators. Consequently, Eq. (5.2) defines canonical transformations, even though $\mathcal{M}$ does not fulfills $\mathcal{M}^\dagger \mathcal{M} = 1$. This information is relevant here because our very first notion of canonical transformations at the quantum level was given by Dirac [42] and Weyl [43], for whom unitary transformations are canonical transformations. This is indeed true, however, canonical transformations can also be non-unitary.

A naive definition of a canonical transformation in quantum mechanics can be formulated from the canonical quantization scheme, where we just replace the Poisson brackets relations with the commutation relation, apart from constant factors. In other words, the canonical quantization scheme strongly suggests that quantum canonical transformations are the ones that leave the fundamental commutation relation

$$[\hat{\rho}_i, \hat{\rho}_j] = i\Omega_{ij} \tag{5.4}$$

invariant. From Eq. (5.4), it is clear that any transformation implemented by an invertible operator $\hat{C}\left(\{\hat{\rho}_i\}\right)$ of the form $\hat{\rho}_i^{'} = \hat{C}\hat{\rho}_i\hat{C}^{-1}$ is a canonical transformation, provided $\boldsymbol{\Omega}$ is a c-number or commutes with $\hat{C}$ (which is true for the fundamental commutation relations). There is no reason to restrict $\hat{C}$ to unitary operators only. Therefore, any similarity transformation as Eq. (5.3) can be interpreted as a canonical transformation at the quantum level.

The main point here is that our well-known notion of unitary transformations is a linear norm-preserving isomorphism (isometry) from a Hilbert space on itself. We commonly write the latter statement as

$$\langle U\psi, U\phi\rangle = \langle \psi, \phi\rangle \;\Rightarrow\; U^{\dagger}U = 1 \; . \tag{5.5}$$

However, in quantum mechanics we need not specify a Hilbert space in defining canonical transformations [29]. Thus, there are circumstances, such as equation (4.24), that we can have an isometry between two different Hilbert spaces with possible different inner products. In this case, we still can employ the concept of canonical transformations from any similarity relation as in Eq. (5.3).

Although $\mathcal{M}$ in Eq. (4.24) does not fulfills $\mathcal{M}^{\dagger}\mathcal{M} = 1$ according to the canonical inner product, but we stress here that $\mathcal{M}$ is unitary. In Hilbert spaces with different inner products, the concept of physical equivalence is implemented by

$$\langle \psi, \psi\rangle_{\eta'} = \langle \mathcal{M}\psi, \mathcal{M}\psi\rangle_{\eta} \tag{5.6}$$

or, using the definition of the η-inner product in terms of the canonical product $\langle , \rangle$

$$\left\langle \psi, \eta'\psi \right\rangle = \langle \mathcal{M}\psi, \eta\mathcal{M}\psi\rangle \tag{5.7}$$

or simply,

$$\eta' = \mathcal{M}^{\dagger}\eta\mathcal{M} \; , \tag{5.8}$$

which is the generalized notion of unitary transformation. That is, a unitary operator $\mathcal{M} : \mathcal{H}_{\eta} \to \mathcal{H}_{\eta'}$, where $\langle , \rangle_{\eta}$ is the inner product in $\mathcal{H}_{\eta}$ and $\langle , \rangle_{\eta'}$ is the inner product in $\mathcal{H}_{\eta'}$, is such that (5.6) holds and. It is then clear from (5.8) that, when the isometry is from $\mathcal{H}$ to itself and $\eta = 1$, then $\mathcal{M}^{\dagger}\mathcal{M}$ =1.

5.2 Pseudo Canonical Quantization Scheme

As it was said in the introduction, in dealing with systems with infinite energy levels, there is a classical-quantum correspondence through the Pseudo-Canonical quantization scheme. In order to better understand what this name means, let us start with two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_I$, such that $\mathcal{H}$ has a metric operator η and $\mathcal{H}_I$ has the canonical inner product. Also, as before, consider $\mathcal{M}$ as the isometry between $\mathcal{H}$ and $\mathcal{H}_I$ that defines η through equation Eq. (4.24)

From Eq. (5.2), if we have a physical problem described by a hermitian operator $\hat{H} : \mathcal{H} \to \mathcal{H}$,[1] then $H_I = \mathcal{M}^{-1}H\mathcal{M}$ is hermitian acting in $\mathcal{H}_I$. This allows us to define $\mathcal{H}$

[1] We also can say that $\hat{H}$ is η-pseudo-hermitian.

as the physical Hilbert space, and then we can establish a set of observables $\hat{O}_i : \mathcal{H} \to \mathcal{H}$ by identifying the physical observables $\hat{o}_i : \mathcal{H}_I \to \mathcal{H}_I$ with the canonical transformations

$$\hat{O}_i = \mathcal{M}\hat{o}_i\mathcal{M}^{-1} \ . \tag{5.9}$$

Considering, for instance, $\mathcal{H} = L^2(\mathbb{R}^n)$ as the Hilbert space for a continuum system,[2] we can take the phase-space coordinates $\rho_i = q_i$ and $\rho_{i+n} = p_i$ in order to define the observables

$$\hat{\varrho}_i = \mathcal{M}\hat{\rho}_i\mathcal{M}^{-1} \ . \tag{5.10}$$

Thus, since a canonical transformation does not change the algebra of the operators, the $2n$ operators $\{\hat{\varrho}_i\}$ satisfy

$$[\hat{\varrho}_i, \hat{\varrho}_j] = i\Omega_{ij} \ . \tag{5.11}$$

Suppose now that we write down the Hamiltonian $\hat{H} : \mathcal{H} \to \mathcal{H}$ using the variables $\hat{\varrho}_i$. A possible physical meaning for this $\hat{H}$ can be established by looking at the limit

$$H_c(\varrho_{ic}) = \lim_{\hbar \to 0} \hat{H}(\hat{\varrho}_i)\Big|_{\hat{\varrho}_i \to \varrho_{ic}} \ , \tag{5.12}$$

where ϱ_{ic} stand for the $2n$ classical coordinates of the phase-space. Assuming H_c does exist, there is a classical theory that we can quantize in order to reproduce the quantum system described by the physical Hilbert space $\mathcal{H}$, using what is called the *η-pseudo-Hermitian canonical quantization scheme* [17, 19, 30]

$$\{\varrho_{ic}, \varrho_{jc}\}_{PB} \to -i\,[\hat{\varrho}_i, \hat{\varrho}_j] \ , \tag{5.13}$$

where $\{,\}_{PB}$ stands for the Poisson brackets and $[,]$ stands for the commutator.

The transformation in $\hat{H}$ can also be performed by using $\hat{H}_I = \mathcal{M}^{-1}\hat{H}\mathcal{M}$. Therefore, $\hat{H}_I : \mathcal{H}_I \to \mathcal{H}_I$ is a hermitian operator, since $\mathcal{H}_I$ has the canonical inner product. The representation of the quantum system based on $\hat{H}_I$ is called, for this reason, *hermitian representation*. Besides, we can also try to find the classical Hamiltonian in the same way as equation (5.12)

$$H_{Ic}(\rho_{ic}) = \lim_{\hbar \to 0} \hat{H}_I(\hat{\rho}_i)\Big|_{\hat{\rho}_i \to \rho_{ic}} \ . \tag{5.14}$$

By examining the problem in section 4.4, we have already expected that both quantum theories for $\hat{H}$ and $\hat{H}_I$ are related through a canonical transformation. However, now we know that this relation can be implemented through a unitary operator as $\mathcal{M}\mathcal{M}$, as well as its relation with the metric operator in $\mathcal{H}$. Furthermore, it turns out that there is a quantum canonical transformation $\mathcal{M}$ that corresponds to the classical canonical transformation R. The latter is exactly what we aimed before, that is, the study of a classical-quantum correspondence for systems with finite energy levels.

[2] $L^2(\mathbb{R}^n)$ denotes the Hilbert space for square integrable functions on $\mathbb{R}^n$.

5.3 η-Classical Limit

At this point of the present work, we hope that it is clear from the last section that we are trying to achieve a connection between two pseudo-classical theories, related by a classical canonical transformation and implemented by $R \in SO(3, \mathbb{C})$, and two quantum theories, related by a quantum canonical transformation implemented by a unitary operator $\mathcal{M}$. Moreover, there is the classical-quantum connection through the canonical quantization scheme, which implies the existence of a classical limit.

So far, we have concluded that a complex field can describe a unitary theory. However, we have also concluded that a complex field has a damped precession equation as a classical limit. For this reason, due the physical equivalence of the canonical transformations, we should expect that the classical limit for pseudo-hermitian quantum theories does not produce damping. In order to see this, we propose a classical limit for pseudo-hermitian operators.

The main point is that, when a system is pseudo-hermitian, there is a metric operator that defines the right inner product. Because of that, we should not take the classical limit as in Eq. (3.22). Rather, we must evaluate

$$n_i\left(t\right) = \left\langle\psi\right| \eta\hat{\sigma}_i \left|\psi\right\rangle \ . \tag{5.15}$$

In this case, since η is time-independent, the time evolution for $\sigma_i\left(t\right)$, as in Eq. (3.23), is

$$\dot{n}_i\left(t\right) = i\left\langle\psi\right| \eta \left[\hat{H}, \hat{\sigma}_i\right] \left|\psi\right\rangle = -\varepsilon_{ijk} n_j\left(t\right) F_k \ ,$$

or in a vector notation,

$$\dot{\boldsymbol{n}} = -\boldsymbol{n} \times \boldsymbol{F} \ . \tag{5.16}$$

It is clear that (5.16) corresponds to the equations of motion of the pseudo-classical theory $\hat{H}$, that is, Eq. (3.11). But the role of Eq. (3.28) has to be better explained. It follows from the fact that (3.28) is obtained when we use the inner product in Eq. (3.22). When $\hat{H}$ is pseudo-hermitian, η should be used, furnishing (5.16). Consequently, the prescription (3.22) can only be used if $\hat{H}$ is not pseudo-hermitian. In this case, condition (3.36) does not hold and the classical theories are not related by a canonical transformation. In other words, $\hat{H}$ and $\hat{H}_I$ in this case are not physically equivalent theories.

6 Rabi Problem

6.1 Semi-Classical Approach

So far we have developed a series of concepts, both in the pseudo-classical framework as well as in the pseudo-hermitian framework, using a formalism that assumes general external fields. In this case, in order to attack a specific problem we can choose a specific form for both $\boldsymbol{B}$ and $\boldsymbol{F}$. This section has the latter as the main goal.

The choice we are about to make in order to explore the consequences of the previously defined set up is somewhat based on the well-known *Rabi problem*. Because of that, we will spend a few pages in order to define what is the Rabi problem, so that we can compare with the theory we want to explore.

The Semi-Classical Rabi problem is defined as a spin $1/2$ particle, confined in a region of space, interacting with an external real oscillating magnetic field given by

$$\boldsymbol{B}_R = (B\cos(\omega t), B\sin(\omega t), B_z)\,. \tag{6.1}$$

The main interest in this problem is to know how the field (6.1) promotes transitions between the eigenstates of σ_3, that is, the states of spin up and spin down, denoted by,

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,, \tag{6.2}$$

with eigenvalues

$$E_+ = \frac{B_z}{2} \qquad \text{and} \qquad E_- = -\frac{B_z}{2}\,. \tag{6.3}$$

We will assume the initial condition $|\psi, 0\rangle = |-\rangle$.

The Hamiltonian operator for this problem is

$$\hat{H}_R = \frac{B}{2}[\cos(\omega t)\hat{\sigma}_1 + \sin(\omega t)\hat{\sigma}_2] + \frac{B_z}{2}\hat{\sigma}_3\,, \tag{6.4}$$

which is explicitly time-dependent. In this case, we can not simply exponentiate $\hat{H}_R$ because there will be ordering problems. Although we can solve this by using the Dyson series, there is a simpler and more elegant way of dealing with this that requires a simple change of reference frame. The main point is that, if we go to the reference frame that rotates with $\boldsymbol{B}_R$, the resulting magnetic field will not be time-dependent, and therefore, the time evolution operator is obtained immediately. Let us see how it works.

The field (6.1) is rotating around the z-axis with frequency ω. Therefore, let us rotate our reference frame around the z-axis with a time-dependent parameter $\theta = \omega t$. This can be achieved by using the operator

$$\hat{R}_z(\omega t) = \exp\left(i\omega t\frac{\hat{\sigma}_3}{2}\right)\,. \tag{6.5}$$

Imposing that the Schrodinger equation must be the same in both reference frames, under a time-dependent linear transformation the Hamiltonian must transforms as

$$\hat{H} = i\frac{\partial \hat{R}_z}{\partial t}\hat{R}_z^\dagger + \hat{R}_z\hat{H}_R\hat{R}_z^\dagger \ , \tag{6.6}$$

or explicitly,

$$\hat{H} = (B_z - \omega)\frac{\hat{\sigma}_3}{2} + \frac{B}{2}\exp\left(i\omega t\frac{\hat{\sigma}_3}{2}\right)\left[\cos(\omega t)\sigma_1 + \sin(\omega t)\sigma_2\right]\exp\left(-i\omega t\frac{\hat{\sigma}_3}{2}\right) . \tag{6.7}$$

Although it is not clear from (6.7), the Hamiltonian in the rotating reference frame is indeed time-independent, as one can check by differentiating $\hat{H}$ with respect to time. This implies that we can choose $t = 0$ in Eq. (6.7) for simplicity. Therefore, the Hamiltonian in the reference frame that rotates with the field (6.1) is given by

$$\hat{H} = \frac{1}{2}\left(\delta\hat{\sigma}_3 + B\hat{\sigma}_1\right) \ \text{, with} \quad \delta = B_z - \omega \ . \tag{6.8}$$

The δ factor is usually called *detuning*.

In the rotating frame, the time-evolution operator $\hat{U}(t)$ can be calculated by simply exponentiating $\hat{H}$. Then,

$$\hat{U}(t) = \cos\left(\frac{\Omega_R}{2}t\right) - \frac{i}{\Omega_R}(\delta\hat{\sigma}_3 + B\hat{\sigma}_1)\sin\left(\frac{\Omega_R}{2}t\right) \ , \tag{6.9}$$

where the quantity Ω_R is called *Rabi frequency* and is defined as

$$\Omega_R = \sqrt{\delta^2 + B^2} \ . \tag{6.10}$$

Recalling that we want to evaluate the transition probability between $|-\rangle$ and $|+\rangle$, assuming $|\psi, 0\rangle = |-\rangle$, we need to know how an observer that rotates with the field sees these states. For this purpose, we act with $\hat{R}_z$ on the states $|\pm\rangle$. Since the rotation is around the z-axis and $|\pm\rangle$ are eigenvectors of $\hat{\sigma}_3$, in the rotating frame these states will only acquire a phase that will not contribute to transition probabilities. In this sense, we can still use $|\pm\rangle$ to calculate the desired probabilities. Therefore,

$$\left|\left\langle +, \hat{U}\left(t\right) - \right\rangle\right|^2 = \frac{B^2}{\Omega_R^2}\sin^2\left(\frac{\Omega_R}{2}t\right) \ . \tag{6.11}$$

There are a few things that should be emphasized. First, when $\delta = 0$, that is $\omega = B_z$, the frequency of the rotating field is exactly the difference between the energy levels of $|\pm\rangle$. This particular value of ω is called *resonance frequency* and the Rabi frequency (6.10) reduces to $\Omega_R = B$. Consequently, equation (6.11) says that, in the resonance frequency $\omega = B_z$, the probabilities oscillate with maximum amplitude. Also, in the absence of the oscillating field, the eigenvalues of $H(B = 0)$ are given by $E_\pm = \pm\frac{1}{2}(B_z - \omega)$, and therefore, at the resonance there is no difference between the energy levels.

6.2 Rabi Problem with Gilbert Damping term

We now give some motivation for choosing a specific form of the external field. First of all, we are particularly interested in systems where the hermiticity is broken continuously, namely by a real-valued parameter α. Secondly, when the hermiticity is broken, we should have a theory with damping whose classical limit yields the LLG equation. These two concepts together can be achieved by choosing the external field to be

$$\boldsymbol{F} = \frac{1+i\alpha}{1+\alpha^2}\boldsymbol{B} \text{ , with } \boldsymbol{B} \in \mathbb{R} \; . \tag{6.12}$$

Furthermore, since we want to use the Rabi problem as a known theory for comparison, we choose the real external field to be the Rabi field (6.1). In this case, we will look at the theory described by

$$\boldsymbol{F}_R = \left(F\cos\left(\omega t\right), F\sin\left(\omega t\right), F_z\right) = \frac{1+i\alpha}{1+\alpha^2}\boldsymbol{B}_R \; . \tag{6.13}$$

Since we do not want to deal with time-dependent external fields, we also want to look at the present theory in the rotating frame. In this case, performing the same rotation (6.5), the external field is given by

$$\boldsymbol{F} = \left(F_1, 0, F_3\right) \; , \tag{6.14}$$

with

$$F_1 = F = \frac{1+i\alpha}{1+\alpha^2}B \text{ and } F_3 = \Delta = \frac{1+i\alpha}{1+\alpha^2}B_z - \omega \; . \tag{6.15}$$

Note that, although this example is not the Rabi problem, it generates the Rabi problem in the limit $\alpha \to 0$. Furthermore, since for $\alpha \neq 0$, in the non-rotating frame, the field is given by (6.13), we can refer to α in this theory as the Gilbert damping.

6.3 The Problem

Let us start with the Hamiltonian (3.32), with the choices (6.15). In general, the external field (6.14) yields a Hamiltonian operator that is neither hermitian nor pseudo-hermitian. However, in this section we are mainly interested in seeing under what circumstances it is pseudo-hermitian.

In this case, the pseudo-classical theory is given by

$$H = -i(F\xi_2\zeta_3 + \Delta\zeta_1\zeta_2) \; . \tag{6.16}$$

Performing the canonical transformation (3.35), we can write

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \frac{1}{\Omega\Omega_R}\begin{pmatrix} FB-\delta\Delta & 0 & F\delta+B\Delta \\ 0 & -\Omega\Omega_R & 0 \\ F\delta+B\Delta & 0 & -\left(FB-\delta\Delta\right) \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \; , \tag{6.17}$$

where

$$\Omega^2 = F_1^2 + F_3^2 \ . \tag{6.18}$$

The new Hamiltonian is given by

$$H_I = -i\tilde{\Omega}\left(B\zeta_2\zeta_3 + \delta\zeta_1\zeta_2\right) \ , \text{with } \tilde{\Omega} = \frac{\Omega}{\Omega_R} \ , \tag{6.19}$$

and the canonical quantization for both theories yields

$$\hat{H} = \frac{1}{2}(F\hat{\sigma}_1 + \Delta\hat{\sigma}_3) \tag{6.20}$$

$$\hat{H}_I = \frac{\tilde{\Omega}}{2}(B\hat{\sigma}_1 + \delta\hat{\sigma}_3) \ . \tag{6.21}$$

As one can see, the limit $\alpha \to 0$ implies $\hat{H} \to \hat{H}_I$.

Recalling that we are interested in the case where $\boldsymbol{F} = (F_1, 0, F_3)$ is a complex field and $\boldsymbol{B} = \tilde{\Omega}\,(B, 0, \delta)$ is a real field, we need $\Omega \in \mathbb{R}$. In this case, (6.17) is indeed canonical and $\hat{H}$ is pseudo-hermitian. Therefore, we can follow the scheme previously discussed to find the metric operator η that renders $\hat{H}$ hermitian,

$$\left|\psi_\pm^\eta\right\rangle = \frac{1}{F}\begin{pmatrix} \Delta \pm \Omega \\ F \end{pmatrix} \text{ with } E_\pm = \pm\frac{\Omega}{2} = \pm\frac{1}{2}\sqrt{F^2 + \Delta^2} \tag{6.22}$$

and $\hat{H}\left|\psi_\pm^\eta\right\rangle = E_\pm\left|\psi_\pm^\eta\right\rangle$. Also

$$\left|\psi_\pm\right\rangle = \frac{1}{B}\begin{pmatrix} \delta \pm \Omega_R \\ B \end{pmatrix} \text{ with } E_{I\pm} = \pm\frac{\Omega}{2} = \pm\frac{\tilde{\Omega}}{2}\sqrt{B^2 + \delta^2} \tag{6.23}$$

and $\hat{H}_I\left|\psi_\pm\right\rangle = E_{I\pm}\left|\psi_\pm\right\rangle$. Following the soft-limit prescription, the isometry $\mathcal{M}$ can be explicitly implemented by the unitary operator

$$\mathcal{M} = \frac{1}{F\Omega_R}\begin{pmatrix} B\Omega & \Delta\Omega_R - \delta\Omega \\ 0 & F\Omega_R \end{pmatrix} \ , \tag{6.24}$$

and the metric operator will be given by

$$\eta = \frac{1}{B^2\Omega^2}\begin{pmatrix} |F|^2\Omega_R^2 & F^*\Omega_R\left(\delta\Omega - \Delta\Omega_R\right) \\ F\left(\delta\Omega - \Delta^*\Omega_R\right) & B^2\left|\Omega\right|^2 + \left|\delta\Omega - \Delta\Omega_R\right|^2 \end{pmatrix} \ . \tag{6.25}$$

Finally, one can check that the operator in (6.25) indeed renders $\hat{H}$ hermitian, since

$$\hat{H}^\dagger = \eta\hat{H}\eta^{-1} \text{ and } \left\langle\psi_\pm^\eta, \eta\psi_\pm^\eta\right\rangle = \delta_{\pm\pm} \ , \tag{6.26}$$

and the soft-limit $\alpha \to 0$ yields $\mathcal{H} \to \mathcal{H}_I$.

Let us consider the dynamics of this problem. Since the Hamiltonian $\hat{H}$ is time-independent, it can be exponentiated to obtain the transition amplitudes. For instance, considering the initial condition $\left|\psi_0^\eta\right\rangle = \left|\psi_-^\eta\right\rangle$, we have

$$\left\langle\psi_+^\eta, \eta\psi_t^\eta\right\rangle = \left\langle\psi_+^\eta, \eta\exp\left(-iHt\right)\psi_0^\eta\right\rangle = -i\frac{B}{\Omega_R}\sin\left(\frac{\Omega}{2}t\right) \ . \tag{6.27}$$

When $\alpha \to 0$, Eq. (6.27) yields the Rabi oscillations for a two-level system. Moreover, equation Eq. (6.27) agree with the calculations in [44], in the sense that the Rabi frequency Ω changes. However, unlike the proposal presented in [44], our states do not lose the normalization condition under time evolution.

We then see that, when we can connect $\hat{H}$ and $\hat{H}_I$ with a canonical transformation, $\Omega \in \mathbb{R}$ and there is no damping on the transition amplitudes. In this case, if we want to go out of the rotating frame, we can not use $\hat{R}_z$ as done in section 6.1. Instead, we must use its equivalent in $\mathcal{H}$, namely,

$$\tilde{\hat{R}}_z = \mathcal{M}^{-1}\hat{R}_z\mathcal{M} = \frac{1}{B\Omega}\begin{pmatrix} B\Omega e^{-\frac{i\omega t}{2}} & 2i\left(\delta\Omega - \Delta\Omega_R\right)\sin\left(\frac{\omega t}{2}\right) \\ 0 & B\Omega e^{\frac{i\omega t}{2}} \end{pmatrix}. \tag{6.28}$$

Following our prescription,

$$\begin{aligned} \hat{H}_R &= i\frac{\partial\hat{R}_z}{\partial t}\hat{R}_z^{-1} + \hat{R}_z\hat{H}\hat{R}_z^{-1} \\ &= \frac{1}{2\Omega_R}\begin{pmatrix} B_z\Omega - \omega\left(\Omega_R - \Omega\right) & B\Omega\exp\left(-it\omega\right) \\ B\Omega\exp\left(i\omega t\right) & -B_z\Omega + \omega\left(\Omega_R - \Omega\right) \end{pmatrix}. \end{aligned} \tag{6.29}$$

In the soft-limit, the Hamiltonian associated to the Rabi problem in the non-rotating frame is recovered. In summary, when condition (3.36) holds, this theory is still unitary, as it should be, and there is no damping in the equations of motion, independent of the reference frame.

It should also be mentioned that the example provided in this section contains a canonical transformation between a complex field $\boldsymbol{F}$ and a real field $\boldsymbol{B}$. This automatically says that the Hamiltonian $\hat{H}_I$ is hermitian, while $\hat{H}$ is pseudo-hermitian. However, the general development also holds for two generic complex fields. In this case either both are non-pseudo-hermitian or both are pseudo-hermitian. That is, the discussion will depends on whether $F_1^2 + F_3^2$ is real or not. Again, there is no inconsistency since we will only have physical equivalence between two theories that are either both unitary or both non-unitary.

7 Measurable Effects

7.1 Unitary configuration of the External Field

Let us turn our attention back to the constraint (3.36), which we rewrite here for convenience

$$B_1^2 + B_3^2 = F_1^2 + F_3^2 \ . \tag{7.1}$$

Using the example provided in the last chapter, the external complex field is

$$\boldsymbol{F} = (F, 0, \Delta) \ . \tag{7.2}$$

Again, complex $\boldsymbol{F}$ describes, in general, a non-unitary theory. However, if it was obtained from a real field $\boldsymbol{B} \in \mathbb{R}$, then the condition (7.1) holds and the theory is pseudo-hermitian (unitary).

The relevant condition for that is

$$F^2 + \Delta^2 \in \mathbb{R} \quad , \tag{7.3}$$

which directly implies the constraint

$$B^2 = B_z \left[\omega\left(1+\alpha^2\right) - B_z\right] \Rightarrow \omega\left(1+\alpha^2\right) B_z > B_z^2 \text{ for } B, B_z, \alpha \neq 0 \ . \tag{7.4}$$

Also, since B is a real number, the condition $B^2 > 0$ implies that either

$$\omega\left(1+\alpha^2\right) < B_z < 0 \text{ or } 0 < B_z < \omega\left(1+\alpha^2\right) \ . \tag{7.5}$$

The main point here is that, for any value of the external field but (7.4), $\hat{H}$ is not pseudo-hermitian and there is damping in the equations of motion. In this case, we interpret this result as the configuration of $\boldsymbol{B}$ such that the external field injects energy in the system at the same rate it dissipates. When this happens, the classical limit is a precession movement, that is Eq. (5.16), instead the LLG equation (2.13).

This is a possible test for the proposed theory, because, if we can produce, this effective external field, we can adjust the parameters so that the damping vanishes.

7.2 Total Suppression of the Transition Amplitude

Another peculiar behavior of the proposed set up is the total suppression of the usual resonant transition. When we explicitly write the constraints (7.4), we obtain the transition amplitude

$$P = \left|\left\langle \psi_+^\eta, \eta \psi_t^\eta \right\rangle\right|^2 = \left|\frac{B}{\Omega_R}\right|^2 \sin^2\left(\frac{t}{2}\sqrt{\omega\left(B_z - \omega\right)}\right) \ . \tag{7.6}$$

The first thing we note is that there is a special frequency, namely the resonance frequency of the undamped Rabi problem

$$\omega = B_z \ , \tag{7.7}$$

where the transition probability vanishes completely. Also, it should be noted that $\omega = B_z$ is an allowed frequency of our theory, since $B_z^2 \left(1 + \alpha^2\right) > B_z^2$. Therefore, in principle, $P \to 0$ is a possible effect for any non-vanishing value of α.

The total suppression of the transition amplitude can be interpreted as follows. Since the external field is, when (7.4) holds, injecting energy in the system so that the evolution is unitary, the energy available for the system to use in order to perform a energy level transition is always less than the total energy that the fields provides. In this case, we interpret the total suppression as the situation when there is no energy left for the system to perform energy level transitions. In other words, the damping is dissipating the same amount of energy that the external field is providing. That is, for a system with a Gilbert damping factor α, the resonance occurs between the field and the damping, rather than between the field and energy level transition.

8 Final Remarks

In this work, the classical-quantum correspondence for a system with finite energy levels and complex external field was analyzed. We showed that a non-unitary canonical transformation on the quantum side can be classically implemented by a rotation $R \in SO\left(n, \mathbb{C}\right)$, where $\mathbb{C}$ stands for the field of the complex numbers rather than the real numbers $\mathbb{R}$. In this regard, we take a real field into a complex field both in classical and quantum theory through a canonical transformation.

The main point in this description was to properly consider the unitarity. If we connect two fields $\boldsymbol{F}$ and $\boldsymbol{B}$, with $\mathrm{Im}(\boldsymbol{F}) \neq 0$ and $\mathrm{Im}(\boldsymbol{B}) \neq 0$, both theories are non-unitary and the classical limit yields a damping in the equations of motion. If we connect two fields $\boldsymbol{F}$ and $\boldsymbol{B}$, with $\mathrm{Im}(\boldsymbol{F}) = 0$ and $\mathrm{Im}(\boldsymbol{B}) = 0$, both theories are unitary and the classical limit yields the undamped precession equation. The latter is also the case when $\mathrm{Im}(\boldsymbol{F}) \neq 0$ and $\mathrm{Im}(\boldsymbol{B}) = 0$ and $B^2 = F^2$ holds. This happens because, for real $\boldsymbol{B}$, the existence of the canonical transformation which changes $\boldsymbol{F}$ to $\boldsymbol{B}$ implies that the theory described by $\boldsymbol{F}$ is pseudo-hermitian. At this point, we know that every pseudo-hermitian theory can be made unitary by a proper choice of inner product. Also, since a complex field describes damping, we interpret the pseudo-hermitian configuration of the external field to be the one which injects energy into the system at the same rate it dissipates, thus, suppressing the damping.

After recalling all these ideas and results, it should be noted that when $\boldsymbol{B}$ is real and $\boldsymbol{F}$ is complex, there is a canonical transformation between a hermitian Hamiltonian and a non-hermitian Hamiltonian. When $\boldsymbol{B}$ and $\boldsymbol{F}$ are time-independent, there is an equality between both Hamiltonians, namely,

$$H\left(\zeta_i\left(\xi\right)\right) = H_I\left(\xi_i\right) \ . \tag{8.1}$$

In this case, we conclude that the hermiticity is not a fundamental requirement of a theory. Instead, the fundamental requirement is unitarity.

Using the pseudo-hermitian framework, we have also provided a schematic construction to explicitly compute the metric operator η in the Hilbert space where $\hat{H}$ is hermitian. Beyond that, $\hat{H}$ has a Hermitian representation, namely $\hat{H}_I$, where $\hat{H}_I$ is hermitian with respect to the canonical inner product. This representation is well-known from [19] for systems with infinite energy levels and defines two physically equivalent Hilbert spaces, namely $\mathcal{H}$ and $\mathcal{H}_I$. However, since our theory has a limit $\alpha \to 0$ in which it becomes unitary and $\hat{H}_I \to \hat{H}$, we emphasize that this interpretation requires a preferred construction of the metric operator that renders the right limit as $\alpha \to 0$. We refer to this as a soft-limit.

Recent applications of the pseudo-hermitian set up suggests interesting perspectives for the developed framework. Topological properties of the theory can be explored, by evaluating topological numbers such as the Berry Phase. A second quantization approach of the Rabi problem can be investigated considering the open system as we have done here. The properties of the group $SO\left(n,\mathbb{C}\right)$ can be explored linking this structure to other physical scenarios. Finally, the developed formalism might be extended to lattice systems, and in this case topological phase transitions at the exceptional points can be explored.

A Metric, Norm and Inner-Product

A.1 Metric

Metric spaces are more general than inner product spaces. Given any set M, which does not need to be a vector space, a metric g in M is a map

$$g : M \times M \to \mathbb{R}_0^+ \tag{A.1}$$

which fulfills

- $d(m, n) = d(m, n)$;
- $d(m, n) \geq 0$ and is only zero when $m = n$;
- $d(m, q) + d(q, n) \geq d(m, n)$;

Also, the metric d should not be confused with a metric tensor from differential geometry, although it also gives the notion of distances. Besides, a metric in a set M induces a topology given by the open balls

$$B_d(x, r) = \{y \mid d(x, y) < r\} \ . \tag{A.2}$$

The pair $(M, B_d(x, r))$ is a topological metric space.

A.2 Norm

A norm on a vector space M is a real valued function which associates to an element $x \in M$, a number $||x||$. A norm must fulfill

- $||x|| \geq 0$ and is only zero when $x = 0$;
- $||\alpha x|| = |\alpha| \, ||x||$;
- $||x + y|| \leq ||x|| + ||y||$;

If the norm is defined, M is a normed vector space. Also, the norm induces a metric $d(x, y) = ||x - y||$. However, not all norms come from a metric.

A.3 Inner Product

A inner product is a notion which we can define on vector spaces. A operation $\langle , \rangle : M \times M \to \mathbb{C}$ is an inner product if it fulfills

- $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;
- $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$
- $\langle x, y\rangle = \overline{\langle y, x\rangle}$
- $\langle x, x\rangle \geq 0$ and is only zero when $x = 0$.

If the inner product is defined on M, then M is an inner product space. An inner product induces a norm $||x|| = \sqrt{\langle x, x\rangle}$, which in turn, induces a metric $d(x, y) = ||x - y|| = \sqrt{\langle x - y, x - y\rangle}$.

A.4 Summarizing

Inner product spaces $\subset$ Normed spaces $\subset$ Metric spaces

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