Universidade Estadual de Londrina

EDGAR JOSE CANDALES DUGARTE

PARTITIONS THEORY IN PHYSICS:
SOME APPLICATIONS ON STRING THEORY AND STATISTICAL MECHANICS

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Master dissertation presented to Physics Department of Londrina State University, as a partial pre-requisite to obtain a Master Degree.
Supervisor: Prof. Dr. Thiago dos Santos Pereira

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"The first principle is that you must not fool yourself and
you are the easiest person to fool."
Richard Phillips Feynman.

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Abstract

The combinatorial power of the Partitions theory allows to calculate thermodynamical parameters of physical systems. In particular, Partitions theory is especially useful to count states that are specified by the application of creation-operators. Also we derived and refined the formula of Hardy-Ramanujan via Statistical Mechanics.

keywords: Partitions Theory. Statistical Mechanics. String Theory. Quantum Harmonic Oscillator. Multipartite Generating Functions.

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Resumo

O poder combinatório da teoria de partições permite calcular os parâmetros termodinâmicos dos sistemas físicos. Em particular, a teoria das partições é especialmente útil para contar os estados que são especificados pela aplicação dos operadores de criação. Também derivamos e refinamos a fórmula de Hardy-Ramanujan através da Mecânica Estatística.

palavras-chave: Teoria de partições. Mecânica estatística. Teoria de cordas. Oscilador harmônico quântico. Função geradora de multipartições.

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1 Introduction

The study of black holes represents one of the best theoretical laboratories for theories such as String Theory and M-theory. In fact, it has been found a relationship between thermodynamics and black holes. For example, the first law of thermodynamics states the conservation of energy, but in General Relativity we know that energy and mass are also related, therefore the mass of black hole is also subjected to the laws of thermodynamics. It has lead to an analogy between black hole physics and thermodynamics [1].

One of the most interesting results that involves black holes and thermodynamics was the discovery of the Hawking radiation. Basically, quantum mechanics (in particular, the uncertainty principle) indicates that the energy in any place cannot be zero (since it would be completely known). This fact leads to the vacuum fluctuations, which can be interpreted as the constant creation and annihilation of particles. When one pair of particles is created close to the horizon of black hole, one member of the pair may fall in the black hole and the other member would escape, appearing as the black hole is radiating. Treating the black hole as a black body, Stephen Hawking calculated the temperature of black hole, obtaining

$$T = \frac{\hbar c^3}{8\pi kGM} \tag{1.1}$$

where M is the mass of the black hole. But not only does a black hole have a temperature, it also behaves as if it has entropy. It was discovered in 1974 by Stephen Hawking that

$$S = \frac{Akc^3}{4\hbar G} \tag{1.2}$$

where A is the area of the horizon of the black hole. This relation shows that the area (a geometric property) is directly related to the entropy (a thermodynamic property), suggesting a deep connection between quantum gravity and thermodynamics. String Theory and M-theory promise to explain the relationship between black holes and thermodynamics.

On the other hand, Statistical Mechanics has become the most powerful tool to understand the thermodynamics of a system. It relies on the partition functions to fully describe (thermodynamically) a system. But what is the partition function of a black hole? It is a problem that physicists and mathematicians are trying to solve looking at the symmetries or simply by guessing an expression.

To determine the entropy of a system, the partition function is essential. In physics, the partition function enumerates the configurations of a system for a given set of quantum numbers, but in mathematics the partition function is a little more subtle. The partition of a number is a way to write that number as a sum of positive integers, for example: $4 = 2+2$ but also $4 = 1+1+1+1$, thus $2+2$ and $1+1+1+1$ represent different partitions of 4. For a long time, mathematicians did not succeed to find a formula that specified the number of partitions of any integer n. That is why Euler developed generating functions, which roughly speaking are polynomials whose coefficients gives the number of partitions.

It has already been used the combinatorial power of partitions theory in quantum mechanics [2], a matter that we will present in detail here.

String Theory is another branch which has benefited a lot from Partitions Theory. The degeneracy of states of string can be calculated through the partition functions. In [3] is shown how to recognize the number of tachyonic states, massless states, etc, using the partition function.

We also explore the applicability of the partitions theory in the Bose-Einstein and Fermi-Dirac statistics. In both cases leading to mathematical difficulties that were not surpassed, mainly because the convergence of partition function is unknown, but it seems to be related to the McMahon function and the Dedekind eta function. Thus, we left as an open problem for further research.

Having in mind the above stated, we begin chapter two with a background in Partitions Theory, stating the most relevant theorems to our work. In chapter 3 we present direct applications of Partitions Theory in Quantum Mechanics, and after that applications of the partitions function in the context of String Theory. Subsequently, in chapter 4 we offer several applications in Statistical Mechanics, a branch in which partitions theory has found fertile terrain. We show how Statistical Physics, despite rude approximations, allows to obtain formulas already known in partitions theory, such as the Formula of Hardy-Ramanujan. The power of Statistical Mechanics has allowed to go beyond, even to improve such a formula, a calculation that we present in detail.

Finally, we present a background of Multipartite Generating Functions (MGF), which represent a generalization of the partition functions to higher dimensionality. We developed a formula to calculate a polynomials related to MGF using Bell polynomials. Unfortunately we did not found direct applications to physics, but it is being widely used in hyperbolic geometry and other areas [4].

2 The Elementary Theory of Partitions

In this first chapter we are going to present an intuitive idea of the theory of partitions and after that we will introduce the theory formally and the terminology involved [5].

2.1 The idea behind

Mathematicians observed that numbers can be decomposed into parts and represented by fundamental blocks. The fundamental theorem of arithmetic establishes the central role of primes in number theory: any integer greater than 1 is either a prime itself or can be expressed as a product of primes that is unique up to ordering [6], allowing to interpret the prime numbers as the basic building blocks of the set $\mathbb{N}$.

Thus, every number can be expressed in terms of a product of prime factors, which represent the basis of the called [7] Multiplicative number theory. There is, however, another way to approach the structure of numbers called Additive Number Theory. The basic idea of the additive number theory is that every number can be expressed as the sum of other numbers, but in this branch it has not been discovered any fundamental blocks or uniqueness in the representation of natural numbers. The way in which any number can be represented by the sum of others is a partition of the number. For example: Partitions of 4 are

$$4 = 3 + 1 \tag{2.1}$$

$$4 = 2 + 2 \tag{2.2}$$

Therefore, we say that $\{3,1\}$ and $\{2,2\}$ are partitions of 4. From this reasoning we might be tempted to say that 1 is the fundamental block in the additive number theory, since every number can be represented by a sum of 1s in a unique way (maybe I am speculating about it, but it seems that 1 is, in fact, the fundamental structure in this context).

Since the age of Euler, mathematicians have tried to figure out the fundamental elements of the additive number theory. It was Euler's idea to introduce the partition function $p(n)$ [8]: defined as the number of ways of writing a positive integer n as a sum of strictly positive integers. Euler observed that

$$\begin{aligned}
(1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^3+x^6+\ldots)\ldots &= 1+x+(x^2+x^2)+\ldots \\
&= 1+x+2x^2+\ldots \\
\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdots &= 1+\sum p(n)x^n
\end{aligned}$$

in other words, the coefficients that result from that product gives exactly the number of partitions of the exponent n.

Goldbach Conjecture is one of the most interesting discoveries, stating that every even $n > 4$ is the sum of two odd primes, giving some connection with prime numbers.

In the next section we are going to introduce the theory formally guided by [5].

2.2 Partitions

The fact that every $n \in \mathbb{N}$ can be represented as the sum of other numbers $\in \mathbb{N}$, that is, the number n can be expressed as the sum of smaller parts suggest the next definition

Definition. The sequence $\lambda \equiv (\lambda_1, \lambda_2, ..., \lambda_r)$ with $\lambda_i \in \mathbb{N}$ is **a partition** of $n \iff$

(a) $\sum_{i=1}^{r} \lambda_i = n$

(b) $\lambda_i \geq \lambda_j \quad \forall i < j \in \{1, ..., r\}$

$\lambda_i \in \lambda$ are called the parts of the partition λ (i.e., the elements of the set λ).

Example: Partition of number 4 is $(3,2)$. Also, 4 can be partitioned in different ways:

(a) $\lambda = (4)$

(b) $\lambda = (3,1)$

(c) $\lambda = (2,1,1)$

(d) $\lambda = (1,1,1,1)$

therefore $p(4) = 5$ (a total of 5 partitions for 4).

Notation: $\lambda \vdash n$ means that λ is a partition of n

We see that for different numbers there are different partitions, so naturally arise the question: what is the number of partitions of a given number n?. This lead us to the next definition

Definition. $p(n)$ is the number of partitions of n, and is called **partition function**

Remark:

- $p(n) \equiv 0 \quad \forall n \in \mathbb{Z}_-$
- $p(0) \equiv 1 \rightarrow \lambda = (\quad)$ empty sequence
- $n = 1 \rightarrow \lambda = (1) \rightarrow p(1) = 1$
- $n = 2 \rightarrow \lambda = (2), \lambda = (1,1) = (1^2) \rightarrow p(2) = 2$
- $n = 4 \rightarrow$

  (a) $\lambda = (4)$

(b) $\lambda = (3,1)$

(c) $\lambda = (2,2) = (2^2)$

(d) $\lambda = (2,1,1) = (2,1^2)$

(e) $\lambda = (1,1,1,1) = (1^4)$

$\rightarrow p(4) = 5$

Through an analogous reasoning, we can prove that

$$p(6) = 11 \tag{2.3}$$
$$p(10) = 42 \tag{2.4}$$
$$p(20) = 627 \tag{2.5}$$
$$p(100) = 190569292 \tag{2.6}$$
$$p(200) = 3972999029388 \tag{2.7}$$

in other words, it increases very rapidly.

It has been discovered relations that the number of partitions of certain sets are related, which justifies the next definitions.

Definition. Let $\mathscr{J}$ denote the set of all partitions

It means that $\forall \lambda \equiv (\lambda_1, \ldots, \lambda_k) \rightarrow \lambda \in \mathscr{J}$,

Definition. Let $p(\mathscr{S}, n)$ denote the number of partitions of n that belong to a subset $\mathscr{S} \subset \mathscr{J}$

Example: Let $\mathscr{O}$ be the set of all partitions with odd parts, and let $\mathscr{D}$ be the set of all partitions with distinct parts, therefore

- $n = 1, \quad \lambda = (1) \rightarrow p(\mathscr{O}, 1) = 1$
- $n = 2, \quad \lambda = (2), \lambda = (1,1) \rightarrow p(\mathscr{O}, 2) = 1$
- $n = 3, \quad \lambda = (3), \lambda = (2,1), \lambda = (1,1,1) \rightarrow p(\mathscr{O}, 3) = 2$
- $n = 1, \quad \lambda = (1) \rightarrow p(\mathscr{D}, 1) = 1$
- $n = 2, \quad \lambda = (2), \lambda = (1,1) \rightarrow p(\mathscr{D}, 2) = 1$
- $n = 3, \quad \lambda = (3), \lambda = (2,1), \lambda = (1,1,1) \rightarrow p(\mathscr{D}, 3) = 2$

Sugesting that the number of partitions of a number with odd parts is equal to the number of partitions with different parts. Immediately we ask: how many partitions have any number n? or in how many ways can be partitioned that number?

2.3 Infinite Product Generating Functions of one variable

The search of a formula to specify the number of partitions of a given $n \in \mathbb{N}$ leads to generating functions of partitions, which is not exactly a formula capable to predict the number of partitions.

We are going to present a historical overview of the development of this branch, in order to understand the generating functions.

2.3.1 Historical Overview.

In September 1740 Euler received a letter from Philippe Naude asking (among other things) how to determine the number of ways in which a given positive integer can be expressed as a sum of positive integers [9]. The problem of expressing a positive number as the sum of others gave birth to the so called *additive number theory*

Historically, it is said that Euler, after many attempts to develop a formula to calculate the number of partitions, observed that the product

$$(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots = 1+\sum p(n)x^n \tag{2.8}$$

expands in a polynomial whose coefficients $p(n)$ match with the number of partitions of the exponent n. And, in particular, he observed that

$$(1-x)(1-x^2)\cdots(1-x^{51}) = \sum_i a_i x^i \tag{2.9}$$

where the coefficients $a_i = +1, 0, -1$, which can be generalized in

$$\prod_{j=1}^{\infty}(1-x^j) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k+1)/2} \tag{2.10}$$

where $k(3k+1)/2 \forall k \in \mathbb{N}$ are called pentagonal numbers (already studied by the ancient Greek mathematicians). This result is known as the Euler´s pentagonal numbers theorem.

It is worth mentioning that those products represent the only method known to calculate the number of partitions. The polynomial generated by that product lead to the Generating Functions.

2.3.2 Formal Construction

The generating functions represent the most powerful tool in partitions theory, since no physicist or mathematician has been able to find a direct procedure to calculate partitions. We are going to present at a high detailed level the construction of generating functions

Definition. The Generating Function [10] of the sequence $\{a_n(x)\}$ of numbers or functions is an expression of the form [1]

$$F(x,w) = \sum_{n=0}^{\infty} a_n(x)w^n. \tag{2.11}$$

Definition. Let H be a set of positive integers, i.e., $H \subset \mathbb{Z}_+$. Thus, we define

$$\text{”}H\text{”} \equiv \{\Gamma_1,...,\Gamma_i|\Gamma_r = (\lambda_1,...,\lambda_j) \rightarrow \lambda_r \in H\}$$

and $p(\text{”}H\text{”},n)$ represents the number of partitions of n that have all their parts in H. Note that ”H” is the set of partitions whose parts are in H.

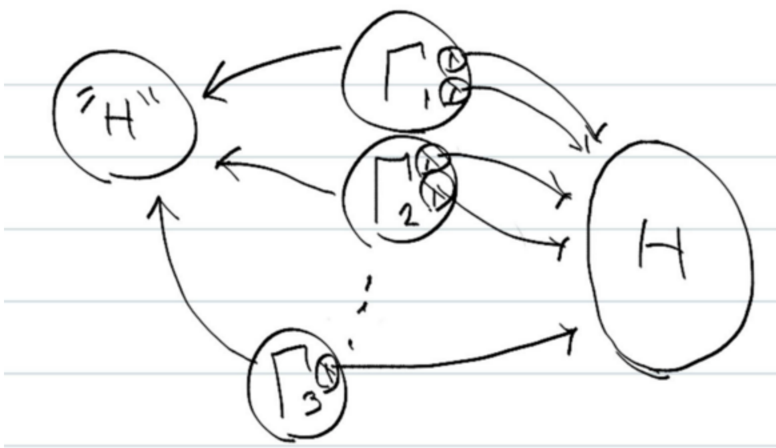
Definition. Let $H \subset \mathbb{Z}_+$. We let ”H”$(\leq d)$ denote the set of all partitions in which no part appears more than d times and each part is in H, that is,

$$\text{”}H\text{”}(\leq d) \equiv \{\Gamma_1,...,\Gamma_n|\Gamma_r = (...\lambda_r^n,...) \rightarrow n \leq d \text{ and } \lambda_r \in H\}$$

See Figure 1. **Example:** Let $\mathbb{N} \subset \mathbb{Z}_+ \rightarrow p[\text{”}N\text{”}(\leq 1),n] = p(\mathscr{D},n)$.

Suppose that $n = 3$, $p[\text{”}N\text{”}(\leq 1),3]$ would represent the number of partitions of 3 whose parts are in N and appears only one time, specifically $(3),(2,1)$, i.e.,$p[\text{”}N\text{”}(\leq 1),3] = 2$, that is going to be the same number of partitions of 3 whose parts are different, as expected, because the partitions of three whose parts are different are $(3),(2,1)$, therefore $p(\mathscr{D},3) = 2$, and finally $p(\mathscr{D},3) = p[\text{”}N\text{”}(\leq 1),3] = 2$.

Figure 1 – Observe that ”H” is made of all the sets whose elements are partitions whose elements are in H



What we are going to present is considered one of the greatest achievement in the partition theory. Warning, we present two versions of the theorem.

[1] Formal power series is a generalization of a polynomial, where the number of terms is allowed to be infinite, one may think of a formal power series as a power series in which we ignore questions of convergence by not assuming that the variable x denotes any numerical value (not even an unknown value)

Theorem 2.3.1. *(Theorem of Euler) For $|x| < 1$ we have*

$$\prod_{m=1}^{\infty} \frac{1}{1-x^m} = \sum_{n=0}^{\infty} p(n)x^n, \tag{2.12}$$

where $p(0) = 1$

Proof: Based on the fact that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad \forall x \in \mathbb{R} : |x| < 1. \tag{2.13}$$

Consider the product

$$\begin{aligned}
\prod_{m=1}^{\infty} \frac{1}{1-x^m} &= \frac{1}{1-x^1} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \\
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \implies \qquad &= \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} x^{2n}\right)\left(\sum_{n=0}^{\infty} x^{3n}\right) \cdots \\
&= (1 + x^{1\cdot 1} + x^{2\cdot 1} + x^{3\cdot 1} + \cdots)(1 + x^{1\cdot 2} + x^{2\cdot 2} + x^{3\cdot 2} + \cdots) \times \cdots \\
&\quad \cdots \times (1 + x^{1\cdot 3} + x^{2\cdot 3} + x^{3\cdot 3} + \cdots) \cdots \\
&= 1 + x^{1\cdot 1} + x^{2\cdot 1} + x^{3\cdot 1} + \cdots + x^{1\cdot 2} + x^{1\cdot 1}x^{1\cdot 2} + x^{2\cdot 1}x^{1\cdot 2} + x^{3\cdot 1}x^{1\cdot 2} + \cdots \\
&\cdots + x^{2\cdot 2} + x^{1\cdot 1}x^{2\cdot 2} + x^{2\cdot 1}x^{2\cdot 2} + x^{3\cdot 1}x^{2\cdot 2} + \cdots
\end{aligned}$$

observe that

$$x^{2\cdot 1} = x^{1+1} \tag{2.14}$$
$$x^{1\cdot 2} = x^2 \tag{2.15}$$
$$x^{3\cdot 1} = x^{1+1+1} \tag{2.16}$$
$$x^{1\cdot 3} = x^3 \tag{2.17}$$
$$x^1 x^{1\cdot 2} = x^{1+2} \tag{2.18}$$
$$\vdots = \vdots \tag{2.19}$$

therefore

$$\prod_{m=1}^{\infty} \frac{1}{1-x^m} = 1 + x^1 + \underbrace{x^{1+1} + x^2}_{\text{parts of 2}} + \underbrace{x^{1+1+1} + x^{1+2} + x^3}_{\text{parts of 3}} + \cdots \tag{2.20}$$

$$\tag{2.21}$$

defining $p(0) \equiv 0$, we conclude that

$$\boxed{\prod_{m=1}^{\infty} \frac{1}{1-x^m} = \sum_{n=0}^{\infty} p(n)x^n.} \tag{2.22}$$

Note that we conjectured (based on the recurrence) that the coefficients of x^n are the number of partitions of n, however there are no guarantees of the convergence of the series Next we present a complete proof.

Theorem 2.3.2. *Let $H \subset \mathbb{Z}_+$ and let*

$$f(q) \equiv \sum_{n\geq 0} p(\text{"}H\text{"}, n) q^n$$

$$f_d(q) \equiv \sum_{n\geq 0} p(\text{"}H\text{"}(\leq d), n) q^n,$$

Therefore, $\forall |q| < 1$

- $f(q) = \prod_{n\in H}(1-q)^{n-1}$
- $f_d(q) = \prod_{n\in H}(1+q^n+...+q^{dn}) = \prod_{n\in H}(1-q^{(d+1)n}(1-q^n)^{-1}$

Proof: Let $H = \{h_1, h_2, ...\}$

$$\prod_{n\in H}(1-q^n)^{-1} = \prod_{n\in H}(1+q^n+q^{2n}+...) \tag{2.23}$$

$$= (1+q^{h_1}+q^{2h_1}+\cdots)(1+q^{h_2}+q^{2h_2}+\cdots)(1+q^{h_3}+q^{2h_3}+\cdots)\cdots \tag{2.24}$$

$$= \sum_{a_1=0}^{\infty}(q^{h_1})^{a_1}\cdot\sum_{a_2=0}^{\infty}(q^{h_2})^{a_2}\cdot\sum_{a_3=0}^{\infty}(q^{h_3})^{a_3}\cdots \tag{2.25}$$

$$= \sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\sum_{a_3=0}^{\infty}\cdots q^{a_1h_1+a_2h_2+a_3h_3+\cdots} \tag{2.26}$$

observe that the exponent of q is precisely the partition $(h_1^{a_1}, h_2^{a_2}, h_3^{a_3}, \cdots)$, thus we can rewrite

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\sum_{a_3=0}^{\infty}\cdots q^{a_1h_1+a_2h_2+a_3h_3+\cdots} = \sum_{n=0}^{\infty} p_n q^n \tag{2.27}$$

being $a_1h_1+a_2h_2+a_3h_3+\cdots$ a partition of n and p_n the number of times that the exponent n appears as a power, therefore

$$p_n = p(\text{"}H\text{"}, n) \implies \prod_{n\in H}(1-q^n)^{-1} = \sum_{n=0}^{\infty} p(\text{"}H\text{"}, n) q^n \tag{2.28}$$

in particular, if $H = \mathbb{Z}_+$ then $p(\text{"}H\text{"}, n) = p(n)$, therefore we can write

$$\prod_{n\in H}(1-q^n)^{-1} = \sum_{n=0}^{\infty} p(n) q^n \tag{2.29}$$

which means that the number of parts of n, i.e., $p(n)$ can be calculated from that expression. To prove the second part of the theorem, we consider

$$f_d(q) = \prod_{n\in H}(1+q^n+\cdots+q^{dn}) \tag{2.30}$$

$$= (1+q^{h_1}+\cdots+q^{dh_1})(1+q^{h_2}+\cdots+q^{dh_2})\cdots \tag{2.31}$$

$$= \sum_{a_1=0}^{d} q^{a_1h_1}\cdot\sum_{a_2=0}^{d} q^{a_2h_2}\cdots \tag{2.32}$$

$$= \sum_{a_1=0}^{d}\sum_{a_2=0}^{d}\cdots q^{a_1h_1+a_2h_2\cdots} = \sum_{n=0}^{\infty} p_n q^n \tag{2.33}$$

Thus, p_n is the number of times that the sum $a_1h_1+a_2h_2+\cdots$ will appear and where $a_i \leq d$, will not appear more than d times. Therefore:

$$\prod_{n\in H}(1+q^n+\cdots+q^{dn}) = \sum_{n=0}^{\infty} p(\text{"}H\text{"}(\leq d),n)q^n \tag{2.34}$$

This theorem is extremely powerful. It allows to calculate the number of partitions of numbers whose parts have specific characteristics.

Generating Function	The number of partitions of n into parts which are
$\prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}$	odd
$\prod_{n=1}^{\infty}\frac{1}{1-x^{2n}}$	even
$\prod_{n=1}^{\infty}\frac{1}{1-x^{n^2}}$	squares
$\prod_{n\in p}\frac{1}{1-x^{p}}$	primes
$\prod_{n=1}^{\infty}(1+x^n)$	unequal
$\prod_{n\in p}(1+x^n)$	distinct primes

Theorem 2.3.3. Corollary of Euler*:*

$$p(\mathscr{O},n) = p(\mathscr{D},n) \quad \forall n\in\mathbb{N}$$

Proof: Given $H_0 \equiv \{\text{all odd integers}\} \implies \mathscr{O} = \text{"}H_0\text{"}$. By the last theorem

$$\sum_{n=0}^{\infty} p(\mathscr{O},n)q^n = \prod_{n\in H_0}(1-q^n)^{-1} \tag{2.35}$$

$$= \prod_{n=1}^{\infty}(1-q^{2n-1})^{-1} \tag{2.36}$$

on the other hand, $\mathscr{D} = \{\text{all partitions with distinct parts}\}$. Thus, $H_1 \equiv \{1,2,3,\ldots\} \implies \text{"}H_1\text{"} = \mathscr{D}$. Taking into account that $d=1$

$$\sum_{n=0}^{\infty} p(\text{"}H_1\text{"}(\leq 1),n)q^n = \prod_{n\in H_1}(1+q^n+\cdots+q^{dn}) = \prod_{n\in H_1}(1+q^n) \tag{2.37}$$

$$\sum_{n=0}^{\infty} p(\mathscr{D},n)q^n = \prod_{n=1}^{\infty}(1+q^n)\cdot\frac{(1-q^n)}{(1-q^n)} = \prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^n} \tag{2.38}$$

considering the fact that

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^n} =^{?} \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}} \tag{2.39}$$

$$\implies \sum_{n\geq 0} p(\mathscr{O},n)q^n = \sum_{n\geq 0} p(\mathscr{D},n)q^n \tag{2.40}$$

since expansion series of a function is unique we conclude that

$$\boxed{p(\mathscr{O},n) = p(\mathscr{D},n).}$$

2.4 The Hardy-Ramanujan Asymptotic Partition Formula

In the search of an exact formula for $p(n)$, Hardy and Ramanujam found a formula which approximates the asymptotic behavior. Details about the formula can be found in [5]. This famous formula is

$$p(n) \approx \frac{1}{4n\sqrt{3}}\exp\left(\pi\sqrt{\frac{2}{3}n}\right). \tag{2.41}$$

We will show later how this formula can be deduced via String Theory and Statistical Mechanics. Also, this formula has been exploited in several areas of physics, especially String Theory and Statistical Mechanics.

It is actually known an exact expression for $p(n)$. According to the theorem 5.1 in [5]

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty}A_k(n)k^{1/2}\left[\frac{d}{dx}\frac{\sinh((\pi/k)(\frac{2}{3}(x-1/24))^{1/2})}{(x-1/24)^{1/2}}\right]_{x=n} \tag{2.42}$$

It was discovered in 1937 by Rademacher. Due to the complexity of the formula it is not widely used compared to the Formula of Hardy-Ramanujan. In fact, theres is still some obscurity in relation to the formula of Rademacher.

3 Applications

We will try to show how the Partition Theory, and in particular, the generating functions can be applied in Theoretical Physics.

This subject has already been considered before, and it is still used, especially to research super-gravity and super-symmetric black holes (see [1] for details). The application of Partition theory has already been used in quantum mechanics. For example, we can see in [2] an analogy between counting the number of partitions and number of micro-states of a quantum system of bosonic harmonic oscillators.

3.1 Harmonic Oscillator

Consider the system of one-dimensional quantum harmonic oscillators with frequency ω obeying the Bose statistics. Let us count the number of ways to distribute the energy E in this system. Each oscillator can occupy energy levels given by $E_j = \hbar\omega(j+1/2)$, where the ground-state energy can be dropped off yielding

$$E_j = \hbar\omega j, \quad j = 0, 1, 2, \ldots$$

A system of oscillators with $E = 3\hbar\omega$ have the next possible configurations

- One particle into the level $j = 3$.
- One particle to the state $j = 2$, and another one into $j = 1$.
- Three particles into the state $j = 1$.

It is interesting to note that the order of the summands is not significant in partitions is equivalent to the indistinguishably of quantum particles. Thus, given a system with a given energy E, the number of particles and its levels of energy are limited by it. The possible combination of creation operators is estimated by the theory of partitions. From the vacuum state $|\Omega >$, the possible states of the system are

- $(a_3^\dagger)|\Omega >$
- $(a_2^\dagger)(a_1^\dagger)|\Omega >$
- $(a_1^\dagger)^3|\Omega >$

Thus, we can see that the way in which the creation operators can be mixed can be estimated from the ways the energy can be partitioned.

Recall that the quantization of the scalar field gives the state of the system in terms of creation and annihilation operators, that is, ([3] pg. 309)

$$(a^\dagger_{p_1})^{n_1}(a^\dagger_{p_2})^{n_2}\cdots(a^\dagger_{p_k})^{n_k}\,|\Omega\rangle \tag{3.1}$$

is a state which contains n_1 particles with momentum $\vec{p}_1$, n_2 with momentum $\vec{p}_2$, and so on. Let us take into account that

$$|n\rangle = (a^\dagger)^n\,|\Omega\rangle \implies H\,|n\rangle = \left(n+\frac{1}{2}\right)\omega\,|n\rangle \tag{3.2}$$

therefore, with $|n\rangle \equiv (a^\dagger_{p_1})^{n_1}(a^\dagger_{p_2})^{n_2}\cdots(a^\dagger_{p_k})^{n_k}\,|\Omega\rangle$

$$\begin{aligned} H\,|n\rangle &= \left[\left(n_1+\frac{1}{2}\right)\omega_{p_1}+\left(n_2+\frac{1}{2}\right)\omega_{p_2}+\cdots+\left(n_k+\frac{1}{2}\right)\omega_{p_k}\right]|n\rangle && (3.3)\\ &= \left[(n_1\omega_{p_1}+n_2\omega_{p_2}+\cdots+n_k\omega_{p_k})+\frac{1}{2}\left(\omega_{p_1}+\omega_{p_2}+\cdots+\omega_{p_k}\right)\right]|n\rangle && (3.4)\end{aligned}$$

if $\omega_{p_1}=\omega_{p_2}\cdots=\omega_{p_k}\equiv\omega$, then

$$H\,|n\rangle = \left[(n_1+n_2+\cdots+n_k)+\frac{1}{2}k\right]\omega\,|n\rangle \tag{3.5}$$

we can see that the total energy is $E=(n_1+n_2+\cdots+n_k)+\frac{1}{2}k$, where the sum of the integers $n_1+n_2+\cdots+n_k$ fix the energy of the system, but not the configuration. Thus, the $n_i s$ are limited by the energy of the system.

3.2 Counting States

How to count the number of states that a string has at any given mass level can be approached via Partitions Theory [3]. There exists an exact formula for $p(n)$ presented in [5] although extremely complicated. However the Theory of Partitions is capable to predict the number of combinations through the generating functions, i.e., an expression of the form

$$f(x)=\sum_{n=0}^{\infty}a(n)x^n \tag{3.6}$$

where $a(n)$ is the number of states with $N^\perp=n$, where $N^\perp$ represents the auto value of the number operator $\hat{N}^\perp\equiv n\hat{a}^\dagger_n\hat{a}_n$

Thus, if we have only one oscillator $a^\dagger_1$, then there is just one state $|0\rangle$ with $N^\perp=0$, one state $a^\dagger_1|0\rangle$ with $N^\perp=1$ and so on. Thus the coefficients of x^k in

$$f(x)=1+x+x^2+x^3\cdots \tag{3.7}$$

gives the number of states with level $N^\perp=k$ (which in this case is $1\forall k$)

Now, suppose a system made of one oscillator $a_2^\dagger$ with mode number two. In this mode number we can have the oscillator in $|0\rangle$ with $N^\perp = 0$, in $a_2^\dagger|0\rangle$ with $N^\perp = 2$, and so on. Through the same reasoning, the coefficients of x^{2k} in

$$f_2(x) = 1 + x^2 + x^4 + \cdots \tag{3.8}$$

gives the number of states with $N^\perp = 2k$ (wich, again, it is $1 \forall k$).

Now, consider a system composed of both oscillators $a_1^\dagger$ and $a_2^\dagger$, what is the number of states for a given mass level $N^\perp = k$? Example: Observe that for $N^\perp = 3$ we have the states $(a_1^\dagger)^3|0\rangle$, $a_1^\dagger a_2^\dagger|0\rangle$, that is to say, two states. To say what is the number of states fo any given $N^\perp$ is not possible, since in theory of partitions does not posses a formula to answer that question, the only way is via the generating functions. Therefore, the generating function capable to say (thorough the coefficients) the number of states with $N^\perp = k$ is

$$f_{12}(x) = f_1(x)f_2(x) = (1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots) \tag{3.9}$$

$$= 1 + x + x^2 + x^3 + \cdots + x^2 + x^3 + x^4 + x^5 + \cdots + \tag{3.10}$$

$$\cdots + x^4 + x^5 + x^6 + x^7 + \cdots \tag{3.11}$$

$$= 1 + x + 2x^2 + 2x^3 + \cdots \tag{3.12}$$

we can see that the coefficient of x^3 (for which $N^\perp = 3$) is two (which is the number of states). Note that the generating function is reduced to

$$f_{12}(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \tag{3.13}$$

To see how mix the oscillator operators consider

$$(1 + a_1^\dagger x + (a_1^\dagger)^2 x^2 + \cdots)(1 + a_2^\dagger x^2 + (a_2^\dagger)^4 x^4 + \cdots) \tag{3.14}$$

Finally, if we have $a_1^\dagger, a_2^\dagger, a_3^\dagger, \ldots$ of all mode numbers, the generating function would be of the kind

$$f(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} \tag{3.15}$$

3.3 Application to Bosonic Open String.

In the theory of the bosonic open string[3] a general state $|\lambda\rangle$ is specified by

$$|\lambda\rangle = \prod_{n=1}^{\infty} \prod_{I=2}^{25} \left(a_n^{I\dagger}\right)^{\lambda_{n,I}} |p^+, \vec{p}_T\rangle \tag{3.16}$$

where $\left|p^+, \vec{p}_T\right\rangle$ is the ground state, and the number $\lambda_{n,I}$ denotes the number of times that the creation operator $a_n^{I\dagger}$ appears.

It was proved that $D = 26$, but with $D-2 = 24$ transverse light-cone directions, with oscillators of all modes for each direction, and taking into account that each specie of oscillator gives a generating function, we have that the total generating function is

$$f(x) = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{24}} \tag{3.17}$$

In a generating function based on $\alpha' M^2$ the coefficient of x^k counts the number of states with $\alpha' M^2 = k$. We know from [3] that for open bosonic strings $\alpha' M^2 = N^{\perp} - 1$ so the $\alpha' M^2$ generating function is obtained by dividing the $N^{\perp}$ generating function by one power of x, therefore

$$f_{os}(x) = \frac{1}{x} \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{24}} \tag{3.18}$$

i.e.,

$$f_{os}(x) = \frac{1}{x} + 24 + 324x + 3200x^2 + 25650x^3 + \cdots \tag{3.19}$$

thus, the coefficient of $x = -1$ in which $\alpha' M^2 = -1$ is 1, one tachyonic state. $\alpha' M^2 = 0 \to M = 0$ have coefficient 24 and therefore 24 massless states (Maxwell field), and 324 states with $\alpha' M^2 = +1$.

3.4 Neveu-Shwarz sector.

We know that the Neveu-Shwarz sector is a sector of the space states of super-symmetric strings, and therefore it contains fermionic oscillators. Any state is a linear combination of

$$|\lambda\rangle = \prod_{I=2}^{9} \prod_{n=1}^{\infty} \left(a_{-n}^{I}\right)^{\lambda_{n,I}} \prod_{J=2}^{9} \prod_{r=\frac{1}{2},\frac{3}{2},\ldots} \left(b_{-r}^{J}\right)^{\rho_{r,J}} |NS\rangle \otimes \left|p^+, \vec{p}_T\right\rangle \tag{3.20}$$

To specify the levels of states of the Neveu-Schwars sector is not an easy task. The first levels are, specifying the eigenvalues of $N^{\perp}$ and mass squared operator, we have

$$\alpha' M^2 = -\frac{1}{2}, \quad N^{\perp} = 0 \quad \to \quad |NS\rangle \otimes \left|p^+, \vec{p}_T\right\rangle \tag{3.21}$$

$$\alpha' M^2 = 0, \quad N^{\perp} = \frac{1}{2} \quad \to \quad b_{-1/2}^{I} |NS\rangle \otimes \left|p^+, \vec{p}_T\right\rangle \tag{3.22}$$

$$\alpha' M^2 = \frac{1}{2}, \quad N^{\perp} = 1 \quad \to \quad \{\alpha_{-1}^{I}, b_{-1/2}^{I} b_{-1/2}^{J}\} |NS\rangle \otimes \left|p^+, \vec{p}_T\right\rangle \tag{3.23}$$

$$\alpha' M^2 = 1, \quad N^{\perp} = \frac{3}{2} \quad \to \quad \{\alpha_{-1}^{I} b_{-1/2}^{J}, b_{-3/2}^{I}, b_{-1/2}^{I} b_{-1/2}^{J} b_{-1/2}^{K}\} |NS\rangle \otimes \left|p^+, \vec{p}_T\right\rangle \tag{3.24}$$

showing first only one tachyonic state, secondly eight massless states. counting the rest can be done using partitions theory.

Containing fermionic oscillators b_{-r}^{J}, which contributes to the operator number $N^{\perp}$. With a single fermionic creation operator b_{-r} the contribution is r to $N^{\perp}$, producing to possible states:

$|0\rangle$ and $b_{-r}|0\rangle$. Therefore, the generating function is

$$f_r(x) = 1 + x^r \tag{3.25}$$

since the Neveu-Schwarz sector contains oscillators $b^I_{-1/2}, b^I_{-3/2}, \ldots$ in eight species, the generating function associated is

$$\left[(1+x^{1/2})(1+x^{3/2})(1+x^{5/2})\cdots\right]^8 = \prod_{n=1}^{\infty}(1+x^{n-\frac{1}{2}})^8 \tag{3.26}$$

taking into account that $\alpha' M^2 = N^{\perp} - \frac{1}{2}$ and that we have eight bosonic coordinates as well, the generating function $f_{NS}(x)$ is given by

$$f_{NS}(x) = \frac{1}{\sqrt{x}} \prod_{n=1}^{\infty} \left(\frac{1+x^{n-\frac{1}{2}}}{1-x^n} \right)^8 \tag{3.27}$$

after expansion we obtain

$$f_{NS}(x) = \frac{1}{\sqrt{x}} + 8 + 36\sqrt{x} + 128x + 402x^{3/2} + 1152x^2 + \cdots \tag{3.28}$$

showing the tachyon at $\alpha' M^2 = -1/2$, eight massless states, and 36 states at $\alpha' M^2 = 1/2$.

Note that the theory of partitions can be used in the context of harmonic oscillator, this is because the auto values of some operators, such as the Hamiltonian or the number operator, have auto values related to integers. Therefore, wherever there are addition of numbers the partitions theory will deal with the problem.

4 Partitions and the Quantum Violin String.

Another interesting application of Partitions Theory arise in the context of String Theory. The quantum violin string is a quantum mechanical non-relativistic string with fixed endpoints. A good background on this subject can be found on[3] [1]

4.1 Quantum Violin String (V.S.) (approximation approach)

The Hamiltonian of the quantum violin string is given by

$$H \equiv \hbar\omega_0 \sum_{l=1}^{\infty} l a_l^\dagger a_l \tag{4.1}$$

where $a_l^\dagger$ and a_l are the well known creation annihilation operators. From the Hamiltonian can be recognize the number operator, which is

$$N \equiv \sum_{l=1}^{\infty} l a_l^\dagger a_l \tag{4.2}$$

Being $|\Omega\rangle$ the vacuum state of the string, a quantum state $|\Psi\rangle$ is obtained by letting creation operators to act

$$|\Psi\rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\cdots(a_l^\dagger)^{n_l}\cdots|\Omega\rangle \tag{4.3}$$

Given an Energy, say $E = 4$, it can be distributed among oscillators in different ways, i.e.,

$$E = 4 \to [4],[3,1],[2,2],[2,1,1],[1,1,1,1] \tag{4.4}$$

Ω represents number of configurations, which in this case is the number of partitions of E. Thus

$$\Omega = P(N) \to S = k\ln[P(N)] \tag{4.5}$$

$H \equiv \hbar\omega_0 N \to E = \hbar\omega_0 N$, therefore

$$S = k\ln[P(E/\hbar\omega_0)] \tag{4.6}$$

4.1.1 Partition Function

Statistical Mechanics tells us that

$$Z = \sum_i e^{-\beta E_i} \tag{4.7}$$

[1] For details about the quantum violin string see: A first course in String Theory, Barto Zwiebach, offering an elementary approach to String Theory.[3]

For Quantum Violin String

$$Z = \sum_N \exp\left[\frac{E_N}{kT}\right] \tag{4.8}$$
$$= \sum_{n_1,n_2,\dots} \exp\left[\frac{\hbar\omega_0}{kT}(n_1+2n_2+\cdots)\right] \tag{4.9}$$
$$= \sum_{n_1}\exp\left[\frac{\hbar\omega_0}{kT}n_1\right]\cdot\sum_{n_2}\exp\left[\frac{\hbar\omega_0}{kT}2n_1\right]\cdots \tag{4.10}$$
$$= \prod_{l=1}^{\infty}\sum_{n_l}\exp\left[\frac{\hbar\omega_0}{kT}ln_l\right] \tag{4.11}$$

$$\boxed{Z = \prod_{l=1}^{\infty}\left[1-\exp\left(-\frac{\hbar\omega_0 l}{kT}\right)\right]^{-1}}. \tag{4.12}$$

Note that this formula has exactly the form of the Euler's theorem if we substitute x for $\exp\left(-\frac{\hbar\omega_0 l}{kT}\right)$.

4.1.2 Free Energy

Here we proceed to carry out an approximation of the free energy

$$F = -kT\ln Z \tag{4.13}$$
$$= kT\sum_{l=1}^{\infty}\ln\left[1-\exp\left(-\frac{\hbar\omega_0 l}{kT}\right)\right] \tag{4.14}$$

If l is continuous

$$F \approx kT\int_1^{\infty} dl\ln\left[1-\exp\left(-\frac{\hbar\omega_0 l}{kT}\right)\right] \tag{4.15}$$

$$x \equiv \frac{\hbar\omega_0}{kT}l \implies F\approx kT\int_{\frac{\hbar\omega_0}{kT}=1}^{\infty}\frac{kT}{\hbar\omega}dx\ln\left[1-e^{-x}\right] \tag{4.16}$$

taking into account that

$$\ln(1-x) = -\left(x+\frac{1}{2}x^2+\cdots\right) \implies \ln(1-e^{-x}) = -\left(e^{-x}+\frac{1}{2}e^{-2x}+\cdots\right) \tag{4.17}$$

therefore

$$F \approx \frac{(kT)^2}{\hbar\omega_0}\int_0^{\infty}dx\left(e^{-x}+e^{-2x}+e^{-3x}\cdots\right) \tag{4.18}$$
$$\approx -\frac{(kT)^2}{\hbar\omega_0}\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] \tag{4.19}$$
$$\approx -\frac{(kT)^2}{\hbar\omega_0}\zeta(2) = -\frac{1}{\hbar\omega_0}\frac{\pi^2}{6}\frac{1}{\beta^2} \tag{4.20}$$

4.1.3 Entropy

Based on the free energy, let us proceed to calculate the entropy, which will be connected to the number of configurations

$$S = -\frac{\partial F}{\partial T} \approx k\frac{\pi^2}{3}\frac{kT}{\hbar\omega_0} \tag{4.21}$$

but taking into account that

$$E = -\frac{\partial \ln Z}{\partial \beta} = \frac{\partial}{\partial \beta}(\beta F) \tag{4.22}$$

$$\Longrightarrow E = \frac{\pi^2}{6}\left(\frac{kT}{\hbar\omega_0}\right)^2 \Longrightarrow \boxed{\left(\frac{kT}{\hbar\omega_0}\right) = \sqrt{\frac{6E}{\pi^2\hbar\omega_0}}} \tag{4.23}$$

substituting in the expression for the entropy

$$\boxed{S \approx k\pi\sqrt{\frac{2}{3}N}} \tag{4.24}$$

But the Boltzmann entropy is given by

$$S = k\ln[p(N)] \implies \ln[p(N)] \approx \pi\sqrt{\frac{2}{3}N} \tag{4.25}$$

$$\boxed{p(N) \approx \exp\left(\pi\sqrt{\frac{2}{3}N}\right)} \tag{4.26}$$

which is the leading term of the Hardy-Ramanujan formula

4.2 Approach without Approximation.

In the last section the approximation was done through the logarithm

$$F = kT\sum_{l=1}^{\infty}\ln\left[1-\exp\left(-\frac{\hbar\omega_0}{kT}l\right)\right] \tag{4.27}$$

$$S = -\frac{\partial F}{\partial T} = -k\sum_{l=1}^{\infty}\ln\left[1-\exp\left(-\frac{\hbar\omega_0}{kT}l\right)\right] + \frac{\hbar\omega_0}{T}\sum_{l=1}^{\infty}\frac{l\exp\left(-\frac{\hbar\omega_0}{kT}l\right)}{1-\exp\left(-\frac{\hbar\omega_0}{kT}l\right)} \tag{4.28}$$

with $x \equiv \hbar\omega_0\beta l$

$$\Longrightarrow \boxed{S = \sum_{\frac{x}{\hbar\omega_0\beta}=1}^{\infty}\left[\frac{xe^{-x}}{1-e^{-x}} - \ln(1-e^{-x})\right]} \tag{4.29}$$

Trying to carry out the same reasoning as before, we consider that

$$E = \frac{\partial}{\partial \beta}(\beta F) = \frac{\partial}{\partial \beta}\sum_{l=1}^{\infty} \ln\left[1-\exp\left(-\frac{\hbar\omega_0}{kT}l\right)\right] \tag{4.30}$$

$$= \sum_{l=1}^{\infty}\left[1-\exp\left(-\hbar\omega_0\beta l\right)\right]^{-1}(-1)\exp\left(-\hbar\omega_0\beta l\right)\cdot\left(-\hbar\omega_0 l\right) \tag{4.31}$$

$$= \hbar\omega_0 \sum_{l=1}^{\infty} \frac{l\exp\left(-\hbar\omega_0\beta l\right)}{1-\exp\left(-\hbar\omega_0\beta l\right)} \tag{4.32}$$

$$= \frac{1}{\beta}\sum_{\frac{x}{\hbar\omega_0\beta}=1}^{\infty} \frac{xe^{-x}}{1-e^{-x}} \tag{4.33}$$

$$\implies S = k\beta E - k\sum_{\frac{x}{\hbar\omega_0\beta}=1}^{\infty} \ln(1-e^{-x}) \tag{4.34}$$

If we consider that

$$E = \frac{\partial}{\partial \beta}\sum_{\frac{x}{\hbar\omega_0\beta}=1}^{\infty} \ln(1-e^{-x}) \implies \sum_{\frac{x}{\hbar\omega_0\beta}=1}^{\infty} \ln(1-e^{-x}) = \int d\beta \cdot E(\beta) \tag{4.35}$$

The entropy with no approximation can be writen as

$$\boxed{S = \frac{\hbar\omega_0}{T}N - k\int d\beta E(\beta)} \tag{4.36}$$

It is interesting to note that in [2] was obtained an expression that relates statistical mechanics and partitions given by

$$s(\beta) = \beta E + \ln Z(\beta) = \beta E - \sum_{j=1}^{\infty} \ln\left(1-e^{-\beta j}\right) \tag{4.37}$$

which is in complete agreement with the expression for the entropy found in the context of the violin string (taking $k = 1$).

4.2.1 Some considerations of the Difficulties.

The entire problem with the no-approximation-approach is

$$\sum_{l=1}^{\infty} \frac{l\exp(-\hbar\omega_0\beta l)}{1-\exp(-\hbar\omega_0\beta l)} =? \tag{4.38}$$

and ultimately is the Free energy, i.e.,

$$\boxed{\sum_{l=1}^{\infty} \ln\left[1-\exp(-\hbar\omega_0\beta l)\right] =?} \tag{4.39}$$

The last expression has been studied and is related to the **Dedekind Eta Function** $\eta(\tau)$**:**

$$\sum_{l=1}^{\infty} \ln\left[1-\exp(-\hbar\omega_0\beta l)\right] = \frac{\hbar\omega_0\beta}{24} + \ln\left[\eta\left(\frac{\hbar\omega_0\beta i}{2\pi}\right)\right] \tag{4.40}$$

Therefore, the entropy can actually be calculated with ”complete" precision

$$S = k\beta E - k\left[\frac{\hbar\omega_0\beta}{24} + \ln\left[\eta\left(\frac{\hbar\omega_0\beta i}{2\pi}\right)\right]\right] \tag{4.41}$$

where

$$\eta(\tau) \equiv e^{\frac{i\pi\tau}{12}}\prod_{n=1}^{\infty}\left(1 - e^{2ni\pi\tau}\right) \tag{4.42}$$

However, the problem also lies on the Partition function

$$\boxed{Z = \prod_{l=1}^{\infty}\left[1 - \exp\left(-\frac{\hbar\omega_0 l}{kT}\right)\right]^{-1}} \tag{4.43}$$

limiting the calculations and physical interpretations.

4.3 Partition Function (statistical Mechanics) and Partition Function (Theory of Partitions)

Here we use directly the generating functions in thermodynamics

$$Z = \prod_{l=1}^{\infty}\left[1 - \exp\left(-\frac{\hbar\omega_0 l}{kT}\right)\right]^{-1} \tag{4.44}$$

It is interesting to note that in the context of the theory of partitions we have that

$$f(x) = \prod_{n=0}^{\infty}(1 - x^n)^{-1} = \sum_{n=0}^{\infty} p(n)x^n \tag{4.45}$$

Thus, taking $x \equiv \exp(-\hbar\omega_0/kT)$ we obtain that the partition function in thermodynamic context is

$$Z = \sum_{n=0}^{\infty} p(n)x^n = \sum_{n=0}^{\infty} p(n)\exp\left(-\frac{\hbar\omega_0}{kT}n\right) \tag{4.46}$$

We might think that it is a simpler way to write and manipulate the partition function, and theoretically It would allow us to obtain thermodynamical results. well, Let us see

4.3.1 Free energy

By definition

$$F = -kT\ln Z = -kT\ln\left[\sum_{n=0}^{\infty} p(n)\exp\left(\frac{-\hbar\omega_0}{kT}n\right)\right] \tag{4.47}$$

therefore the entropy is

$$S = -\frac{\partial F}{\partial T} = k\ln\left[\sum_{n=0}^{\infty} p(n)\exp\left(\frac{-\hbar\omega_0}{kT}n\right)\right] + kT\frac{\partial}{\partial T}\ln\left[\sum_{n=0}^{\infty} p(n)\exp\left(\frac{-\hbar\omega_0}{kT}n\right)\right] \tag{4.48}$$

and the energy

$$E = \frac{\partial}{\partial \beta}(\beta F) = -\frac{\partial}{\partial \beta} \ln \left[\sum_{n=0}^{\infty} p(n) \exp\left(\frac{-\hbar\omega_0}{kT} n \right) \right] \tag{4.49}$$

if we take into account that $S = k\ln(p(N)$, then we have that

$$\ln(p(N)) = \ln \left[\sum_{n=0}^{\infty} p(n) \exp\left(\frac{-\hbar\omega_0}{kT} n \right) \right] + T\frac{\partial}{\partial T} \ln \left[\sum_{n=0}^{\infty} p(n) \exp\left(-\frac{\hbar\omega_0}{kT} n \right) \right] \tag{4.50}$$

or

$$p(N) = \left[\sum_{n=0}^{\infty} p(n) \exp\left(\frac{-\hbar\omega_0}{kT} n \right) \right] \cdot \exp\left[T\frac{\partial}{\partial T} \ln \left[\sum_{n=0}^{\infty} p(n) \exp\left(-\frac{\hbar\omega_0}{kT} n \right) \right] \right] \tag{4.51}$$

It appears to give **no clue, but it is very interesting** the relation between $p(N)$ and $p(n)$.

4.4 Alternative approach to the Formula of Hardy-Ramanujan

Here we basically reproduce the deduction of the formula of Hardy-Ramanudam based on [2]. Let $\Gamma(E)$ be the number of micro-states with energy E, then

$$Z(\beta) = \sum_j e^{-\beta E_j} = \int_0^{\infty} \Gamma(E) e^{-\beta E} dE \tag{4.52}$$

Note in the last expression that $Z(E)$ is the Laplace transform of $\Gamma(E)$. Thus, the inverse Laplace transform is

$$\Gamma(E) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} Z(\beta) e^{\beta E} d\beta \tag{4.53}$$

$$\begin{aligned} \rightarrow S(\beta) &= k \ln \Gamma(E) && (4.54) \\ &= k \ln \left[\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} Z(\beta) e^{\beta E} d\beta \right] && (4.55) \end{aligned}$$

The strategy lies on the fact that

$$S(\beta) = \beta E + \ln Z(\beta) \implies Z(\beta) = e^{S(\beta) - \beta E} \tag{4.56}$$

and the number of micro-states is obtained by

$$\Gamma(E) = \frac{e^{S(\beta_0)}}{\sqrt{2\pi S''(\beta_0)}} \tag{4.57}$$

and through this approach, we take the substitution

$$E \rightarrow n, \qquad \Gamma(E) \rightarrow p(n) \tag{4.58}$$

The partition function of a system of one-dimensional harmonic oscillators obeying the Bose statistics

$$Z = \prod_{j=1}^{\infty} \frac{1}{1 - e^{\beta\hbar\omega j}} \tag{4.59}$$

taking $\hbar\omega \equiv 1$. Therefore, the entropy is

$$S(\beta) = \beta E - \sum_{j=1}^{\infty} \ln\left(1 - e^{-\beta j}\right) \tag{4.60}$$

using the Euler-Maclaurin Formula

$$\sum_{j=1}^{\infty} f(j) = \int_1^{\infty} f(x)dx + \frac{f(1)}{2} - \frac{1}{12}f'(1) + \dots \tag{4.61}$$

in the approximation that $\beta \to 0$ the first term becomes

$$\ln\left(1 - e^{-\beta x}\right) = -\left(e^{-\beta x} + \frac{1}{2}e^{-2\beta x} + \frac{1}{3}e^{-3\beta x} + \dots\right) \tag{4.62}$$

$$\int_0^{\infty} \ln\left(1 - e^{-\beta x}\right) dx = -\int_0^{\infty} \left(e^{-\beta x} + \frac{1}{2}e^{-2\beta x} + \frac{1}{3}e^{-3\beta x} + \dots\right) dx \tag{4.63}$$

$$= -\frac{1}{\beta}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) = -\frac{1}{\beta}\zeta(2) = -\frac{\pi^2}{6\beta} \tag{4.64}$$

next term

$$f(x) = \ln\left(1 - e^{-\beta x}\right) \tag{4.65}$$

$$= \ln\beta x - \frac{\beta x}{2} + \dots \tag{4.66}$$

$$\Longrightarrow f(1) = \ln\beta - \frac{1}{2}\beta \tag{4.67}$$

the last term

$$f'(x) = \frac{1}{x} - \frac{\beta}{2} \tag{4.68}$$

$$\Longrightarrow f'(1) = 1 - \frac{1}{2}\beta \tag{4.69}$$

substituting we obtain that

$$\sum_{j=1}^{\infty} \ln\left(1 - e^{-\beta j}\right) = -\frac{\pi^2}{6\beta} + \frac{1}{2}\ln\beta - \frac{1}{4}\beta - \frac{1}{12} + \frac{1}{24}\beta \tag{4.70}$$

$$= -\frac{\pi^2}{6\beta} + \frac{1}{2}\ln\beta - \frac{5}{24}\beta - \frac{1}{12} \tag{4.71}$$

Therefore the entropy is reduced to

$$S(\beta) = \beta E + \frac{\pi^2}{6\beta} - \frac{1}{2}\ln\beta + \frac{5}{24}\beta + \frac{1}{12}\dots \tag{4.72}$$

$$= \frac{\pi^2}{6\beta} - \frac{1}{2}\ln\beta + \beta\left(E + \frac{5}{24}\right) \tag{4.73}$$

If we consider $E >> 0$ we get

$$S(\beta) = \frac{\pi^2}{6\beta} - \frac{1}{2}\ln\beta + \beta E \tag{4.74}$$

$$S'(\beta) = -\frac{\pi^2}{6\beta^2} - \frac{1}{2\beta} + E = 0 \tag{4.75}$$

as $\beta \to 0$ we can write

$$E = \frac{\pi^2}{6\beta^2} \implies \beta_0 = \frac{\pi}{\sqrt{6E}} \tag{4.76}$$

and therefore

$$S''(\beta_0) = \frac{3\pi^2}{12\beta_0^3} = \frac{2\sqrt{6}}{\pi} E^{3/2} \tag{4.77}$$

Substituting for $\Gamma(E)$ we obtain

$$\Gamma(E) = \frac{e^{S(\beta_0)}}{\sqrt{2\pi S''(\beta_0)}} \tag{4.78}$$

$$S(\beta_0) = 2\pi\sqrt{\frac{E}{6}} \implies \quad \Gamma(E) = \frac{e^{\pi\sqrt{2E/3}}}{4\sqrt{3}E} \tag{4.79}$$

changing $\Gamma(E) \to p(n)$ and $E \to n$, we finally obtain

$$p(n) = \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n} \tag{4.80}$$

Which is exactly the formula of Hardy-Ramanujam, obtained after a very rude approximation of Statistical Mechanics.

4.5 Correction of the Formula of Hardy-Ramanujan

Here we present an approximation of $p(n)$ that surpassed the Formula of Hardy-Ramanujan. Taking the expression for $Z(\beta)$ in $\Gamma(E)$ we obtain that

$$\Gamma(E) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{S(\beta)} d\beta \tag{4.81}$$

Let β_0 be a stationary point, such that $S'(\beta_0) = 0$, we consider the Taylor series in the vicinity of β_0

$$S(\beta) \approx S(\beta_0) + \frac{1}{2!}S''(\beta_0)(\beta-\beta_0)^2 + \frac{1}{3!}S'''(\beta_0)(\beta-\beta_0)^3 \tag{4.82}$$

substituting the approximation of $S(\beta)$ in the expression for $\Gamma(E)$

$$\Gamma(E) \approx \frac{e^{S(\beta_0)}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \exp\left[\frac{1}{2!}S''(\beta_0)(\beta-\beta_0)^2 + \frac{1}{3!}S'''(\beta_0)(\beta-\beta_0)^3\right] d\beta \tag{4.83}$$

taking $\beta = ix + \beta_0$ the integral is reduced to

$$\Gamma(E) \approx \frac{e^{S(\beta_0)}}{2\pi i}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2!}S''(\beta_0)x^2 - \frac{1}{3!}S'''(\beta_0)ix^3\right] dx \tag{4.84}$$

$$\approx \frac{e^{S(\beta_0)}}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2!}S''(\beta_0)x^2}\left[\cos\left(\frac{S'''(\beta_0)}{3!}x^3\right) - i\sin\left(\frac{S'''(\beta_0)}{3!}x^3\right)\right] dx \tag{4.85}$$

$$\approx \frac{e^{S(\beta_0)}}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2!}S''(\beta_0)x^2}\cos\left(\frac{S'''(\beta_0)}{3!}x^3\right) dx \tag{4.86}$$

taking for granted that

$$\int_{-\infty}^{\infty} e^{-bx^2} cos(ax^3)dx = \frac{2be^{2b^3/27a^2} K_{1/3}\left(\frac{2b^3}{27a^2}\right)}{3\sqrt{3}|a|} \tag{4.87}$$

with $a = S'''(\beta_0)/6$ and $b = \frac{1}{2}S''(\beta_0)$ we obtain

$$\Gamma(E) = \frac{e^{S(\beta_0)}}{2\pi} \frac{2S''(\beta_0)}{\sqrt{3}|S'''(\beta_0)|} \exp\left[\frac{(S''(\beta_0))^3}{3\,(S'''(\beta_0))^2}\right] K_{1/3}\left(\frac{(S''(\beta_0))^3}{3\,(S'''(\beta_0))^2}\right) \tag{4.88}$$

where $K_\nu(x)$ is the Bessel function of the second kind, in complete agreement with [11]. As stated before

$$S(\beta) = S(\beta_0) + \frac{1}{2!}S''(\beta_0)(\beta-\beta_0)^2 + \frac{1}{3!}S'''(\beta_0)(\beta-\beta_0)^3 \tag{4.89}$$

and taking into account that

$$S(\beta_0) = 2\pi\sqrt{\frac{E}{6}} \tag{4.90}$$

$$S''(\beta_0) = \frac{2\sqrt{6}}{\pi}E^{3/2} \tag{4.91}$$

$$S'''(\beta_0) = -\frac{36}{\pi^2}E^2 \tag{4.92}$$

therefore

$$S(\beta) = 2\pi\sqrt{\frac{E}{6}} + \frac{\sqrt{6}}{\pi}E^{3/2}\left(\beta - \frac{\pi}{\sqrt{6E}}\right)^2 + \frac{9}{\pi^2}E^2\left(\beta - \frac{\pi}{\sqrt{6E}}\right)^3 \tag{4.93}$$

the argument of the Bessel function becomes

$$\frac{(S''(\beta_0))^3}{3\,(S'''(\beta_0))^2} = \frac{\left(\frac{2\sqrt{6}}{\pi}E^{3/2}\right)^3}{3\left(-\frac{36}{\pi^2}E^2\right)^2} = \frac{48\pi\sqrt{6}}{3888}E^{1/2} = \frac{\pi}{27}\sqrt{\frac{2}{3}E} \tag{4.94}$$

Now, the argument of the exponential

$$S(\beta_0) + \frac{(S''(\beta_0))^3}{3\,(S'''(\beta_0))^2} = 2\pi\sqrt{\frac{E}{6}} + \frac{\pi}{27}\sqrt{\frac{2}{3}E} \tag{4.95}$$

$$= \pi\sqrt{\frac{2}{3}E} + \frac{\pi}{27}\sqrt{\frac{2}{3}E} \tag{4.96}$$

$$= \frac{28}{27}\pi\sqrt{\frac{2}{3}E} \tag{4.97}$$

substituting, we obtain

$$\Gamma(E) = \frac{1}{18\sqrt[4]{E^{3/4}}} e^{\frac{28}{27}\pi\sqrt{\frac{2}{3}E}} K_{1/2}\left(\frac{1}{27}\pi\sqrt{\frac{2}{3}E}\right) \tag{4.98}$$

changing $E \to n$ and $\Gamma(E) \to p(n)$

$$p(n) = \frac{1}{18\sqrt[4]{n^{3/4}}} e^{\frac{28}{27}\pi\sqrt{\frac{2}{3}n}} K_{1/2}\left(\frac{1}{27}\pi\sqrt{\frac{2}{3}n}\right) \tag{4.99}$$

offering a better approximation of the of $p(n)$.

5 Multipartite Generating Functions

In this chapter we present an extension of the Partition theory to higher dimensions[5], a subject extensively used in [12]. Basically, instead of considering numbers and studying its partitions we are going to consider vectors made of integers, and study the partitions associated to these vectors.

5.1 Vector Composition

Definition. A partition of $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a set of vectors $(\beta_1^{(i)}, \beta_2^{(i)}, \ldots, \beta_r^{(i)}), 1 \leq i \leq s$ such that

$$\sum_{i=1}^{s} (\beta_1^{(i)}, \beta_2^{(i)}, \ldots, \beta_r^{(i)}) = (\alpha_1, \alpha_2, \ldots, \alpha_r)$$

if the order of the parts is taken into account we call $(\beta_1^{(1)}, \ldots, \beta_r^{(1)}), \ldots, (\beta_1^{(s)}, \ldots, \beta_r^{(s)})$ a composition of $(\alpha_1, \ldots, \alpha_r)$, that is,

$$(\alpha_1, \alpha_2, \ldots, \alpha_r) \equiv \{(\beta_1^{(i)}, \beta_2^{(i)}, \ldots, \beta_r^{(i)}), 1 \leq i \leq s | \sum_{i=1}^{s} (\beta_1^{(i)}, \ldots, \beta_r^{(i)}) = (\alpha_1, \ldots, \alpha_r)\} \quad (5.1)$$

Definition. We let $P = (\alpha_1, \ldots, \alpha_r; m)$ denote the number of partitions with m parts, and we let $c(\alpha_1, \ldots, \alpha_r; m)$ denote the number of compositions of $(\alpha_1, \ldots, \alpha_r)$ with m parts. **Example:** $P(2,1,1;2) = 5$, since there are five partitions of $(2,1,1)$ into two parts

(a) $(2,1,0)(0,0,1)$

(b) $(2,0,1)(0,1,0)$

(c) $(2,0,0)(0,1,1)$

(d) $(1,1,0)(1,0,1)$

(e) $(1,1,1)(1,0,0)$

Definition. We let $P(\alpha_1, \ldots, \alpha_r) \equiv$ total number of partitions of $(\alpha_1, \ldots, \alpha_r)$

Note that

$$P(\alpha_1, \ldots, \alpha_r) = \sum_{m \geq 1} P(\alpha_1, \ldots, \alpha_r; m)$$

and let $c(\alpha_1, \ldots, \alpha_r) \equiv$ total number of compositions. Note that

$$c(\alpha_1, \ldots, \alpha_r) = \sum_{m \geq 1} c(\alpha_1, \ldots, \alpha_r; m)$$

5.2 Multipartite Generating Functions

In an analogous way, with the purpose to answer the number of partitions of a given vector, it has been developed the Multipartite Generating functions.

Definition. Let $P(\mathbf{n}) = P(n_1,\dots,n_r)$ denote the number of partitions of $\mathbf{n} \equiv (n_1,...,n_r)$, i.e.,

$$\mathbf{n} = \xi^{(1)} + \xi^{(2)} + \dots + \xi^{(r)} \tag{5.2}$$

$$\xi^{(i)} = (\xi_1^{(i)}, \xi_2^{(i)}, \dots, \xi_r^{(i)}), \quad \xi^{(i)} \geq \xi^{(j)} \quad \forall i,j \in \mathbb{R} \tag{5.3}$$

$,\forall \xi^{(i)} \leq j \to P_{\leq}(\mathbf{n};j)$, where $P_{\leq}(\mathbf{n};j)$ represents the number of parts of the partition with elements less than j.

Let $Q(\mathbf{n})$ denote the number of partitions of $\mathbf{n}$ and into distinct parts, and $Q(n,j)$ the number of such partitions with j parts. Thus, we can announce a generalized theorem based on Euler´s pentagonal theorem.

Theorem 5.2.1.

$$\sum_{n_1,\dots,n_r \geq 0} P(\mathbf{n}) x_1^{n_1} \cdots x_r^{n_r} = \prod_{n_1 \geq 0,\cdots,n_r \geq 0} (1 - x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r})^{-1} \tag{5.4}$$

$$\sum_{n_1,\dots,n_r \geq 0} Q(\mathbf{n}) x_1^{n_1} \cdots x_r^{n_r} = \prod_{n_1 \geq 0,\cdots,n_r \geq 0} (1 - x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}) \tag{5.5}$$

with $|x_l| \leq 1$

5.3 Cheema's Extension of Euler's Theorem.

Theorem 5.3.1. *For every* $\mathbf{n}$*,* $Q_+(\mathbf{n})$ *equals* $\mathcal{O}(\mathbf{n})$*, the number of partitions of* $\mathbf{n}$ *in which each part* $(\xi_1^{(i)},\dots,\xi_r^{(i)})$ *has at least one odd component* Proof.

$$1 + \sum_{(\mathbf{n} > (0))} Q_+(\mathbf{n}) x_1^{n_1} \cdots x_r^{n_r} = \prod_{(\mathbf{n}) > (0)} (1 + x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}) \tag{5.6}$$

$$= \prod_{(\mathbf{n}) > (0)} \frac{\left(1 - x_1^{2n_1} x_2^{2n_2} \cdots x_r^{2n_r}\right)}{(1 - x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r})} \tag{5.7}$$

$$= \sum_{(\mathbf{n} > (0))} (1 - x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r})^{-1} \tag{5.8}$$

$$= \sum_{(\mathbf{n} > (0))} \mathcal{O}(\mathbf{n}) x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r} \tag{5.9}$$

$\to Q_+(\mathbf{n}) = \mathcal{O}(\mathbf{n}) \quad \forall \mathbf{n}$

5.4 Bell Polynomials

Bell Polynomials arise naturally from differentiating a composite function n-times. Suppose that $h(t) \equiv f(g(t))$, hence

$$\frac{dh}{dt} = \frac{d}{dt} f(g(t)) = \frac{df}{dg}\frac{dg}{dt} \tag{5.10}$$

$$\frac{d^2h}{dt^2} = \frac{d}{dt}\left(\frac{df}{dg}\frac{dg}{dt}\right) \tag{5.11}$$

$$= \frac{d}{dt}\left(\frac{df}{dg}\right)\frac{dg}{dt} + \frac{df}{dt}\frac{d^2g}{dt^2} \tag{5.12}$$

$$= \frac{d^2f}{dg^2}\left(\frac{dg}{dt}\right)^2 + \frac{df}{dt}\frac{d^2g}{dt^2} \tag{5.13}$$

Defining

$$h_n \equiv \frac{d^nh}{dt^n}, \qquad f_n \equiv \frac{d^2f}{dg^n}, \qquad g_n \equiv \frac{d^ng}{dt^n} \tag{5.14}$$

we can write

$$h_1 = f_1g_1 \tag{5.15}$$

$$h_2 = f_2g_1^2 + f_1g_2 \tag{5.16}$$

$$h_3 = g_3f_1 + (3g_1g_2)f_2 + g_1^3f_3 \tag{5.17}$$

$$\vdots \tag{5.18}$$

$$h_n = \sum_{k=1}^{n} B_{nk}(g_1,\ldots,g_{n-k+1})f_k \tag{5.19}$$

where the coefficients $B_{nk}(g_1,\ldots,g_{n-k+1})$ of the derivatives f_k are the partial Bell Polynomials.

Definition. The complete Bell Polynomial is

$$Y_n(g_1,\ldots,g_n) = \sum_{k=1}^{n} B_{nk}(g_1,\ldots,g_{n-k+1}) \tag{5.20}$$

taking $h(t) \equiv e^{g(t)}$ then

$$\boxed{Y_n(g_1,\ldots,g_n) = e^{-g(t)}\frac{d^ne^{g(t)}}{dt^n}} \tag{5.21}$$

which completely specified the Bell's Polynomials, providing the important recurrence formula:

$$Y_{n+1}(g_1,\ldots,g_{k+1}) = e^{-g}D^n(De^g) \tag{5.22}$$

$$= e^{-g}D^n(g_1De^g) \tag{5.23}$$

$$= \sum_{k=0}^{n}\binom{n}{k}(e^{-g}D^{n-k}e^g)D^kg_1 \tag{5.24}$$

$$= \sum_{k=0}^{n}\binom{n}{k}Y_{n-k}(g_1,\ldots,g_{k+1} \tag{5.25}$$

The first complete Bell Polynomials are:

$$B_0 = 1, \tag{5.26}$$

$$B_1(x_1) = x_1, \tag{5.27}$$

$$B_2(x_1,x_2) = x_1^2 + x_2, \tag{5.28}$$

$$B_3(x_1,x_2,x_3) = x_1^3 + 3x_1x_2 + x_3, \tag{5.29}$$

$$B_4(x_1,x_2,x_3,x_4) = x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4, \tag{5.30}$$

$$B_5(x_1,x_2,x_3,x_4,x_5) = x_1^5 + 10x_2x_1^3 + 15x_2^2x_1 + 10x_3x_1^2 + 10x_3x_2 + 5x_4x_1 + x_5 \tag{5.31}$$

$$\tag{5.32}$$

This expression allows to obtain the generating function of the Bell polynomials [5]

$$\mathfrak{B} = \sum_{n=0}^{\infty} \frac{Y_n u^n}{n!} \tag{5.33}$$

Leading to

$$\log \mathfrak{B} = \sum_{n=1}^{\infty} \frac{g_n u^n}{n!} \tag{5.34}$$

If we exponentiate and expand the infinite product of exponential functions, we obtain

$$Y_n(g_1,\ldots,g_n) = \sum_{(\mathbf{k})\vdash n} \frac{n!}{k_1!\cdots k_n!} \left(\frac{g_1}{1!}\right)^{k_1} \left(\frac{g_2}{2!}\right)^{k_2} \cdots \left(\frac{g_n}{n!}\right)^{k_n} \tag{5.35}$$

an expression that we will use in the next section.

5.5 Bell Polynomials in Multipartite Partition Problems

Based on [4], let

$$\mathscr{P}_j(x_1,\ldots,x_r) = \mathscr{P}_j = 1 + \sum_{(\mathbf{n})>0} P_{\leq}(\mathbf{n};j) x_1^{n_1} \cdots x_r^{n_r} \tag{5.36}$$

$$\mathscr{Q}_j(x_1,\ldots,x_r) = \mathscr{Q}_j = 1 + \sum_{(\mathbf{n})>0} Q(\mathbf{n};j) x_1^{n_1} \cdots x_r^{n_r} \tag{5.37}$$

$$F(u) = 1 + \sum_{j=1} \mathscr{P}_j u^j \tag{5.38}$$

$$G(u) = 1 + \sum_{j=1}^{\infty} \mathscr{Q}_j u^j \tag{5.39}$$

where the coefficients $\mathscr{P}_j$ are given by [4]

$$\mathscr{P}_j = Y_j\left(0!\beta_r(1), 1!\beta_r(2), 2!\beta_r(3), 3!\beta_r(4), ..., (j-1)!\beta_r(j)\right)/j! \tag{5.40}$$

where

$$\beta_j(m) \equiv \prod_{i=1}^{j}(1-x_i^m)^{-1}. \tag{5.41}$$

Now, consider

$$F(u) \equiv \prod_{\mathbf{n}\geq 0}(1-ux_1^{n_1}x_2^{n_2}\cdots x_r^{n_r})^{-1} \tag{5.42}$$

$$G(u) \equiv \prod_{\mathbf{n}\geq 0}(1+ux_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}) \tag{5.43}$$

Therefore

$$\begin{aligned}
\log F(u) &= -\sum_{\mathbf{n}\geq 0}\log(1-ux_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}) && (5.44)\\
&= \sum_{m=1}^{\infty}\sum_{\mathbf{n}\geq 0}\frac{u^m x_1^{n_1 m}x_2^{n_2 m}\cdots x_r^{n_r m}}{m} && (5.45)\\
&= \sum_{m=1}^{\infty}\frac{u^m}{m}(1-x_1^m)^{-1}(1-x_2^m)^{-1}\cdots(1-x_r^m)^{-1} && (5.46)\\
&= \sum_{m=1}\frac{u^m}{m!}(m-1)!\beta_r(m) && (5.47)
\end{aligned}$$

Showing manifestly a generating function form.

5.6 General Formula

According to [12], the generating function $F(z,X)$ can be written as

$$F(z,X) = 1+\sum_{j=1}^{\infty}\mathscr{P}_j(x_1,x_2,\ldots,x_m)z^j \tag{5.48}$$

We developed a formula to calculate the $\mathscr{P}_j$ coefficients.

It is actually possible to write a generalized formula for $\mathscr{P}_j$. We take for granted from [5] (equation (12.3.7)), that

$$Y_n(g_1,\ldots,g_n) = \sum_{(k)\vdash n}\frac{n!}{k_1!\cdots k_n!}\left(\frac{g_1}{1!}\right)^{k_1}\left(\frac{g_2}{2!}\right)^{k_1}\cdots\left(\frac{g_n}{n!}\right)^{k_n} \tag{5.49}$$

using the expression of $\mathscr{P}_j$ given by the theorem we can write

$$\begin{aligned}
\mathscr{P}_j &= \frac{1}{j!}Y_j\left(0!\beta_r(1),1!\beta_r(2),2!\beta_r(3),3!\beta_r(4),...,(j-1)!\beta_r(j)\right) && (5.50)\\
&= \frac{1}{j!}\sum_{(k)\vdash j}\frac{j!}{k_1!\cdots k_j!}\left(\frac{0!\beta_r(1)}{1!}\right)^{k_1}\left(\frac{1!\beta_r(2)}{2!}\right)^{k_1}\cdots\left(\frac{(j-1)!\beta_r(j)}{j!}\right)^{k_j} && (5.51)
\end{aligned}$$

and substituting explicitly $\beta_r(m)$

$$\mathscr{P}_j = \sum_{(k)\vdash j} \frac{1}{k_1!\cdots k_j!} \left(\frac{0!\beta_r(1)}{1!}\right)^{k_1} \left(\frac{1!\beta_r(2)}{2!}\right)^{k_2} \cdots \left(\frac{(j-1)!\beta_r(j)}{j!}\right)^{k_j} \tag{5.52}$$

$$= \sum_{(k)\vdash j} \frac{1}{k_1!\cdots k_j!} \left(\frac{0!}{1!}\prod_{i=1}^{r}(1-x_i^1)^{-1}\right)^{k_1} \left(\frac{1!}{2!}\prod_{i=1}^{r}(1-x_i^2)^{-1}\right)^{k_2} \cdots \tag{5.53}$$

$$\cdots \left(\frac{(j-1)!}{j!}\prod_{i=1}^{r}(1-x_i^j)^{-1}\right)^{k_j} \tag{5.54}$$

$$= \sum_{(k)\vdash j} \prod_{n=1}^{j} \frac{1}{k_n!} \left(\frac{(n-1)!}{n!}\prod_{i=1}^{r}(1-x_i^n)^{-1}\right)^{k_n} \tag{5.55}$$

$$= \sum_{(k)\vdash j} \prod_{n=1}^{j} \frac{1}{k_n!}\frac{1}{n^{k_n}} \prod_{i=1}^{r}(1-x_i^n)^{-k_n} \tag{5.56}$$

$$\boxed{\mathscr{P}_j = \sum_{(k)\vdash j} \prod_{n=1}^{j} \frac{1}{k_n!}\frac{1}{n^{k_n}} \frac{1}{\prod_{i=1}^{r}(1-x_i^n)^{k_n}},} \tag{5.57}$$

representing a very compact formula. We present examples about how to use it in the appendix.

5.7 $\mathscr{P}_j$ in terms of $\mathscr{R}(s)$.

Based on [4], $\mathscr{R}(s)$ is the so called Ruelle spectral function, where they showed that

$$\beta_m(n) = \frac{\mathscr{R}(s = n(m+1)(1-i\rho(\vartheta))+1-n)}{\mathscr{R}(s = n(1-i\rho(\vartheta))+1-n)}. \tag{5.58}$$

As we know from eq. (6,48)

$$\mathscr{P}_j = \sum_{(k)\vdash j} \frac{1}{k_1!\cdots k_j!} \left(\frac{0!\beta_r(1)}{1!}\right)^{k_1} \left(\frac{1!\beta_r(2)}{2!}\right)^{k_2} \cdots \left(\frac{(j-1)!\beta_r(j)}{j!}\right)^{k_j} \tag{5.59}$$

$$= \sum_{(k)\vdash j} \prod_{n=1}^{j} \frac{1}{k_n!} \left(\frac{\beta_r(n)}{n}\right)^{k_n} \tag{5.60}$$

$$\boxed{\mathscr{P}_j = \sum_{(k)\vdash j} \prod_{n=1}^{j} \frac{1}{k_n!\cdot n^{k_n}} \left[\frac{\mathscr{R}(s = n(r+1)(1-i\rho(\vartheta))+1-n)}{\mathscr{R}(s = n(1-i\rho(\vartheta))+1-n)}\right]^{k_n}.} \tag{5.61}$$

Expressions of this kind is being used in the context of hiperbolic geometry and eliptic genera.

6 Conclusion and Perspectives

We showed how Partition Theory can be applied in Quantum Mechanics, Statistical Physics, and String Theory. We observed how Quantum Mechanics and String Theory benefited form partitions theory. In the case of Quantum Mechanics, it uses partitions directly to estimate the degeneracy of the states, particularly the harmonic oscillator. String Theory uses the combinatorial power of the partition theory, in particular it uses the Generating Functions to estimate the number of states of the string. Recall that the state of the string depends on a set of creation-operators, as the energy of the string increases the number of operators and their combination also increases, making difficult to count the number of states of the string in each level. But Partitions theory offers a quick solution.

The case of Statistical Mechanics resulted to be the opposite. It appears that Statistical Mechanics helps Partitions Theory. We showed how after rude approximations in the partition function and free energy leads to the formula of Hardy-Ramanujan. What is more, it was possible to improve the Hardy-Ramanujan approximation via Statistical Mechanics.

On the other hand, we tried to exploit the Bose-Einstein and Fermi-Dirac Statistics to obtain results about Partitions Theory. We did not succeed in that purpose, but we hope that future researchers, considering our approach and taking into account our difficulties, may obtain fruitful results.

Finally, after exploring the Multipartite Generating functions, we developed a formula to produce the polynomials that are the coefficients of the generating functions. The polynomials are generated by the Bell's polynomials, where a general formula was presented with examples.

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Appendix

APPENDIX A – Examples of $\mathscr{P}_j$ the coefficients.

Here we present first a calculation of $\mathscr{P}_j$ using facts of chapter 5. In section 2 we generate a compact formula to calculate $\mathscr{P}_j$, testing it with additional examples.

A.1 Examples

$$\mathscr{P}_j = Y_j\left(0!\beta_r(1), 1!\beta_r(2), 2!\beta_r(3), 3!\beta_r(4), ..., (j-1)!\beta_r(j)\right) / j! \tag{A.1}$$

where

$$\beta_j(m) \equiv \prod_{i=1}^{j}(1-x_i^m)^{-1} \tag{A.2}$$

A.1.1 Example 1: $\mathscr{P}_2$

$$\mathscr{P}_2 = \frac{Y_2(0!\beta_r(1), 1!\beta_r(2))}{2!} \tag{A.3}$$

But

$$Y_n(y_1, y_2, ...) = e^{-y}\frac{d^n}{dt^n}e^y \tag{A.4}$$

$$\rightarrow Y_2(y_1, y_2) = e^{-y}\frac{d^2}{dt^2}e^y \tag{A.5}$$

$$= e^{-y}\frac{d}{dt}\left(e^y y_1\right) \tag{A.6}$$

$$= e^{-y}\left(e^y y_1^2 + e^y y_2\right) \tag{A.7}$$

$$= y_1^2 + y_2 \tag{A.8}$$

Therefore

$$\mathscr{P}_2 = \frac{\beta_r^2(1)+\beta_r(2)}{2} \tag{A.9}$$

$$= \frac{1}{2}\left[\left(\prod_{i=1}^{r}(1-x_i)^{-1}\right)^2 + \prod_{i=1}^{r}(1-x_i^2)^{-1}\right] \tag{A.10}$$

$$= \frac{1}{2}\left[\prod_{i=1}^{r}\frac{1}{(1-x_i)^2} + \prod_{i=1}^{r}\frac{1}{(1-x_i)(1+x_i)}\right] \tag{A.11}$$

$$= \frac{1}{2}\prod_{i=1}^{r}\frac{1}{(1-x_i)}\left[\frac{1}{\prod_{i=1}^{r}(1-x_i)} + \frac{1}{\prod_{i=1}^{r}(1+x_i)}\right] \tag{A.12}$$

$$\mathscr{P}_2 = \frac{1}{2}\frac{\prod_{i=1}^{r}(1-x_i)+\prod_{i=1}^{r}(1+x_i)}{\prod_{i=1}^{r}(1-x_i)(1-x_i^2)} \tag{A.13}$$

A.1.2 Example 2: $\mathscr{P}_3$

Through the same procedure

$$\mathscr{P}_3 = Y_3(0!\beta_r(1), 1!\beta_r(2), 2!\beta_r(3))/3! \tag{A.14}$$

But

$$Y_n(y_1, y_2, ...) = e^{-y}\frac{d^n}{dt^n}e^y \tag{A.15}$$

$$\rightarrow Y_3(y_1, y_2, y_3) = e^{-y}\frac{d^3}{dt^3}e^y \tag{A.16}$$

$$= e^{-y}\frac{d^2}{dt^2}\left(e^y y_1\right) \tag{A.17}$$

$$= e^{-y}\frac{d}{dt}\left(e^y y_1^2 + e^y y_2\right) \tag{A.18}$$

$$= e^{-y}\left(e^y y_1^3 + e^y 2y_1y_2 + e^y y_1y_2 + e^y y_3\right) \tag{A.19}$$

$$= y_1^3 + 3y_1y_2 + y_3 \tag{A.20}$$

Therefore

$$Y_3(0!\beta_r(1), 1!\beta_r(2), 2!\beta_r(3)) = \beta_r(1)^3 + 3\beta_r(1)\beta_r(2) + 2!\beta_r(3) \tag{A.21}$$

$$\mathscr{P}_3 = \frac{1}{6}\left[\left(\prod_{i=1}^{r}(1-x_i)^{-1}\right)^3 + 3\left(\prod_{i=1}^{r}(1-x_i)^{-1}\right)\left(\prod_{i=1}^{r}(1-x_i^2)^{-1}\right) + 2\left(\prod_{i=1}^{r}(1-x_i^3)^{-1}\right)\right] \tag{A.22}$$

$$= \frac{1}{6}\left[\prod_{i=1}^{r}\frac{1}{(1-x_i)^3} + 3\prod_{i=1}^{r}\frac{1}{1-x_i}\frac{1}{1-x_i^2} + 2\prod_{i=1}^{r}\frac{1}{1-x_i^3}\right] \tag{A.23}$$

$$= \frac{1}{6}\left[\frac{1}{\prod_{i=1}^{r}(1-x_i)^3} + \frac{3}{\prod_{i=1}^{r}(1-x_i)^2(1+x_i)} + \frac{2}{\prod_{i=1}^{r}(1-x_i^3)}\right] \tag{A.24}$$

$$= \frac{1}{6}\frac{1}{\prod_{i=1}^{r}(1-x_i)^2}\left[\frac{1}{\prod_{i=1}^{r}(1-x_i)} + \frac{3}{\prod_{i=1}^{r}(1+x_i)}\right] + \frac{1}{3\prod_{i=1}^{r}(1-x_i^3)} \tag{A.25}$$

$$= \frac{1}{6}\frac{1}{\prod_{i=1}^{r}(1-x_i)^2}\left[\frac{3\prod_{i=1}^{r}(1-x_i) + \prod_{i=1}^{r}(1+x_i)}{\prod_{i=1}^{r}(1-x_i^2)}\right] + \frac{1}{3\prod_{i=1}^{r}(1-x_i^3)} \tag{A.26}$$

$$= \frac{1}{6}\left[\frac{3\prod_{i=1}^{r}(1-x_i) + \prod_{i=1}^{r}(1+x_i)}{\prod_{i=1}^{r}(1-x_i^2)(1-x_i)^2}\right] + \frac{1}{3\prod_{i=1}^{r}(1-x_i^3)} \tag{A.27}$$

A.1.3 Example 3: $\mathscr{P}_4$

Again, through the same procedure

$$\mathscr{P}_4 = Y_3(0!\beta_r(1), 1!\beta_r(2), 2!\beta_r(3), 3!\beta_r(4))/4! \tag{A.28}$$

But

$$Y_n(y_1, y_2, \ldots) = e^{-y}\frac{d^n}{dt^n}e^y \tag{A.29}$$

$$\to Y_4(y_1, y_2, y_3, y_4) = e^{-y}\frac{d^4}{dt^4}e^y \tag{A.30}$$

$$= e^{-y}\frac{d^3}{dt^3}\left(e^y y_1\right) \tag{A.31}$$

$$= e^{-y}\frac{d^2}{dt^2}\left(e^y y_1^2 + e^y y_2\right) \tag{A.32}$$

$$= e^{-y}\frac{d}{dt}\left(e^y y_1^3 + e^y 2y_1y_2 + e^y y_1y_2 + e^y y_3\right) \tag{A.33}$$

$$= e^{-y}\frac{d}{dt}\left(y_1^3 + 3y_1y_2 + y_3\right)e^y \tag{A.34}$$

$$= e^{-y}\left[(3y_1^2y_2 + 3y_2^2 + 3y_1y_3 + y_4)e^y + \left(y_1^3 + 3y_1y_2 + y_3\right)e^y y_1\right] \tag{A.35}$$

$$= y_4 + 4y_3y_1 + 6y_1^2y_2 + 3y_2^2 + y_1^4 \tag{A.36}$$

$$= y_1^4 + 6y_1^2y_2 + 3y_2^2 + 4y_1y_3 + y_4 \tag{A.37}$$

$$Y_4(0!\beta_r(1),1!\beta_r(2),2!\beta_r(3),3!\beta_r(4)) = \beta_r(1)^4+6\beta_r(1)^2\beta_r(2)+3\beta_r(2)^2+... \tag{A.38}$$

$$...+4\cdot 2\beta_r(1)\beta_r(3)+6\beta_r(4) \tag{A.39}$$

$$= \beta_r(1)^4+6\beta_r(1)^2\beta_r(2)+3\beta_r(2)^2+... \tag{A.40}$$

$$...+8\beta_r(1)\beta_r(3)+6\beta_r(4) \tag{A.41}$$

$$\tag{A.42}$$

Therefore

$$\mathscr{P}_4 = \frac{1}{4!}\left[\left(\prod_{i=1}^{r}(1-x_i)^{-1}\right)^4+6\left(\prod_{i=1}^{r}(1-x_i)^{-1}\right)^2\left(\prod_{i=1}^{r}(1-x_i^2)^{-1}\right)+... \right. \tag{A.43}$$

$$\left. ...+3\left(\prod_{i=1}^{r}(1-x_i^2)^{-1}\right)^2+8\left(\prod_{i=1}^{r}(1-x_i)^{-1}\right)\left(\prod_{i=1}^{r}(1-x_i^3)^{-1}\right)+6\left(\prod_{i=1}^{r}(1-x_i^4)^{-1}\right)\right] \tag{A.44}$$

APPENDIX B – Calculation of $\mathscr{P}_j$ through the General Formula.

B.1 Calculation of $\mathscr{P}_2$

$$\mathscr{P}_2 = \sum_{(k)\vdash 2} \prod_{n=1}^{2} \frac{1}{k_n!} \frac{1}{n^{k_n}} \frac{1}{\prod_{i=1}^{r}(1-x_i^n)^{k_n}} \tag{B.1}$$

$$= \sum_{(k)\vdash 2} \frac{1}{k_1!} \frac{1}{1^{k_1}} \frac{1}{\prod_{i=1}^{r}(1-x_i^1)^{k_1}} \frac{1}{k_2!} \frac{1}{2^{k_2}} \frac{1}{\prod_{i=1}^{r}(1-x_i^2)^{k_2}} \tag{B.2}$$

$$\tag{B.3}$$

observe that the sum runs over the partitions of 2. From the first chapter of Andrew's book

$$p(2) = 2: \quad 1+1 = (1^2), \quad 2 = (2) \tag{B.4}$$

from which is clear that $(k) = (k_1 = 2, k_2 = 0), (k_1 = 0, k_2 = 1)^1$, therefore

$$\mathscr{P}_2 = \frac{1}{2!} \frac{1}{1^2} \frac{1}{\prod_{i=1}^{r}(1-x_i)^2} \frac{1}{0!} \frac{1}{2^0} \frac{1}{\prod_{i=1}^{r}(1-x_i^2)^0} + \frac{1}{0!} \frac{1}{1^0} \frac{1}{\prod_{i=1}^{r}(1-x_i)^0} \frac{1}{1!} \frac{1}{2^1} \frac{1}{\prod_{i=1}^{r}(1-x_i^2)^1} \tag{B.5}$$

$$= \frac{1}{2}\left[\frac{1}{\prod_{i=1}^{r}(1-x_i)^2} + \frac{1}{\prod_{i=1}^{r}(1-x_i^2)}\right] \tag{B.6}$$

rearranging terms in exactly the same way as the first part we obtain:

$$\boxed{\mathscr{P}_2 = \frac{1}{2} \frac{\prod_{i=1}^{r}(1-x_i) + \prod_{i=1}^{r}(1+x_i)}{\prod_{i=1}^{r}(1-x_i)(1-x_i^2)}} \tag{B.7}$$

which is **exactly the same expression shown by [5] eq. (12.3.8)**

B.2 Calculation of $\mathscr{P}_3$ (again)

with $j = 3$ we can write that

$$\mathscr{P}_3 = \sum_{(k)\vdash 3} \prod_{n=1}^{3} \frac{1}{k_n!} \frac{1}{n^{k_n}} \frac{1}{\prod_{i=1}^{r}(1-x_i^n)^{k_n}} \tag{B.8}$$

$$= \sum_{(k)\vdash 3} \frac{1}{k_1!} \frac{1}{1^{k_1}} \frac{1}{\prod_{i=1}^{r}(1-x_i^1)^{k_1}} \frac{1}{k_2!} \frac{1}{2^{k_2}} \frac{1}{\prod_{i=1}^{r}(1-x_i^2)^{k_2}} \frac{1}{k_3!} \frac{1}{3^{k_3}} \frac{1}{\prod_{i=1}^{r}(1-x_i^3)^{k_3}} \tag{B.9}$$

[1] the sum follows a convention in which the base represents the value of n for k_n and the exponent the value of k_n itself. Example: $(1^2) = (1^2 2^0) = (k_1 = 2, k_2 = 0)$

with a similar procedure, the sum runs over the partitions of 3, which are

$$1+1+1=(1^3)\rightarrow(k_1=3,k_2=0,k_3=0) \tag{B.10}$$
$$2+1=(1^1 2^1)\rightarrow(k_1=1,k_2=1,k_3=0) \tag{B.11}$$
$$3=(3^1)\rightarrow(k_1=0,k_2=0,k_3=1) \tag{B.12}$$

therefore, a sum of three terms

$$\mathscr{P}_3=\frac{1}{3!}\frac{1}{1^3}\frac{1}{\prod_{i=1}^r(1-x_i^1)^3}\frac{1}{0!}\frac{1}{2^0}\frac{1}{\prod_{i=1}^r(1-x_i^2)^0}\frac{1}{0!}\frac{1}{3^0}\frac{1}{\prod_{i=1}^r(1-x_i^3)^0}+... \tag{B.13}$$
$$...+\frac{1}{1!}\frac{1}{1^1}\frac{1}{\prod_{i=1}^r(1-x_i^1)^1}\frac{1}{1!}\frac{1}{2^1}\frac{1}{\prod_{i=1}^r(1-x_i^2)^1}\frac{1}{0!}\frac{1}{3^0}\frac{1}{\prod_{i=1}^r(1-x_i^3)^0}+... \tag{B.14}$$
$$...+\frac{1}{0!}\frac{1}{1^0}\frac{1}{\prod_{i=1}^r(1-x_i^1)^0}\frac{1}{0!}\frac{1}{2^0}\frac{1}{\prod_{i=1}^r(1-x_i^2)^0}\frac{1}{1!}\frac{1}{3^1}\frac{1}{\prod_{i=1}^r(1-x_i^3)^1} \tag{B.15}$$
$$=\frac{1}{6}\frac{1}{\prod_{i=1}^r(1-x_i)^3}+\frac{1}{2}\frac{1}{\prod_{i=1}^r(1-x_i)}\frac{1}{\prod_{i=1}^r(1-x_i^2)}+\frac{1}{3}\frac{1}{\prod_{i=1}^r(1-x_i^3)} \tag{B.16}$$
$$=\frac{1}{6}\left[\frac{1}{\prod_{i=1}^r(1-x_i)^3}+3\frac{1}{\prod_{i=1}^r(1-x_i)(1-x_i^2)}+2\frac{1}{\prod_{i=1}^r(1-x_i^3)}\right] \tag{B.17}$$

which is exactly the same result obtained through the other way (compare to eq. 22)

B.3 Calculation of $\mathscr{P}_4$ (again)

Now we are going to face the beast.

$$\begin{aligned}\mathscr{P}_4&=\sum_{(k)\vdash 4}\prod_{n=1}^{4}\frac{1}{k_n!}\frac{1}{n^{k_n}}\frac{1}{\prod_{i=1}^r(1-x_i^n)^{k_n}}\\&=\sum_{(k)\vdash 4}\frac{1}{k_1!}\frac{1}{1^{k_1}}\frac{1}{\prod_{i=1}^r(1-x_i^1)^{k_1}}\frac{1}{k_2!}\frac{1}{2^{k_2}}\frac{1}{\prod_{i=1}^r(1-x_i^2)^{k_2}}\frac{1}{k_3!}\frac{1}{3^{k_3}}\\&\qquad\frac{1}{\prod_{i=1}^r(1-x_i^3)^{k_3}}\frac{1}{k_4!}\frac{1}{4^{k_4}}\frac{1}{\prod_{i=1}^r(1-x_i^4)^{k_4}}\end{aligned}$$

The partitions of 4

$$1+1+1+1=(1^4)\rightarrow(k_1=4,k_2=0,k_3=0,k_4=0) \tag{B.18}$$
$$2+1+1=(1^2 2)\rightarrow(k_1=2,k_2=1,k_3=0,k_4=0) \tag{B.19}$$
$$2+2=(2^2)\rightarrow(k_1=0,k_2=2,k_3=0,k_4=0) \tag{B.20}$$
$$3+1=(13)\rightarrow(k_1=1,k_2=0,k_3=1,k_4=0) \tag{B.21}$$
$$4=(4)\rightarrow(k_1=0,k_2=0,k_3=0,k_4=1) \tag{B.22}$$

Expanding the sum

$$\mathscr{P}_4 = \frac{1}{4!}\frac{1}{1^4}\frac{1}{\prod_{i=1}^r(1-x_i)^4} + \frac{1}{2!}\frac{1}{1^2}\frac{1}{\prod_{i=1}^r(1-x_i)^2}\frac{1}{1!}\frac{1}{2^1}\frac{1}{\prod_{i=1}^r(1-x_i^2)^1} + ... \tag{B.23}$$

$$... + \frac{1}{2!}\frac{1}{2^2}\frac{1}{\prod_{i=1}^r(1-x_i^2)^2} + \frac{1}{1!}\frac{1}{1^1}\frac{1}{\prod_{i=1}^r(1-x_i)^1}\frac{1}{1!}\frac{1}{3^1}\frac{1}{\prod_{i=1}^r(1-x_i^3)^1} + ... \tag{B.24}$$

$$... + \frac{1}{1!}\frac{1}{4^1}\frac{1}{\prod_{i=1}^r(1-x_i^4)^1} \tag{B.25}$$

$$= \frac{1}{24}\frac{1}{\prod_{i=1}^r(1-x_i)^4} + \frac{1}{4}\frac{1}{\prod_{i=1}^r(1-x_i)^2(1-x_i^2)} + \frac{1}{8}\frac{1}{\prod_{i=1}^r(1-x_i^2)^2} + ... \tag{B.26}$$

$$... + \frac{1}{3}\frac{1}{\prod_{i=1}^r(1-x_i)(1-x_i^3)} + \frac{1}{4}\frac{1}{\prod_{i=1}^r(1-x_i^4)} \tag{B.27}$$

$$= \frac{1}{24}\left[\frac{1}{\prod_{i=1}^r(1-x_i)^4} + 6\frac{1}{\prod_{i=1}^r(1-x_i)^2(1-x_i^2)} + 3\frac{1}{\prod_{i=1}^r(1-x_i^2)^2} + ...\right. \tag{B.28}$$

$$\left.... + 8\frac{1}{\prod_{i=1}^r(1-x_i)(1-x_i^3)} + 6\frac{1}{\prod_{i=1}^r(1-x_i^4)}\right] \tag{B.29}$$

Comparing to eq. 44, we can see that is exactly the same result.

B.4 $\mathscr{P}_2$ in terms of $\mathscr{R}(s)$.

$$\mathscr{P}_2 = \sum_{(k)\vdash 2}\prod_{n=1}^{2}\frac{1}{k_n!\cdot n^{k_n}}\left[\frac{\mathscr{R}(s=n(r+1)(1-i\rho(\vartheta))+1-n)}{\mathscr{R}(s=n(1-i\rho(\vartheta))+1-n)}\right]^{k_n} \tag{B.30}$$

$$= \sum_{(k)\vdash 2}\frac{1}{k_1!\cdot 1^{k_1}}\left[\frac{\mathscr{R}(s=n(r+1)(1-i\rho(\vartheta))+1-1)}{\mathscr{R}(s=1(1-i\rho(\vartheta))+1-1)}\right]^{k_1} \tag{B.31}$$

$$\cdot\frac{1}{k_2!\cdot 2^{k_2}}\left[\frac{\mathscr{R}(s=2(r+1)(1-i\rho(\vartheta))+1-2)}{\mathscr{R}(s=2(1-i\rho(\vartheta))+1-2)}\right]^{k_2} \tag{B.32}$$

Partitions of 2

$$1+1=(1^2)\to(k_1=2,k_2=0) \tag{B.33}$$

$$2=(2^1)\to(k_1=0,k_2=1) \tag{B.34}$$

$$\tag{B.35}$$

therefore

$$\mathscr{P}_2 = \frac{1}{2!}\left[\frac{\mathscr{R}(s=n(r+1)(1-i\rho(\vartheta)))}{\mathscr{R}(s=1(1-i\rho(\vartheta)))}\right]^2\cdot\frac{1}{0!}\left[\frac{\mathscr{R}(s=2(r+1)(1-i\rho(\vartheta))+1-2)}{\mathscr{R}(s=2(1-i\rho(\vartheta))+1-2)}\right]^0 + ... \tag{B.36}$$

$$... + \frac{1}{0!}\left[\frac{\mathscr{R}(s=n(r+1)(1-i\rho(\vartheta)))}{\mathscr{R}(s=1(1-i\rho(\vartheta)))}\right]^0\cdot\frac{1}{1!\cdot 2^1}\left[\frac{\mathscr{R}(s=2(r+1)(1-i\rho(\vartheta))+1-2)}{\mathscr{R}(s=2(1-i\rho(\vartheta))+1-2)}\right]^1 \tag{B.37}$$

$$= \frac{1}{2}\left[\frac{\mathscr{R}(s=n(r+1)(1-i\rho(\vartheta)))}{\mathscr{R}(s=1-i\rho(\vartheta))}\right]^2 + \frac{1}{2}\frac{\mathscr{R}(s=2(r+1)(1-i\rho(\vartheta))-1)}{\mathscr{R}(s=2(1-i\rho(\vartheta))-1)} \tag{B.38}$$

B.5 $\mathscr{P}_3$ in terms of $\mathscr{R}(s)$.

$$\mathscr{P}_3 = \sum_{(k)\vdash 3}\prod_{n=1}^{3}\frac{1}{k_n!\cdot n^{k_n}}\left[\frac{\mathscr{R}(s=n(r+1)(1-i\rho(\vartheta))+1-n)}{\mathscr{R}(s=n(1-i\rho(\vartheta))+1-n)}\right]^{k_n}$$
$$= \sum_{(k)\vdash 3}\frac{1}{k_1!\cdot 1^{k_1}}\left[\frac{\mathscr{R}(s=1(r+1)(1-i\rho(\vartheta))+1-1)}{\mathscr{R}(s=1(1-i\rho(\vartheta))+1-1)}\right]^{k_1}$$
$$\frac{1}{k_2!\cdot 2^{k_2}}\left[\frac{\mathscr{R}(s=2(r+1)(1-i\rho(\vartheta))+1-2)}{\mathscr{R}(s=2(1-i\rho(\vartheta))+1-2)}\right]^{k_2}\cdots$$
$$\cdots\frac{1}{k_3!\cdot 3^{k_3}}\left[\frac{\mathscr{R}(s=3(r+1)(1-i\rho(\vartheta))+1-3)}{\mathscr{R}(s=3(1-i\rho(\vartheta))+1-3)}\right]^{k_3}$$

taking the partitions of 3

$$1+1+1=(1^3)\to(k_1=3,k_2=0,k_3=0) \tag{B.39}$$
$$2+1=(1^1 2^1)\to(k_1=1,k_2=1,k_3=0) \tag{B.40}$$
$$3=(3^1)\to(k_1=0,k_2=0,k_3=1) \tag{B.41}$$

therefore

$$\mathscr{P}_3 = \sum_{(k)\vdash 3}\frac{1}{k_1!}\left[\frac{\mathscr{R}(s=(r+1)(1-i\rho(\vartheta)))}{\mathscr{R}(s=(1-i\rho(\vartheta)))}\right]^{k_1}\frac{1}{k_2!\cdot 2^{k_2}}\left[\frac{\mathscr{R}(s=2(r+1)(1-i\rho(\vartheta))-1)}{\mathscr{R}(s=2(1-i\rho(\vartheta))-1)}\right]^{k_2}\cdots \tag{B.42}$$

$$\cdots\frac{1}{k_3!\cdot 3^{k_3}}\left[\frac{\mathscr{R}(s=3(r+1)(1-i\rho(\vartheta))-2)}{\mathscr{R}(s=3(1-i\rho(\vartheta))-2)}\right]^{k_3} \tag{B.43}$$

$$=\frac{1}{3!}\left[\frac{\mathscr{R}(s=(r+1)(1-i\rho(\vartheta)))}{\mathscr{R}(s=(1-i\rho(\vartheta)))}\right]^{3}\frac{1}{0!}\left[\frac{\mathscr{R}(s=2(r+1)(1-i\rho(\vartheta))-1)}{\mathscr{R}(s=2(1-i\rho(\vartheta))-1)}\right]^{0}\cdots \tag{B.44}$$

$$\cdots\frac{1}{0!}\left[\frac{\mathscr{R}(s=3(r+1)(1-i\rho(\vartheta))-2)}{\mathscr{R}(s=3(1-i\rho(\vartheta))-2)}\right]^{0}+... \tag{B.45}$$

$$...+\frac{1}{1!}\left[\frac{\mathscr{R}(s=(r+1)(1-i\rho(\vartheta)))}{\mathscr{R}(s=(1-i\rho(\vartheta)))}\right]^{1}\frac{1}{1!\cdot 2}\left[\frac{\mathscr{R}(s=2(r+1)(1-i\rho(\vartheta))-1)}{\mathscr{R}(s=2(1-i\rho(\vartheta))-1)}\right]^{1}\cdots \tag{B.46}$$

$$\cdots\frac{1}{0!}\left[\frac{\mathscr{R}(s=3(r+1)(1-i\rho(\vartheta))-2)}{\mathscr{R}(s=3(1-i\rho(\vartheta))-2)}\right]^{0}+... \tag{B.47}$$

$$...+\frac{1}{0!}\left[\frac{\mathscr{R}(s=(r+1)(1-i\rho(\vartheta)))}{\mathscr{R}(s=(1-i\rho(\vartheta)))}\right]^{0}\frac{1}{0!}\left[\frac{\mathscr{R}(s=2(r+1)(1-i\rho(\vartheta))-1)}{\mathscr{R}(s=2(1-i\rho(\vartheta))-1)}\right]^{0}\cdots \tag{B.48}$$

$$\cdots\frac{1}{1!\cdot 3}\left[\frac{\mathscr{R}(s=3(r+1)(1-i\rho(\vartheta))-2)}{\mathscr{R}(s=3(1-i\rho(\vartheta))-2)}\right]^{1} \tag{B.49}$$

$$\mathscr{P}_3 = \frac{1}{6}\left[\frac{\mathscr{R}(s=(r+1)(1-i\rho(\vartheta)))}{\mathscr{R}(s=(1-i\rho(\vartheta)))}\right]^3 \tag{B.50}$$

$$+\frac{1}{2}\frac{\mathscr{R}(s=(r+1)(1-i\rho(\vartheta)))}{\mathscr{R}(s=(1-i\rho(\vartheta)))}\frac{\mathscr{R}(s=2(r+1)(1-i\rho(\vartheta))-1)}{\mathscr{R}(s=2(1-i\rho(\vartheta))-1)} \tag{B.51}$$

$$...+\frac{1}{3}\frac{\mathscr{R}(s=3(r+1)(1-i\rho(\vartheta))-2)}{\mathscr{R}(s=3(1-i\rho(\vartheta))-2)}. \tag{B.52}$$