Oton Henrique Marcori

Theoretical methods in spatially anisotropic cosmologies

Universidade Estadual de Londrina, Londrina, PR

May 24, 2019

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PhD thesis presented as partial requirement for granting the title of Doctor in Physics by the Physics Department of Universidade Estadual de Londrina

Supervisor: Thiago Pereira dos Santos

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All the clouds turn to words
All the words float in sequence
No one knows what they mean
Everyone just ignores them
(Brian Eno)

Abstract

A cornerstone of the standard cosmological model is the Copernican Principle, i.e., the assumption that the universe is, on average, homogeneous and isotropic. CMB measurements show that deviations from isotropy are minuscule, and a scenario where homogeneity is broken is highly unlikely. Nonetheless, the CMB alone can only assert that the early universe was highly isotropic, hence not excluding the possibility of late time anisotropies. Also, an isotropic temperature distribution could arise from non-isotropic spacetimes. This motivates investigations into the hypothesis of isotropy. Relaxing this assumption leads to a class of spacetimes: the Bianchi models. They are spatially homogeneous but anisotropic, and one of the simplest generalizations of FLRW spacetimes, thus possessing strong cosmological appeal. This thesis presents two novel theoretical methods to be applied in the context of spatially anisotropic cosmological models. The first is related to the study of correlation functions and how they are connected to spacetime symmetries. We show that requiring correlators to be invariant under symmetry transformations fix their functional dependence. A general solution is found for Bianchi spaces, which allows us to analyze the impact of primordial anisotropies on the CMB covariance matrix in the large angle limit. We also show how our results apply to both Gaussian and non-Gaussian correlators. The second method deals with the drift of cosmological observables in the context of real time cosmology. Highly precise astrometric measurements, allowed by recent technological advancements, can in principle detect time variations of the redshift and direction of observation of distant stars in a relatively short time span. This can be used to assess cosmological observables like the Earth's peculiar velocity with respect to the CMB rest frame and, especially, late time anisotropies of Bianchi I models. The original method developed to evaluate the drifts is applied to various cosmological scenarios and can, in principle, be extended to other spacetimes.

Keywords: Cosmology; Killing vectors; Bianchi models; two-point correlation functions; CMB anisotropies; real time cosmology; redshift drift; peculiar velocity; cosmological aberration drift.

Resumo

O princípio Copernicano (a hipótese de que o universo é, em média, homogêneo e isotrópico) é a base do modelo padrão da cosmologia. Medidas da RCF mostram que desvios da isotropia são minúsculos, e um cenário onde homogeneidade é quebrada é altamente improvável. Entretanto, a RCF permite somente afirmar que o universo primordial era muito isotrópico, de forma a não excluir a possibilidade de anisotropias tardias. Ademais, uma distribuição de temperatura isotrópica pode surgir de espaços-tempo não isotrópicos. Isso motiva investigações sobre a hipótese de isotropia. Relaxar esse requerimento leva a uma classe de espaços-tempo: os modelos de Bianchi. Eles são espacialmente homogêneos mas anisotrópicos, e uma das generalizações mais simples de espaços-tempo FLRW, possuindo portanto forte apelo cosmológico. Essa tese apresenta dois novos métodos teóricos para serem aplicados no contexto de modelos cosmológicos espacialmente anisotrópicos. O primeiro é relacionado ao estudo de funções de correlação e como elas estão conectadas com as simetrias do espaço-tempo. Nós mostramos que exigir que as correlações sejam invariantes por transformações de simetrias fixa sua dependência funcional. Uma solução geral foi encontrada para espaços de Bianchi, o que permitiu analisar o impacto de anisotropias primordiais na matriz de covariância da RCF no limite de grandes ângulos. Também mostramos como os resultados se aplicam para tanto correlações Gaussianas como não-Gaussianas. O segundo método lida com o *drift* dos observáveis cosmológicos no contexto de cosmologia em tempo real. Medidas astrométricas altamente precisas, permitidas por avanços tecnológicos recentes, podem em princípio detectar variações temporais no *redshift* e direção de observação de estrelas distantes em um intervalo de tempo relativamente curto. Isso pode ser usado para inferir outros observáveis cosmológicos, como a velocidade peculiar da Terra com relação ao referencial da RCF, e, especialmente, anisotropias tardias de modelos tipo Bianchi I. Esse método originial desenvolvido para avaliar os *drfits* é aplicado a diversos cenários cosmológicos e pode, em princípio, ser estendido para outros espaços-tempo.

Palavras-chave: Cosmologia; vetores de Killing; modelos de Bianchi; funções de correlação de dois pontos; anisotropias da RCF; cosmologia em tempo real; drift do redshift; velocidade peculiar; drift da aberração cósmica.

Contents

Overview

The standard model of cosmology is the so called ΛCDM model (the acronym stands for *Lambda-Cold-Dark-Matter*) and it is built from a series of hypothesis about the behavior of the universe, together with advancements in observational cosmology which allowed for precise measurements of cosmological parameters. One fundamental hypothesis is that an initial phase of rapid expansion in the first 10^{-32} seconds of the universe's existence took place, and that it is responsible for producing the seeds which would generate, via gravitational instability mechanisms, astrophysical structures whose spatial scales are of order of 100Mpc [1, 2]. These seeds are produced by quantum fluctuations and inherit their statistical properties, which then are propagated to the statistical distribution of cosmological observables [3]. This implies that the details of the universe's early physics are preserved in the statistics of measurable cosmological quantities, being the gaussianity and isotropy of the Cosmic Microwave Background (CMB) radiation two of the most important examples.

Another fundamental hypothesis of the ΛCDM model is that our universe is, on average, spatially homogeneous and isotropic. The isotropy hypothesis has strong observational backing since the temperature anisotropies of the CMB are smaller than one part in 10^5 [4, 5]. On the other hand, the homogeneous hypothesis is harder to be put to test, given that all known observations are made in the small neighborhood of the Earth's orbit. In principle, one cannot discard the possibility that our universe be isotropic only around a privileged observation point, although many studies show this is a highly unlikely situation [6, 7].

Recent works using new data from the CMB radiation suggest that the geometry of the universe obeys the isotropy hypothesis with only small deviations [8, 9]. However, there are some subtleties in asserting that is indeed the case. First, the CMB is an image of the early universe, when it was around 300 thousand years old. Thus, results based on CMB measurements can only tell that our early universe was highly isotropic, which does not exclude mechanisms that could induce anisotropy later in the universe's evolution, affecting, e.g., the spatial distribution of galaxies [10, 11, 12]. And second, although assuming an isotropic cosmological model implies that the temperature fluctuations are equally isotropic, the opposite is not true. There are theoretical counterexamples showing that spatially anisotropic universes can exhibit a perfectly isotropic CMB spectrum [13, 14, 15]. These factors justify deeper theoretical studies about the isotropy hypothesis of the universe and its impacts on the statistics of cosmological data.

In the ΛCDM model of cosmology, the hypothesis of homogeneity and isotropy fixes, up to the curvature of the spatial sections, the geometry of the universe: it is given by the famous Friedmann-Lemaître-Robertson-Walker (FLRW) metric [16]. A second family of metrics, with strong cosmological appeal, can be obtained by relaxing the isotropy hypothesis. In this case, one assumes the universe is only spatially homogeneous (i.e., the metric is invariant under spatial translations), but allows the possibility for spatial anisotropies to arise. By doing so, one arrives at a family of metrics, which gives origin to the Bianchi models in general relativity [17, 18]. They were first classified, on purely algebraic grounds, by Luigi Bianchi, and named after him. The Bianchi classification is based on the algebra of the Killing vectors of these spaces, and there are nine different classes, with two being classified by a continous parameter [18]. These solutions were introduced into the context of cosmology by G. Ellis and M. MacCallum through the study of solutions of Einstein equations in universes with a perfect fluid matter content [19]. John D. Barrow and collaborators studied the temperature distribution of the CMB in a subclass of Bianchi spaces, namely, those which can be taken to a FLRW metric through an appropriate limit [20, 21]. Recent works use Bianchi spaces as a proxy to probe the universe's isotropy through analyzes of CMB temperature and polarization maps [8, 22]. Also, the theory of perturbations in Bianchi spaces was studied by A. Pontzen and A. Challinor [23] in the context of "near-FLRW" spaces, and in a perturbative and gauge invariant manner by T. S. Pereira and collaborators [14, 24, 25].

Given the importance and relevancy of this theme, this thesis presents two different theoretical methods that show promising applications in Bianchi spacetimes. The first corresponds to a novel and systematic way of finding the functional dependence of two point correlation functions. It can be applied, in principle, to any spacetime that presents continuous symmetries, i.e., isometries. It is, however, especially useful for Bianchi spacetimes, since we found a particular solution for this case. Two point correlation functions are fundamental tools for cosmology and many other areas of physics, and the approach presented here is completely general. We were also able to successfully apply this method to an inhomogeneous model and extend it to N-point functions, presenting original results for both cases. The first part of this thesis comprises the theoretical basis, description, application and discussions of this novel method for anisotropic spacetime cosmology.

The other topic this thesis tackles is related to a rising field within cosmology that has been recently named as real time cosmology [26]. The development in technological capabilities of precision measurements of distant stars and galaxies have enabled us to measure the time variation of cosmological observables in real time. This provides a new way to probe different cosmological models. In this light, we present here a general derivation of the time drift of two important cosmological observables: the gravitational redshift and direction of observation of distant objects. One can then apply this derivation to any spacetime of interest in order to probe different cosmological quantities. We apply it to different contexts, including the possibilities of non-inertial observers, and specially, to the context of Bianchi spacetimes. This provides us with a mean to constrain the anisotropy of the current universe. Thus, the second part of this thesis is dedicated to demonstrate and apply this general derivation of the drifts of cosmological observables.

List of publications

The work presented in this thesis has resulted in the publications of the following original research articles:

- Two-point correlation functions in inhomogeneous and anisotropic cosmologies, by the authors Marcori, Oton H.; Pereira, Thiago S., on 02/2017 in Journal of Cosmology and Astroparticle Physics, Issue 02.

- Direction and redshift drifts for general observers and their applications in cosmology, by the authors Marcori, Oton H.; Pitrou, Cyril; Uzan, Jean-Philippe; Pereira, Thiago S., on 07/2018 in Physical Review D, Volume 98, Issue 2.

Part I

Two point correlation functions in cosmology

1 Introduction

A central assumption of the standard cosmological model is that the universe we observe is a fair sample of an (hypothetical) ensemble of universes. This hypothesis has far reaching consequences, but it also brings along a whole statistical framework from which cosmological observables are to be computed. It follows in particular that cosmological parameters are not deduced directly from physical fields – which in this framework are viewed as one realization of random variables – but rather from their statistical moments, such as the one, two, and higher N-point correlation functions. When using perturbation theory to describe the clumpy universe, the one-point function is usually defined to be zero, since one is actually interested in the fluctuations of physical fields around their mean values. Thus, the first non-trivial statistical moment is the two-point (or Gaussian) correlation function (we will abbreviate it as 2pcf from now on).

Two-point functions are ubiquitous tools in modern physics. In field theory they are disguised as Green's functions (or the propagator), whereas in general relativity they could be simply a distance function, a bitensor or Synge's world function [27, 28, 29] – just to mention a few examples. In cosmology, two-point *correlation* functions are a cornerstone of the standard ΛCDM model. Once it arises as the quantization of a free field in the early inflationary universe [30, 31, 32] (see ref. [33] for an up-to-date review), it propagates to virtually all cosmological and astrophysical computations one might be interested in – most popularly in its Fourier (i.e., the power spectrum) version. The same reasoning holds for higher-order correlation functions in connection with "Beyond-ΛCDM" approaches [34, 35]. For example, in the context of CMB anisotropies, the observable of interest is the temperature variation of the CMB as a function of the direction of observation on the sky, i.e., $\frac{\Delta T}{T}(\hat{\boldsymbol{n}})$. Then, the 2pcf associated with this observable is given by $\left\langle \frac{\Delta T}{T}(\hat{\boldsymbol{n}}_1)\frac{\Delta T}{T}(\hat{\boldsymbol{n}}_2)\right\rangle$, where the average is being evaluated in an ensemble (since, however, we only have one universe at our disposal, this averaging is then substituted by the spatial average through the ergodic theorem [16]). In a given spacetime, its symmetries are carried along to the 2pcf, more specifically, to the 2pcf functional dependence. For example, the hypothesis of isotropy restricts the 2pcf above to depend only on the angle between $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$. Therefore, knowledge of the functional dependence of the 2pcf is crucial, since it alone can tell a lot about the statistical properties of cosmological observables, potentially allowing one to disentangle cosmological signals from systematical effects in real data.

There are essentially two independent routes to find the functional dependence of the 2pcf in cosmology. In the first, one uses heuristic symmetry arguments (or its lack thereof) to fix this functional dependence. This idea has been successfully applied in cosmology, mainly in connection with CMB physics, in refs. [36, 37, 38, 39, 40, 41].

However straightforward, the phenomenological quality of this approach prevents one to link the resulting 2pcf to the statistics of a field in a well-defined background geometry. Alternatively, one can deploy the full machinery of perturbation theory in the desired spacetime. After dealing with known issues of gauge invariance and mode decomposition, the full set of Einstein equations can be solved and the statistics of the 2pcf can be computed [11, 14, 15, 24, 42]. This option is clearly more expensive, but is certain to lead to statistics with known spacetime symmetries.

These considerations lead us to ask whether one can systematically find the functional properties of correlation functions given the spacetime symmetries, and without the need to resort to expensive computations involving perturbation theory. In fact, when metric and fluid perturbations are small, they can be seen as external fields evolving over a fixed background, regardless of their dynamics. By expanding such fields in an appropriate set of basis eigenfunctions, one ensures that their statistical properties will inherit the symmetries of the background metric. Thus, in a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, for example, the 2pcf of a random field can only depend on the invariant distance between the two points, since this is the only combination allowed by the symmetries of the FLRW metric. Analogous ideas were explored in refs. [43, 44], where the conformal invariance of the de Sitter spacetime has been used to find the shape of two- and three-point correlation functions in dark-energy dominated universes.

In this first part of the thesis, we systematically develop the idea of using the symmetries of the background metric to fix the *functional* form of the 2pcf. First, we review the necessary mathematical tools needed to proceed, like symmetries in general relativity (expressed either through Killing vectors or isometries) and the Bianchi classification. Then, starting from the definition of a two-point function in a general manifold, we show that the imposition of isometric invariance on the 2pcf leads to a set of coupled first order partial differential equations which can be solved by means of well known techniques – but most easily through the method of characteristic curves [45] – to fix the functional form of the 2pcf. We illustrate the method with different examples, where we show how it correctly recovers the 2pcf in the spatially flat FLRW spacetime and in Minkowski spacetime. We then construct a formal solution to the aforementioned set of differential equations which holds for any spacetime having at least one Killing vector, but that is particularly useful to Bianchi spacetimes. Further, we apply the formalism to obtain the 2pcf in different classes of spacetimes, including the case of an off-center inhomogeneous but spherically symmetric spacetime and in the class of homogeneous but spatially flat anisotropic geometries of the Bianchi family. We then derive the CMB temperature covariance matrix for the Sachs-Wolfe effect in the limit of almost Friedmannian symmetry for the Bianchi cases, and comment on their multipolar signatures. We also show how the method can be easily generalized to include any N-point correlation function. Finally, we comment on extending the method to other Bianchi models and conclude.

2 An introduction to Bianchi cosmology

One of the core assumptions of modern cosmology is the homogeneity and isotropy of the universe. However, one can be interested in spacetimes that do not fall into this restrict category, especially if one is interested in testing how homogeneous and isotropic the universe is. Relaxing the assumption of isotropy is an interesting theoretical exercise but also provides us with means to compare our universe with an anisotropic cosmological model, and, e.g., constrain the anisotropic shear at different epochs of the universe's evolution. By doing so, one arrives at a class of spaces, called the Bianchi spaces, named after the mathematician who first classified them. A Bianchi spacetime has spatial sections which are anisotropic but homogeneous. My purpose in the following is to show how this classification comes to be and how one can construct a cosmological model from the Bianchi spaces. To do so, some introductory knowledge on Lie groups and Lie algebra is necessary. The key points will be presented in the next section, followed by the Bianchi classification. Most of this material is referenced from refs. [18, 17, 46, 47].

2.1 Isometries, the Lie derivative and Killing vectors

The notions of isometries and Lie derivatives are essential to understand the Bianchi classification, and thus will be reviewed in brief here. Let $\mathcal{M}$ be an n-dimensional manifold and ϕ be a map[1] $\phi : \mathcal{M} \mapsto \mathcal{M}$. If ϕ is smooth, one-to-one and has a smooth inverse, then it is called a diffeomorphism. When one states that a map between points in a manifold is *smooth*, what is meant is that the local representative of this map, which is the induced map between points in the $\Re^n$ local patches of the manifold, is infinitely differentiable.

A diffeomorphism can be used to pull back and push forward objects in a manifold. Let us illustrate how this comes to be with a real function. Let f be a function on $\mathcal{M}$ that maps points $p \in \mathcal{M}$ to real numbers $f(p)$, i.e., $f : \mathcal{M} \mapsto \Re$. One can compose f and ϕ to create a new function $\phi^* f(p) = f(\phi(p))$ which is the pull back of f. Since ϕ is a diffeomorphism and hence invertible, the push forward of f can also be constructed by making use of ϕ^{-1}. We call $\phi_* f(p) = f(\phi^{-1}(p))$ the push forward of f. Once the pull back and push forward of real functions are defined, one can extend these notions to vectors, one forms and general tensors on manifolds. Recall that a vector is defined through its action on real functions of $\mathcal{M}$. Let v be a vector in the tangent space V_p at $p \in \mathcal{M}$. Then,

[1] One can also define more general maps $\phi : \mathcal{M} \mapsto \mathcal{N}$, but for our purposes this will not be necessary.

the push forward of v is a vector $\phi_* v$ in the tangent space $V_{\phi(p)}$ at $\phi(p)$. It is defined as

$$(\phi_* v)(f) = v(\phi^* f), \tag{2.1.1}$$

i.e., the push forward of v acts on f the same way v acts on the pull back of f. The pull back of v can be defined in analogy using the push forward of f (which makes use of ϕ^{-1}). And in the same way one can define the pull back and push forward of one forms and tensors.

The discussion above shows that a diffeomorphism (and its inverse) can induce maps on objects of a manifold. One can then compare, e.g., a tensor field under the action of such a map with the original tensor field. If one finds that, for a tensor T, $\phi_* T = T$, then ϕ is a symmetry transformation for the tensor T. There is a special name for the diffeomorphism that is a symmetry transformation of the metric tensor g. If $\phi_* g = g$, then ϕ is called an isometry.

To move forward, we need to introduce the concept of a one-parameter family of diffeomorphisms. Let us parametrize continuously a family of diffeomorphisms by a real parameter t. We will write this family as ϕ_t. One can think of it as describing a curve on $\mathcal{M}$ when acting on a fixed point p. For example, $\phi_{t=0} = p$, and for every t, $\phi_t(p)$ corresponds to a point on this curve. The composition of two diffeomorphisms is given by $\phi_t \circ \phi_s = \phi_{t+s}$. Now, at each point of the curve defined by $\phi_t(p)$ there will be a tangent vector. Thus, ϕ_t defines a vector field $v \in V_{\phi_t(p)}$ of vectors tangent to $\phi_t(p)$. One can conversely use a vector field v on $\mathcal{M}$ to generate a one-parameter family of diffeomorphisms through its integral curve, i.e., the curve whose tangent vector field is v. In this sense, a vector field can be seen as a generator of a one parameter family of diffeomorphisms.

We are now able to define a Lie derivative. Given a vector field v on $\mathcal{M}$, one can ask how a general tensor T changes along the integral curves of v. The Lie derivative is the formal tool to evaluate that. Let ϕ_t be the one-parameter family of diffeomorphisms generated by v. Then, the Lie derivative of a tensor T along the integral curves of v is given by

$$\mathcal{L}_v T = \lim_{t\to 0} \frac{\phi_* T - T}{t}. \tag{2.1.2}$$

Note that the Lie derivative is a more primitive notion than the covariant derivative since it is independent of the metric; it depends only on the existence of a diffeomorphism generator vector field v. Since its definition is similar to ordinary derivatives, it is not hard to see that it is linear and obeys the Leibniz rule. In the case where T is a real function f, we have that

$$\mathcal{L}_v f = \lim_{t\to 0} \frac{f(\phi_t(p)) - f(\phi_0(p))}{t} \tag{2.1.3}$$

$$= \frac{df(\phi_t(p))}{dt}. \tag{2.1.4}$$

Since v is the tangent vector to the curve defined by $\phi_t(p)$, the Lie derivative of a scalar function can be written as $\mathcal{L}_v f = v(f)$. In an arbitrary coordinate system, this is reduced simply to $\mathcal{L}_v f = v^\mu \partial_\mu f$, where v^μ are the components of v in this coordinate system. From here on, Greek indexes are used as general coordinate indexes. To find the action of a Lie derivative on a vector w, let us make use of an adapted coordinate system. We will work in a coordinate system $x^\mu = (x^0, x^1, ...)$ such that x^0 is the parameter along the integral curves of v. Then, $v = \partial_0$, or $v = (1, 0, ...)$. Thus, the action of ϕ_t on a vector w at a point $(x^0, x^1, ...)$ is simply given by w at the point $(x^0 + t, x^1, ...)$. It is then straightforward that in this coordinate system

$$\mathcal{L}_v w^\mu = \frac{\partial w^\mu}{\partial x^0}. \tag{2.1.5}$$

It is also true in this coordinate system that $[v, w]^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu = \frac{\partial w^\mu}{\partial x^0}$. We thus have the covariant expression

$$\mathcal{L}_v w^\mu = [v, w]^\mu, \tag{2.1.6}$$

which must be valid in any coordinate system. The expressions for Lie derivatives of one-forms and general tensors can then be found using the results above and the Leibniz rule. Let us illustrate this with an important example: the metric tensor. To evaluate the Lie derivative of $g_{\mu\nu}$, we start by noting that

$$\mathcal{L}_v \left(g_{\mu\nu} w^\mu u^\nu\right) = v^\rho \nabla_\rho \left(g_{\mu\nu} w^\mu u^\nu\right). \tag{2.1.7}$$

If we now expand $\mathcal{L}_v \left(g_{\mu\nu} w^\mu u^\nu\right)$ using the Leibniz rule, we get

$$\mathcal{L}_v \left(g_{\mu\nu} w^\mu u^\nu\right) = \left(\mathcal{L}_v g\right)_{\mu\nu} w^\mu u^\nu + g_{\mu\nu} \left(\mathcal{L}_v w\right)^\mu u^\nu + g_{\mu\nu} w^\mu \left(\mathcal{L}_v u\right)^\nu. \tag{2.1.8}$$

After some simple algebra, comparing (2.1.7) and (2.1.8) gives

$$\mathcal{L}_v g_{\mu\nu} = v^\rho \nabla_\rho g_{\mu\nu} + g_{\rho\nu} \nabla_\mu v^\rho + g_{\mu\rho} \nabla_\nu v^\rho \tag{2.1.9}$$
$$= \nabla_\mu v_\nu + \nabla_\nu v_\mu, \tag{2.1.10}$$

where we used that ∇ is the derivative operator compatible with the metric g.

Now, suppose that a one-parameter family of diffeomorphisms satisfies $\phi_{t*} g = g$, that is, it is a one-parameter family of isometries. The vector field K which generates ϕ_t is then called a Killing vector field. Since its integral curves are isometries, it must be that

$$\mathcal{L}_K g = 0. \tag{2.1.11}$$

Thus, from eq. (2.1.10), we have

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{2.1.12}$$

This is called Killing's equation. It is important to note that Killing vectors (and hence, isometries) are the language in which symmetries are represented in general relativity. If a given spacetime presents one (or more) continuous symmetry, there is a Killing vector field associated with it, and having it at hand is a helpful tool in many calculations and insights. For example, if $\mathcal{M}$ possesses a geodesic curve and a Killing vector field, there is a conserved quantity which can be easily evaluated. Let t be the tangent vector to the geodesic, and K be the Killing vector field. Then, the quantity $K_\mu t^\mu$ is conserved along the geodesic. This can be easily proved:

$$t^\nu \nabla_\nu \left(K_\mu t^\mu\right) = t^\nu t^\mu \nabla_\nu K_\mu + K_\mu t^\nu \nabla_\nu t^\mu \tag{2.1.13}$$

$$= 0, \tag{2.1.14}$$

where the first term is zero due to Killing's equation and the last due to the geodesic equation. This result will be used in the second part of this thesis extensively to compute the drift of the direction of observation of distant light sources. In this scenario, t is the tangent vector to a null geodesic and the conserved quantity is the massless particle's 4-momentum in the direction of K.

Another property of a Killing vector field is that it is completely defined by the values of K^μ and $\nabla_\mu K_\nu$ at a given point, i.e., if one knows these quantities at a certain point $p \in \mathcal{M}$ then one can calculate K^μ for any other point in $\mathcal{M}$ [17]. This allows us to count the maximum set of independent Killing vector fields of a manifold. In an n-dimensional manifold, K^μ provides us with n parameters, while $\nabla_\mu K_\nu$, being antisymmetric due to Killing's equation, has $\frac{n^2-n}{2}$ parameters. This amounts to $\frac{n(n+1)}{2}$ independent parameters, which is then the maximum number of Killing vector fields (or isometries, or even, symmetries) an n-dimensional manifold can have (for a pedagogical proof of this result, see ref. [48]).

Other useful properties of Killing vector fields are:

- linear combinations of Killing vectors are also Killing, which follows trivially from the linearity of the Lie derivative;
- if K_1 and K_2 are Killing, then $K_3 = [K_1, K_2]$ is also a Killing vector, since Lie derivatives satisfy $\mathcal{L}_{[K_1,K_2]} = \mathcal{L}_{K_1}\mathcal{L}_{K_2} - \mathcal{L}_{K_2}\mathcal{L}_{K_1}$ (which is a direct consequence of the Jacobi identity).

With these basics covered, we now follow to more advanced topics in order to understand the Bianchi classification of spatially anisotropic cosmologies.

2.2 Lie groups and Lie algebra for Bianchi spaces

Let us begin with the notion of a group. A group $\mathcal{G}$ is a mathematical set of objects $\{g_1, g_2, ..., g_n\}$ possessing a map $\mathcal{G} \times \mathcal{G} \mapsto \mathcal{G}$, called the multiplication map (and from here on represented by the multiplication operation), and an identity element e obeying the following properties. For every element $g \in \mathcal{G}$ we have that

- $g_1 (g_2 g_3) = (g_1 g_2) g_3$;
- $eg = ge = g$;
- there exists an inverse object $g^{-1} \in \mathcal{G}$ such that $gg^{-1} = g^{-1}g = e$.

An example of a group is the collection of diffeomorphisms of a manifold $\mathcal{M}$. This is a group of infinite objects $\{\phi_1, \phi_2, ...\}$ which are mappings of points $p \in \mathcal{M}$ into the points $\phi_i(p)$ $(i = 1, 2, ...)$. The multiplication map is simply the composition function, i.e., $\phi_1\phi_2(p) = \phi_1(\phi_2(p))$ and the identity element is the diffeomorphism e such that $e(p) = p$.

Now, let us move onto the notion of a Lie group. We will restrict ourselves to Lie groups of finite dimension which are enough for our purposes. By definition, a Lie group of dimension m is a group which is also an m-dimensional manifold such that the inverse map $i(g) = g^{-1}$ and the multiplication map $f(g_1, g_2) = g_1 g_2$ are smooth, i.e., they are locally infinitely differentiable. Note that since a Lie group is also a manifold, each object g of the Lie group is locally associated with a set of m parameters, and thus i and f must depend smoothly on these parameters. The set of diffeomorphisms of a manifold is not a finite dimensional Lie group because it is too big a set, i.e., there are infinitely many diffeomorphisms and as such they cannot be characterized by m parameters. The set of isometries, however, is a Lie group, since an n-dimensional manifold can have at most $\frac{n(n+1)}{2}$ isometries.

Now, we are about to show that a Lie group uniquely defines a Lie algebra (whose definition is stated further). To do so, we need to familiarize ourselves with the notion of left and right translation maps. Let $\mathcal{G}$ be a Lie group of m dimensions. The left translation map ψ for every $g, h \in \mathcal{G}$ is given by

$$\psi_h(g) = hg. \tag{2.2.1}$$

Its inverse is given by $\psi_h^{-1}(g) = h^{-1}g$ such that $\psi_h^{-1}(\psi_h(g)) = g$. Note that ψ_h, for each $h \in \mathcal{G}$, is a diffeomorphism of the manifold $\mathcal{G}$ since the multiplication and inverse maps are smooth. As a diffeomorphism, it induces maps on vectors, one forms and tensors on $\mathcal{G}$ through pull backs and push forwards. For example, the push forward of a vector v by ψ_h is

$$(\psi_{h*}v)(f) = v(\psi_h^* f), \tag{2.2.2}$$

where $\psi_h^* f$ is the pull back of a real function f defined as before, i.e., $\psi_h^* f(g) = f(\psi_h(g))$. If a vector v satisfies $\psi_{h*}v = v$ for all $h \in \mathcal{G}$, then it is called a left invariant vector, i.e., it is invariant under left translations. An important property is that a left invariant vector field is defined by its value at the identity element $e \in \mathcal{G}$. This can be seen through the following. Let v_g be a vector field on $\mathcal{G}$. Here, the index g denotes an arbitrary point on the manifold $\mathcal{G}$ at which the vector is taken. Then, if v_g is a left invariant vector field, we have that $\psi_{h*}(v_g) = v_{hg}$. Note that the point changes due to the push forward of the vector field (a vector at g "pushed forward" by ψ_h is now a vector at $\psi_h(g) = hg$). Thus, for $g = e$, we have that

$$v_h = \psi_{h*}(v_e)\,. \tag{2.2.3}$$

Another property which is easy to check is that a linear combination of left invariant vector fields is also left invariant. These properties show that the set of all left invariant vector fields forms a vector space that is also m-dimensional. This last fact follows because all left invariant vector fields are in the tangent space $V_e \in \mathcal{G}$, which shares the same dimension of $\mathcal{G}$.

The next step to show how a Lie algebra arises from a Lie group is to show that, for any diffeomorphism ϕ of a manifold, we have

$$\phi_*[v,u] = [\phi_* v, \phi_* u]\,. \tag{2.2.4}$$

A proof of the above can be shown through coordinate expressions of the push forward. Suppose ϕ is a diffeomorphism taking a point p of a manifold to the point $\phi(p) = q$. Then $(\phi_* v)^\mu = \frac{\partial y^\mu}{\partial x^\nu} v^\nu$, where x^μ is the local coordinate patch of the point p and y^μ is the local coordinate patch of q [46]. A straightforward but tedious calculation gives $(\phi_*[v,u])^\mu = [\phi_* v, \phi_* u]^\mu$. Since the commutators are defined independently of coordinates, eq. (2.2.4) follows. A consequence of the above is that a commutator of left invariant vector fields is itself a left invariant vector field. Thus, if one takes a basis of left invariant vector fields $\{v_A, A = 1, ..., m\}$, the commutator of two basis vectors must be a linear combination of the left invariant vectors of the basis, thus satisfying

$$[v_A, v_B] = C^C_{AB} v_C, \tag{2.2.5}$$

where the C^C_{AB} are called the structure constants of the algebra. Note that the uppercase Latin indexes are not coordinate indexes, but labels for the basis vectors. The structure constants are by definition antisymmetric in its lower indexes, i.e., $C^C_{AB} = -C^C_{BA}$. If eq. (2.2.5) is plugged into the Jacobi identity for the commutators of any three vectors

$$[[v_A, v_B], v_C] + [[v_B, v_C], v_A] + [[v_C, v_A], v_B] = 0 \tag{2.2.6}$$

one finds that the structure constants satisfy

$$C^E_{[AB} C^F_{C]E} = 0. \tag{2.2.7}$$

Now, onto the definition of a Lie algebra: a finite dimensional vector space V equipped with a bilinear operation $[\ \ ,\ \ ] : V \times V \mapsto V$ with elements $v, u \in V$ such that $[v, v] = 0$, and $[v, u] = -[u, v]$ and that also satisfies Jacobi's identity is a Lie algebra. Thus, we've shown that an m-dimensional Lie group generates an m-dimensional Lie algebra through its left invariant vectors and whose bilinear operation is the usual commutator.

Now, we can also define a right translation map χ_h. It acts on elements of $\mathcal{G}$ through $\chi_h(g) = gh$, as expected. Note that ψ_h and χ_h commute:

$$\psi_h(\chi_i(g)) = hgi = \chi_i(\psi_h(g)). \tag{2.2.8}$$

The same properties of left translations extend to right translations: they are also diffeomorphisms, and also define a set of right invariant vector fields which can be used to generate a Lie algebra. Then, $\{\xi_A, A = 1, ..., m\}$ is a set of right invariant vector basis with an algebra given by

$$[\xi_A, \xi_B] = D^C_{AB}\xi_C. \tag{2.2.9}$$

Let ϕ_t be a one-parameter family of diffeomorphisms generated by the integral curves of a right invariant vector field ξ on $\mathcal{G}$. Let us define the action of ϕ_t on the identity element by $\phi_t(e) \equiv h(t)$, where $h(t = 0) = e$. Note that, since ξ is a right invariant vector, ϕ_t and χ_h commute, for any h. Let us now evaluate the action of ϕ_t on an arbitrary point $g \in \mathcal{G}$:

$$\phi_t(g) = \phi_t(\chi_g(e)) = \chi_g(\phi_t(e)) = \chi_g(h(t)) = h(t)g = \psi_{h(t)}(g). \tag{2.2.10}$$

What one can check in the above is that the diffeomorphism ϕ_t constructed from a right invariant vector field ξ is generating a left translation. Thus, right invariant vector fields are generators of left translations through their integral curves, and vice versa (it should be easy to see why). This has an important consequence for the left and right invariant vector basis:

$$\mathcal{L}_{v_A}\xi_B = [v_A, \xi_B] = 0 \quad \forall \quad A, B. \tag{2.2.11}$$

This is expressing the fact that "dragging" ξ_A, a right invariant vector, along the integral curves of v_A, which are right translations, does not change ξ_A. Note that the inverse is also true, i.e., $\mathcal{L}_{\xi_A} v_B = 0$.

Now, one might wonder if the Lie algebras generated by left invariant vectors and right invariant vectors are just different representations of the same underlying algebra. The answer is yes, and this is seen by making a suitable choice of $\{v_A\}$ and $\{\xi_A\}$ at e, the identity point of $\mathcal{G}$. This can always be done since they are linearly independent and span the tangent space at every point of $\mathcal{G}$. The choice we make is that $v_A|_e = \xi_A|_e$ for every A. Then, there must be an invertible position-dependent matrix M^A_B [47] which satisfies

$$\xi_A = M^B_A v_B \tag{2.2.12}$$

and $M^B_A|_e = \delta^B_A$. Expanding the commutator $[v_A, \xi_B] = \left[(M^{-1})^C_A \xi_C, \xi_B\right] = 0$ one is able to find after some simple algebra that

$$D^C_{AB} = -M^D_B \xi_A \left((M^{-1})^C_D\right), \tag{2.2.13}$$

where here the vector ξ_A is meant to be acting on the (position-dependent) matrix $(M^{-1})^C_D$. Now, using $v_A = (M^{-1})^B_A \xi_B$ on eq. (2.2.5) and expanding, one is able to arrive at

$$[\xi_C, \xi_D] = -M^A_C M^B_D C^E_{AB} (M^{-1})^F_E \xi_F. \tag{2.2.14}$$

In comparison with eq. (2.2.9), it is easy to see that

$$D^F_{CD} = -M^A_C M^B_D C^E_{AB} (M^{-1})^F_E. \tag{2.2.15}$$

Now, while the matrix M^A_B depends on position, the structure constants do not. So it is only necessary to evaluate the above equation at one point to find the structure constant D^F_{CD}. Thus, evaluating it at the identity point, we arrive finally at

$$D^C_{AB} = -C^C_{AB}, \tag{2.2.16}$$

that is, apart from a minus sign, these structure constants represent the same underlying Lie algebra.

After this concise exposition about Lie groups and Lie algebras, we can now apply these mathematical tools to study homogeneous spacetimes, which is what we do in the following.

2.3 Bianchi spacetimes and their classification

In here we make the connection between the latest sections and general relativity, more specifically, to homogeneous spaces. Let us consider a manifold $\mathcal{M}$ and a metric g on this manifold, i.e., a spacetime $(\mathcal{M}, g)$. Let us call $\mathrm{Isom}(\mathcal{M})$ the group of isometries of this spacetime, defined by

$$\mathrm{Isom}(\mathcal{M}) = \{\phi : \mathcal{M} \mapsto \mathcal{M} \,|\, \phi_* g = g\}, \tag{2.3.1}$$

i.e., it is the set of all diffeomorphisms whose induced action on the metric tensor does not alter it. The group of isometries of a spacetime, as was pointed before, forms a Lie group.

A spatially homogeneous spacetime is one in which there is a family of spacelike hypersurfaces[2] Σ_t such that, for any two points $p, q \in \Sigma_t$ there exists an isometry $\phi \in \mathrm{Isom}(\mathcal{M})$ that takes $\phi(p) = q$. This means that $\mathrm{Isom}(\mathcal{M})$ acts transitively on Σ_t. To properly define the notion of transitiveness, let us first understand the concept of an orbit

[2] A spatial hypersurface Σ_t is a submanifold of $\mathcal{M}$ which is everywhere orthogonal to the 4-velocity of the fundamental cosmological observers. Usually, one chooses these observers to be aligned with the matter flow of the spacetime. The existence of these observers allows one to make a covariant 1+3 split of the spacetime with respect to their 4-velocity, foliating the spacetime in spatial hypersurfaces Σ_t. The precise definition of the parameter t with which we foliate it will be given below. For more on this topic, see refs. [46, 49].

of Isom($\mathcal{M}$). The orbit O_p of the group of isometries of a given point $p \in \mathcal{M}$ is the set of all points q that p can be taken to through the isometries of Isom($\mathcal{M}$). More precisely,

$$O_p = \{q \,|\, q = \phi(p) \,\text{for all}\, \phi \in \text{Isom}(\mathcal{M})\}. \tag{2.3.2}$$

If $O_p = \Sigma_t$, then the action of the group of isometries on Σ_t is said to be transitive. Let us now define the isotropy subgroup of isometries of a given point p, $\mathcal{I}_p(\mathcal{M})$:

$$\mathcal{I}_p(\mathcal{M}) = \{\phi \in \text{Isom}(\mathcal{M}) \,|\, \phi(p) = p\}. \tag{2.3.3}$$

Note that an isometry associated with isotropy is one which keeps the point it acts on fixed, as is expected with, e.g., a rotation. Now, let n be the dimension of Isom($\mathcal{M}$) and m be the dimension of $\mathcal{I}_p(\mathcal{M})$. An additional condition for a spacetime to be spatially homogeneous is that n be greater than or equal to the dimension of Σ_t (which in standard cosmology is 3). The Bianchi spacetimes, which are spatially homogeneous but anisotropic, have $n = 3$ and $m = 0$. In this case, the action of Isom($\mathcal{M}$) on Σ_t is said to be simply transitive. This means that for all $p, q \in \Sigma_t$, there is a unique element $\phi \in \text{Isom}(\mathcal{M})$ such that $\phi(p) = q$. If otherwise, a homogeneous spacetime is called multiply transitive (e.g., the FLRW spacetime).

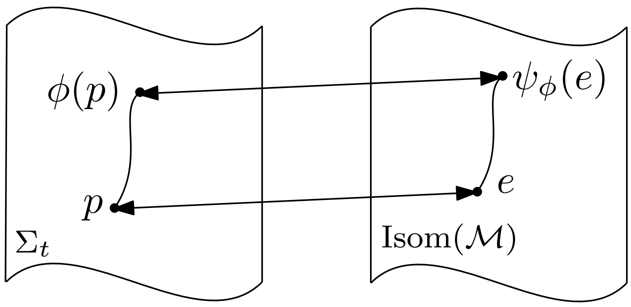

Figure 1 – Schematic representation of the identification between the points on Σ_t with the isometries on Isom($\mathcal{M}$).

Since the isometries of Bianchi spacetimes acts simply transitively on Σ_t, it is possible to put the elements of Isom($\mathcal{M}$) in a one to one correspondence with points on Σ_t. This correspondence is made by associating an isometry ϕ with a point $\phi(p)$. The identity isometry e, e.g., is associated with an arbitrary point p. Once this identification is made, we can now recognize the action of an isometry $\phi \in \text{Isom}(\mathcal{M})$ on Σ_t, which takes a point p to a point $\phi(p)$, as a left translation on Isom($\mathcal{M}$), which is taking the identity isometry e to the isometry $\psi_\phi(e) = \phi$ (see Figure 1). Then, we can apply our knowledge of Lie groups and Lie algebras to Bianchi spacetimes. Tensor fields on Σ_t invariant under isometries now correspond to the left invariant tensor fields on Isom($\mathcal{M}$). This means that there are, in particular, vector fields invariant under isometries on Σ_t which correspond to the left invariant vector fields of Isom($\mathcal{M}$). Thus, there must be a 3-dimensional basis of vectors on Σ_t satisfying the Lie algebra of eq. (2.2.5). Let us call for the time being this invariant basis on Σ_t as the set $\{e_1, e_2, e_3\}$, which will, as we know, satisfy

$$[e_i, e_j] = -C^k_{ij} e_k, \tag{2.3.4}$$

where the minus sign was put there for convenience. Note that we have switched to lowercase Latin indexes, which run from 1 to 3. We also have, in view of this correspondence, that the generators of isometries on Σ_t are the generators of left translations on Isom$(\mathcal{M})$. The generators of isometries, as we have learned, are the Killing vector fields of Σ_t, and the generators of left translations are the right invariant basis vectors which satisfy the algebra of eq. (2.2.9). Thus, a Bianchi spacetime has a 3-dimensional basis of Killing vector fields, which we will call $\{K_1, K_2.K_3\}$, that satisfy

$$[K_i, K_j] = C^k_{ij} K_k. \tag{2.3.5}$$

Also, since generators of left translations are right invariant, we have that

$$\mathcal{L}_{K_i} e_j = [K_i, e_j] = 0. \tag{2.3.6}$$

As before, the algebra satisfies Jacobi's identity and hence the structure constants satisfy $C^m_{[ij} C^n_{k]m} = 0$. We show further that the structure constants are also constant in time, i.e., they do not depend on t. The above are the defining characteristics of the Bianchi spacetimes.

One might wonder what are the possibilities of the spatial nature of the Σ_t sections given these conditions. This depends on how many inequivalent Lie algebras one can find which satisfy eq. (2.3.5). Thus, we are going to work with the algebra of the Killing vectors to classify all possible 3-dimensional Lie algebras. This is called the Bianchi classification, and for historical reasons is labeled by roman numerals from I to IX. By choosing one of these algebras, one can then build a spatially homogeneous spacetime and corresponding cosmological model, which are called the Bianchi models.

We start by analyzing the structure constants of the Killing vector algebra. The structure constants C^k_{ij} are anti symmetric in the lower indexes. Thus, C^k_{ij} has 9 independent components, and can be written in terms of a symmetric tensor n_{ij} and a vector a_i. The decomposition reads

$$C^k_{ij} = \epsilon_{ijm} n^{mk} + \delta^k_j a_i - \delta^k_i a_j. \tag{2.3.7}$$

If one plugs eq. (2.3.7) in Jacobi's identity for the structure constant, $C^m_{[ij} C^n_{k]m} = 0$, after a tedious but straightforward calculation one gets

$$n^{ij} a_j = 0. \tag{2.3.8}$$

Since we have the freedom to perform rotations of the Killing vector fields at a given point, i.e., $K_i \to K_i' = \gamma_i^j K_j$, we can choose a rotation such that n^{ij} is diagonal:

$$n^{ij} = \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix}. \tag{2.3.9}$$

Note that basis changes that permute $1, 2, 3$ are still allowed. From condition (2.3.8), we can see that either $a_i = 0$ or n^{ij} has at least one zero eigenvalue. Through permutations one can always move this eigenvalue to the upper left corner of n^{ij}, that is, we can assume that $n_1 = 0$ given that $a_i \neq 0$. Thus, in view of condition (2.3.8), a_i must assume the form

$$a_i = (a, 0, 0). \tag{2.3.10}$$

Note that this also covers the case of $a_i = 0$ if we then rewrite condition (2.3.8) as

$$a n_1 = 0. \tag{2.3.11}$$

With eqs. (2.3.9) and (2.3.10) we can now rewrite the commutation relations of the Killing vectors as

$$[K_1, K_2] = (aK_2 + n_3 K_3), \qquad [K_2, K_3] = n_1 K_1, \qquad [K_3, K_1] = (n_2 K_2 - aK_3). \tag{2.3.12}$$

Note that we are still in full generality, so the above is true for all Bianchi spaces. To proceed with the classification, we must exhaust all the possible values of (n_1, n_2, n_3, a). A systematic way of doing that is achieved by using the freedom to scale the Killing vectors, as is shown in ref [18]. Thus, we will perform the transformation $K_i \to K_i' = C_i K_i$, where C_i are the scaling constants (here the indexes are not summed over). This casts eq. (2.3.12) into

$$\begin{gathered} [K_1, K_2] = \left(\frac{a}{C_1} K_2 + \frac{C_3}{C_1 C_2} n_3 K_3\right), \qquad [K_2, K_3] = \frac{C_1}{C_2 C_3} n_1 K_1, \\ [K_3, K_1] = \left(\frac{C_2}{C_1 C_3} n_2 K_2 - \frac{a}{C_1} K_3\right). \end{gathered} \tag{2.3.13}$$

Now one can start to exhaust the cases. For example, the case where $n_i = 0$ and $a = 0$ is Bianchi I. In Bianchi I, all structure constants are zero and all Killing vectors commute with one another. Let us analyze the non-trivial case of Bianchi VII_0. This case is achieved when $n_3 = a = 0$ but $n_{1,2} \neq 0$. The commutation relations are thus

$$[K_1, K_2] = 0, \qquad [K_2, K_3] = \frac{C_1}{C_2 C_3} n_1 K_1, \qquad [K_3, K_1] = \frac{C_2}{C_1 C_3} n_2 K_2. \tag{2.3.14}$$

We have the freedom to choose the scaling factors as we please, thus we pick $C_1 = \frac{C_2 C_3}{n_1}$. We then have

$$[K_1, K_2] = 0, \qquad [K_2, K_3] = K_1, \qquad [K_3, K_1] = \frac{n_2 n_1}{C_3^2} K_2. \tag{2.3.15}$$

Now, the choice of C_3 will affect the resulting Bianchi type. If $n_2 n_1 > 0$, we then choose $C_3 = \sqrt{n_2 n_1}$. This gives

$$[K_1, K_2] = 0, \qquad [K_2, K_3] = K_1, \qquad [K_3, K_1] = K_2. \tag{2.3.16}$$

These are the commutation relations that define the Bianchi type VII$_0$ model. Note that, from (2.3.14), had we defined $n'_1 = \frac{C_1}{C_2 C_3} n_1$ and $n'_2 = \frac{C_2}{C_1 C_3}$ we would have ended up having $n'_1 = 1$ and $n'_2 = 1$. This is compatible with the standard table of Bianchi classifications, where Bianchi VII$_0$ appears with $(n_1, n_2, n_3, a) = (1, 1, 0, 0)$. Going back to our choice of C_3, we could have chosen it to be $C_3 = \sqrt{-n_1 n_2}$ had we considered the case where $n_1 n_2 < 0$. We then would have ended up in Bianchi VI$_0$, with $(n_1, n_2, n_3, a) = (1, -1, 0, 0)$. Another interesting case to analyze is the Bianchi type VII$_h$, where the continuous parameter h appears naturally in this type's classification[3]. This case corresponds to $n_1 = 0$ but $a \neq 0$ and $n_{2,3} \neq 0$. The commutators then are

$$[K_1, K_2] = \frac{a}{C_1} K_2 + \frac{C_3}{C_1 C_2} n_3 K_3, \qquad [K_2, K_3] = 0, \qquad [K_3, K_1] = -\frac{a}{C_1} K_3 + \frac{C_2}{C_1 C_3} n_2 K_2. \tag{2.3.17}$$

Choosing $C_2 = \frac{C_1 C_3}{n_2}$ then gives

$$[K_1, K_2] = \frac{a}{C_1} K_2 + \frac{n_2 n_3}{C_1^2} K_3, \qquad [K_2, K_3] = 0, \qquad [K_3, K_1] = -\frac{a}{C_1} K_3 + K_2. \tag{2.3.18}$$

Now, if $n_2 n_3 > 0$, we choose $C_1 = \sqrt{n_1 n_3}$ and we fall into the Bianchi VII$_h$ type. This gives the following commutators:

$$[K_1, K_2] = \frac{a}{\sqrt{n_2 n_3}} K_2 + K_3, \qquad [K_2, K_3] = 0, \qquad [K_3, K_1] = -\frac{a}{\sqrt{n_2 n_3}} K_3 + K_2. \tag{2.3.19}$$

Since we already fixed all possible scaling factors, there is nothing we can do to rescale a. So it remains an arbitrary parameter, and the types corresponding to $(n_1, n_2, n_3, a) = (0, 1, 1, a)$ are actually a family of types that depend on a. It is standard to define another parameter, $h = \frac{a^2}{n_2 n_3}$. These types are all accommodated into the Bianchi type VII$_h$, whose commutators are given by

$$[K_1, K_2] = \sqrt{h} K_2 + K_3, \qquad [K_2, K_3] = 0, \qquad [K_3, K_1] = -\sqrt{h} K_3 + K_2. \tag{2.3.20}$$

Notice that we recover the previous case by making $h = 0$.

The same procedure described above can be applied until all possibilities are exhausted and the standard Bianchi classification is constructed. The results are summarized in Table 1.

Now one can use a Bianchi type to build a cosmological model. Let us recapitulate. A Bianchi spacetime $(\mathcal{M}, g)$ has homogeneous (in the sense defined before) spatial

[3] Not to be confused with the left translation map parameter h used in the previous section.

Bianchi type	n_1	n_2	n_3	a	FLRW limit
I	0	0	0	0	Flat
II	1	0	0	0	-
III	0	1	-1	1	-
IV	0	0	1	1	-
V	0	0	0	1	Open
VI_h	0	1	-1	$\sqrt{-h}$	-
VII_0	0	1	1	0	Flat
VII_h	0	1	1	$\sqrt{h}$	Open
VIII	1	1	-1	0	-
IX	1	1	1	0	Closed

Table 1 – Standard table of Bianchi types classification according to the values of (n_1, n_2, n_3, a) after rescaling as described in the text. Remember that permutations of (n_1, n_2, n_3) are allowed so the table may look different from one source to another. The Biachi type III is often missing in textbooks since it is a subcase of Bianchi VI_h. We also include the FLRW limit for the Bianchi types this is possible (a dash means the limit is not clearly defined). Open and closed mean, respectively, negatively curved FLRW and positively curved FLRW. For more information on this topic, see refs. [23, 47].

hypersurfaces Σ_t with a set $\{K_i\}$ of Killing vectors which forms a 3-dimensional Lie algebra. There also is a natural set of invariant basis vectors, $\{e_i\}$ on Σ_t that forms too a 3-dimensional Lie algebra that, when a suitable choice of the basis is made, shares the same structure constants as the algebra of the Killing vectors. This invariant basis satisfy $\mathcal{L}_{K_i} e_j = 0$. Note that both the set of Killing vectors and the invariant basis vectors are tangent to Σ_t. Let us define precisely what is the parameter t with which we foliate $\mathcal{M}$. We start at a spatial hypersurface Σ_0, and let p be a point on Σ_0. Now, let n be the unit vector normal to Σ_0 at p. Note that it will be orthogonal to both $\{K_i\}$ and $\{e_i\}$ at p. There is a geodesic λ whose initial point is p and tangent vector is n. Given that $K_i^\mu n_\mu = 0$ at p, for all i, then λ will be orthogonal to all spatial hypersurfaces it crosses, since, if K_i is a Killing vector and n a tangent vector to a geodesic, the quantity $K_i^\mu n_\mu$ is conserved along it, as we have shown in eq. (2.1.14). Then, the parameter t is naturally the proper time along the geodesic λ at the hypersurface Σ_t it crosses. Also, by spatial homogeneity, n will be everywhere orthogonal to each Σ_t. At each Σ_t, we additionally require that $\mathcal{L}_{K_i} n = 0$, or $[K_i, n] = 0$. Note that the integral curves of the vector field n will be timelike geodesics orthogonal to each Σ_t at every point, and being unit, n must satisfy $n^\mu n_\mu = -1$. We can then extend the invariant vector basis at Σ_0 to the other spatial sections by noting that $\mathcal{L}_n e_i = [n, e_i] = 0$ is also satisfied by applying the Jacobi identity to the vectors (n, k_i, e_j).

Now, we can build an orthonormal basis to $(\mathcal{M}, g)$ using the set of orthogonal basis vectors (n, e_1, e_2, e_3). Applying the Jacobi identity to the vectors (n, e_i, e_j), one can show, after straightforward algebraic manipulations, that $n(C_{ij}^k) = 0$, i.e., the structure constants are constant from one spatial hypersurface to the other, thus not depending on t. Now, we define a set of invariant one-forms $\{\tilde{e}^i\}$ which satisfies $e_i^\mu \tilde{e}_\mu^j = \delta_i^j$. From these one-forms we can build the metric tensor of a Bianchi model. Let $\tilde{n}$ be the one-form of n. Then, the metric of a Bianchi model is given by

$$g = -\tilde{n} \otimes \tilde{n} + g_{ij}(t)\tilde{e}^i \otimes \tilde{e}^j. \tag{2.3.21}$$

Note that g_{ij} can only depend on t since the spatial hypersurfaces are homogeneous. A metric tensor built in the manner above describes the most general homogeneous and spatially anisotropic spacetime. The peculiarities of each Bianchi model are then contained in its algebraic structure. If needed, one can use the commutation relations and a suitable coordinate system to find coordinate expressions for $\{\tilde{e}^i\}$ and cast the metric in a more familiar form (this is done, for example, in [50]). Summarizing, to build a spatially homogeneous cosmological model, we choose a Bianchi type, which then leads to a certain algebra of Killing vectors and invariant basis. We define a timelike vector orthogonal to the spatial hypersurfaces, and build an orthonormal basis which allows us to construct the metric (2.3.21).

A modern approach to the Bianchi classification applicable to Bianchi spaces with FLRW limit is given by the work in ref. [23]. It starts with one of the three possible 3-dimensional maximally symmetric spaces: Euclidean (flat), hyperbolic (open, negatively curved) or spherical (closed, positively curved). In these spaces, one can identify six Killing vector fields, since they are maximally symmetric. These six Killing vectors are split into three translational Killing vectors $\{T_i\}$ and three rotational Killing vectors $\{R_i\}$. Then, for each of the spaces one identifies the algebra of these vectors, i.e., $[T_i, T_j]$, $[T_i, R_j]$ and $[R_i, R_j]$. For the vector fields associated with homogeneity, there is an ambiguity in their definition. Demanding $\{T_i\}$ to have a close algebra still leaves the freedom to add rotations to them, i.e., we can perform

$$T_i \to K_i = T_i + \rho_i^j R_j, \tag{2.3.22}$$

where ρ_i^j are constants. Then, for each maximally symmetric space, one exhaust all possible values of ρ_i^j which leads to a closed algebra, i.e., $[K_i, K_j] = C_{ij}^k K_k$. Up to reparametrizations, in Euclidean spaces one has, e.g., that the allowed values of ρ_i^j are either $\rho_i^j = 0$ for all i, j, or $\rho_3^3 = 1$ and all other $\rho_i^j = 0$. The first case leads to a closed algebra $[K_i, K_j] = C_{ij}^k K_k$ of the Bianchi I type, while the second leads to a Bianchi VII$_0$ type algebra. Demanding the space to be invariant under only $\{K_i\}$ breaks the rotational symmetries, and one then arrives at Bianchi spaces according to their maximally symmetric limits (see Table 1). The metric of these spaces must now be written using the invariant basis associated with $\{K_i\}$, i.e., the aforementioned $\{e_i\}$. This approach allows one to see how Bianchi models with FLRW limit naturally rise from maximally symmetric spaces.

3 Two point correlation functions in Bianchi spaces

We start with an informal description of what is meant by a two-point correlation function in a spacetime. For a rigorous and mathematically complete description of two-point functions in Riemannian spaces, see ref. [51].

3.1 Basic formalism

A *two-point function* f on a manifold $\mathcal{M}$ is simply a map that takes a pair of points $(p, q) \in \mathcal{M} \times \mathcal{M}$ to a real number. Known examples in physics are Green's functions or the geodesic distance between two points. Here we shall be mainly interested in *correlation* functions, so we also demand f to be symmetric

$$f(p, q) = f(q, p) \tag{3.1.1}$$

since correlation is clearly a pairwise concept. In most interesting situations in cosmology one is dealing with the correlation of random variables in spacetimes with some symmetries. Whenever these variables can be viewed as external fields over a fixed background (e.g., in linear cosmological perturbation theory), their statistical properties will inherit the symmetries of the underlying space. We would thus like to define an invariant correlation function with respect to these symmetries. As we have seen, symmetries are reflected in the existence of an isometry, which is a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ that keeps the metric of $\mathcal{M}$ unchanged. A vector field $\boldsymbol{K}$ is capable of generating a one-parameter family of diffeomorphisms ϕ_t through its integral curves. From here on we use boldface on any vectors, one-forms and tensors to avoid confusion and differentiate them from their components. As we have seen in the previous chapter, if ϕ_t is an isometry, then $\boldsymbol{K}$ is a Killing vector field. Now, suppose that $\mathcal{M}$ possesses an isometry represented by a one-parameter family of diffeomorphisms ϕ_t that maps any point $p \in \mathcal{M}$ to the point $\phi_t(p) \in \mathcal{M}$ such that $\phi_0(p) = p$. Clearly, f will be invariant under this symmetry if

$$f(p, q) = f(\phi_t(p), \phi_t(q)). \tag{3.1.2}$$

In practice, though, one ultimately has to work in a specific coordinate system. Suppose ψ is a chart in an open interval in $\mathcal{M}$. Let us define the local representative ξ of f through

$$f \circ \psi^{-1} = f(\psi^{-1}(x_1^\mu), \psi^{-1}(x_2^\mu)) \equiv \xi(x_1^\mu, x_2^\mu), \tag{3.1.3}$$

where clearly $x_1^\mu = [\psi(p)]^\mu$ and $x_2^\mu = [\psi(q)]^\mu$. In this way, the isometry maps the points $x_{1,2}^\mu$ to (see Figure 2)

$$x_1^\mu(t) = [\psi \circ \phi_t(p)]^\mu, \qquad x_2^\mu(t) = [\psi \circ \phi_t(q)]^\mu. \tag{3.1.4}$$

Therefore, locally the condition (3.1.2) can be written as

$$\xi(x_1^\mu, x_2^\mu) = \xi(x_1^\mu(t), x_2^\mu(t)). \tag{3.1.5}$$

For an infinitesimal displacement along the isometry, we have that

$$x_{1,2}^\mu(t) = x_{1,2}^\mu(0) + \frac{dx_{1,2}^\mu}{dt}dt. \tag{3.1.6}$$

The vector $\frac{dx_{1,2}^\mu}{dt}$ is tangent to the curve defined by the isometry, so $K^\mu = \frac{dx^\mu}{dt}$ is the Killing vector associated with the isometry ϕ_t. Plugging (3.1.6) in (3.1.5) and Taylor expanding to first order takes us to the condition

$$K^\mu \partial_\mu \xi|_1 + K^\mu \partial_\mu \xi|_2 = 0. \tag{3.1.7}$$

Notice that to find this condition we assumed that the points p and q in $\mathcal{M}$ can be covered by the same coordinate chart. The condition above can be written in an equivalent manner recalling the action of a vector $\boldsymbol{K}$ on a scalar $\xi(x_1^\mu, x_2^\mu)$, i.e., (3.1.7) is equivalent to

$$\boldsymbol{K}(\xi) = 0. \tag{3.1.8}$$

Since a manifold usually presents several independent isometries, we generalize the above result to the set of equations

$$K_A^\mu \partial_\mu \xi|_1 + K_A^\mu \partial_\mu \xi|_2 = 0, \tag{3.1.9}$$

where A runs from 1 to m, m being the dimension of Isom($\mathcal{M}$), the set of all isometries of $\mathcal{M}$. As we will see, this set of equations completely determines the functional dependence of the 2pcf.

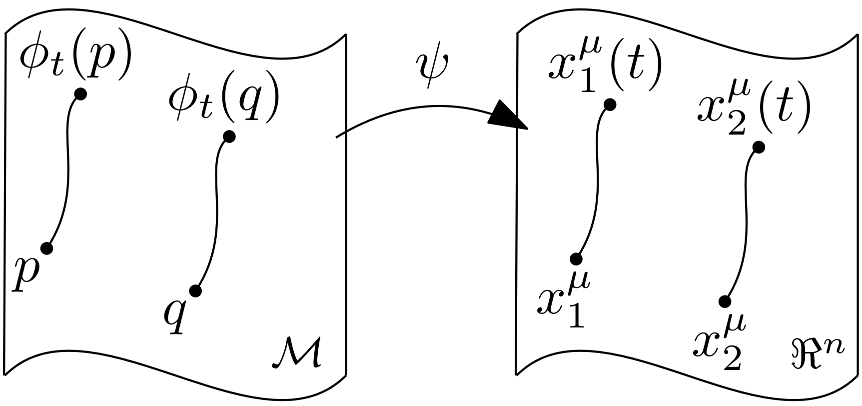

Figure 2 – Representation of the map between the manifold $\mathcal{M}$ and a local patch in $\Re^n$. See text for details.

3.2 Examples: spatially flat FLRW and Minkowski spacetimes

Equations (3.1.9) form the core of our formalism. They will lead to a set of coupled first order partial differential equations which can be implicitly solved by means of the method of *characteristics curves* [45]. In order to illustrate the method let us consider a two-point function in a spatially flat FLRW universe; it could be, for example, the ensemble average of the gravitational potential at two points on the same time slice. Since the spatial slices of FLRW universes are maximally symmetric they possess six independent Killing vectors: three of translation ($\boldsymbol{T}_i$) and three of rotation ($\boldsymbol{R}_i$). In Cartesian coordinates these vectors read

$$\boldsymbol{T}_i = \partial_i, \qquad \boldsymbol{R}_i = \epsilon_{ijk} x^j \partial^k. \tag{3.2.1}$$

The two point function in FLRW depends initially on six coordinates, i.e., on $\xi = \xi(x_1, x_2, ..., z_2)$. For the application of our method, it is more convenient to work with coordinates $\boldsymbol{x}_\pm = (x_\pm, y_\pm, z_\pm)$, defined as

$$x_\pm = x_2 \pm x_1, \tag{3.2.2}$$

and equivalently for $y_\pm$ and $z_\pm$, in such a way that now $\xi = \xi(x_-, x_+, ..., z_+)$. Let us start by analyzing the symmetry associated with the Killing vector $\boldsymbol{T}_x = \partial_x = (1, 0, 0)$. For this vector, the condition (3.1.9) reads

$$\frac{\partial \xi}{\partial x_1} + \frac{\partial \xi}{\partial x_2} = 0. \tag{3.2.3}$$

When written in the $\boldsymbol{x}_\pm$ coordinates, (3.2.3) turns into

$$\frac{\partial \xi}{\partial x_+} = 0. \tag{3.2.4}$$

From the above it is trivial to conclude that ξ cannot depend on x_+. This conclusion, however, will not be immediate for more complicated cases, so we will illustrate how this result follows from the method of characteristic curves. Let t[1] be the parameter along the integral curves of $\boldsymbol{T}_x$, i.e., of the isometry generated by $\boldsymbol{T}_x$. Then, we can write $\boldsymbol{T}_x = \frac{d}{dt}$, and from eq. (3.1.8) we have

$$\frac{d\xi}{dt} = \dot{x}_- \frac{\partial \xi}{\partial x_-} + \dot{x}_+ \frac{\partial \xi}{\partial x_+} + ... + \dot{z}_+ \frac{\partial \xi}{\partial z_+} = 0, \tag{3.2.5}$$

where the overdot means partial derivatives with respect to t. Comparing eqs. (3.2.4) and (3.2.5) term by term, we see that

$$\dot{x}_- = \dot{y}_- = ... = \dot{z}_+ = 0, \qquad \dot{x}_+ = 1, \tag{3.2.6}$$

[1] We will eventually use t or r as parameters along isometries to perform calculations; however, they are not related to neither the time coordinate nor the radial distance.

i.e., all coordinates are constant along the isometry except for x_+. Thus, since we want to impose invariance of the 2pcf with respect to the isometry generated by $\boldsymbol{T}_x$, ξ cannot depend on x_+. The same procedure can be easily applied to the other Killing vectors $\boldsymbol{T}_y$ and $\boldsymbol{T}_z$, and then lead to the conclusion that ξ cannot depend either on y_+ or on z_+. We can finally conclude that imposing translational invariance for a 2pcf in FLRW leads to the functional form $\xi = \xi(x_-, y_-, z_-)$.

Now, let us consider the rotational Killing vector $\boldsymbol{R}_z = -y\partial_x + x\partial_y = (-y, x, 0)$. The condition (3.1.9), already in $\boldsymbol{x}_\pm$ coordinates, is given by

$$-y_-\frac{\partial \xi}{\partial x_-} + x_-\frac{\partial \xi}{\partial y_-} = 0, \tag{3.2.7}$$

where we made use of the previous conditions $\frac{\partial \xi}{\partial x_+} = \frac{\partial \xi}{\partial y_+} = 0$ to arrive at the above. Let r be the parameter along the isometry generated by $\boldsymbol{R}_z$, i.e., $\boldsymbol{R}_z = \frac{d}{dr}$. Again, from (3.1.8) we have that

$$\frac{d\xi}{dr} = \dot{x}_-\frac{\partial \xi}{\partial x_-} + \dot{y}_-\frac{\partial \xi}{\partial y_-} + \dot{z}_-\frac{\partial \xi}{\partial z_-} = 0, \tag{3.2.8}$$

where now the overdot means $\frac{\partial}{\partial r}$. Comparing these last two equations leads us to

$$\dot{x}_- = -y_-, \qquad \dot{y}_- = x_-, \qquad \dot{z}_- = 0. \tag{3.2.9}$$

From the set of equations above, it is easy to see that z_- does not depend on r, but that, in a general manner, x_- and y_- do. However, it is possible, in this case, to find a combination of x_- and y_- independent of the parameter r. To see that, let us decouple the above equations noting that $\ddot{x}_- = -\dot{y}_-$ and $\ddot{y}_- = \dot{x}_-$. We thus have

$$\ddot{x}_- = -x_-, \qquad \ddot{y}_- = -y_-. \tag{3.2.10}$$

After an arbitrary phase choice, a general solution to the above equations is

$$x_- = A\cos r, \qquad y_- = A\sin r, \tag{3.2.11}$$

where A is not necessarily a constant since it can still depend on the parameters related to other isometries. We can then see that the combination $x_-^2 + y_-^2 = A^2$ does not depend on r, in a way that imposing invariance to the 2pcf with respect to the isometry generated by $\boldsymbol{R}_z$ restricts its functional form to $\xi = \xi\left(\sqrt{x_-^2 + y_-^2}, z_-\right)$. The square root was put there in order to maintain the correct length dimension, and, as the above method illustrates, $\sqrt{x_-^2 + y_-^2} = A$ also does not depend on r. It remains to apply the other rotational Killing vectors. Let us continue with $\boldsymbol{R}_y = z\partial_x - x\partial_z = (z, 0, -x)$. Following the same steps as before we have

$$z_-\frac{\partial \xi}{\partial x_-} - x_-\frac{\partial \xi}{\partial z_-} = 0. \tag{3.2.12}$$

Notice, however, that due to our previous calculations we know that x_- and y_- must appear in the specific combination $(x_-^2 + y_-^2)$ to be invariant by the already considered isometries. Thus, let us define $u_-^2 \equiv x_-^2 + y_-^2$. In this manner, the above equation turns into

$$\frac{z_- x_-}{u_-}\frac{\partial \xi}{\partial u_-} - x_- \frac{\partial \xi}{\partial z_-} = 0. \tag{3.2.13}$$

Again, let s be the parameter along the integral curves of $\boldsymbol{R}_y$. Expanding $\frac{d\xi}{ds}$ as a total derivative and comparing it term by term to eq. (3.2.13), we have that

$$u_- \dot{u}_- = z_- x_-, \qquad \dot{z}_- = -x_-. \tag{3.2.14}$$

To find the s independent combination, let us rewrite the above equations as $u_- \dot{u}_- = -z_- \dot{z}_-$, which then leads to

$$\frac{\partial}{\partial s}\left(u_-^2 + z_-^2\right) = 0. \tag{3.2.15}$$

Thus, $u_-^2 + z_-^2 = x_-^2 + y_-^2 + z_-^2$ does not depend on the parameter s. Finally, the 2pcf that is invariant under the isometries of FLRW spacetime is given by

$$\xi_{FL} = \xi_{FL}\left(\sqrt{x_-^2 + y_-^2 + z_-^2}\right), \tag{3.2.16}$$

which is the expected result. Notice that we did not need to use the third rotational Killing vector $\boldsymbol{R}_x$ to fix the functional dependence. This happens because invariance under the isometry associated with $\boldsymbol{R}_x$ is guaranteed by the algebra of the rotational Killing vectors ($\boldsymbol{R}_x = [\boldsymbol{R}_z, \boldsymbol{R}_y]$). Let us introduce the notation $\boldsymbol{r}_{1,2} = (x_{1,2}, y_{1,2}, z_{1,2})$ so we can write $\xi_{FL} = \xi_{FL}\left(|\boldsymbol{r}_2 - \boldsymbol{r}_1|\right)$, or, in a more compact manner, $\xi_{FL} = \xi_{FL}(r_-)$. While we are in the topic of correlation functions in FLRW, it is worth mentioning that in the context of CMB perturbations for large angles, the solution above reads

$$\xi_{FL} = \xi_{FL}\left(\Delta\eta\sqrt{2 - 2\cos\gamma}\right), \tag{3.2.17}$$

where γ is the angle between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ and $\Delta\eta$ is the distance to the last scattering surface.

We would like to illustrate that the above method also works for symmetries associated with temporal translations. To do that, let us extend what was done for FLRW to the Minkowski spacetime, where, apart from the six Killing vectors present on FLRW, there are four more Killing vectors, one associated with time translations ($\boldsymbol{T}_t$, where t represents the time coordinate) and three corresponding to the boost symmetries ($\boldsymbol{B}_i$). Let us use a standard Cartesian coordinate system and notice that, imposing invariance with respect to spatial translations and rotations already leads to

$$\xi = \xi\left(t_1, t_2, \sqrt{x_-^2 + y_-^2 + z_-^2}\right). \tag{3.2.18}$$

We hope it is clear that imposing invariance with respect to time translations leads to $\xi = \xi\left(t_-, , \sqrt{x_-^2 + y_-^2 + z_-^2}\right)$. Let us now impose invariance with respect to the boost symmetries. We start with the Killing vector $\boldsymbol{B}_x = x\partial_t + t\partial_x = (x, t, 0, 0)$. Equation (3.1.9) for this case is given by

$$x_-\frac{\partial\xi}{\partial t_-} + t_-\frac{\partial\xi}{\partial x_-} = 0. \tag{3.2.19}$$

Plugging $r_-^2 = x_-^2 + y_-^2 + z_-^2$ in the above gives

$$x_-\frac{\partial\xi}{\partial t_-} + \frac{t_- x_-}{r_-}\frac{\partial\xi}{\partial r_-} = 0. \tag{3.2.20}$$

Let u be the parameter along the integral curve of $\boldsymbol{B}_x$. Once more, expanding the total derivative $\frac{d\xi}{du}$ and comparing term by term with the above gives

$$\dot{t}_- = x_-, \qquad r_-\dot{r}_- = t_- x_-. \tag{3.2.21}$$

Both equations above can be rewritten as $r_-\dot{r}_- = t_-\dot{t}_-$, which leads to

$$\frac{\partial}{\partial u}\left(-t_-^2 + r_-^2\right) = 0. \tag{3.2.22}$$

Thus, the 2pcf in Minkowski spacetime assumes the form

$$\xi_M = \xi_M\left(\sqrt{-t_-^2 + x_-^2 + y_-^2 + z_-^2}\right). \tag{3.2.23}$$

It is expected that the 2pcf invariant under Minkowsky symmetries only depends on the geodesic distance between two points in this spacetime since it is a maximally symmetric one. Once more, the algebra of the boost Killing vectors allows fixing the functional dependence of the 2pcf demanding invariance with respect to only one of them.

What has been done so far allows us to fix only the *functional* dependence of the 2pcf. However, when a spacetime has a scaling (or conformal) symmetry, the associated Killing vectors further constrain the 2pcf, and completely fix its *shape*, up to a global constant. This is the case, e.g., for Minkowski spacetime. A conformal Killing vector K will satisfy the conformal Killing equation

$$\mathcal{L}_K g = \lambda g \tag{3.2.24}$$

for some function λ. In Minkowski spacetime, one can easily check that $K = x^\mu\partial_\mu$ is a conformal Killing vector with $\lambda = 1$. A 2pcf will be conformal invariant if it satisfies

$$K^\mu\partial_\mu\xi|_1 + K^\mu\partial_\mu\xi|_2 = -\omega\xi \tag{3.2.25}$$

for some constant weight ω. Let us write the geodesic distance in Minkowski as μ_-. Then, the above reduces to

$$\mu_-\frac{d\xi_M}{d\mu_-} = -\omega\xi_M, \tag{3.2.26}$$

a solution of which is

$$\xi_M = \frac{C(\omega)}{\mu_-^\omega}, \tag{3.2.27}$$

where C is a constant of integration. Notice that, apart from C and ω, we were able to fully fix the shape of the 2pcf. In cosmology, a de Sitter phase leads to exact scaling symmetry of the spacetime, which can be used to constrain Gaussian [43] and non-Gaussian correlators [44].

3.3 General solution

We will present here a general solution to eq. (3.1.7). For the sake of simplicity, we will treat first the case of a manifold possessing only one Killing vector field. First, notice that the Lie derivative of the 2pcf (which is a scalar) in the direction of the Killing vector $\boldsymbol{K}$ is given by

$$\mathcal{L}_{\boldsymbol{K}}\xi = \boldsymbol{K}(\xi). \tag{3.3.1}$$

Therefore, the condition (3.1.8) is equivalent to $\mathcal{L}_{\boldsymbol{K}}\xi = 0$. Let us suppose for the moment that there exists a vector $\boldsymbol{e}$ that commutes with $\boldsymbol{K}$, i.e.,

$$[\boldsymbol{K}, \boldsymbol{e}] = \mathcal{L}_{\boldsymbol{K}}\boldsymbol{e} = 0. \tag{3.3.2}$$

If this is the case, then we can show that

$$\xi = \xi\left(\int_{r_1}^{r_2} \sqrt{g_{\mu\nu}e^\mu e^\nu} dr\right) \tag{3.3.3}$$

is a general solution to eq. (3.1.8), where r is the parameter along the curve tangent to $\boldsymbol{e}$, i.e., along the integral curves of $\boldsymbol{e}$. Since $\mathcal{L}_{\boldsymbol{K}}\boldsymbol{e} = 0$, $\boldsymbol{e}$ is a vector that connects two close points on different curves generated by $\boldsymbol{K}$ (Figure 3). This is called a connecting vector.

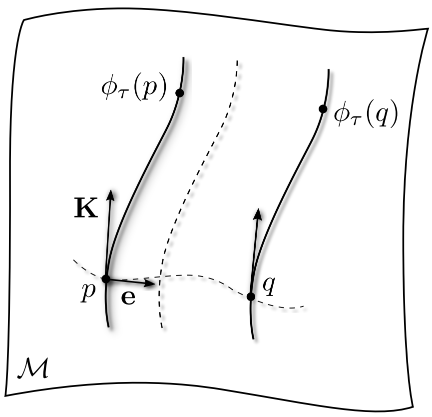

Figure 3 – Schematic representation of two points p, q on the manifold $\mathcal{M}$ connected by the vector $\boldsymbol{e}$ and dragged by the isometry ϕ_t, which in turn is generated by the Killing vector $\boldsymbol{K}$.

To show that (3.3.3) is indeed a solution to (3.1.8), notice that $\int_{r_1}^{r_2}\sqrt{g_{\mu\nu}e^{\mu}e^{\nu}}dr$ is a scalar, therefore the Lie derivative (which acts on scalars simply as a partial derivative) can trivially go through the integration sign. It remains to show that $\mathcal{L}_{\boldsymbol{K}}\left(g_{\mu\nu}e^{\mu}e^{\nu}\right)=0$, from which it follows that $\mathcal{L}_{\boldsymbol{K}}\left[\xi\left(\int_{r_1}^{r_2}\sqrt{g_{\mu\nu}e^{\mu}e^{\nu}}dr\right)\right]=0$ (once more, since ξ is also a scalar, the chain rule trivially applies here), and then condition (3.1.8) will be satisfied. Since the Lie derivative also abides by the product rule, we have that

$$\mathcal{L}_{\boldsymbol{K}}\left(g_{\mu\nu}e^{\mu}e^{\nu}\right)=\left(\mathcal{L}_{\boldsymbol{K}}\boldsymbol{g}\right)_{\mu\nu}e^{\mu}e^{\nu}+2g_{\mu\nu}\left(\mathcal{L}_{\boldsymbol{K}}\boldsymbol{e}\right)^{\mu}e^{\nu}=0 \tag{3.3.4}$$

where the first term is zero because, since $\boldsymbol{K}$ is a Killing vector, $\mathcal{L}_{\boldsymbol{K}}\boldsymbol{g}=0$, and the second term is also zero by means of eq. (3.3.2). However, we depend on our ability to find a vector $\boldsymbol{e}$ which satisfies (3.3.2). This is not trivial for most spacetimes but as we have discussed in the section on Lie algebras it can be easily done for Bianchi spacetimes. In general, a manifold usually has more than one Killing vector field. In these cases, it will be necessary to find a set of independent $\{\boldsymbol{e}_A\}$ commuting with the Killing vectors $\{\boldsymbol{K}_B\}$. Therefore, a general solution to the set of equations (3.1.9) is given by

$$\xi=\xi\left(\int_{r_1}^{r_2}\sqrt{g_{\mu\nu}e_1^{\mu}e_1^{\nu}}dr,\int_{s_1}^{s_2}\sqrt{g_{\mu\nu}e_2^{\mu}e_2^{\nu}}ds,\ldots\right), \tag{3.3.5}$$

where there are as many arguments as many vectors $\boldsymbol{e}_A$ one can find satisfying (3.3.2).

This solution is particularly useful for the Bianchi class of spacetimes. As we have seen, Bianchi geometries are spatially homogeneous but anisotropic, possessing a set of three Killing vector fields $\{\boldsymbol{K}_i\}$ that satisfies the closed algebra $[\boldsymbol{K}_i,\boldsymbol{K}_j]=C^k_{ij}\boldsymbol{K}_k$. The metric of these spaces is usually built from an invariant basis of vectors $\{\boldsymbol{e}_i\}$ which satisfies the condition $[\boldsymbol{K}_i,\boldsymbol{e}_j]=0$. We shall write the general Bianchi metric from eq. (2.3.21) now with a standard parametrization for the spatial part of the metric $g_{ij}(t)$ [19, 52]. It is given by

$$ds^2=-dt\otimes dt+e^{2\alpha(t)}\left(e^{2\beta(t)}\right)_{ij}\tilde{\boldsymbol{e}}^i\otimes\tilde{\boldsymbol{e}}^j, \tag{3.3.6}$$

where t is the parameter along the unit timelike, hypersurface orthogonal vector $\boldsymbol{n}$, $\{\tilde{\boldsymbol{e}}^i\}$ are the duals of $\{\boldsymbol{e}_i\}$, $e^{\alpha(t)}$ is the average scale factor and $\left(e^{2\beta(t)}\right)_{ij}$ is a 3×3 symmetric and traceless matrix whose diagonal elements are directional scale factors. In this parametrization, alpha represents the shape-preserving volume expansion, while beta is the volume-preserving shape expansion. Using the above, we have that for Bianchi spacetimes

$$g_{\mu\nu}e_i^{\mu}e_i^{\nu}=e^{2\alpha(t)}e^{2\beta_{ii}(t)}, \tag{3.3.7}$$

where in the above there is no sum in the index i. This implies that

$$\int_{r_1}^{r_2}\sqrt{g_{\mu\nu}e_1^{\mu}e_1^{\nu}}dr=e^{\alpha(t)}e^{\beta_{11}(t)}(r_2-r_1), \tag{3.3.8}$$

with similar expressions for the two other arguments. Thus, we have arrived at a formal expression for the 2pcf which is invariant under the symmetries of any Bianchi spacetime:

$$\xi = \xi\left(e^{\alpha(t)}e^{\beta_{11}(t)}(r_2 - r_1), e^{\alpha(t)}e^{\beta_{22}(t)}(s_2 - s_1), e^{\alpha(t)}e^{\beta_{33}(t)}(u_2 - u_1)\right). \tag{3.3.9}$$

To convert this function to one valid in an specific coordinate system we have to find the parametric curves of the vectors $\boldsymbol{e}_i$ in the desired coordinates and invert these relations to obtain the parameters as a function of the coordinates. Of course, the success of this procedure depends on the coordinate system chosen. We will illustrate this method with explicit examples in next section, where we find ξ for the geometries of Bianchi I and VII_0 universes using the general solution presented here.

3.4 Applications

We are now in position to put the above formalism to practical use. We start with the example of an inhomogeneous universe with an off-center special point around which it is spherically symmetric. This could be seen as an off-center LTB spacetime, though in reality any spherically symmetric spacetime with a privileged point (e.g., Schwarzschild spacetime, whose privileged point is usually the origin) will lead to the same 2pcf functional dependence. Then, we consider two anisotropic spacetimes with spatially flat spatial sections – namely, the models of Bianchi I and VII_0. They were chosen initially for they are spatially flat, which makes calculations easier and clearer, and also because in an appropriate limit they fall into a flat FLRW universe (note that not all Bianchi models have FLRW limit - see Table 1). We derive the FLRW limit of the 2pcf with first order corrections in both cases. At last, we connect these results with the temperature covariance matrix of CMB fluctuations in the large angle limit, where the Sachs-Wolfe effect is enough to account for the CMB spectrum.

3.4.1 Universe with a special point

The 2pcf of an universe with a special point was studied from a phenomenological point of view in ref. [38]. Recently, the effect of a spherically symmetric off-centered void on the frequency and polarization of CMB photons was investigated in ref. [53]. One of the main interests in LTB-like universes is that they can mimic dark energy effects without the need to modify General Relativity [54]; however, recent investigations do not seem to support this scenario [6, 7].

Here, we model a spherically symmetric universe with an off-centered[2] privileged point through its Killing vectors. Suppose $\boldsymbol{w} = (a, b, c)$ represents the coordinates of the special, off-centered point with respect to the origin of a coordinate chart on this universe.

[2] We allow the special point to be away from the origin since, if we were to live in such a universe, the Earth could as well be away from the privileged point. Hence, observers at Earth would be stuck to their off-centered position to perform measurements.

Then, the only symmetries present are three rotations about the special point $\boldsymbol{w}$. They are represented by the Killing vectors

$$\boldsymbol{R}_i = \epsilon_{ijk}\left(x^j - w^j\right)\partial^k. \tag{3.4.1}$$

Let us apply condition (3.1.9) initially for the Killing vector $\boldsymbol{R}_z = (-y+b, x-a, 0)$. Already in $\boldsymbol{x}_\pm$ coordinates, we have

$$-(y_- - 2b)\frac{\partial\xi}{\partial x_+} - y_-\frac{\partial\xi}{\partial x_-} + (x_+ - 2a)\frac{\partial\xi}{\partial y_+} + x_-\frac{\partial\xi}{\partial y_-} = 0. \tag{3.4.2}$$

Let r be the parameter along the integral curves of $\boldsymbol{R}_z$. Comparing the above equation with the total derivative expansion of $\frac{d\xi}{dr} = 0$ we have

$$\dot{x}_+ = -y_- + 2b \qquad \dot{y}_+ = x_+ - 2a, \qquad \dot{x}_- = y_-, \qquad \dot{y}_- = x_-, \qquad \dot{z}_\pm = 0. \tag{3.4.3}$$

Decoupling and solving these equations in the same manner as before, we find that combinations $u_-^2 = x_-^2 + y_-^2$ and $v_+^2 = (x_+ - 2a)^2 + (y_+ - 2b)^2$ are independent from parameter r. With these results at hand, we apply condition (3.1.9) to the Killing vector $\boldsymbol{R}_y = (z-c, 0, -x+a)$. We arrive at

$$\frac{z_- x_-}{u_-}\frac{\partial\xi}{\partial u_-} + \frac{(z_+ - 2c)(x_+ - 2a)}{v_+}\frac{\partial\xi}{\partial v_+} - (x_+ - 2a)\frac{\partial\xi}{\partial z_+} - x_-\frac{\partial\xi}{\partial z_-} = 0. \tag{3.4.4}$$

Let s be the parameter along the curves of $\boldsymbol{R}_y$. Comparing the above with $\frac{d\xi}{ds} = 0$ gives

$$u_-\dot{u} = z_- x_-, \qquad \dot{z}_- = -x_-, \qquad v_+\dot{v}_+ = (z_+ - 2c)(x_+ - 2a), \qquad \dot{z}_+ = -x_+ + 2a. \tag{3.4.5}$$

Using the first and second equations above, we find that $u_-^2 + z_-^2$ is an s-independent combination, and using the third and fourth we find $v_+^2 + (z_+ - 2c)^2$ is another. Thus, the invariant 2pcf is dependent on

$$\sqrt{u_-^2 + z_-^2} = \sqrt{x_-^2 + y_-^2 + z_-^2}, \tag{3.4.6}$$

$$\sqrt{v_+^2 + (z_+ - 2c)^2} = \sqrt{(x_+ - 2a)^2 + (y_+ - 2b)^2 + (z_+ - 2c)^2}. \tag{3.4.7}$$

Recalling the definition of $\boldsymbol{r}_{1,2} = (x_{1,2}, y_{1,2}, z_{1,2})$ and with a handful of algebra maneuvers, we rewrite the functional dependence of the 2pcf as

$$\xi_w = \xi_w\left(|\boldsymbol{r}_2 - \boldsymbol{r}_1|, \sqrt{|\boldsymbol{r}_2 - \boldsymbol{w}|^2 + |\boldsymbol{r}_1 - \boldsymbol{w}|^2 + 2(\boldsymbol{r}_2 - \boldsymbol{w})\cdot(\boldsymbol{r}_1 - \boldsymbol{w})}\right), \tag{3.4.8}$$

or, in a more simplified manner, as

$$\xi_w = \xi_w\left(|\boldsymbol{r}_2 - \boldsymbol{r}_1|, |\boldsymbol{r}_2 + \boldsymbol{r}_1 - 2\boldsymbol{w}|\right). \tag{3.4.9}$$

This result is compatible with the one found heuristically by the authors of ref. [38], however, our formalism shows that the 2pcf obeying the LTB symmetries are more restrict than the one found by the authors of this reference. Still, this difference does not affect their conclusions since most of them were made using the Fourier transformed version of the 2pcf (i.e., the power spectrum).

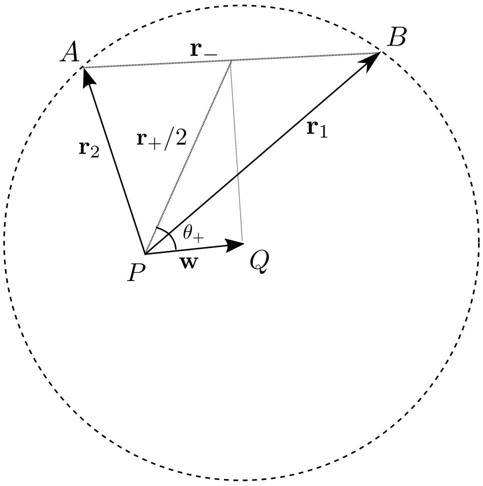

Figure 4 – Schematic representation of an universe with a special off-centered point Q away from the origin P by a distance $\boldsymbol{w}$. In this universe, the correlation between two photons coming from positions A and B can only depend on $|\boldsymbol{r}_2 - \boldsymbol{r}_1|$ and $|\boldsymbol{r}_2 + \boldsymbol{r}_1 - 2\boldsymbol{w}|$.

Before moving forward, we would like to make two comments regarding the solution (3.4.9). First, it may not seem at first glance that this solution recovers the FLRW case (eq. (3.2.16)) when $\boldsymbol{w} = 0$. That is because (3.4.9) is only invariant with respect to rotations. To recover the FLRW solution one must, after making $\boldsymbol{w} = 0$, impose invariance with respect to the translational Killing vectors $\boldsymbol{T}_i$. As we have seen, this imposition prohibits the coordinates (x_+, y_+, z_+) from appearing in the functional dependence of the 2pcf. However, we can easily write the 2pcf of an inhomogeneous spacetime that is spherically symmetric around the origin. That is done by making $\boldsymbol{w} = 0$ in eq. (3.4.9) and gives

$$\xi_0 = \xi_0 \left(|\boldsymbol{r}_2 - \boldsymbol{r}_1|, |\boldsymbol{r}_2 + \boldsymbol{r}_1|\right), \tag{3.4.10}$$

which is functionally equivalent to $\xi_0 = \xi_0 \left(r_1, r_2, \hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_2\right)$, since the only angle appearing in (3.4.10) is the angle between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. This last functional form is more commonly found in the LTB related literature. Second, as mentioned in [38], the solution (3.4.9) respects a global translational symmetry of the form

$$\boldsymbol{r}_{1,2} \to \boldsymbol{r}_{1,2} + \boldsymbol{a}, \qquad \boldsymbol{w} \to \boldsymbol{w} + \boldsymbol{a} \tag{3.4.11}$$

for any vector $\boldsymbol{a}$. This symmetry will prove to be useful further.

We have concluded the task of finding the 2pcf in an universe with a special, off-centered point. The question of whether we actually live close to an off-center spherically symmetric universe can be tested by measuring off-diagonal terms in the covariance matrix of CMB temperature fluctuations. Since we know that our universe is very close to FLRW, we can test this hypothesis by deriving the FLRW limit of eq. (3.4.9), including leading order corrections. To do that we first introduce the variables

$$\boldsymbol{r}_\pm = \boldsymbol{r}_2 \pm \boldsymbol{r}_1. \tag{3.4.12}$$

Now, notice that the desired limit of eq. (3.4.9) involves the expansion of two independent parameters, $|\boldsymbol{w}|$ and $r_+ = |\boldsymbol{r}_+|$. Let us start with the first. Assuming $|\boldsymbol{w}| \ll 1$, we have that

$$|\boldsymbol{r}_+ - 2\boldsymbol{w}| = \sqrt{r_+^2 - 4\boldsymbol{r}_+ \cdot \boldsymbol{w} + 4|\boldsymbol{w}|^2} \tag{3.4.13}$$

$$= r_+ - \frac{2\boldsymbol{r}_+ \cdot \boldsymbol{w}}{r_+} + \mathcal{O}(|\boldsymbol{w}|^2), \tag{3.4.14}$$

that is, to first order

$$|\boldsymbol{r}_+ - 2\boldsymbol{w}| \approx r_+ - 2\hat{\boldsymbol{n}}_+ \cdot \boldsymbol{w}, \tag{3.4.15}$$

where $\hat{\boldsymbol{n}}_+$ is the unitary vector in the direction of $\boldsymbol{r}_+$. Thus,

$$\xi_w \approx \xi_w\left(r_-, r_+ - 2\hat{\boldsymbol{n}}_+ \cdot \boldsymbol{w}\right) \tag{3.4.16}$$

$$= \xi_0\left(r_-, r_+\right) - 2\frac{\partial \xi_0\left(r_-, r_+\right)}{r_+}\hat{\boldsymbol{n}}_+ \cdot \boldsymbol{w} + \cdots \tag{3.4.17}$$

Moving on, let us assume that $\xi_0\left(r_-, r_+\right)$ changes weakly with respect to the parameter r_+, i.e., the deviation of $\xi_0\left(r_-, r_+\right)$ with respect to homogeneity is small. We then have

$$\xi_0\left(r_-, r_+\right) = \xi_0\left(r_-, 0\right) + \frac{\partial \xi_0\left(r_-, 0\right)}{\partial r_+} r_+ + \cdots \tag{3.4.18}$$

It is important to notice that, first, we are also treating r_- and r_+ as independent parameters, and second, we are not assuming r_+ is small. In fact, in the context of CMB temperature fluctuations, this will hardly be the case since for coinciding points on the CMB last scattering surface we have that $r_+ = 2\Delta\eta$, where $\Delta\eta$ is the distance to the last scattering surface. Assuming ξ_0 changes weakly with respect to r_+ implies, as mentioned, that the deviation from homogeneity is small, i.e., that

$$\frac{\partial \xi_0\left(r_-, 0\right)}{\partial r_+} = \frac{\partial x_+}{\partial r_+}\frac{\partial \xi_0\left(r_-, 0\right)}{\partial x_+} + \frac{\partial y_+}{\partial r_+}\frac{\partial \xi_0\left(r_-, 0\right)}{\partial y_+} + \frac{\partial z_+}{\partial r_+}\frac{\partial \xi_0\left(r_-, 0\right)}{\partial z_+} \tag{3.4.19}$$

$$= \frac{r_+}{2}\left\{\frac{1}{x_+}\boldsymbol{T}_x\left[\xi_0\left(r_-, 0\right)\right] + \frac{1}{y_+}\boldsymbol{T}_y\left[\xi_0\left(r_-, 0\right)\right] + \frac{1}{z_+}\boldsymbol{T}_z\left[\xi_0\left(r_-, 0\right)\right]\right\}, \tag{3.4.20}$$

where $\boldsymbol{T}_i\left[\xi_0\left(r_-, 0\right)\right]$ is meant to represent the action of the vector $\boldsymbol{T}_i$ on the scalar function $\xi_0\left(r_-, 0\right)$. Thus, we are in fact assuming that $\boldsymbol{T}_i\left[\xi_0\left(r_-, 0\right)\right] = \mathcal{L}_{\boldsymbol{T}_i}\xi_0\left(r_-, 0\right) \ll 1$ for all i, i.e., dragging $\xi_0\left(r_-, 0\right)$ along translation isometries changes it only slightly. Since $\xi_0\left(r_-, 0\right)$ does not depend on r_+ (neither on $|\boldsymbol{w}|$), we can write $\xi_0\left(r_-, 0\right) = \xi_{FL}(r_-)$. Finally, collecting the expansion terms to first order gives

$$\xi_w = \xi_{FL}(r_-) + \frac{\partial \xi_0\left(r_-, 0\right)}{\partial r_+}\left(r_+ - 2\hat{\boldsymbol{n}}_+ \cdot \boldsymbol{w}\right), \tag{3.4.21}$$

which is the desired result.

To extract the amplitude of the corrections we still need the specific form of the function $\xi_0(r_-, r_+)$, which can only be fixed from first principles [42, 53, 55]. Notice, however, that the first correction in (3.4.21) (which is the term proportional to r_+) does not bring us any new information when compared to (3.2.17), since its angular dependence is associated with the angle between $\boldsymbol{r}_2$ and $\boldsymbol{r}_1$. This term is a monopole-type contribution and can only alter the amplitude of the isotropic temperature spectrum; it will not induce off-diagonal terms in the CMB correlation matrix. On the other hand, the second correction above leads to a dipolar modulation in the 2pcf due to presence of the angle between $\hat{\boldsymbol{n}}_+$ and $\boldsymbol{w}$; this implies non-zero ℓ,$\ell+1$ correlations in the CMB covariance matrix.

3.4.2 Bianchi I and VII$_0$

To find the functional dependence of the 2pcf in the Bianchi universe, we use the general solution (3.3.9) which we showed satisfies the condition (3.1.9). Solving the set of equations (3.1.9) directly would generate the same results, apart from the dependence on the directional scale factors we find in the following.

Let us begin with Bianchi I. Since it is spatially flat (this will also apply to Bianchi VII$_0$), we can use the simple Cartesian coordinate system of Euclidean space. The three translational Killing vectors of this space are then given by

$$\boldsymbol{T}_x = (1,0,0), \qquad \boldsymbol{T}_y = (0,1,0), \qquad \boldsymbol{T}_z = (0,0,1). \tag{3.4.22}$$

The set of invariant basis vectors $\{\boldsymbol{e}_i\}$ that satisfy $[\boldsymbol{T}_i, \boldsymbol{e}_j] = 0$ are simply [23, 50]

$$\boldsymbol{e}_1 = (1,0,0), \qquad \boldsymbol{e}_2 = (0,1,0), \qquad \boldsymbol{e}_3 = (0,0,1). \tag{3.4.23}$$

A systematic way of finding them for any Bianchi model is to choose a coordinate system and then solve the equations $[\boldsymbol{T}_i, \boldsymbol{e}_j] = 0$ subject to the initial conditions $\boldsymbol{T}_i|_0 = \boldsymbol{e}_i|_0$.

Let us, then, solve the integral curves of $\boldsymbol{e}_1$. To do that, we write $\boldsymbol{e}_1 = \frac{d\boldsymbol{x}}{dr}$ and solve it for $\boldsymbol{x}(r)$ component by component. It is not hard to see that the solution will be $x = r$, $y = z = \text{const}$. In this case, it is trivial to invert the relation between the parameters and the coordinates, and we find that $r_2 - r_1 = x_2 - x_1 = x_-$. Solving for $\boldsymbol{e}_{2,3}$ follows similarly, allowing us to write $s_2 - s_1 = y_-$ and $t_2 - t_1 = z_-$. Plugging these results in the general solution (3.3.9) gives us the 2pcf invariant under Bianchi I symmetries:

$$\xi_I = \xi_I\left(e^{\alpha(t)}e^{\beta_{11}(t)}x_-, e^{\alpha(t)}e^{\beta_{22}(t)}y_-, e^{\alpha(t)}e^{\beta_{33}(t)}z_-\right). \tag{3.4.24}$$

Observational evidences show that our universe is very close to isotropic [4, 5, 9]. Therefore, we can find the FLRW limit of ξ_I expanding the 2pcf above around $\beta_{ii} = 0$.

First, let us make $e^{\beta_{ii}} \approx 1 + \beta_{ii}$ inside the arguments of (3.4.24). Performing a Taylor expansion in the resulting 2pcf gives

$$\xi_I = \xi_I(x_-, y_-, z_-) + \left(\beta_{11} x_- \frac{\partial}{\partial x_-} + \beta_{22} y_- \frac{\partial}{\partial y_-} + \beta_{33} z_- \frac{\partial}{\partial z_-}\right) \xi_I(x_-, y_-, z_-) + \cdots \quad (3.4.25)$$

where, from now on, we will omit the functional dependence on α for the sake of simplicity since this factor does not enter our calculations. To go on, we must notice that in the above expression the failure to satisfy rotational invariance is entirely dependent on β_{ii} factors, in the sense that complete rotational invariance is recovered exactly when $\beta_{ii} = 0$. Thus, for example, if we apply $\boldsymbol{R}_z$ to the 2pcf above we find

$$\boldsymbol{R}_z(\xi_I) = \boldsymbol{R}_z(\xi_I(x_-, y_-, z_-)) + \beta_{11}(\cdots) + \beta_{22}(\cdots) + \beta_{33}(\cdots), \quad (3.4.26)$$

where the terms in parenthesis are of the form $x_- \partial_{y_-}(x_- \partial_{x_-} \xi_I)$ and so on, but which are irrelevant to this discussion. The important point is that the right hand side of (3.4.26) is linear in β_{ii}. Therefore, for ξ_I to be invariant under rotations to zeroth order, it is necessary that $\boldsymbol{R}_z(\xi_I(x_-, y_-, z_-)) = 0$. Repeating the same analysis for $\boldsymbol{R}_{x,y}$ leads us to the condition

$$\xi_I(x_-, y_-, z_-) = \xi_{FL}(r_-). \quad (3.4.27)$$

Thus, the FLRW limit of (3.4.24), including first order anisotropic corrections, is given by

$$\xi_I = \xi_{FL} + \frac{1}{r_-} \frac{\partial \xi_{FL}}{\partial r_-} \left(\beta_{11} x_-^2 + \beta_{22} y_-^2 + \beta_{33} z_-^2\right). \quad (3.4.28)$$

As we shall see, this function will introduce quadrupole corrections on the CMB statistics, which is a well known fact in the literature [23, 24, 56, 25]. However, the 2pcf above does not alter the isotropic spectrum.

The space of Bianchi VII$_0$ also presents a spatially flat FLRW limit when $\beta_{ii} = 0$. The isometries of this space can be seen as two orthogonal displacements on the xy plane, and a displacement on the z axis followed by a rotation on the xy plane [23] (see Figure 5). In Cartesian coordinates, the three Killing vectors are given by [50]

$$\boldsymbol{T}_x = (1, 0, 0), \qquad \boldsymbol{T}_y = (0, 1, 0), \qquad \boldsymbol{T}_z = (-y, x, 1). \quad (3.4.29)$$

The set of invariant basis vectors that commute with $\boldsymbol{T}_i$ are [23, 50]

$$\boldsymbol{e}_1 = (\cos z, \sin z, 0), \qquad \boldsymbol{e}_2 = (-\sin z, \cos z, 0), \qquad \boldsymbol{e}_3 = (0, 0, 1). \quad (3.4.30)$$

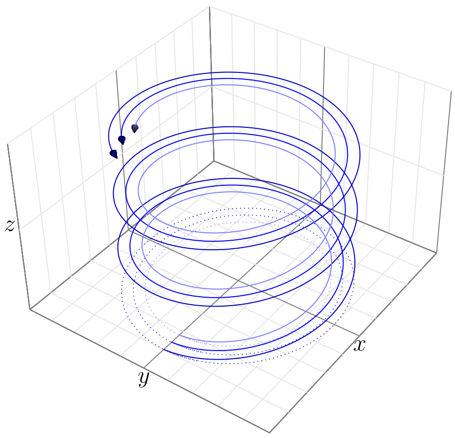

Figure 5 – Integral curves of the Killing vectors (i.e., the isometries) in Bianchi VII_0.

The integral curves of the first invariant vector $\boldsymbol{e}_1$ are given by

$$x(r) = x_0 + r\cos z_0, \qquad y(r) = y_0 + r\sin z_0, \qquad z(r) = z_0. \tag{3.4.31}$$

It is simple to invert the relations above to find the parameter r as a function of the coordinates. First, notice that $x_- = (r_2 - r_1)\cos z_0$ and $y_- = (r_2 - r_1)\sin z_0$. Then,

$$r_2 - r_1 = \sqrt{x_-^2 + y_-^2}. \tag{3.4.32}$$

The integral curves of $\boldsymbol{e}_2$ are similar, yielding $s_2 - s_1 = \sqrt{x_-^2 + y_-^2}$, and for $\boldsymbol{e}_3$ we have seen the result is trivially $t_2 - t_1 = z_-$. Thus, the invariant 2pcf in Bianchi VII_0 is

$$\xi_{VII_0} = \xi_{VII_0}\left(e^{\alpha(\tau)}e^{\beta_{11}(\tau)}\sqrt{x_-^2 + y_-^2}, e^{\alpha(\tau)}e^{\beta_{22}(\tau)}\sqrt{x_-^2 + y_-^2}, e^{\alpha(\tau)}e^{\beta_{33}(\tau)}z_-\right). \tag{3.4.33}$$

The isotropic limit of ξ_{VII_0} follows from the same discussion in the Bianchi I case, with the slight difference that, at lowest order, ξ_{VII_0} is already invariant under $\boldsymbol{R}_z$ rotations due to the isometries of Bianchi VII_0. Then,

$$\xi_{VII_0}\left(\sqrt{x_-^2 + y_-^2}, \sqrt{x_-^2 + y_-^2}, z_-\right) = \xi_{FL}(r_-), \tag{3.4.34}$$

and the expansion around $\beta_{ii} = 0$ gives

$$\xi_{VII_0} = \xi_{FL} + \frac{1}{r_-}\frac{\partial \xi_{FL}}{\partial r_-}\left[(\beta_{11} + \beta_{22})(x_-^2 + y_-^2) + \beta_{33}z_-^2\right]. \tag{3.4.35}$$

3.4.3 CMB covariance matrix

The equations (3.4.9), (3.4.24) and (3.4.33), together with their FLRW limits, are the first main results from the previous sections. We emphasize that these results are completely general and can be equally applied to large scale structure as well as to CMB physics. Here, we shall be interested in the latter, henceforth we would like to derive the multipolar expansion of eqs. (3.4.21), (3.4.25) and (3.4.35).

For the case of an LTB universe with an off-centered point, performing this expansion is not trivial due to the mutual dependence on the parameters r_- and r_+ in eq. (3.4.21), therefore we postpone this analysis to future works. As we shall see, for the Bianchi cases it will be possible to find these expansions and associate them to the temperature fluctuation covariance matrix of the CMB in the limit of large angles. In these scales, the dominant contribution is given by the Sachs-Wolfe temperature fluctuation effect. This effect is due to a gravitational redshift that photons experience on the last scattering surface, and it is given by [57]

$$\frac{\Delta T}{T}(\boldsymbol{r}) = \frac{\Phi(\boldsymbol{r})}{3}, \tag{3.4.36}$$

where Φ is the gravitational potential of a perturbed FLRW metric. To compute all the effects of anisotropic geometries on the CMB, one must also take into account the time evolution of the gravitational potential, photon collisional effects and secondary processes as reionization. Clearly, the formalism presented here does not allow for a treatment of such effects since it is a formalism based only on a space's geometries and symmetries. On the other hand, we can imagine a scenario in which the asymmetries of the early universe are erased through inflation, but where the quantum fluctuations preserve these asymmetries in the statistics of the gravitational potential Φ. The works in refs. [24, 37, 25] make use of the same scenario to tackle different issues. Thus, in this scenario, the CMB statistics preserve signals from these primordial asymmetries, and we assume the subsequent evolution of the universe is isotropic.

To extract multipoles from the 2pcf in eqs. (3.4.25) and (3.4.35), let us call the correlation functions of Bianchi I and VII_0 collectively as $\xi(\boldsymbol{r}_-)$, since in both cases the dependence rests on the terms r_-, x_-,y_- and z_-. We can thus expand them collectively in spherical harmonics as

$$\xi(\boldsymbol{r}_-) = \sum_{\ell m} \xi_{\ell m}(r_-) Y_{lm}(\hat{\boldsymbol{n}}_-), \tag{3.4.37}$$

where $\xi_{\ell m}(r_-)$ are the expansion coefficients, $Y_{\ell m}(\hat{\boldsymbol{n}}_-)$ are the usual spherical harmonics and $\hat{\boldsymbol{n}}_-$ is the unit vector in the $\boldsymbol{r}_-$ direction. To continue, let us introduce spherical coordinates such that

$$\boldsymbol{r}_- = r_- \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right), \tag{3.4.38}$$

which allows us to extract the coefficients $\xi_{\ell m}(r_-)$ from equations (3.4.25) and (3.4.35). The results are collected in Table 2.

Geometry	$\xi_{00}/\sqrt{4\pi}$	ξ_{20}	ξ_{22}
Bianchi I	ξ_{FL}	$\beta_{33}\sqrt{\frac{4\pi}{5}}r_-\partial_{r_-}\xi_{FL}$	$(\beta_{11}-\beta_{22})\sqrt{\frac{2\pi}{15}}r_-\partial_{r_-}\xi_{FL}$
Bianchi VII_0	$\xi_{FL}-\frac{1}{3}\beta_{33}r_-\partial_{r_-}\xi_{FL}$	$\frac{4\beta_{33}}{3}\sqrt{\frac{4\pi}{5}}r_-\partial_{r_-}\xi_{FL}$	0

Table 2 – Non-zero multipolar coefficients of the 2pcf in the anisotropic universes Bianchi I and VII_0. The traceless condition of β_{ij} was used to simplify the results. Note that, in Bianchi I, ξ_{00} is not altered, whereas it is in Bianchi VII_0. Thus, in the latter, not only there is a quadrupolar anisotropic contribution, the anisotropies also modify the isotropic spectrum, i.e., the monopole.

We now want to convert eq. (3.4.37) into an expression for the CMB covariance matrix in harmonic space. In order to do that, it is standard to expand the temperature fluctuations $\frac{\Delta T}{T}(\boldsymbol{r})$ in spherical harmonics as

$$\frac{\Delta T}{T}(\boldsymbol{r}) = \sum_{\ell m} a_{\ell m}(r) Y_{\ell m}(\hat{\boldsymbol{n}}), \tag{3.4.39}$$

where $a_{\ell m}(r)$ are the expansion coefficients. The above can be inverted using the orthogonality of the spherical harmonics to give the coefficients in terms of the temperature fluctuation:

$$a_{\ell m}(r) = \int d^2\boldsymbol{n} \frac{\Delta T}{T}(\boldsymbol{r}) Y^*_{\ell m}(\hat{\boldsymbol{n}}). \tag{3.4.40}$$

For temperature fluctuations measured at two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, the CMB covariance matrix is then given by

$$\left\langle a_{\ell_1 m_1} a^*_{\ell_2 m_2} \right\rangle = \int d^2\boldsymbol{n}_1 \int d^2\boldsymbol{n}_2 \left\langle \frac{\Delta T}{T}(\boldsymbol{r}_1) \frac{\Delta T}{T}(\boldsymbol{r}_2) \right\rangle Y^*_{\ell_1 m_1}(\hat{\boldsymbol{n}}_1) Y_{\ell_2 m_2}(\hat{\boldsymbol{n}}_2), \tag{3.4.41}$$

where $\langle\ \rangle$ represents an ensemble average.

In the scenario we are contemplating, the dominant effect is the Sachs-Wolfe effect, and thus we can write the covariance matrix as

$$\left\langle a_{\ell_1 m_1} a^*_{\ell_2 m_2} \right\rangle = \frac{1}{9} \int d^2\boldsymbol{n}_1 \int d^2\boldsymbol{n}_2 \left\langle \Phi(\boldsymbol{r}_1)\Phi(\boldsymbol{r}_2) \right\rangle Y^*_{\ell_1 m_1}(\hat{\boldsymbol{n}}_1) Y_{\ell_2 m_2}(\hat{\boldsymbol{n}}_2), \tag{3.4.42}$$

therefore, the correlation function of interest is given by the ensemble average of the gravitational potential, i.e.,

$$\xi(\boldsymbol{r}_-) = \left\langle \Phi(\boldsymbol{r}_1)\Phi(\boldsymbol{r}_2) \right\rangle . \tag{3.4.43}$$

In a homogeneous and isotropic spacetime (i.e., FLRW), $\xi(\boldsymbol{r}_-) = \xi(r_-)$, that is, there is no dependence on the directions $\hat{\boldsymbol{n}}_{1,2}$. In this case, the covariance matrix reduces to

$$\left\langle a_{\ell_1 m_1} a^*_{\ell_2 m_2} \right\rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} C_{\ell_1}, \tag{3.4.44}$$

where the C_ℓ is the isotropic CMB angular power spectrum. This is not the case for the anisotropic Bianchi spacetimes. Usually, deviations from isotropy are calculated in terms of the power spectrum, which is the Fourier transform of the 2pcf. It is given by

$$P(\boldsymbol{k}) = \int d^3\boldsymbol{r}_- e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_-}\xi(\boldsymbol{r}_-). \tag{3.4.45}$$

Note that, since both Bianchi I and VII_0 have spatially flat Euclidean sections, we are entitled to use plane waves in our Fourier expansions. It is possible to obtain the power spectrum explicitly from the two point functions derived for the Bianchi cases, as is shown in Appendix A. Here, however, we shall expand $P(\boldsymbol{k})$ in spherical harmonics: $P(\boldsymbol{k}) = \sum_{\ell m} P_{\ell m}(k) Y_{\ell m}(\hat{\boldsymbol{k}})$. Notice that (3.4.45) can be inverted

$$\xi(\boldsymbol{r}_-) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_-} P(\boldsymbol{k}) \tag{3.4.46}$$

$$= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)} P(\boldsymbol{k}) \tag{3.4.47}$$

Using the expansion of $P(\boldsymbol{k})$ in spherical harmonics and the Rayleigh expansion

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi \sum_{\ell m} i^\ell j_\ell\left(kr\right) Y^*_{\ell m}(\hat{\boldsymbol{k}}) Y_{\ell m}\left(\hat{\boldsymbol{n}}\right) \tag{3.4.48}$$

in eq. (3.4.47) gives

$$\xi(\boldsymbol{r}_-) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \sum_{\ell m} P_{\ell m}(k) Y_{\ell m}(\hat{\boldsymbol{k}}) (4\pi)^2 \sum_{\ell_1 m_1} (-i)^{\ell_1} j_{\ell_1}\left(kr_1\right) Y_{\ell_1 m_1}(\hat{\boldsymbol{k}}) Y^*_{\ell_1 m_1}\left(\hat{\boldsymbol{n}}_1\right) \times \tag{3.4.49}$$

$$\times \sum_{\ell_2 m_2} i^{\ell_2} j_{\ell_2}\left(kr_2\right) Y^*_{\ell_2 m_2}(\hat{\boldsymbol{k}}) Y_{\ell_2 m_2}\left(\hat{\boldsymbol{n}}_2\right)$$

$$= \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} \left[\frac{2}{\pi} \sum_{\ell m} i^{\ell_2-\ell_1} \int k^2 dk P_{\ell m}(k) j_{\ell_1}\left(k\Delta\eta\right) j_{\ell_2}\left(k\Delta\eta\right) \mathcal{G}^{\ell,\ell_1,\ell_2}_{-m,m_1,m_2}\right] \times \tag{3.4.50}$$

$$\times Y^*_{\ell_1 m_1}\left(\hat{\boldsymbol{n}}_1\right) Y_{\ell_2 m_2}\left(\hat{\boldsymbol{n}}_2\right),$$

where we have made the identification $r_1 = r_2 = \Delta\eta$ for points on the last scattering surface, and j_ℓ are spherical Bessel functions. Also,

$$\mathcal{G}^{\ell_1,\ell_2,\ell_3}_{m_1,m_2,m_3} \equiv \int d^2\hat{\boldsymbol{n}} Y_{\ell_1 m_1}(\hat{\boldsymbol{n}}) Y_{\ell_2 m_2}(\hat{\boldsymbol{n}}) Y_{\ell_3 m_3}(\hat{\boldsymbol{n}}) \tag{3.4.51}$$

are called the Gaunt coefficients. Recalling (3.4.42), one can read the covariance matrix from eq. (3.4.50). It will then be given by [34, 37]:

$$\left\langle a_{\ell_1 m_1} a^*_{\ell_2 m_2} \right\rangle = i^{\ell_2-\ell_1} \frac{2}{9\pi} \sum_{\ell m} \int k^2 dk P_{\ell m}(k) j_{\ell_1}(k\Delta\eta) j_{\ell_2}(k\Delta\eta) \left(-1\right)^m \mathcal{G}^{\ell,\ell_1,\ell_2}_{-m,m_1,m_2}, \tag{3.4.52}$$

Therefore, using eq. (3.4.52), one can convert a characteristic of the power spectrum directly into a characteristic of the covariance matrix. We can then extract $P_{\ell m}$ from the coefficients on Table 2 making use of the Hankel transform, which, given eq. (3.4.46) and the multipolar expansions of both $\xi(\boldsymbol{r}_-)$ and $P(\boldsymbol{k})$, gives the coefficients of one in terms of the other (see ref. [58] for the use of Hankel transforms in cosmology):

$$P_{\ell m} = 4\pi i^{-\ell} \int_0^\infty r_-^2 dr_- j_\ell(kr_-)\xi_{\ell m}(r_-). \tag{3.4.53}$$

However, it is interesting to obtain an expression for the covariance matrix directly as a function of the coefficients $\xi_{\ell m}$. This can be done by inserting (3.4.53) in (3.4.52), which results in

$$\left\langle a_{\ell_1 m_1} a^*_{\ell_2 m_2} \right\rangle = \frac{8}{9} \sum_{\ell m} \int_0^{2\Delta\eta} r_-^2 dr_- \xi_{\ell m}(r_-) J_{\ell_1,\ell_2,\ell}(r_-) \left(-1\right)^m \mathcal{G}^{\ell,\ell_1,\ell_2}_{-m,m_1,m_2}, \tag{3.4.54}$$

where the coefficients J_{ℓ_1,ℓ_2,ℓ_3} are defined through the following integral

$$J_{\ell_1,\ell_2,\ell_3}(R, r_1, r_2) = i^{\ell_2-\ell_1-\ell_3} \int_0^\infty k^2 dk j_{\ell_1}(kr_1) j_{\ell_2}(kr_2) j_{\ell_3}(kR). \tag{3.4.55}$$

The integral above can be solved analitically [59]. A relevant property of these coefficients is that they are zero whenever R is outside the range

$$|r_1 - r_2| \leq R \leq r_1 + r_2. \tag{3.4.56}$$

In the context of CMB physics, $r_1 = r_2 = \Delta\eta$, and thus $r_- = \Delta\eta\sqrt{2 - 2\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2}$. So, with the identification $R = r_-$, we have

$$0 \leq r_- \leq 2\Delta\eta. \tag{3.4.57}$$

This explains why the domain of the integral in (3.4.54) is finite, and it makes perfect sense since it is impossible to correlate two points on the CMB last scattering surface whose distance is bigger than $2\Delta\eta$, i.e., bigger than twice the distance to the last scattering surface.

Returning to Table 2, we see that each geometry leaves its characteristic fingerprints on the CMB temperature spectrum. In lower orders in β_{ii}, both Bianchi models produce quadrupolar anisotropies, which implies non-zero ℓ,ℓ+2 correlations in the covariance matrix. However, only Bianchi VII_0, through its monopolar contribution, is capable of altering the isotropic temperature spectrum, i.e., the C_ℓs. Higher order corrections in β_{ii} produces high order multipoles, but we will not consider those here. Notice that the parity symmetries of these two Bianchi models prohibits even-odd couplings of the harmonic coefficients [61].

We would like to emphasize that in the specific scenario considered here, only asymmetries from the initial conditions can affect the covariance matrix, after which we assume the universe to be pure FLRW. In particular, contributions resulting from integrated effects from the last scattering surface to us – like the effect induced by the lensing potential in anisotropic universes [11, 12] – cannot be extracted from this formalism in its present form. On the other hand, the multipolar features resulting from the coefficients in Table 2 would still be preserved – perhaps in an integrated version – as long as perturbations are functions of the background coordinates. Indeed this is corroborated by the results of [11, 12], where quadrupolar corrections in the correlation of weak-lensing convergence of large-scale structure in a Bianchi I spacetime was found. Furthermore, since (3.1.9) was designed to work on scalar functions, it cannot be directly applied to the cross-correlations between scalar, vector and tensor perturbations, which are known to couple dynamically through the evolution of the background shear [62, 63]. Nevertheless, since tensor fields are still seen as external fields in a fixed background, tensor correlators should be expected to obey a similar formalism as the one presented here (see also [27]).

3.5 Non-Gaussian correlations

It is straightforward to extend this formalism to non-Gaussian correlation functions (i.e., functions of N points where $N > 2$). Let φ be any scalar N point correlation function. Following the same arguments that lead to the set of eqs. (3.1.9) gives the condition

$$\sum_{j=1}^{N} K^{\mu}_{A} \partial_\mu \varphi|_j = 0. \tag{3.5.1}$$

The first non-Gaussian correlation function of interest is the three point correlation function (3pcf). Let us consider this function in a flat FLRW universe, where there are translational and rotational symmetries. Imposing invariance with respect to $\boldsymbol{T}_x = (1, 0, 0)$, for example, leads to the condition

$$\frac{\partial \varphi}{\partial x_1} + \frac{\partial \varphi}{\partial x_2} + \frac{\partial \varphi}{\partial x_3} = 0. \tag{3.5.2}$$

This can be solved for any φ with an arbitrary dependence on the coordinate X defined as

$$X = l x_1 + m x_2 + n x_3, \tag{3.5.3}$$

where l, m and n are constants that satisfy $l + m + n = 0$. We use this fact to write X as

$$X = l(x_1 - x_3) + m(x_2 - x_3), \tag{3.5.4}$$

which shows that φ must depend on the base combinations $(x_1 - x_3)$ and $(x_2 - x_3)$. Applying the same reasoning to the Killing vectors $\boldsymbol{T}_{y,z}$ gives

$$\varphi(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = \varphi(\boldsymbol{r}_1 - \boldsymbol{r}_3, \boldsymbol{r}_2 - \boldsymbol{r}_3). \tag{3.5.5}$$

This is the functional dependence of a 3pcf in a universe with three translational Killing vectors given by $\boldsymbol{T}_i = \partial_i$ [64]. To obtain the expression that is also invariant under rotations, let us introduce $\boldsymbol{u} = \boldsymbol{r}_1 - \boldsymbol{r}_3$ and $\boldsymbol{v} = \boldsymbol{r}_2 - \boldsymbol{r}_3$. Then, we must find the functional dependence of $\varphi(\boldsymbol{u}, \boldsymbol{v})$ with respect to the rotation vectors $\boldsymbol{R}_i$. Notice however that a rotation around the i axis of the vectors $\boldsymbol{r}_{1,2,3}$ are equivalent to rotations around the same axis of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. This can be seen by calculating the components of the vectors in the $\boldsymbol{u}, \boldsymbol{v}$ coordinates. For example, for $\boldsymbol{R}_z$ we have

$$\begin{aligned}\boldsymbol{R}_z &= (x_1\partial_{y_1} - y_1\partial_{x_1}) + (x_2\partial_{y_2} - y_2\partial_{x_2}) + (x_3\partial_{y_3} - y_3\partial_{x_3}) \\ &= \left(x_1\partial_{u_y} - y_1\partial_{u_x}\right) + \left(x_2\partial_{v_y} - y_2\partial_{v_x}\right) + x_3\left(-\partial_{u_y} - \partial_{v_y}\right) - y_3\left(-\partial_{u_x} - \partial_{v_x}\right) \\ &= \left(u_x\partial_{u_y} - u_y\partial_{u_x}\right) + \left(v_x\partial_{v_y} - v_y\partial_{v_x}\right).\end{aligned}$$

Equivalent results hold for $\boldsymbol{R}_{x,y}$. Therefore, the task of finding the $\varphi(\boldsymbol{u}, \boldsymbol{v})$ which is invariant under rotations follows the same steps that lead to eq. (3.4.10). The solution then must have the form $\varphi\left(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v}\right)$. In terms of the original coordinates, we have

$$\varphi_{FL} = \varphi_{FL}\left(|\boldsymbol{r}_1 - \boldsymbol{r}_2|, |\boldsymbol{r}_1 + \boldsymbol{r}_2 - 2\boldsymbol{r}_3|\right), \tag{3.5.6}$$

which is completely equivalent to $\varphi_{FL}\left(|\boldsymbol{r}_1 - \boldsymbol{r}_3|, |\boldsymbol{r}_2 - \boldsymbol{r}_3|, (\boldsymbol{r}_1 - \boldsymbol{r}_2)\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right)$, a format more common in the literature [64].

We would like to make an interesting comment about the solution (3.5.6). Notice that under the identification $\boldsymbol{r}_3 = \boldsymbol{w}$, the 3pcf has exactly the same functional dependence as the 2pcf in eq. (3.4.9) - the 2pcf of a universe with a special, off-centered point. This suggests that the bispectrum (i.e., the Fourier transform of the 3pcf) in an FLRW universe can mimic the power spectrum (the Fourier transform of the 2pcf) of an off-centered LTB universe. A similar result has been found by the authors of ref. [65], who have pointed out that a non-zero statistically homogeneous and isotropic bispectrum in the strong squeezed limit - i.e., the limit in which the three points form a triangle with one side much smaller than the other two - can induce statistical anisotropies in the power spectrum. The similarity between the 3pcf in FLRW universe and a 2pcf in a LTB universe we have found reflects the fact that the third point in the 3pcf can be seen as a "special" point with which the two other points must correlate. Thus, in a sense, any 3pcf in a homogeneous universe acts like a 2pcf in a universe with a special point. Analogously, the privileged point in an off-centered LTB universe plays the role of a third point with which any two other points must correlate. This explains why these two correlators have the same functional structure.

As one more application, let us consider the 3pcf in an LTB universe, one which presents only rotational symmetries about the origin. Imposing invariance with respect to the Killing vector $\boldsymbol{R}_z$, for example, leads us to

$$\left(x_1\frac{\partial\varphi}{\partial y_1} + x_2\frac{\partial\varphi}{\partial y_2} + x_3\frac{\partial\varphi}{\partial y_3}\right) - \left(y_1\frac{\partial\varphi}{\partial x_1} + y_2\frac{\partial\varphi}{\partial x_2} + y_3\frac{\partial\varphi}{\partial x_3}\right) = 0. \tag{3.5.7}$$

We could try and solve this equation above with the introduction of new variables $X = lx_1 + mx_2 + nx_3$ and $Y = ly_1 + my_2 + ny_3$. This takes the above equation into

$$X\frac{\partial\varphi}{\partial Y} - Y\frac{\partial\varphi}{\partial X} = 0. \tag{3.5.8}$$

By the method of characteristic curves, we would find that the combination $X^2 + Y^2$ solves the above. However, this is not the most general solution. To see why, notice that in the absence of translational symmetry, the constants l, m and n are not constrained anymore. Thus, let us write

$$l + m + n = 2p \tag{3.5.9}$$

for a constant p. We can then rewrite the variable X as

$$X = l(x_1 - x_3) + m(x_2 - x_3) + p(x_1 + x_3) + p(x_2 + x_3) - p(x_1 + x_2), \tag{3.5.10}$$

with an analogous expression for Y. This shows us that in this case there are five base combinations on which φ will depend on. Repeating the same analysis for $\boldsymbol{R}_{x,y}$ we finally have that

$$\varphi_0 = \varphi_0 \left(|\boldsymbol{r}_1 - \boldsymbol{r}_3|, |\boldsymbol{r}_2 - \boldsymbol{r}_3|, |\boldsymbol{r}_1 + \boldsymbol{r}_3|, |\boldsymbol{r}_2 + \boldsymbol{r}_3|, |\boldsymbol{r}_1 + \boldsymbol{r}_2|\right). \tag{3.5.11}$$

Notice in particular that the combination $|\boldsymbol{r}_1 + \boldsymbol{r}_2|$ cannot be ignored, as we would expect with an immediate comparison with eq. (3.4.10). This is so because the vector $\boldsymbol{r}_1 + \boldsymbol{r}_2$ is linearly independent from $\boldsymbol{r}_1 + \boldsymbol{r}_3$ and $\boldsymbol{r}_2 + \boldsymbol{r}_3$ (the plane made by any two of them would not contain the third).

In possession of the solution (3.5.11) we can build by inspection the 3pcf of an LTB universe with a special, off-centered point $\boldsymbol{w}$. Recall that this universe must respect the global symmetry (3.4.11). Then, it is easy to see that the 3pcf below respects rotational symmetry and the mentioned global symmetry:

$$\varphi_w = \varphi_w \left(|\boldsymbol{r}_1 - \boldsymbol{r}_3|, |\boldsymbol{r}_2 - \boldsymbol{r}_3|, |\boldsymbol{r}_1 + \boldsymbol{r}_3 - 2\boldsymbol{w}|, |\boldsymbol{r}_2 + \boldsymbol{r}_3 - 2\boldsymbol{w}|, |\boldsymbol{r}_1 + \boldsymbol{r}_2 - 2\boldsymbol{w}|\right). \tag{3.5.12}$$

3.6 Comments on extending the method to other Bianchi spaces

With the successful application of the formalism on the spaces of Bianchi I and VII_0, we would expect to be able to apply it to other Bianchi spaces, given the solution (3.3.9). Of cosmological interest are the Bianchi spaces which have clear FLRW limits, which are the spaces of Bianchi V and VII_h (whose limits are negatively curved FLRW), and Bianchi IX (whose limit is positively curved FLRW) [22]. Applying the method to both Bianchi V and VII_h allowed us to find the functional dependence of the 2pcf in these spaces, as we show in the following.

We start with the Bianchi V case. Adopting the same coordinate system as in ref. [21], the three Killing vectors of Bianchi V are given by [50]

$$\boldsymbol{T}_x = (1, 0, 0), \qquad \boldsymbol{T}_y = (0, 1, 0), \qquad \boldsymbol{T}_z = (x, y, 1), \tag{3.6.1}$$

and the invariant basis vectors are

$$\boldsymbol{e}_1 = (e^z, 0, 0), \qquad \boldsymbol{e}_2 = (0, e^z, 0), \qquad \boldsymbol{e}_3 = (0, 0, 1). \tag{3.6.2}$$

It is not hard to solve the integral curves of $\boldsymbol{e}_1$, which provides the result $r_2 - r_1 = e^{-\frac{z_+}{2}} x_-$. For $\boldsymbol{e}_2$ the result is analogously $s_2 - s_1 = e^{-\frac{z_+}{2}} y_-$, while for $\boldsymbol{e}_3$ it is trivially $t_2 - t_1 = z_-$. Thus, the 2pcf invariant under Bianchi V symmetries is

$$\xi_V = \xi_V \left(e^{\beta_{11}(t)} e^{-\frac{z_+}{2}} x_-, e^{\beta_{22}(t)} e^{-\frac{z_+}{2}} y_-, e^{\beta_{33}(t)} z_- \right). \tag{3.6.3}$$

One may think a term like z_+ on the functional dependence of the 2pcf breaks translational invariance, but this is only apparent. The translations on the x and y directions are obviously respected. For the translation associated with $\boldsymbol{T}_z$, notice that eq. (3.1.9) for this Killing vector is

$$x_- \frac{\partial \xi}{\partial x_-} + y_- \frac{\partial \xi}{\partial y_-} + 2 \frac{\partial \xi}{\partial z_+} = 0. \tag{3.6.4}$$

Let u be the parameter along the isometry of $\boldsymbol{T}_z$; then, comparing the above with a total derivative expansion of $\frac{d\xi}{du} = 0$ we get

$$\dot{x}_- = x_-, \qquad \dot{y}_- = y_-, \qquad \dot{z}_+ = u, \qquad \dot{z}_- = 0. \tag{3.6.5}$$

The solutions to the above are

$$x_-(u) = Ae^u, \qquad y_-(u) = Be^u, \qquad z_+(u) = 2u, \qquad z_- = \text{const}, \tag{3.6.6}$$

where A and B are constant. Therefore, a translation along the parameter u by a factor $v/2$ implies that

$$x_- \to e^{-\frac{v}{2}} x_-, \qquad y_- \to e^{-\frac{v}{2}} y_-, \qquad z_+ \to z_+ + v, \qquad z_- \to z_-, \tag{3.6.7}$$

in such a way that, if plugged into eq. (3.6.3), it keeps ξ_V invariant.

Let us reintroduce the curvature factor $k^2 \sim \frac{1}{R_c}$ in eq. (3.6.3), where R_c is the universe's curvature radius (note that initially k was set to minus one). This can be done now through a simple dimensional analysis, which leads to

$$\xi_V = \xi_V \left(e^{\beta_{11}(t)} e^{-\frac{\sqrt{k} z_+}{2}} x_-, e^{\beta_{22}(t)} e^{-\frac{\sqrt{k} z_+}{2}} y_-, e^{\beta_{33}(t)} z_- \right). \tag{3.6.8}$$

The next objective would be to take the flat FLRW limit of the above solution, including first order corrections in the directional scale factors β_{ii} and in the curvature. Let us first take the limit of eq. (3.6.8) for small β_{ii}. As, to first order, we do not allow for curvature-anisotropy couplings, the result is

$$\xi_V = \xi_V \left(e^{-\frac{\sqrt{k} z_+}{2}} x_-, e^{-\frac{\sqrt{k} z_+}{2}} y_-, z_- \right) + \beta_{11} \bar{x}_- \frac{\partial \xi_{FL}}{\partial \bar{x}_-} + \beta_{22} \bar{y}_- \frac{\partial \xi_{FL}}{\partial \bar{y}_-} + \beta_{33} \bar{z}_- \frac{\partial \xi_{FL}}{\partial \bar{z}_-}. \tag{3.6.9}$$

The overbar on the coordinates are to indicate they are coordinates of flat FLRW (as opposed to negatively curved FLRW), as will be explained further. The 2pcf $\xi_V|_{\beta=0}$ (which is the first term above) must now be invariant under rotations of the background FLRW space, since it no longer depends on β_{ii}. However, we have yet to expand on curvature, thus, the background FLRW space is the negatively curved FLRW, which has the shape of a 3-dimensional hyperbolic $\boldsymbol{H}^3$ space. So, $\xi_V|_{\beta=0}$ must depend only on the invariant distance between two points in this space (since it is maximally symmetric), which we will call r_-, as we have done before, but now it is given implicitly by [50]

$$\frac{1}{k}\cosh\left(\sqrt{k}r_-\right) = \frac{1}{2}\left[e^{-\sqrt{k}z_+}\left(x_-^2 + y_-^2\right) + \frac{2}{k}\cosh\left(\sqrt{k}z_-\right)\right]. \tag{3.6.10}$$

Let us call $\xi_V|_{\beta=0} = \xi_{\boldsymbol{H}^3}(r_-)$. Our job now, to find the first order curvature correction to (3.6.9), is to expand $\xi_{\boldsymbol{H}^3}(r_-)$ for small curvature and collect the first order term. Notice, however, that the coordinates on $\boldsymbol{H}^3$ have an implicit dependence on curvature, which can be seen through the eqs.

$$x = \frac{1}{\sqrt{k}}\frac{\sinh(\sqrt{k}r)\sin\theta\cos\phi}{\cosh(\sqrt{k}r) - \sinh(\sqrt{k}r)\cos\theta}, \tag{3.6.11}$$

$$y = \frac{1}{\sqrt{k}}\frac{\sinh(\sqrt{k}r)\sin\theta\sin\phi}{\cosh(\sqrt{k}r) - \sinh(\sqrt{k}r)\cos\theta}, \tag{3.6.12}$$

$$z = -\frac{1}{\sqrt{k}}\ln\left[\cosh(\sqrt{k}r) - \sinh(\sqrt{k}r)\cos\theta\right], \tag{3.6.13}$$

which are just the transformations from the adopted coordinate system to a spherical one [21]. Thus, Taylor expanding the above to first order in $\sqrt{k}$ they are related to the background (overbarred) coordinates by

$$x = \bar{x} + \sqrt{k}\bar{x}\bar{z} + \mathrm{O}(k), \tag{3.6.14}$$

$$y = \bar{y} + \sqrt{k}\bar{y}\bar{z} + \mathrm{O}(k), \tag{3.6.15}$$

$$z = \bar{z} + \frac{1}{2}\sqrt{k}(\bar{z}^2 - \bar{r}^2) + \mathrm{O}(k). \tag{3.6.16}$$

We can now plug this back into eq. (3.6.10) and Taylor expand both sides to find r_- as a function of $\bar{r}_-$ (which is the invariant distance on flat FLRW). This gives

$$r_- = \bar{r}_- + \mathcal{O}(k). \tag{3.6.17}$$

i.e., there is no first order curvature correction to the invariant distance between two points on $\boldsymbol{H}^3$. This in turn means that, to first order in curvature, $\xi_{\boldsymbol{H}^3}(r_-) = \xi_{FL}(\bar{r}_-)$, and thus the 2pcf in Bianchi V is, in this limit

$$\xi_V = \xi_{FL}(\bar{r}_-) + \beta_{11}\bar{x}_-\frac{\partial\xi_{FL}}{\partial\bar{x}_-} + \beta_{22}\bar{y}_-\frac{\partial\xi_{FL}}{\partial\bar{y}_-} + \beta_{33}\bar{z}_-\frac{\partial\xi_{FL}}{\partial\bar{z}_-}, \tag{3.6.18}$$

i.e., to first order in the directional scale factor and curvature it is completely equivalent to the invariant 2pcf of Bianchi I. To find new characteristics for the Bianchi V 2pcf one must go to second order in β_{ii} and curvature, which would also allow curvature-anisotropy couplings. As both curvature and anisotropies are estimated to be very small in our current universe, we set those calculations aside, and, for the purposes of this work, the Bianchi V result is equivalent to that of Bianchi I.

The Bianchi VII_h case follows a similar pattern. Using its Killing vectors and invariant basis we were able to find the invariant 2pcf, which is given by

$$\xi_{VII_h} = \xi_{VII_h}\left(e^{\beta_{11}(\tau)}e^{-\frac{\sqrt{h}z_+}{2}}\sqrt{x_-^2+y_-^2}, e^{\beta_{22}(\tau)}e^{-\frac{\sqrt{h}z_+}{2}}\sqrt{x_-^2+y_-^2}, e^{\beta_{33}(\tau)}z_-\right). \tag{3.6.19}$$

If we proceed in the same manner as described for Bianchi V to evaluate the first order corrections in anisotropy and curvature, one will find that the Bianchi VII_h case is completely equivalent to Bianchi VII_0.

The other space of cosmological interest is the Bianchi IX case. The invariant basis vectors of this space are given by [50]

$$\boldsymbol{e}_1 = (1,0,0)\,,\ \boldsymbol{e}_2 = (\sin x\tan y, \cos x, -\sin x\sec y)\,,\ \boldsymbol{e}_3 = (-\cos x\tan y, \sin x, \cos x\sec y)\,. \tag{3.6.20}$$

We were not able, however, to solve the integral curves of (3.6.20) analytically. Even if it were possible, we infer that inverting the relation between the parameters and a chosen coordinate system would not be an easy task.

4 Conclusions and perspectives

Correlation functions belong to the core of modern cosmology. The perspective of extending the ΛCDM model to inhomogeneous, anisotropic, and non-Gaussian universes depends crucially on our abilities to model and measure such functions with increasing levels of sophistication. In this work we have introduced a novel formalism which allows us to fix the functional dependence of correlation functions in terms of the underlying spacetime (continuous) symmetries. Given a set of Killing vectors, we have found a set of first order partial differential equations which can be solved for the functional dependence of the correlation function. The method works for arbitrary N-point correlators as long as cosmological perturbations can be treated as external fields on a fixed background. We have also provided a general solution to the two point correlation function which naturally introduces the time dependence, provided one finds a set of triad vectors commuting the Killing vector fields. This solution is particularly useful in applications to Bianchi cosmologies, where such triad of vectors are naturally present.

We have successfully applied the formalism to the two point function in three different cosmological spacetimes, namely, the anisotropic and spatially flat solutions of Bianchi types I and VII_0, and to the case of an off-center LTB universe, which includes the standard LTB model as a special case. Our results for an LTB universe correct an earlier result appearing in the literature based on incomplete heuristic arguments [38]. We have provided asymptotic expansions of these correlation functions around the known Friedmannian case. We have analyzed, in the context of large angle CMB fluctuations, the multipolar expansions of the 2pcf in the Bianchi cases. Each spacetime leaves its own multipolar fingerprint on the CMB covariance matrix $\left\langle a_{\ell_1 m_1} a^*_{\ell_2 m_2}\right\rangle$. To the lowest order in the expansion parameters, we have found that Bianchi I spacetimes lead to quadrupolar couplings $\left\langle a_{\ell_1 m_1} a^*_{\ell_1 \pm 2, m_2}\right\rangle$ while preserving the isotropic angular spectrum C_ℓ. Bianchi VII_0 models, on the other hand, lead to quadrupole couplings as well as suppression of the C_ℓ's at large angles (i.e., low ℓ's).

We have also applied the method to infer the functional dependence of (non-Gaussian) three point correlation functions to the (well-known) case of a FLRW universe, and also to the case of an off-center LTB universe. As a byproduct of our formalism we have found a formal link between the three-point correlation function in an FLRW universe and a Gaussian 2pcf in an off-center LTB universe. This link results from the fact that a universe with a strong dependence on a special point is formally degenerate with a non-Gaussian universe with non-zero 3-point correlation function.

At last, we have made some remarks about the possibility of extending our formalism to other Bianchi spaces. We have found that, for Bianchi V and VII_h, which have a negatively curved FLRW limit, to first order in β and curvature, their results are equivalent to that of Bianchi I and VII_0, respectively. We also show how to explicitly evaluate the power spectrum from the 2pcf found for Bianchi I and VII_0.

We would like to end with some remarks on the limitations and possible extensions of the formalism. First, we stress that, although the method can be used provide quick access to the multipolar structure of CMB correlations in a given geometry, it cannot be expected to fix the amplitude of the correlations without any further geometrical or physical input. In the case of CMB physics, for example, a full account of the relativistic and collisional Boltzmann transport equation is required to fix the amplitude of the C_ℓ's. The case of Bianchi I is a clear example. While the quadrupolar couplings we found here are compatible with the result of more in-depth analysis, the present formalism cannot predict the oscillations in the power spectrum resulting from linear perturbation theory [25, 62] nor the correlation between scalar and tensor modes arising from the dynamical couplings with the shear [24, 63]. Nonetheless, it is worth mentioning that scaling invariance of the spacetime, when it exists, can be used to fully fix the shape of scale invariant 2-point correlation functions. The connection between conformal invariance of the de Sitter spacetime and the origin of CMB fluctuations has been a topic of intensive investigation in the recent years [43, 44] and, in this respect, our findings offer a possibility of extending such results to less symmetric spacetimes.

Second, we have not considered the case of spin functions, which are of central importance to the physics of polarization and weak-lensing of the CMB. The case of vector two-point functions in de Sitter spacetimes have been addressed in [27] using a different formalism, where it was found that it also has the same symmetries of the background space. In the present formalism this conclusion is not immediate since it was developed particularly for a scalar 2pcf. We postpone such analysis to future works. Nonetheless, we emphasize that the method developed here is general, and can be equally useful in applications to quantum field theory in curved spacetime.

Part II

Redshift and direction drifts in cosmology

5 Introduction

Over the past decades, the developments of observational cosmology, among which are the measurements of cosmic microwave background (CMB), baryonic acoustic oscillations (BAO), type Ia supernovae and cosmic shear, have led to a robust model of our universe in the framework of Friedmann-Lemaître-Robertson-Walker spacetimes. Through a set of well-measured parameters one can then define the *concordance model* of cosmology: the ΛCDM model. The need for a dark sector has triggered the necessity to develop tests of the hypothesis on which this model lies (see refs. [54, 66] for their description and existing tests). Any data which is not strictly located on our past light cone brings sharp constraints, in particular on the Copernican principle [67].

The first evaluation of the time drift of the cosmic redshift dates back to the 60s [68, 69]. Since it offers a direct measurement of the local cosmic expansion rate [70], it evolved to the idea of "real time" cosmology [26], based on the time drifts of both the cosmic redshift and the direction as new cosmological observables. For example, the possibility of measuring the redshift drift was thought both as a way of measuring the instantaneous cosmic expansion rate as a function of redshift, $H(z)$, [71] and hence to better constrain dark energy models [72, 73, 74] in the framework of Friedmann-Lemaître cosmologies. On a more fundamental level, the redshift drift was shown to offer a way to test of the Copernican principle in [67, 75, 76], while lensing-type effects of a local void were investigated in [53]. Direction drift effects were also investigated in the test of the Copernican principle in Refs. [77], while in Refs. [78, 79] it was used to constrain anisotropic (Bianchi I) cosmological models. More recently, the aberration drift was used as a redshift-independent tool to extract the proper acceleration of the earth with respect to CMB [80]. From a more formal perspective, several investigations of optical drift effects in general relativity provide a covariant derivation of cosmic parallax for a pair of sources in general spacetimes [81, 82, 83, 84, 85].

From an observational perspective, the technical advancements in the field of precision astrometry have allowed the Gaia space mission to measure the parallax of astrophysical objects with unprecedented precision [86, 87]. Forthcoming experiments carrying state-of-the-art spectrography on the E-ELT aim to reach the sensitivity to measure the redshift drift by monitoring Ly-α absorption lines of distant quasars in a time span of a decade [88] and it has been demonstrated that the use of many spectral lines and quasars is an important measure to reduced the variance in $\dot{z}$ induced by linear cosmological perturbations [89].

The second part of this thesis revisits both the redshift and direction drifts in several cosmological frameworks. It starts by providing an overview on astrometry, highlighting significant historical moments and describing this work's approach. Then, we move on to the evaluation of the drifts. First, we present a simple analysis in Minkowski spactime in order to emphasize the need for a two-worldlines analysis, hence non-local in time. A heuristic argument allows one to grasp the physical meaning and typical amplitude of all the different contributions. It takes into account a general accelerated observer and source. Then, we focus on the computation of the redshift and direction drifts for general observers in FLRW spacetime. A multipolar decomposition of this result is presented and compared to the literature. The order of magnitude of the aberration and parallax parts of the direction drift are then estimated. Next, the same drift effects are computed but now for a linearly perturbed FLRW spacetime, including scalar, vector and tensor perturbations. This fully characterizes the imprints of cosmological drifts on the large scale structure. The analysis is finally extended to the case of an anisotropic Bianchi I universe. As such it offers a new way to test the late time isotropy of the cosmic expansion around us, complementary to methods based on either cosmic shear [11, 24, 90] or supernovae [91, 92] observations. We then conclude summarizing our results and their implications.

6 An introduction to astrometry

6.1 An historical overview

Astrometry is one of the oldest fields of study known to man. It comprises the measurement of the position of stars and other celestial objects as they appear on the night sky. As it turns out, the unmovable position of a star is not of ultimate importance. Instead, the tiny variations of its position on the sky over large periods of time are able to offer great insight on a myriad of topics, as we shall see. In what follows, we will present a brief historical overview of astrometry, highlighting some key discoveries and presenting its state in the modern age. Most of this material was based on refs. [93, 94, 95, 96].

Since the ancient times, several civilizations across the globe had early thinkers and philosophers interested in understanding and charting celestial objects. They were, however, limited by the lack of instruments to do so besides the human eye and rough auxiliary objects. Around 1000 BCE, Babylonians in Mesopotamia were the first recorded astronomers of humanity, who systematically observed objects on the night sky and recognized, e.g., the periodicity of the appearance of Venus. Ancient Greece also had its fair share of early astronomers. Around the third century BCE, Greek philosophers already recognized the stars as "fixed" objects, maintaining their relative position while the skies turned above, whereas other objects "wandered" among them (planet, from the Greek *planetes*, means wanderer), apart from the Moon and the Sun. The consensus at that time was that the Earth was fixed relative to the skies (and hence to the Sun and Moon), and the convoluted motion of the planets on the sky bothered many Greek thinkers. While some, in ancient Greece, hinted at the idea of a heliocentric view of the world, it was not supported by the community at the time. Hipparchus, around the second century BCE, pioneered much of the cataloging of stars, adopting the Babylonian system of 360 degrees, 60 minutes and 60 seconds of arc to measure their angular position on the sky, and developing a classification of star brightness which is used up until modern days. His catalog comprised an astonishing figure of approximately 850 stars, measured mostly with the naked eye, with a precision of about one degree.

While during the Dark Ages most of Europe stalled in scientific development, astronomy was a flourishing topic of study in the Chinese and Indian empires, and also among Islamic scholars. Ulugh Beg was one of such scholars who, most notably, built an observatory in 1428 (located in nowadays Uzbekistan) comprised of a 36 meter radius sextant. This allowed him to catalog 994 stars with a precision slightly better than one degree.

What shone a light on Dark Ages' scientific stall was Copernicus' proposition that the Earth is not fixed in space, but rather revolving around the Sun, published in 1543. It would take the works of Galileo, Kepler and Newton to fully understand the motion of the planets around the solar system and to push heliocentrism forward, both within and outside the scientific community. While, after much struggle, the heliocentric view prevailed, together with it came the search for an inevitable effect which was certain to be observed were Copernicus' system true. Such effect is the parallax of distant stars. Parallax is the difference in apparent position of an object due to a change in the observer's line of sight (see Figure 6). Thus, if the Earth was indeed moving around the sun, the stars could not remain fixed on the sky, but rather wobble slightly while the Earth revolved. With the certainty that such effect should be detectable, many scientists set out, with the technology available at the time, to measure the parallax of distant stars. Apart from asserting the heliocentric view, measuring the parallax of a distant star could allow for an evaluation of its distance from Earth through a method known as triangulation. However, scientists were not aware of how far stars could be from Earth, which would render parallax too small to be measured at the time. This even led some scientists in disbelief of the heliocentric model, e.g., Tycho Brahe, who attempted but failed to detect any parallax. Despite technological advancements and improvements in precision, stellar parallax would only be measured around the 1830s; the search for it, however, led to the seminal discovery of the phenomenon of aberration of light.

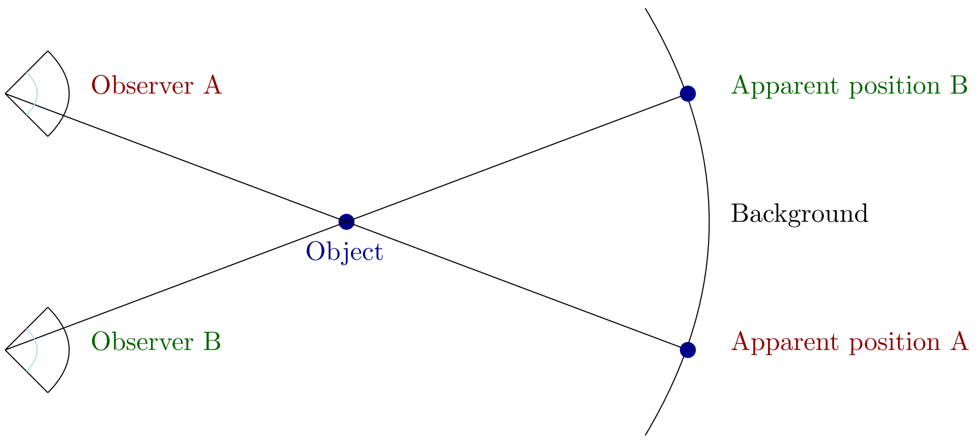

Figure 6 – Simple illustration of the parallax effect.

The aberration of light was discovered by James Bradley when he set out to measure the stellar parallax of the star γ Draconis with an angular precision of about 1arcsec. This star was of particular interest since it passes nearly overhead on London's sky, thus minimizing the refraction of light caused by the atmosphere. Throughout a year of patient measurements, Bradley set out to detect γ Draconis' parallax; however, the angular displacement he found had a different pattern from the one predicted for stellar parallax. The largest values of displacements were delayed by three months when compared to ones expected for parallax. He then had an insight that he could be observing an angular displacement due simply to the fact that the Earth was moving with respect to the star (see Figure 7). An anecdote tells this came to him while riding on a boat through River Thames, analyzing how a wind vane was affected by the wind and the motion of the boat. A simple example to understand aberration is to consider the direction of falling rain. If a person is at rest and the rain is falling vertically, then for a person moving forwards the rain will appear to be coming at an angle. While in the first case the person holds an umbrella vertically, the person moving forwards would have to tilt its umbrella forwards to avoid getting wet. In this analogy, the person moving forwards corresponds to the Earth, and the falling rain corresponds to the light coming from a distant star. By treating light in a particle-like manner, Bradley was able to theoretically explain this phenomenon, which he then presented to the public in 1729. The classical result is that a distant star (considered static) viewed from Earth (moving at a velocity V) is at an apparent angle α' related to the its true angular position α by

$$\tan\alpha' = \frac{\sin\alpha}{\cos\alpha + V/c}, \tag{6.1.1}$$

where c is the velocity of light. In the next chapter we provide a modern derivation of this result. It turned out that the measurement of aberration provided proof that the Earth was indeed moving through space, supporting Copernicus' heliocentric model. The detection of stellar aberration is considered today one of the most important discoveries in astronomy. It confirmed that light travels at a finite speed, and also allowed Bradley to estimate the time light takes to travel from the Sun to the Earth, which he evaluated to be 8 minutes and 12 to 13 seconds, an incredibly precise estimate for the time. And finally, since he failed to detect any parallax, it asserted that stars were indeed enormously distant from the Earth and that detecting it would still be a technological challenge, requiring even more precise instruments.

The first measurements of stellar parallax came through around the 1830s. Astronomers were searching for closer stars, selecting them by their apparent brightness and proper motion. The first reliable measurement of stellar parallax was published in 1838 by Friedrich Bessel, due to the star 61 Cygni, the parallax of which was measured to be 0.314arcsec. This also allowed to evaluate its distance from Earth, which was estimated to be of three parsecs, the first reliable measurement of the distance of a star. Then, in 1939, Thomas Henderson published his results on the parallax of Alpha Centauri, which was of 0.742arcsec, placing it at a distance of 1.35parsec. Now known to be a triple system, Alpha Centauri is among the closest stars to Earth, exhibiting one of the largest parallax displacements. In 1940 the German astronomer Whilhelm Struve also published the detection of a parallax of 0.130arcsec for the star Vega. The measurement of parallax of these stars, and hence of the distances they are to Earth, was fundamental in shaping our understanding of the scales of the Universe.

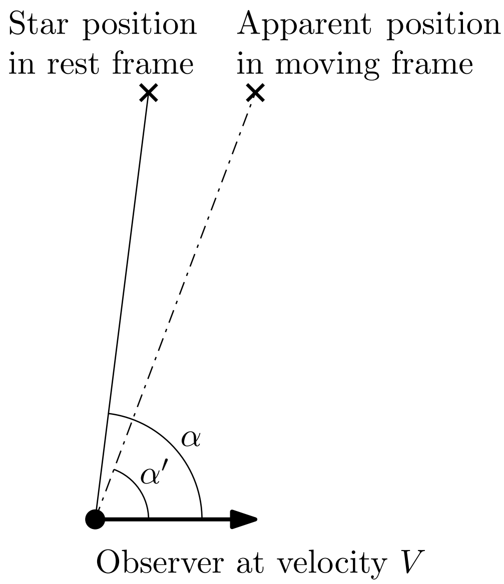

Figure 7 – Simple illustration of the aberration effect.

Since these first measurements of stellar parallax, many technological advancements have been made which allowed scientists to measure the position of stars and other celestial objects with increasing precision. Photography is one of such advancements, through which much larger and more precise catalogs were produced. However, the Earth's atmosphere posed a limit on the precision one could achieve on ground based telescopes. Therefore, in 1980 the project for the first space telescope was proposed: the Hipparcos satellite. It was launched in 1989 to measure the position and proper motion of more than 100,000 stars with a then unprecedented accuracy of 1mas (milliarcsecond). The data from the Hipparcos mission had enormous impact in many areas of physics, from cosmology to the study of exoplanets. The latest advancement in astrometry is due to the space telescope Gaia [95, 96]. The successor of Hipparcos, it is expected to measure one billion stars and other celestial objects with a precision 100 times better than that of Hipparcos, i.e., up to 10μas. The mission was launched in 2013 and is still ongoing, and the data it gathers will undoubtedly revolutionize our knowledge of the universe. The impressive precision of this mission motivated the proposal of a novel field in cosmology called real time cosmology. In the following section, we present and review the main ideas behind real time cosmology.

6.2 The rising field of real time cosmology

The proposal of real time cosmology is to use highly precise measurements, such as those of Gaia, spaced out over a significant time period such that detection of small time variations of cosmological observables is possible. It is expected that a period of a few decades is enough for Gaia to measure a non-zero temporal variation in positions and velocities of stars [26]. In the context of cosmology, this means one can conceive the observation of what is called the *drift* of cosmological observables. The (radial and angular) positions and velocities of celestial objects are related to important cosmological observables, such as the gravitational redshift. One can then use different cosmological models to theoretically evaluate the drifts of such observables and put these predictions side to side with measurements made within a relatively short period. This allows us to obtain direct information about the current universe. For a comprehensive review, see ref. [26].

The most relevant observables in the context of real time cosmology are the drifts of the redshift and direction of observation (the latter comprising both the parallax and aberration drifts, as we shall see). These drifts are sensitive to a wide range of parameters in cosmological models, and thus can be used to assess, e.g., Earth's peculiar motion with respect to the CMB rest frame [80], a possible late time anisotropic expansion of the universe [97, 98], and the effects of the gravitational potential on galaxy clusters [99, 100]. It is also expected that a time variation of cosmological observables can be detected on CMB measurements [101, 102].

Many works have considered the topic of cosmological drifts from a formal theoretical perspective in general spacetimes [81, 82, 83, 84, 85]; however, these results are not readily applicable. The approach in this work is more pragmatic in nature, providing a method that can, in principle, be extended to a range of spacetimes in a step by step fashion. While the calculation of the redshift drift is straightforward, for the direction drift special care is necessary since it does not involve simply a time derivative. We start by considering an observer at Earth (which we allow to possess a non-zero peculiar velocity) receiving from a distant source two light signals spaced over an infinitesimal time interval. Then, we calculate the variation of the direction of observation vector n^i in an orthonormal frame defined by the observer. To do so in a rigorous manner, we show that an approach of comparing null geodesics is necessary; after taking into account the difference in time lapses for each case, we are able to evaluate the direction drift. The results can be then interpreted as a contribution from parallax and another from aberration, apart from the ones specific to curvature. In the following, we briefly review the geodesic equation in the context of General Relativity, which is an essential tool of the approach presented here.

6.3 Some important tools - the geodesic equation

The concept of a straight line, straightforward in Euclidean geometry, is less intuitive in curved spaces, and one needs to replace it with the closest possible definition; this is achieved through the notion of a geodesic. Recall that, given a spacetime $(\mathcal{M}, g)$, there is a derivative operator ∇_μ associated with the metric $g_{\mu\nu}$. Now, let $\gamma(t)$ be a curve on $\mathcal{M}$ (which is a map $\gamma : \Re \mapsto \mathcal{M}$ from a real parameter t to a point $p \in \mathcal{M}$) and T^μ its tangent vector. The curve $\gamma(t)$ will be a geodesic if its tangent vector is parallel transported along itself, i.e., if it satisfies

$$T^\nu \nabla_\nu T^\mu = 0. \tag{6.3.1}$$

This is the closest definition of a straight line in a curved geometry. In a coordinate basis, the above equation has the form

$$\frac{dT^\mu}{dt} + \Gamma^\mu_{\nu\rho} T^\nu T^\rho = 0, \tag{6.3.2}$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols. In the coordinate chart, the geodesic $\gamma(t)$ will be mapped to a curve $x^\mu(t)$ given by the equation $T^\mu = dx^\mu/dt$. Then, (6.3.2) is rewritten as

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0. \tag{6.3.3}$$

Note that if the connection coefficients are everywhere zero, the solution to the above is a straight line in the Euclidean sense (as it should be in the absence of curvature). The set of equations above has a unique solution given the initial conditions for x^μ and dx^μ/dt [17], which asserts that, given a point $p \in \mathcal{M}$ and a tangent vector T^μ at p, a unique geodesic (with tangent vector T^μ) passes through p.

Geodesics can be categorized as timelike, null or spacelike according to the nature of their tangent vector. Timelike and spacelike geodesics are the curves which extremize length between two points in curved geometries. This provides natural parameters for both timelike and spacelike geodesics (respectively, proper time and spatial length of the geodesic). For null geodesics, there is no preferred parameter, though it is convenient to set it such that $k^\mu = dx^\mu/dt$, where k^μ is the four-momentum vector of the massless particle. In the context of General Relativity, geodesics replace the concept of gravitational force: freely falling massive test particles move along timelike geodesic paths, which are curved due to the presence of an external source of energy or momentum. The gravitational interaction is made through the curvature of spacetime, which in turn affects how geodesics behave. The same is true for massless particles, except they move through null geodesic paths.

The overall approach to calculate the direction drift involves finding the null path $x^\mu(t)$ for the light of a given astronomical source. This means we would need to explicitly solve eq. (6.3.3). However, for most spacetimes this need not be done. If a spacetime possesses a symmetry, one can exploit it to find the geodesic equation through a simpler path. As was shown before, if ξ^μ is a Killing vector field then a conserved quantity exists, namely, $\xi^\mu T_\mu$ is constant along the geodesic. In most of the cases below, we shall be working in homogeneous spacetimes, and exploiting this symmetry allows us to easily derive the geodesic equation $x^\mu(t)$. In one case, namely, the perturbed FLRW spacetime, no symmetry is present and we must solve eq. (6.3.3) explicitly.

7 Evaluating the redshift and direction drifts

In the following, we present the calculations for the redshift and direction drifts in different cosmological scenarios.

7.1 General approach

In order to get some intuition on the direction drift, let us start the computation with an heuristic approach. Then, a rigorous calculation is done in Minkowski spacetime to elucidate our approach.

7.1.1 Heuristic argument

In 1728, Bradley obtained an expression for stellar aberration using a corpuscular description of light with Newton optics theory. It is today well-understood in the framework of special relativity that aberration is related to the Lorentz boost associated with a change of referential (see, e.g., ref. [103] for details). Consider an inertial frame S and a second inertial frame S' moving with speed V with respect to S along the X axis. For a light source in the XOY plane of S whose position vector makes an angle α with the OX axis, the photons emitted are observed in the direction $n^i = (\cos\alpha, \sin\alpha, 0)$ and the associated wave-vector is $k^\mu = \omega(1, -n^i)$, where the minus sign in k^i takes into account that the photon is approaching the observer. However, in the frame S' the wave-vector is $k'^\mu = \omega'(1, -n'^i)$, where $n'^i = (\cos\alpha', \sin\alpha', 0)$. Using the Lorentz transformation law for a four-vector, one immediately arrives at

$$\cos\alpha' = \frac{\cos\alpha + V}{1 + V\cos\alpha}, \tag{7.1.1}$$

and

$$\sin\alpha' = \frac{\sqrt{1-V^2}\sin\alpha}{1 + V\cos\alpha}, \tag{7.1.2}$$

or equivalently

$$\tan\alpha' = \frac{\sqrt{1-V^2}\sin\alpha}{\cos\alpha + V}, \tag{7.1.3}$$

where the speed of light has been set to unity by a proper choice of units. This is the relativistic result, which reduces to Bradley's, eq. (6.1.1), in the Newtonian approximation. Bradley considered S to be the quasi-inertial frame of the Solar system and S' attached to the Earth. Over a $\Delta t = 6$ months period, $\Delta V = 2V$ so that the aberration is given by

$$|\cos(\alpha' + \Delta\alpha') - \cos\alpha'| \sim \sin\alpha'\Delta\alpha' \sim 2V\sin^2\alpha', \tag{7.1.4}$$

where we used that $d\alpha'/dV = -\sin\alpha'$, which can be found by deriving eq. (7.1.1). The aberration is just an effect of perspective due to a change of (Lorentz) frame, and the question of whether or not to take into account the source velocity does not arise, which was however a difficult question in corpuscular optics. It follows that the drift is of order

$$\frac{\Delta\alpha'}{\Delta t} \sim \frac{\Delta V}{\Delta t}\sin\alpha' \,. \tag{7.1.5}$$

There is however a second effect to take into account. It is a simple geometric effect due to the fact that in Δt the star and the observer have moved by a typical relative distance $|\mathbf{v}_* - \mathbf{v}_\mathrm{o}|\Delta t$ so that it is expected that

$$\Delta\alpha' \sim \frac{|\mathbf{v}_* - \mathbf{v}_\mathrm{o}|\Delta t}{|\mathbf{r}_* - \mathbf{r}_\mathrm{o}|}, \tag{7.1.6}$$

where $\mathbf{r}_*$ and $\mathbf{r}_\mathrm{o}$ are the star and observer's positions relative to a properly chosen reference frame. In conclusion, we expect the typical total drift to behave as

$$\frac{\Delta\alpha'}{\Delta t} = \frac{|\mathbf{v}_* - \mathbf{v}_\mathrm{o}|}{|\mathbf{r}_* - \mathbf{r}_\mathrm{o}|} + \frac{\Delta V}{\Delta t}|\sin\alpha| \tag{7.1.7}$$

We shall refer to the total drift as *direction drift*; the term inversely proportional to the source-observer distance is the *parallax drift* and the term proportional to the observer's acceleration is the *aberration drift*. Indeed, the arguments above are just an informal sketch of the different contributions. For a rigorous and fully relativistic account of these effects, the angles have to be replaced by directions on the celestial sphere (i.e., unit 2-vectors) and we need to properly define the angles, motions, wave-vectors, etc., as described by non-inertial observers and sources. However, it emphasizes a key issue on which our formalism is built: the direction drift, and similarly the redshift drift, involves two sets of null-geodesic compared before and after a given lapse of observer proper time. It also shows that the total direction drift cannot be reduced to a time derivative (coordinate or proper) of the aberration.

7.1.2 Minkowski spacetime

As a warm up, let us consider a Minkowski spacetime with metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ in Cartesian coordinates. Thanks to the three Killing vectors associated with spatial translations, $\xi^\mu_{(i)} = \delta^\mu_i$, the vector k^μ tangent to a null geodesic $x^\mu(\lambda)$ satisfies

$$k_\mu\xi^\mu_{(i)} = k_i = \mathrm{const} \tag{7.1.8}$$

along the geodesic, according to eq. (2.1.14). By exploiting the constancy of k_i we are able to find the geodesic equation in this (and other spatially homogeneous) spacetimes. Time translations are also symmetries in Minkowski spacetime, and thus, the existence of a timelike Killing vector $\xi^\mu_{(0)} = \delta^\mu_0$ ensures that

$$k_\mu\xi^\mu_{(0)} = k_0 = \mathrm{const.} \tag{7.1.9}$$

As we will see below, k_0 is the frequency of the photon as measured by static observers.

Let us first consider a static observer with 4-velocity $u^\mu \equiv \delta^\mu_0$. We decompose k^μ into a frequency ω and a direction n^μ as

$$k^\mu = \omega(u^\mu - n^\mu), \tag{7.1.10}$$

where n^μ is a spatial vector (with respect to the spatial hypersurfaces orthogonal to u^μ) satisfying $n^\mu u_\mu = 0$. The photon's frequency as seen by the static observer is given by $\omega = -u^\mu k_\mu$, and given condition (7.1.9), it is constant along the null geodesic.

Considering the condition (7.1.8), one can raise the spatial index freely in Minkowski spacetime and rewrite it as $k^i = k^i_{\rm o}$, where, from now one, the subscript "o" denotes the value of k^i taken at the observer's position. Then, let us recall that the null geodesic $x^\mu(\lambda)$ is defined by $k^\mu = dx^\mu/d\lambda$, i.e., k^μ is its tangent vector and λ is an affine parameter. Changing the parameter from λ to t gives

$$k^\mu = \frac{dx^\mu}{d\lambda}, \tag{7.1.11}$$

$$= \frac{dt}{d\lambda}\frac{dx^\mu}{dt}, \tag{7.1.12}$$

$$= \omega\frac{dx^\mu}{dt}. \tag{7.1.13}$$

The above change of parameters will be done in the next sections recurrently. Now, condition (7.1.8) can be cast to

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = -n^i_{\rm o}\,, \tag{7.1.14}$$

Let us integrate this equation considering a null geodesic connecting a source and an observer. This is easily done and gives

$$x^i_s - x^i_{\rm o} = -(t_s - t_{\rm o})n^i_{\rm o}\,, \tag{7.1.15}$$

where the subscript "s" stands for source. Note that, given conditions (7.1.8) and (7.1.9), n^i is constant along the geodesic. However, in general, it is not constant from one geodesic to the other. Let us now consider a second geodesic in which the photon was emitted at $t_s + \delta t_s$ and received by the observer at $t_{\rm o} + \delta t_{\rm o}$. We also assume that the source and observer are static for the moment. Now, we want to write eq. (7.1.14) for this second null geodesic. Let us note that from the first to the second geodesic the positions $x^i_{s,{\rm o}}$ do not change. The time coordinates change according to what was described above, and for the change in the direction of observation, we introduce the general definition

$$\delta_{12}\mathcal{O} = \mathcal{O}|_2 - \mathcal{O}|_1, \tag{7.1.16}$$

for any observable $\mathcal{O}$ compared at two geodesics, separated in general by a lapse $\delta\tau$ of the proper time of the observer. Note that it is a non-local variation which involves two geodesics. For a static observer in Minkowski spacetime we have $\delta\tau = \delta t$. Thus, the direction of observation changes by the factor $\delta_{12} n^i_{\rm o}$. The time variation of this non-local observable is what we define as the direction drift. Considering all of the above, we can now write the equation for the second geodesic:

$$x^i_s - x^i_{\rm o} = -\left(t_s - t_{\rm o} + \delta t_s - \delta t_{\rm o}\right)\left(n^i_{\rm o} + \delta_{12} n^i_{\rm o}\right). \tag{7.1.17}$$

It is expected that in Minkowski spacetime for static source and observer that $\delta_{12} n^i_{\rm o} = 0$. Let us check that this is indeed the case as an illustration of our method. First, we must find the relationship between δt_s and $\delta t_{\rm o}$. Again, in Minkowski we intuitively expect $\delta t_s = \delta t_{\rm o}$. In most spacetimes of interest, however, this is not immediate. To find this relationship in general, we contract both eqs. (7.1.14) and (7.1.17) with $n_{i{\rm o}}$ and subtract them. Being n^i normalized, this gives

$$0 = -\left(t_s - t_{\rm o}\right)\delta_{12} n^i_{\rm o} n_{i{\rm o}} + \left(\delta t_s - \delta t_{\rm o}\right). \tag{7.1.18}$$

We then use that, since n^i is normalized to 1, it satisfies $n^i_{\rm o}\delta_{12} n_{i{\rm o}} = 0$ to arrive at $\delta t_s = \delta t_{\rm o}$, as expected. If we now compare eqs. (7.1.15) and (7.1.17) to extract the direction drift, it is easy to find

$$\delta_{12} n^i_{\rm o} = 0, \tag{7.1.19}$$

i.e., there is no direction drift for static source and observer in Minkowski spacetime, as expected.

Let us now consider a general observer with 4-velocity $\tilde{u}^\mu$. We assume that its peculiar velocity v^μ is small compared to the speed of light, but otherwise arbitrary. Hence, we shall work at linear order in the spatial velocity and write

$$\tilde{u}^\mu = u^\mu + v^\mu, \tag{7.1.20}$$

with $v^\mu u_\mu = 0$. Note that, to first order in the spatial velocity, $\tilde{u}^\mu \tilde{u}_\mu = u^\mu u_\mu = -1$, i.e., it is normalized. From now on quantities with a tilde are associated to the general observer (or a moving source). The wave-vector can be either decomposed as $k^\mu = \omega(u^\mu - n^\mu)$ for the static observer, or as $k^\mu = \tilde{\omega}(\tilde{u}^\mu - \tilde{n}^\mu)$ for the general observer. The frequency $\tilde{\omega}$ for the moving observer is now given by $\tilde{\omega} = -\tilde{u}^\mu k_\mu$. One can use this to find $\tilde{\omega}$ as a function of ω:

$$\tilde{\omega} = -\omega \tilde{u}^\mu \left(u_\mu - n_\mu\right), \tag{7.1.21}$$

$$= \omega\left(1 + v^i n_i\right), \tag{7.1.22}$$

where we used the fact that both v^μ and n^μ are purely spatial vectors. With the above equation, one can also find $\tilde{n}^i$ as a function of n^i. To do that, we compare k^i written as $k^i = -\omega n^i$ with $k^i = \tilde{\omega}\left(v^i - \tilde{n}^i\right)$. This gives

$$-\omega n^i = \omega\left(1 + v^j n_j\right)\left(v^i - \tilde{n}^i\right), \tag{7.1.23}$$

$$= \omega\left(v^i - \tilde{n}^i - v^j n_j n^i\right), \tag{7.1.24}$$

where we used that $v^j n_j \tilde{n}^i = v^j n_j n^i$. This is true to first order since, at this order, n^i and $\tilde{n}^i$ must differ by a factor proportional to v^i. One can then easily rearrange the above to obtain $\tilde{n}^i$ as a function of n^i. We summarize the above results into the set of equations

$$k^i = \tilde{\omega}\left(v^i - \tilde{n}^i\right), \tag{7.1.25a}$$

$$\tilde{\omega} = \omega\left(1 + v^i n_i\right), \tag{7.1.25b}$$

$$\tilde{n}^i = n^i + \perp^i_j v^j, \tag{7.1.25c}$$

where

$$\perp^i_j \equiv \delta^i_j - n^i n_j \tag{7.1.26}$$

is the perpendicular projector with respect to the direction of observation of the static observer, n^i. We shall find a similar set of equations for a general observer in FLRW spacetime.

Now, the proper time of the general observer is related to the coordinate time by $\mathrm{d}\tau^2 = (1 - v^2)\mathrm{d}\tilde{t}^2$, hence $\mathrm{d}\tau = \mathrm{d}\tilde{t}$ at linear order in v. Since eq. (7.1.8) is observer independent, we can use the same trick as before to find the geodesic equation as a function of $\tilde{n}^i$. We set the right side of eq. (7.1.8) at the point of observation, and write $k^i|_\mathrm{o}$ as given by eq. (7.1.25a). This leads to

$$\omega \frac{\mathrm{d}x^i}{\mathrm{d}t} = \tilde{\omega}_\mathrm{o}\left(v^i_\mathrm{o} - \tilde{n}^i_\mathrm{o}\right). \tag{7.1.27}$$

Using eq. (7.1.25b), it takes the form

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \perp^i_j v^j_\mathrm{o} - \tilde{n}^i_\mathrm{o}. \tag{7.1.28}$$

To write this expression, we have again used that, to first order in spatial velocity, $v^j n_j \tilde{n}^i = v^j n_j n^i$. The projector $\perp^i_j$ above should be taken at the value of observation, but since n^i is constant along the geodesic we bother not denoting it to avoid messy notation. Once integrated, the above leads to

$$x^i_s - x^i_\mathrm{o} = (t_s - t_\mathrm{o})\left(\perp^i_j v^j_\mathrm{o} - \tilde{n}^i_\mathrm{o}\right), \tag{7.1.29}$$

since, as just mentioned, n^i and thus $\perp^i_j = \delta^i_j - n^i n_j$ are constants along the null geodesic. This is the generalization of eq. (7.1.15) for a general observer.

Let us now consider a second light ray emitted at $t_s + \delta\tilde{t}_s$ and received at $t_\mathrm{o} + \delta\tilde{t}_\mathrm{o}$. We also allow the source to have a general velocity. The tildes in $\delta\tilde{t}_{s,\mathrm{o}}$ denote that for moving observers and sources, the lapses that connect the two geodesics are not the same as in the static case; they are now velocity dependent. We also have now that $\delta\tilde{t}_\mathrm{o} \neq \delta\tilde{t}_s$, since both the observer and source are moving at different velocities. The end points of the second null-geodesic are now given by $x^i_{s,\mathrm{o}} + v^i_{s,\mathrm{o}}\delta\tilde{t}_{s,\mathrm{o}}$, and the velocity of the observer is then $v^i_\mathrm{o} + \dot{v}^i_\mathrm{o}\delta\tilde{t}_\mathrm{o}$, where the overdot denotes a derivative with respect to t. Figure 8 illustrates the scheme of this calculation. The integration of the second geodesic is then given by

$$x_s^i - x_{\rm o}^i + v_s^i\delta\tilde{t}_s - v_{\rm o}^i\delta\tilde{t}_{\rm o} = \left(t_s - t_{\rm o} + \delta\tilde{t}_s - \delta\tilde{t}_{\rm o}\right)\left(\perp_j^i v_{\rm o}^j - \tilde{n}_{\rm o}^i + \perp_j^i \dot{v}_{\rm o}^j \delta\tilde{t}_{\rm o} - \delta_{12}\tilde{n}_{\rm o}^i\right). \quad (7.1.30)$$

The relation between $\delta\tilde{t}_s$ and $\delta\tilde{t}_{\rm o}$ is obtained by subtracting the contractions of eqs. (7.1.29) and (7.1.30) with $\tilde{n}_{i{\rm o}}$ as before to get

$$\frac{\delta\tilde{t}_s}{\delta\tilde{t}_{\rm o}} = 1 - \left(v_s^i - v_{\rm o}^i\right) n_{{\rm o}i}. \quad (7.1.31)$$

Once inserted in eq. (7.1.30), we then subtract it from eq. (7.1.29) to get the direction drift

$$\frac{\delta_{12}\tilde{n}_{\rm o}^i}{\delta t_{\rm o}} = \frac{\perp_j^i\left(v_s^j - v_{\rm o}^j\right)}{D_{s{\rm o}}} + \perp_j^i \dot{v}_{\rm o}^j, \quad (7.1.32)$$

where $D_{s{\rm o}} \equiv t_{\rm o} - t_s$ is the (positive) radial distance between the observer and the source (in units where $c = 1$). Note that the perpendicular projector multiplies spatial velocities, and as we work at linear order in velocities, it is equivalent to consider the projector built either from n^i or $\tilde{n}^i$.

The expression (7.1.32) matches the heuristic argument. The first term (the parallax drift) is related to the change of apparent position of the source due to the motion of the source and the observer. This is a simple perspective effect that is proportional to the inverse of the distance. The second term (the aberration drift) is the mean peculiar acceleration of the observer perpendicular to the line of sight on the time scale of the observation.

7.1.3 Summary

This first insight shows that it is indeed not correct to start from the Bradley formula for the aberration and simply take its time derivative to obtain the full direction drift. It also defines clearly the general strategy of the computation: (1) consider 2 null geodesics connecting the observer and the source, (2) relate the lapses of time between the two geodesics at the source and at the observer. This provides a well-defined and physically well-under control derivation of both the direction and redshift drifts that can, in principle, be extended to any spacetime. It also emphasizes that we have to pay attention to the fact that ω may not be constant along the geodesics and to properly define the direction of observation for an observer at Earth. To this last purpose, given a metric $g_{\mu\nu}$, we shall introduce a tetrad basis $\epsilon^\mu_{(A)}$ defined by

$$g_{\mu\nu}\epsilon^\mu_{(A)}\epsilon^\nu_{(B)} = \eta_{AB}. \quad (7.1.33)$$

If we choose $\epsilon^\mu_{(0)}$ to be the 4-velocity u^μ of an inertial observer, any vector field V^μ can then be decomposed as

$$V^\mu = (-V^\alpha u_\alpha)u^\mu + V^{(i)}\epsilon^\mu_{(i)}. \quad (7.1.34)$$

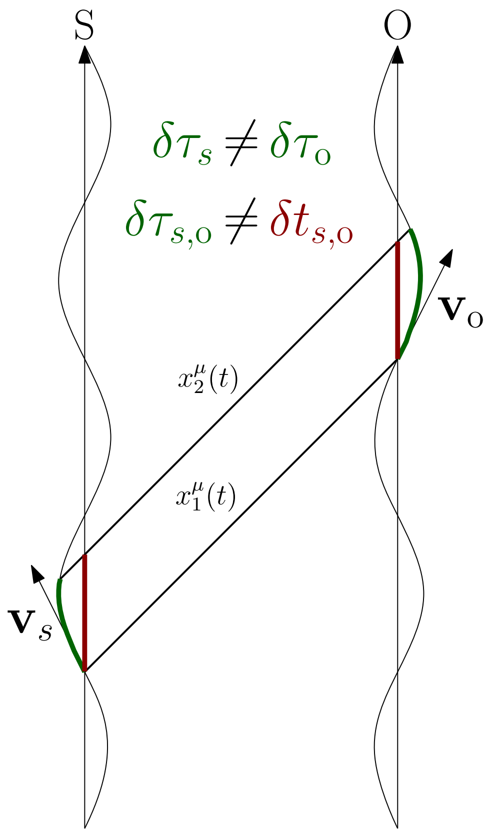

Figure 8 – General definitions for the two null geodesics between a general observer and source. The differences between the times contribute to the parallax part of the drift. Note that at first order in velocities $\delta\tau_{s,o} = \delta\tilde{t}_{s,o}$.

7.2 Friedmann-Lemaître-Robertson-Walker spacetime

7.2.1 Static observers and tetrad decomposition

The construction from the previous section is easily generalized to a cosmological spacetime. Let us assume a cosmology described by a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) solution with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{7.2.1}$$

where t is the cosmic (or coordinate) time, i.e. the proper time of comoving observers with 4-velocity $u^\mu \equiv \delta^\mu_0$ and $a(t)$ is the scale factor. We also introduce the conformal time[1] defined by $\mathrm{d}\eta = a^{-1}\mathrm{d}t$ and define the Hubble function by $H \equiv \mathrm{d}\ln a/\mathrm{d}t$. Let us show here that, when written in tetrad components, the geodesic equation reduces to the same as the Minkowski case. We start by considering a comoving observer, and move on to the general case in the following section. Consider a photon with wave-vector $k^\mu = \omega\,(u^\mu - n^\mu)$. Note that in FLRW spacetime there is no timelike Killing vector field,

[1] Using conformal time usually eases intermediate computations; the final results, however, we shall write in terms of coordinate time (or proper time, when suitable) since they prove more intuitive when done so.

and thus ω is not constant along the null geodesic. To find how ω evolves along it, one needs only to solve the 0-component of the geodesic equation. That is, we must solve

$$\frac{dk^0}{d\lambda} + \Gamma^0_{\mu\nu} k^\mu k^\nu = 0. \tag{7.2.2}$$

With $\Gamma^0_{0\mu} = 0$ and $\Gamma^0_{ij} = a\dot{a}\delta_{ij}$ the above turns into

$$\omega \frac{d\omega}{dt} + a\frac{da}{dt}\delta_{ij}k^i k^j = 0. \tag{7.2.3}$$

Now, since $k^\mu k_\mu = 0$, we have that $\delta_{ij}k^i k^j = (\omega/a)^2$. We plug this into the above, and after some simple algebra, it can be rewritten as

$$\frac{d(a\omega)}{dt} = 0, \tag{7.2.4}$$

i.e., $a\omega$ is constant along the null geodesic.

Now, the same trick used before for homogeneous spacetimes can be used here to find the spatial part of the geodesic equation. Note that in FLRW, however, raising and lowering indexes must be done with care, such that $k_i = \text{const}$ is equivalent to $a^2 k^i = \text{const}$. As before, the l.h.s. is chosen at the point of observation, giving

$$a^2 \frac{dx^i}{d\lambda} = -a_{\rm o}^2 \omega_{\rm o} n_{\rm o}^i. \tag{7.2.5}$$

The above is easily solved when written using conformal time and tetrad components. We use the tetrad field defined by

$$\boldsymbol{\epsilon}_{(0)} = \partial_t, \qquad \boldsymbol{\epsilon}_{(i)} = a^{-1}\partial_i\,. \tag{7.2.6}$$

This means that the tetrad component of the direction vector, $n^{(i)}$, is related to the coordinate component n^i through $n^{(i)} = \epsilon^{(i)}_j n^j = a n^i$. With these modifications, eq. (7.2.5) reads

$$a\omega \frac{dx^i}{d\eta} = -a_{\rm o}\omega_{\rm o} n_{\rm o}^{(i)}. \tag{7.2.7}$$

We now use the fact that $a\omega$ is constant along the geodesic to rewrite it as

$$\frac{dx^i}{d\eta} = -n_{\rm o}^{(i)}. \tag{7.2.8}$$

This is similar to the Minkowski result, eq. (7.1.14), except for the use of conformal time and tetrad component of the direction vector. It is easy to see that, given $a\omega = \text{const}$, $a^2 k^i = \text{const}$ implies that $n^{(i)}$ is constant along the geodesic (whereas in Minkowski we had that n^i was constant). Integration of eq. (7.2.8) follows similarly, as does writing a second geodesic lapsed by $\delta\eta_{s,\rm o}$. It is easy to show through the same procedures as before that the lapses in FLRW are related through $\delta\eta_s = \delta\eta_{\rm o}$. Note that in coordinate time the lapses are not the same due to the evolution of the scale factor. The calculations follows the same path as in the Minkowski static case, which allows us to conclude that, for the comoving FLRW case we have

$$\delta_{12} n_{\mathrm{o}}^{(i)} = 0, \tag{7.2.9}$$

i.e., there is also no direction drift.

It is also a good opportunity now to show how to evaluate the gravitational redshift drift, which was absent in the case of a Minkowski spacetime since it has no curvature. The redshift z is defined as

$$1 + z = \frac{(u^\mu k_\mu)\,|_s}{(u^\mu k_\mu)\,|_{\mathrm{o}}}. \tag{7.2.10}$$

For comoving observer and source, it is simply $1 + z = (\omega_s/\omega_{\mathrm{o}})$. We use that $a\omega = \mathrm{const}$ along a null geodesic to rewrite it as

$$1 + z = \frac{a_{\mathrm{o}}}{a_s}. \tag{7.2.11}$$

On a second geodesic lapsed by $\delta t_{s,\mathrm{o}}$, the redshift is given by

$$1 + z + \delta_{12} z = \frac{a(t_{\mathrm{o}} + \delta t_{\mathrm{o}})}{a(t_s + \delta t_s)} \tag{7.2.12}$$

Expanding the above to first order in the lapses gives, after some simple algebra,

$$\frac{\delta_{12} z}{\delta t_{\mathrm{o}}} = (1 + z)\left(H_{\mathrm{o}} - H_s \frac{\delta t_s}{\delta t_{\mathrm{o}}}\right). \tag{7.2.13}$$

With the lapses in conformal time related by $\frac{\delta \eta_s}{\delta \eta_{\mathrm{o}}} = 1$, in coordinate time we have[2]

$$\frac{\delta t_s}{\delta t_{\mathrm{o}}} = \frac{a_s}{a_{\mathrm{o}}} = \frac{1}{1 + z}. \tag{7.2.14}$$

Plugging the above into eq. (7.2.13) finally gives the standard result

$$\frac{\delta_{12} z}{\delta t_{\mathrm{o}}} = (1 + z) H_{\mathrm{o}} - H_s. \tag{7.2.15}$$

We now move on to the case of observers and sources in general motion

7.2.2 General observers

We again consider a general observer with 4-velocity

$$\tilde{u}^\mu = u^\mu + v^\mu, \tag{7.2.16}$$

where v^μ is spatial ($v^\mu u_\mu = 0$). Again, the tilde denotes quantities associated with a general observer (or moving source). We assume the observer is non-relativistic and work at first order in v^μ.

[2] There is a way to generalize this result when written in terms of the redshift z. This is explained in Appendix B and will be used further.

The wave-vector is decomposed equivalently either as $k^\mu = \omega(u^\mu - n^\mu)$ or as $k^\mu = \tilde{\omega}(\tilde{u}^\mu - \tilde{n}^\mu)$. Again, it follows that

$$\begin{aligned} k^i &= \tilde{\omega}\left(v^i - \tilde{n}^i\right), && (7.2.17a)\\ \tilde{\omega} &= \omega\left(1 + v^i n_i\right), && (7.2.17b)\\ \tilde{n}^i &= n^i + \perp^i_j v^j, && (7.2.17c) \end{aligned}$$

where $\perp^i_j = \delta^i_j - n^i n_j$ is the perpendicular projector with respect to n^i. Again, we can make use of

$$a^2 k^i = k_i = \text{const} \tag{7.2.18}$$

to find the geodesic equation. After decomposing n^i, $\tilde{n}^i$ and the velocities on the tetrad (7.2.6) as

$$n^i = n^{(j)}\epsilon^i_{(j)}, \quad \tilde{n}^i = \tilde{n}^{(j)}\epsilon^i_{(j)}, \quad v^i = v^{(j)}\epsilon^i_{(j)}, \tag{7.2.19}$$

the condition (7.2.18) leads to

$$a\omega\frac{\mathrm{d}x^i}{\mathrm{d}\eta} = a_\mathrm{o}\tilde{\omega}_\mathrm{o}\left(v^{(i)}_\mathrm{o} - \tilde{n}^{(i)}_\mathrm{o}\right). \tag{7.2.20}$$

With the use of eq. (7.2.17b), and the fact that $a\omega = a_\mathrm{o}\omega_\mathrm{o}$ for comoving observers, it reduces to

$$\frac{\mathrm{d}x^i}{\mathrm{d}\eta} = \perp^i_j v^{(j)}_\mathrm{o} - \tilde{n}^{(i)}_\mathrm{o}. \tag{7.2.21}$$

It can be integrated easily since $n^{(i)}$ is constant along the null geodesic, and so is $\perp^i_j$, once expressed as $\perp^i_j = \delta^i_j - n^{(i)}n_{(j)}$. We get

$$x^i_s - x^i_\mathrm{o} = (\eta_s - \eta_\mathrm{o})\left(\perp^i_j v^{(j)}_\mathrm{o} - \tilde{n}^{(i)}_\mathrm{o}\right). \tag{7.2.22}$$

Again, after shifting to conformal time and tetrad components, this is similar to the derivation in Minkowski spacetime.

We now consider the second geodesic describing the observation at a conformal time $\eta_\mathrm{o} + \delta\tilde{\eta}_\mathrm{o}$. Note that the lapse $\delta\tilde{\eta}_\mathrm{o}$ is related to the proper time τ_o of the observer by $\delta\tau_\mathrm{o} = \delta\tilde{t}_\mathrm{o} = a_\mathrm{o}\delta\tilde{\eta}_\mathrm{o}$ as long as we work at first order in velocities. The photon was emitted by the source at a conformal time $\eta_s + \delta\tilde{\eta}_s$. As explained in the previous section, we need once more to find the relationship between $\delta\tilde{\eta}_s$ and $\delta\tilde{\eta}_\mathrm{o}$. Compared to the first geodesic, the positions of the source and the observer are now given by $x^i_{s,\mathrm{o}} + v^{(i)}_{s,\mathrm{o}}\delta\tilde{\eta}_{s,\mathrm{o}}$; the velocity of the observer is $v^{(i)}_\mathrm{o} + v'^{(i)}_\mathrm{o}\delta\tilde{\eta}_\mathrm{o}$ (where a prime refers to a derivative with respect to the conformal time). As long as we work in first order in velocities, we need only the background value of $\delta_{12}n^{(i)}_\mathrm{o} = 0$ so that $\perp^i_j$ remains unchanged. Plugging these modifications in eq. (7.2.21) we get the equation of the second geodesic

$$x^i_s - x^i_\mathrm{o} + v^{(i)}_s\delta\tilde{\eta}_s - v^{(i)}_\mathrm{o}\delta\tilde{\eta}_\mathrm{o} = (\eta_s - \eta_\mathrm{o} + \delta\tilde{\eta}_s - \delta\tilde{\eta}_\mathrm{o})\left(\perp^i_j v^{(j)}_\mathrm{o} - \tilde{n}^{(i)}_\mathrm{o} + \perp^i_j v'^{(j)}_\mathrm{o}\delta\tilde{\eta}_\mathrm{o} - \delta_{12}\tilde{n}^{(i)}_\mathrm{o}\right). \tag{7.2.23}$$

By contracting eqs. (7.2.22) and (7.2.23) with $\tilde{n}_{(i)\text{o}}$ and subtracting them, one gets the relation between $\delta\tilde{\eta}_s$ and $\delta\tilde{\eta}_\text{o}$:

$$\frac{\delta\tilde{\eta}_s}{\delta\tilde{\eta}_\text{o}} = 1 - \left(v_s^{(i)} - v_\text{o}^{(i)}\right) n_{(i)}. \tag{7.2.24}$$

To finish, we just need to combine eqs. (7.2.22), (7.2.23) and (7.2.24) to get the total direction drift, i.e., the change of direction in units of proper time of the observer, which is

$$\frac{\delta_{12}\tilde{n}_\text{o}^{(i)}}{\delta t_\text{o}} = \frac{\perp^i_j \left(v_s^{(j)} - v_\text{o}^{(j)}\right)}{a_\text{o}\chi_{so}} + \perp^i_j \dot{v}_\text{o}^{(j)}, \tag{7.2.25}$$

where $\chi_{so} \equiv \eta_\text{o} - \eta_s$ is the comoving radial distance between the observer and the source. This expression is similar to eq. (7.1.32) derived in a Minkowski spacetime, and its implications will be discussed below.

Let us now turn to the redshift drift. From eq. (7.2.10), the redshift for a moving observer and source is given by

$$1 + \tilde{z} = \frac{\tilde{\omega}_s}{\tilde{\omega}_\text{o}}. \tag{7.2.26}$$

Using eq. (7.2.17b) and recalling that $a\omega = \text{const}$, we can express the redshift as

$$1 + \tilde{z} = \frac{a_\text{o}}{a_s}\left[1 + \left(v_s^{(i)} - v_\text{o}^{(i)}\right) n_{(i)}\right]. \tag{7.2.27}$$

Using the same procedure as for the static case, it is straightforward to write the redshift $1 + \tilde{z} + \delta\tilde{z}$ for a second geodesic. Since $n_{(i)}$ is unchanged from one geodesic to the other, we have

$$1 + \tilde{z} + \delta_{12}\tilde{z} = \frac{a_\text{o}}{a_s}\left(1 + H_\text{o}\delta\tilde{t}_\text{o} - H_s\delta\tilde{t}_s\right)\left[1 + \left(v_s^{(i)} - v_\text{o}^{(i)}\right) n_{(i)} + \left(\dot{v}_s^{(i)}\delta\tilde{t}_s - \dot{v}_\text{o}^{(i)}\delta\tilde{t}_\text{o}\right) n_{(i)}\right]. \tag{7.2.28}$$

It is now a matter of tedious but straightforward algebraic manipulations to arrive at the redshift drift. Only terms to first order on the lapses and velocities should be kept, and the relation between the lapses in coordinate time can be easily obtained from eq. (7.2.24); it is simply

$$\frac{\delta\tilde{t}_s}{\delta\tilde{t}_\text{o}} = \frac{1}{1 + \tilde{z}}. \tag{7.2.29}$$

This can also be obtained by making use of eq. (B.0.9) derived in Appendix B. Finally, the redshift drift for moving observers and sources is given by

$$\frac{\delta_{12}\tilde{z}}{\delta t_\text{o}} = (1 + \tilde{z})\left(H_\text{o} - \dot{v}_\text{o}^{(i)} n_{(i)}\right) - \left(H_s - \dot{v}_s^{(i)} n_{(i)}\right). \tag{7.2.30}$$

7.2.3 Multipolar decomposition

The angular dependence, i.e. in $n^{(i)}$, of the redshift and aberration drifts can be decomposed in multipoles either using symmetric trace-free tensors or spherical harmonics [104]. For the redshift drift, it takes the form

$$\begin{aligned}\frac{\delta_{12}\tilde{z}}{\delta t_{\rm o}} &= \mathcal{W} + \mathcal{W}_i n^{(i)} + \mathcal{W}_{ij} n^{(i)} n^{(j)} + \dots \\ &= \sum_{\ell,m} \mathcal{W}_{\ell m} Y^{\ell m}(n^{(i)})\end{aligned} \tag{7.2.31}$$

where the $\mathcal{W}_{i_1 \dots i_n}$ are symmetric trace-free tensors. From eq. (7.2.30), we read that the monopole and the dipole of the redshift drift are

$$\mathcal{W} = (1+\tilde{z})H_{\rm o} - H_s \tag{7.2.32a}$$

$$\mathcal{W}_i = -(1+\tilde{z})\dot{v}_{\rm o}^{(i)} + \dot{v}_s^{(i)} \,. \tag{7.2.32b}$$

The direction drift can be written in terms of a gradient and curl as

$$\frac{\delta_{12}\tilde{n}_{\rm o}^{(i)}}{\delta t_{\rm o}} = D^{(i)}\mathcal{E}(n^{(l)}) + \epsilon^{(i)(j)} D_{(j)} \mathcal{H}(n^{(l)}) \tag{7.2.33}$$

where $D^{(i)}$ is the covariant derivative on the 2-sphere of unit radius and $\epsilon^{(i)(j)}$ is the Levi-Civita tensor on the unit 2-sphere. The angular dependence of $\mathcal{E}(n^{(i)})$ and $\mathcal{H}(n^{(i)})$ is in turn decomposed in symmetric trace-free tensors or in spherical harmonics. For $\mathcal{E}$ (and similarly for $\mathcal{H}$) this decomposition is

$$\begin{aligned}\mathcal{E}(n^{(l)}) &= \mathcal{E}_i n^{(i)} + \mathcal{E}_{ij} n^{(i)} n^{(j)} + \dots \\ &= \sum_{\ell \geq 1,\, m} \mathcal{E}_{\ell m} Y^{\ell m}(n^{(i)})\,.\end{aligned} \tag{7.2.34}$$

From eq. (7.2.25) we infer that $\mathcal{H}(n^{(i)}) = 0$ and the dipolar components of $\mathcal{E}(n^{(i)})$ are

$$\mathcal{E}_i = \frac{v_s^{(i)} - v_{\rm o}^{(j)}}{a_{\rm o}\chi_{so}} + \dot{v}_{\rm o}^{(i)}\,. \tag{7.2.35}$$

Note that a dipole in $\mathcal{H}(n^{(i)})$ corresponds to a global infinitesimal rotation, hence the direction drift generated by our local velocity cannot be mixed with such effect.

Finally, we note that one could have chosen to decompose the direction drift directly in terms of spin-weighted spherical harmonics; this is actually equivalent to what we are doing since the covariant derivative on the unit 2-sphere $D^{(i)}$ coincides with the spin-raising operator spherical harmonics [105, 106].

7.2.4 Discussion on velocities

The previous sections derive the redshift and direction drifts for both comoving and general observers in a strictly spatially homogeneous and isotropic FLRW spacetime. Let us have a closer look to our result (7.2.25) concerning the aberration drift. It contains two contributions: the first one is the parallax drift, which encompasses the change of parallax due to the motions of the source and observer, and the second, the aberration drift, is similar to the Bradley aberration. While the first depends on the distance of the source, the second depends only on the acceleration of the observer, a property that we have already explained in our heuristic argument. Other works, such as refs. [84, 85], have also computed in more general settings these two contributions to the total variation of direction, employing a different terminology.

To compare the two contributions, we must describe the different motions involved, which comprise the motion of our Local Group of galaxies with respect to the CMB, the motion of the Milky Way with respect to the Local Group, the motion of the Sun around the Milky Way, and finally the motion of the Earth around the Sun. To this purpose, we write

$$v_{\rm o}^{(i)} = v_{\rm LG}^{(i)} + v_{\rm MW}^{(i)} + v_{\odot}^{(i)} + v_{\oplus}^{(i)} \tag{7.2.36}$$

and analyze each motion separately. We also provide rough estimates of the order of magnitude of the aberration and parallax drifts, assuming that the motions of the sources are averaged out.

- Local group velocity, $v_{\rm LG}^{(i)}$

Let us start by considering the motion of our Local Group of galaxies. The Local Group is a comoving observer in the FLRW universe, following the geodesic flow. Thus, this motion is given by solving the geodesic equation for a point particle with 4-velocity $\tilde{u}^\mu$ in a flat FLRW universe. To first order in spatial velocity, its i-component is given by

$$\frac{dv^i}{dt} + \Gamma^i_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = 0. \tag{7.2.37}$$

Given that in flat FLRW $\Gamma^i_{j0} = H\delta^i_j$ is the non-zero contribution to the above, we have

$$\frac{dv^i}{dt} + 2Hv^i = 0. \tag{7.2.38}$$

Rewriting the above using tetrad components, i.e., using $v^{(i)} = av^i$, we get, at the position of the observer,

$$\dot{v}_{\rm o}^{(i)} = -H_{\rm o}v_{\rm o}^{(i)}. \tag{7.2.39}$$

The comoving radial distance from the observer to the source satisfies [57]

$$\frac{a_{\rm o}\chi_{so}}{D_{H_{\rm o}}} = \int_0^z \frac{{\rm d}z'}{E(z')}, \tag{7.2.40}$$

where $E(z) = H(z)/H_{\rm o}$. Using both eqs. (7.2.39) and (7.2.40) in eq. (7.2.25) allows it to be rewritten as

$$\frac{\delta_{12}\tilde{n}_{\rm o}^{(i)}}{\delta t_{\rm o}} = \frac{\perp^i_j}{D_{H_{\rm o}}}\left[\frac{\left(v_s^{(j)} - v_{\rm o}^{(j)}\right)}{\int_0^z \frac{{\rm d}z'}{E(z')}} - v_{\rm o}^{(j)}\right], \tag{7.2.41}$$

where $D_{H_o} = H_o^{-1}$ is the Hubble radius today. Now, $E(z)$ can be evaluated through the Friedmann equations, and in a ΛCDM scenario it is given by [57]

$$E(z) = \sqrt{\Omega_m^o (1+z)^3 + \Omega_\Lambda^o}, \tag{7.2.42}$$

where Ω_m^o and Ω_Λ^o are the energy density parameters for matter and dark energy today, respectively. Note that since the parallax drift is redshift dependent through $E(z)$, it can be distinguished from the aberration drift, which does not depend on z. To compare their magnitude, we use the fiducial cosmology $\Omega_m^o = 0.31$ and $\Omega_\Lambda^o = 0.69$, following the latest Planck values [107].

Assuming the velocity of the sources are averaged out, the parallax drift is comparable to the aberration drift when $\int_0^z \frac{\mathrm{d}z'}{E(z')} \approx 1$. Using $E(z)$ given by (7.2.42), we find this happens for $z \approx 1.48$, which corresponds to sources such that $a_o \chi_{so} \sim D_{H_o}$ (as can be easily seen from eq. (7.2.40)). At smaller redshifts, the parallax drift dominates. Since $1/\int_0^z \frac{\mathrm{d}z'}{E(z')}$ saturates[3] to approximately 0.31 at large redshifts (Figure 9), the parallax drift will always contribute to more than 25% of the total signal. This saturation was already noted in ref. [84]. Considering the LG has an approximate velocity of 620 km/s with respect to the CMB rest frame [108], we expect the aberration drift to be of order 0.03 μas/yr. To perform this estimation, we use the LG velocity and the current D_H value in the aberration contribution of eq. (7.2.41) and then perform a conversion of units from rad/s to μas/yr (we also put back the necessary c factors). The following estimations are also done in a similar manner. For sources with comoving radial distance of D_{H_o}, the parallax drift is of the same magnitude, and remains so even for further sources. For closer sources, e.g., at $0.1 D_{H_o}$, the parallax drift is more significant, at roughly 0.3 μas/yr.

Our estimates can be compared to ref. [80], and essentially its eq. (2). To that purpose we pick up a system of coordinates on the celestial 2-sphere such that

$$\tilde{n}_o^{(i)} = \left(\sin\tilde{\theta}\cos\tilde{\phi}, \sin\tilde{\theta}\sin\tilde{\phi}, \cos\tilde{\theta}\right) \tag{7.2.43}$$

again with the convention that a tilde denotes quantities as seen by the general observer. Following ref. [80], we align the z-axis with the velocity of the observer. While not explicit, this can be deduced from comparisons to our relation $\tilde{n}^{(i)} = n^{(i)} + \perp^i_j v^{(j)}$ at first order in velocities. Hence $v_o^{(i)} = (0, 0, v_o)$. In this coordinates system, the projection of eq. (7.2.25) on the z-axis gives, to first order in velocities,

$$\frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}t} = \frac{1}{a_o \chi_{so}} v_o \sin\theta - \dot{v}_o \sin\theta - \frac{1}{a_o \chi_{so}} \left(\frac{v_s^{(z)} - \cos\theta\, n_{(i)} v_s^{(i)}}{\sin\theta} \right). \tag{7.2.44}$$

Since we are in a pure FLRW, we have to set $\Phi = \Psi = 0$ in the results of ref. [80]. Hence, at first order in velocity, the proper time of the general observer and the cosmic time

[3] For large redshifts, $E(z) \approx \sqrt{\Omega_m^o (1+z)^3}$, such that $1/\int_0^z \frac{\mathrm{d}z'}{E(z')} \approx \sqrt{\Omega_m^o}/2 = 0.28$. The 0.31 value mentioned on the text is found when taking into account the nonzero value of Ω_Λ^o.

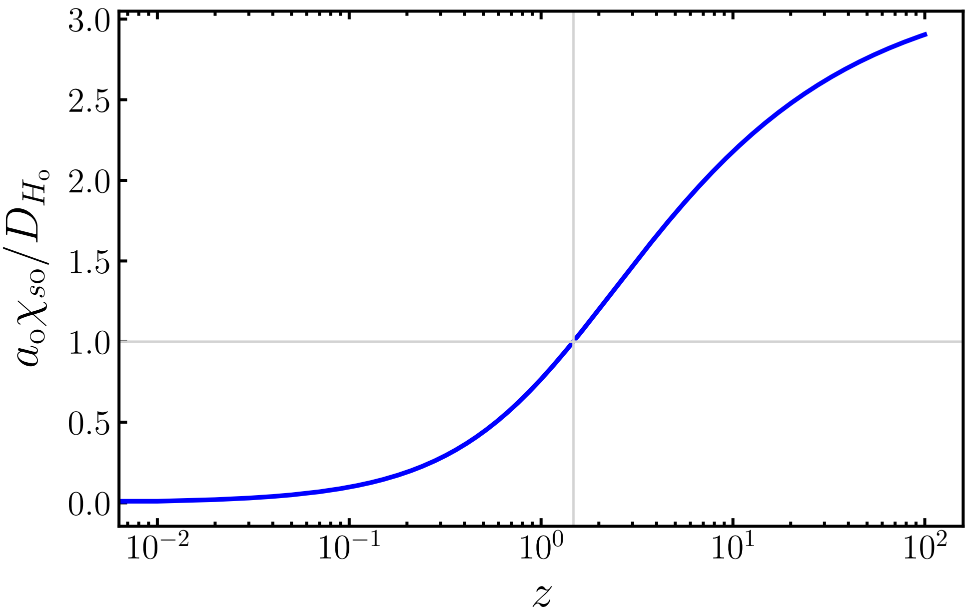

Figure 9 – Behavior of the function $\frac{a_{\rm o}\chi_{so}}{D_{H_{\rm o}}} = \int_0^z \frac{{\rm d}z'}{E(z')}$ as a function of z. Note that it saturates for large z. The vertical line corresponds to $z = 1.48$.

coincide. Comparing the above to eq. (2) of ref. [80], the aberration contribution to the drift matches with our result. The $\dot{\theta}$ of their eq. (2) accounts for the parallax drift due to the motions of the source, which can be suppressed by averaging over many sources. Here, this effect is explicit in the last term of eq. (7.2.44). The authors of ref. [80] are concerned with the aberration drift effect, and argue that the parallactic effects from the motion of the observer can be subtracted from observations. The first term of eq. (7.2.44) is thus not present in ref. [80]. If the effect from the velocities of the sources is averaged out, we are indeed left with the parallax drift due to the motion of the observer and the aberration drift. The total drift thus reads

$$\frac{{\rm d}\tilde{\theta}}{{\rm d}t} = \frac{1}{a_{\rm o}\chi_{so}} v_{\rm o} \sin\theta - \dot{v}_{\rm o} \sin\theta. \tag{7.2.45}$$

While the parallax drift can be treated as a systematic effect to be removed through its z dependence, our method of comparing two infinitesimally close geodesics has the advantage of making explicit the effects for both the source and observer. Also, it can be easily extended to other spacetimes. We show how to apply it in less symmetric spacetimes in the following sections.

- Milky Way velocity, $v^{(i)}_{\rm MW}$

The main source of acceleration for the Milky Way inside the Local Group is the gravitational potential from M31 (Andromeda). Thus, we approximate the acceleration of the Milky Way by $\dot{v}_{\rm MW} = GM_{\rm M31}/R^2_{\rm M31}$. This gives the magnitude of the aberration drift: with $R_{\rm M31} \approx 0.8\,{\rm Mpc}$ and $M_{\rm M31} \approx 1.3 \times 10^{12} {\rm M}_\odot$ [109], it gives a typical magnitude of $0.006\,\mu{\rm as/yr}$ for aberration drift.

The contribution of the parallax drift is estimated from $v_{\rm MW}/a_{\rm o}\chi_{so}$. The Milky Way's velocity with respect to the local group is estimated to be of order 135 km/s [108]. For sources at a distance $a_{\rm o}\chi_{so} = D_{H_{\rm o}}$, the parallax drift is then of the order 0.006 μas/yr. For sources further than $a_{\rm o}\chi_{so} \gtrsim D_{H_{\rm o}}$, the total direction drift due to the motion of the MW is roughly one order of magnitude smaller than the drift due to the LG motion.

- Sun velocity, $v_\odot^{(i)}$

Let us estimate the direction drift due to the motion of the Sun around the Galactic Center. From Kepler's laws, the acceleration of the Sun around the Galactic Center is $\dot{v}_\odot = v_\odot^2/R_{\rm GC}$, where $R_{\rm GC}$ is the distance from the Sun to the Galactic Center. The parallax drift is simply estimated from $v_\odot/a_{\rm o}\chi_{so}$. Considering $v_\odot \approx 230$km/s [108] and the distance to the Galactic Center to be $R_{\rm GC} \approx 8$kpc, the aberration drift and the parallax drift are found to be of the same order for sources such that $a_{\rm o}\chi_{so} \simeq cR_{\rm GC}/v_\odot \approx 10$Mpc. This shows that both the parallax and aberration drifts may contribute significantly to the total signal. The aberration drift is of order 4 μas/yr and, at $a_{\rm o}\chi_{so} = D_{H_{\rm o}}$, the parallax drift is of order 0.01 μas/yr. Thus, the aberration drift contribution from $v_\odot$ due to the motion of the Sun is 100 times larger than the aberration drift due to the motion of the LG.

- Earth velocity, $v_\oplus^{(i)}$

The direction drift contribution from the motion of the Earth around the Sun comprises an annual modulation to the total signal which is not cumulative. Since the motion of the Earth around the Sun is well understood, it is expected that this signal can be subtracted.

- Final remarks on velocities

Table 3 summarizes the previous estimations of the aberration and parallax drifts. To ease comparisons, we consider different comoving radial distances to the source for the parallax drift: $3D_{H_{\rm o}}$, which corresponds to $z \gtrsim 100$, at which $a_{\rm o}\chi_{so}$ saturates; $D_{H_{\rm o}}$, which corresponds to $z \approx 1.48$; and $0.1D_{H_{\rm o}}$, which corresponds to $z \approx 0.1$.

Quasars are found in a range of redshifts up to $z = 5$, but are mostly around $z = 1$ and $z = 2$ [110]. This means that for both the Local Group and the Milky Way motions, the aberration and parallax drifts are of the same order of magnitude. For the motion of the Sun, however, the aberration drift is 100 times larger than the parallax drift.

We must finally note that our estimations completely ignore the direction of the different velocities, which must be taken into account for a precise analysis.

Drifts	LG	MW	$\odot$
Aberration	0.03	0.006	4.0
Parallax at $3D_{H_o}$	0.001	0.002	0.003
Parallax at D_{H_o}	0.03	0.006	0.01
Parallax at $0.1D_{H_o}$	0.3	0.06	0.1

Table 3 – Estimation of aberration and parallax drifts for different motions. The values are expressed in units of μas/yr.

7.3 Perturbed FLRW spacetimes

The main results of the previous section, concerning redshift and direction drifts in a FLRW universe, ignore the existence of small perturbations in the distribution of mass and energy of the universe. Clearly, such fluctuations will contribute to the evaluation of cosmological drifts, and a proper assessment of these quantities in a realistic model of the universe requires that we include such perturbations in our formalism. Thus, in this section we set to solve the important question of deriving cosmological drifts in a linearly perturbed cosmological spacetime.

To this end, let us perturb our metric around flat FLRW spacetime as $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$, where the overbar denotes the background value. The universe is then described by a perturbed FLRW spacetime with geometry

$$\mathrm{d}s^2 = a^2 \left[-\left(1+2\Phi\right)\mathrm{d}\eta^2 + \left(\delta_{ij} + h_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j\right] , \tag{7.3.1}$$

with

$$h_{ij} = -2\Psi\delta_{ij} + 2\partial_{(i}E_{j)} + 2E_{ij} \, . \tag{7.3.2}$$

This defines our choice of Newtonian gauge where the scalar modes are described by the two gravitational potentials, Φ and Ψ, vector perturbations are described by a transverse vector E_i ($\partial_i E^i = 0$) and E_{ij} represents the traceless and transverse tensor perturbation ($E^i_i = 0 = \partial_i E^{ij}$) describing gravitational waves. More information on perturbed FLRW spacetimes can be found in ref. [57], however, the essential formulae for evaluating the drifts can be found in Appendix C. We shall be evaluating the drifts for scalar and tensor perturbations, but the vector perturbation results can be found by making the identification $E_{ij} \to \partial_{(i}E_{j)}$ between the tensor and vector modes.

The gravitational potentials have two effects on the direction drift. The first is a direct effect on the drift, i.e., the direction drift due to these potentials, which we will evaluate further. The second is an effect on the motion of the observer. More precisely, it will affect the geodesic motion of the Local Group with respect to the CMB rest frame. To that end let us evaluate the geodesic equation for a point particle with 4-velocity $\tilde{u}^\mu$. The Christoffel symbols of the metric (7.3.1) can be also found in ref. [57]. To first order in perturbations and in spatial velocity, the i-component of the timelike geodesic equation reads

$$\frac{\mathrm{d}v^i}{d\tau} + 2Hv^i + \frac{1}{a^2}\partial^i\Phi = 0, \tag{7.3.3}$$

where τ is the observer's proper time. In tetrad components[4], the above turns into

$$\frac{\mathrm{d}v_{\mathrm{o}}^{(i)}}{\mathrm{d}\tau} = -H_{\mathrm{o}}v_{\mathrm{o}}^{(i)} - \partial^{(i)}\Phi_{\mathrm{o}}. \tag{7.3.4}$$

Ref. [80] describes how measurements of the cosmological aberration drift could be used to evaluate the contribution from $\partial^{(i)}\Phi_{\mathrm{o}}$. In fact, one can show that the contribution from the gravitational potential is proportional to $H_{\mathrm{o}}v_{\mathrm{o}}^{(i)}$, with the proportionality factor being given by a model-dependent parameter (authors of ref. [80] allow for departure from standard GR). In standard GR, this parameter is of order one, and the contribution of $\partial^{(i)}\Phi_{\mathrm{o}}$ is then of the same order of magnitude as the cosmological aberration drift [80].

7.3.1 Null geodesics

The study of the null geodesics is simpler once one uses the standard trick that they are conformally invariant. The metric (7.3.1) can be rescaled as $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a^2\hat{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$, where from now on the overhat denotes quantities evaluated in the conformal space. If k^μ is the tangent vector to a null geodesic of $g_{\mu\nu}$, and if $\hat{k}^\mu$ is the tangent vector to a null geodesic of $\hat{g}_{\mu\nu}$, then $k^\mu\nabla_\mu k^\nu = 0 = \hat{k}^\mu\nabla_\mu\hat{k}^\nu$. It is easy to check that for this to be satisfied we must have $\hat{k}_\mu = k_\mu$ and $\hat{k}^\mu = a^2k^\mu$.

The conformal null geodesic vector $\hat{k}^\mu$ can be decomposed as

$$\hat{k}^\mu = \hat{\omega}\left(1, n^i\right). \tag{7.3.5}$$

We define $\hat{\bar{\omega}}_{\mathrm{o}}$ and $\bar{n}^i$ as the (constant) background values of $\hat{\omega}$ and and n^i, respectively. To first order in perturbations, the 0-component of the geodesic equation can be found using the Christoffel symbols from Appendix C, and it is given by

$$\frac{1}{\hat{\omega}}\frac{\mathrm{d}\hat{\omega}}{\mathrm{d}\eta} = \Phi' - 2\frac{\mathrm{d}\Phi}{\mathrm{d}\eta} - \frac{1}{2}\bar{n}^i\bar{n}^jh'_{ij}. \tag{7.3.6}$$

where the prime denotes the partial derivative with respect to η and we used that $\frac{\mathrm{d}}{\mathrm{d}\eta} \equiv \frac{1}{\hat{\omega}}\hat{k}^\mu\partial_\mu = \frac{\partial}{\partial\eta} + n^i\partial_i$. Note that in homogeneous spaces $\frac{\mathrm{d}}{\mathrm{d}\eta} = \frac{\partial}{\partial\eta}$, but since perturbed FLRW has no symmetries, one must take into account the spatial variations along the geodesic path. The above can be integrated from the source (s) to the observer (o) to give

$$\hat{\omega}_s - \hat{\omega}_{\mathrm{o}} = \hat{\bar{\omega}}_{\mathrm{o}}\left[-2\left(\Phi_s - \Phi_{\mathrm{o}}\right) + \int_{\eta_{\mathrm{o}}}^{\eta_s}\left(\Phi' - \frac{1}{2}\bar{n}^i\bar{n}^jh'_{ij}\right)\mathrm{d}\eta\right]. \tag{7.3.7}$$

[4] The tetrad basis for perturbed FLRW is given below, but to first order the usual FLRW tetrad used before is enough here.

Note that the above is not an usual integration on a parameter η, rather, it is an integral along the null geodesic curve.

Now, to evaluate the energy of the photon, it is required that the 4-velocity of a comoving observer in perturbed FLRW universe to be computed. Let us write $u^\mu = \bar{u}^\mu + \delta u^\mu$, where $\bar{u}^\mu = a^{-1}(1, \mathbf{0})$ is the background 4-velocity. Then, by requiring that $g_{\mu\nu} u^\mu u^\nu = -1$ one arrives at

$$2\bar{g}_{\mu\nu}\bar{u}^\mu \delta u^\nu + \delta g_{\mu\nu}\bar{u}^\mu \bar{u}^\nu = 0. \tag{7.3.8}$$

From the above it is easy to find $\delta u^0 = -a^{-1}\Phi$, from which we can find $\delta u_0 = \bar{g}_{00}\delta u^0 + \delta g_{00}\bar{u}^0 = -a\Phi$. We then write $\delta u_i = a v_i$ such that, the 4-velocity for an observer in perturbed FLRW is

$$u_\mu = a(-1 - \Phi, v_i). \tag{7.3.9}$$

However, as long as we are in first order in perturbations, the quantity v^i can be seen as a field on the top of an unperturbed FLRW background, for which the results of our previous section applies. Thus, since linear perturbations evolve independently, we can ignore v^i here and, at the end of our computations, add the results of the previous section. Thus, we have that

$$u_\mu = a(-1 - \Phi, \mathbf{0}). \tag{7.3.10}$$

Now, since in conformal space we must also have $\hat{g}_{\mu\nu}\hat{u}^\mu \hat{u}^\nu = -1$, one can deduce that $\hat{u}_\mu = a^{-1}u_\mu$, such that $\hat{u}_\mu = (-1 - \Phi, \mathbf{0})$. Then, together with $\hat{k}^\mu = a^2 k^\mu$, the energy of the photon as measured by an observer with 4-velocity u^μ is

$$\omega = -k^\mu u_\mu = -\frac{1}{a}\hat{k}^\mu \hat{u}_\mu = \frac{\hat{\omega}}{a}\left(1 + \Phi\right). \tag{7.3.11}$$

From the above it is straightforward to deduce the redshift

$$1 + z = \frac{\omega_s}{\omega_\text{o}} = \frac{a_\text{o}}{a_s}\frac{\hat{\omega}_s}{\hat{\omega}_\text{o}}\left[1 + (\Phi_s - \Phi_\text{o})\right]. \tag{7.3.12}$$

Now, $\hat{\omega}_s/\hat{\omega}_\text{o}$ can be obtained from eq. (7.3.7) so that

$$1 + z = \frac{a_\text{o}}{a_s}\left[1 - (\Phi_s - \Phi_\text{o}) + \int_{\eta_\text{o}}^{\eta_s}\left(\Phi' - \frac{1}{2}\bar{n}^i\bar{n}^j h'_{ij}\right)\mathrm{d}\eta\right]. \tag{7.3.13}$$

It decomposes into irreducible scalar and tensor contributions as $1 + \bar{z} + \delta z^{(S)} + \delta z^{(T)}$, with

$$\delta z^{(S)} = \frac{a_\text{o}}{a_s}\left[-(\Phi_s - \Phi_\text{o}) + \int_{\eta_\text{o}}^{\eta_s}\Theta'\mathrm{d}\eta\right], \tag{7.3.14a}$$

$$\delta z^{(T)} = -\frac{a_\text{o}}{a_s}\int_{\eta_\text{o}}^{\eta_s} E'_{ij}\bar{n}^i\bar{n}^j\,\mathrm{d}\eta\,, \tag{7.3.14b}$$

where we have defined

$$\Theta = \Phi + \Psi. \tag{7.3.15}$$

Note that the scalar contributions are simply the Sachs-Wolfe and integrated Sachs-Wolfe effects.

7.3.2 Redshift drift

Evaluating the redshift drift is more involved in a perturbed FLRW universe since the constant time hypersurfaces are no longer homogeneous. Considering a second geodesic corresponding to an observation at $t_\mathrm{o} + \delta t_\mathrm{o}$, the change in redshift $\delta_{12} z$ is obtained from eq. (7.3.13) to be

$$\frac{\delta_{12} z}{1+z} = (H_\mathrm{o}\delta t_\mathrm{o} - H_s \delta t_s) + \delta_{12}\Upsilon \tag{7.3.16}$$

where

$$\Upsilon = -(\Phi_s - \Phi_\mathrm{o}) + \int_{\eta_\mathrm{o}}^{\eta_s} \left(\Phi' - \frac{1}{2}\bar{n}^i \bar{n}^j h'_{ij}\right) \mathrm{d}\eta \tag{7.3.17}$$

and $\delta_{12}\Upsilon$ stands for the difference of Υ between its value on the second and first geodesics.

Looking at the above, the drift of the terms on fixed endpoints is trivially done following the same steps as before. However, to evaluate the drift of the integrated terms in Υ, one has to take into account how they vary spatially from one geodesic to the next as space is no longer homogeneous. For this purpose, let us evaluate $\delta_{12}\Xi$ where Ξ is a general term of the form $\Xi = \int_{\eta_\mathrm{o}}^{\eta_s} \xi[\eta, x^i]\mathrm{d}\eta$, and ξ is a perturbation. One has to take into account how $\xi[\eta, x^i]$ changes from one geodesic to the other. Thus, there will be a contribution due to the fixed endpoints of the integral but also a contribution from the spatial change of the integrand. More explicitly,

$$\delta_{12}\Xi = \int_{\eta_\mathrm{o}+\delta\eta_\mathrm{o}}^{\eta_s+\delta\eta_s} \xi[\eta, x_2^i]\mathrm{d}\eta - \int_{\eta_\mathrm{o}}^{\eta_s} \xi[\eta, x_1^i]\mathrm{d}\eta. \tag{7.3.18}$$

The second line of sight $x_2^i(\eta)$ is given by

$$\begin{aligned} x_2^i(\eta) &= x_1^i(\eta) + \delta x^i(\eta) \\ &= x_1^i(\eta) - \bar{n}^i \delta\eta. \end{aligned} \tag{7.3.19}$$

This allows us to compute the contribution from the endpoints, which, to first order in $\delta\eta$, is simply

$$\int_{\eta_\mathrm{o}+\delta\eta_\mathrm{o}}^{\eta_s+\delta\eta_s} \xi[\eta, x_2^i]\mathrm{d}\eta = \int_{\eta_\mathrm{o}}^{\eta_s} \xi[\eta, x_2^i]\mathrm{d}\eta + \delta\eta_s \xi_s[x_1^i] - \delta\eta_\mathrm{o}\xi_\mathrm{o}[x_1^i]. \tag{7.3.20}$$

Now, we use eq. (7.3.19) to write

$$\xi[\eta, x_2^i] = \xi[\eta, x_1^i] - \bar{n}^i \partial_i \xi[\eta, x_1^i]\delta\eta\,. \tag{7.3.21}$$

Let us recall that $\bar{n}^i\partial_i = \frac{\mathrm{d}}{\mathrm{d}\eta} - \frac{\partial}{\partial\eta}$. Also, since ξ is a perturbation, any term multiplying it is evaluated at the background, and at this level $\delta\eta_s = \delta\eta_\mathrm{o}$. We conclude that

$$\delta_{12}\Xi = \delta\eta_\mathrm{o} \int_{\eta_\mathrm{o}}^{\eta_s} \xi'[\eta, x_1^i]\mathrm{d}\eta\,. \tag{7.3.22}$$

Therefore, our final result is

$$\delta_{12}\Upsilon = -\left(\dot{\Phi}_s\delta t_s - \dot{\Phi}_o\delta t_o\right) + \int_{\eta_o}^{\eta_s}\left(\Phi'' - \frac{1}{2}\bar{n}^i\bar{n}^j h''_{ij}\right)\mathrm{d}\eta\,\delta\eta_o.$$

Plugging this back into eq. (7.3.16), we get

$$\frac{\delta_{12}z}{1+z} = (H_o\delta t_o - H_s\delta t_s) - \left(\dot{\Phi}_s\delta t_s - \dot{\Phi}_o\delta t_o\right) + \int_{\eta_o}^{\eta_s}\left(\Phi'' - \frac{1}{2}\bar{n}^i\bar{n}^j h''_{ij}\right)\mathrm{d}\eta\,\delta\eta_o. \quad (7.3.23)$$

To obtain the redshift drift, we must also take into account the difference between the observer's proper time τ and conformal time η, which is $\delta\tau = \sqrt{-g_{00}}\delta\eta = a\,(1+\Phi)\,\delta\eta$ (or equivalently $\delta\tau = (1+\Phi)\,\delta t$), and then use eq. (B.0.9) to conclude that

$$\frac{\delta_{12}z}{\delta\tau_o} = (1+z)\left[H_o(1-\Phi_o) + \dot{\Phi}_o\right] - \left[H_s(1-\Phi_s) + \dot{\Phi}_s\right] + \frac{(1+z)}{a_o}\int_{\eta_o}^{\eta_s}\left(\Phi'' - \frac{1}{2}\bar{n}^i\bar{n}^j h''_{ij}\right)\mathrm{d}\eta. \quad (7.3.24)$$

7.3.3 Direction drift

To evaluate the direction drift, we start with the i-component of the geodesic equation in conformal space. Using the Christoffel symbols from Appendix C and the photon 4-vector given by eq. (7.3.5) it is

$$\frac{\mathrm{d}n^i}{\mathrm{d}\eta} - \bar{n}^i\frac{\mathrm{d}\Phi}{\mathrm{d}\eta} + \perp^i_j\partial^j\Phi - \frac{1}{2}\perp^i_j\partial^j(\bar{n}^k\bar{n}^l h_{kl}) + \bar{n}^j\frac{\mathrm{d}h^i_j}{\mathrm{d}\eta} - \frac{1}{2}\bar{n}^i\bar{n}^j\bar{n}^k\frac{\mathrm{d}h_{jk}}{\mathrm{d}\eta} = 0, \quad (7.3.25)$$

where $\perp^i_j = \delta^i_j - \bar{n}^i\bar{n}_j$. Now, we must change to tetrad components. The tetrad basis for the conformal space is defined by

$$\boldsymbol{\epsilon}_{(0)} = (1-\Phi)\,\partial_\eta, \quad \boldsymbol{\epsilon}_{(i)} = \left(\delta^j_i - \frac{1}{2}h^j_i\right)\partial_j. \quad (7.3.26)$$

Thus, the direction vector is decomposed in tetrad components as

$$n^{(i)} = \frac{\hat{k}^{(i)}}{\hat{k}^{(0)}}, \quad (7.3.27)$$

so that

$$n^i = n^{(i)} + \Phi n^{(i)} - \frac{1}{2}h^i_j n^{(j)}. \quad (7.3.28)$$

Plugging this decomposition into eq. (7.3.25) gives

$$\frac{\mathrm{d}n^{(i)}}{\mathrm{d}\eta} = -\perp^i_j\partial^j\Phi + \frac{1}{2}\perp^i_j\partial^j\left[\bar{n}^{(k)}\bar{n}^{(l)}h_{kl}\right] - \frac{1}{2}\perp^i_j\bar{n}^{(k)}\frac{\mathrm{d}h^j_k}{\mathrm{d}\eta}. \quad (7.3.29)$$

Integrating this equation and then using $n^{(i)} = n^i - \Phi n^{(i)} + \frac{1}{2}h^i_j n^{(j)}$ and $n^i = \frac{\mathrm{d}x^i}{\mathrm{d}\eta}$, one gets the null geodesic equation,

$$\frac{\mathrm{d}x^i}{\mathrm{d}\eta} = n^{(i)}_o + \bar{n}^{(i)}_o\Phi - \frac{1}{2}\bar{n}^{(j)}_o h^i_j - \frac{1}{2}\perp^i_j\bar{n}^{(k)}_o(h^j_k - h^j_{ok}) - \perp^i_j\partial^j\int_{\eta_o}^{\eta}\left\{\Phi - \frac{1}{2}\left[\bar{n}^{(k)}_o\bar{n}^{(l)}_o h_{kl}\right]\right\}\mathrm{d}\eta. \quad (7.3.30)$$

After integration from η_o to η_s, its scalar and tensor parts are

$$\begin{aligned} x_s^i - x_o^i =& (\eta_s - \eta_o) n_o^{(i)} + \bar{n}_o^{(i)} \int_{\eta_o}^{\eta_s} \Theta \, d\eta - \perp_j^i \int_{\eta_o}^{\eta_s} (\eta_s - \eta) \partial^j \Theta \, d\eta + (\eta_s - \eta_o) \perp_j^i \bar{n}_o^{(k)} E_{ok}^j \\ & - \perp_j^i \bar{n}_o^{(k)} \int_{\eta_o}^{\eta_s} E_k^j \, d\eta - \bar{n}_o^{(j)} \int_{\eta_o}^{\eta_s} E_j^i \, d\eta + \perp_j^i \int_{\eta_o}^{\eta_s} (\eta_s - \eta) \bar{n}_o^{(k)} \bar{n}_o^{(l)} \partial^j E_{kl} \, d\eta, \end{aligned} \tag{7.3.31}$$

where we have performed an integration by parts to express double integrals as single integrals[5].

To find the direction drift, what is left is to compare the equation of a second nearby geodesic with eq. (7.3.31). The calculation is tedious but straightforward (see Appendix D for details). On the second geodesic, the terms evaluated at fixed endpoints are simple to compute, while the integrated terms follow the same calculations done before for the redshift drift, yielding results of the form (7.3.22). Also, it is needed to take into account the difference between proper time of the observer and conformal (or cosmic) time, and also make use of eq. (B.0.9). Splitting in scalar and tensor modes, the direction drift is then finally given by

$$\frac{\delta_{12} n_o^{(i)\,(S)}}{\delta \tau_o} = -\frac{\perp_j^i}{a_o} \int_{\eta_o}^{\eta_s} \frac{(\eta_s - \eta)}{\chi_{so}} \partial^j \Theta' d\eta \,, \tag{7.3.32a}$$

$$\frac{\delta_{12} n_o^{(i)\,(T)}}{\delta \tau_o} = -\frac{1}{a_o} \perp_j^i \bar{n}_o^{(k)} E_{ok}'^j - \frac{2 \perp_j^i \bar{n}_o^{(k)}}{a_o \chi_{so}} \int_{\eta_o}^{\eta_s} E_k'^j d\eta + \frac{\perp_j^i}{a_o} \int_{\eta_o}^{\eta_s} \frac{(\eta_s - \eta)}{\chi_{so}} \bar{n}_o^{(k)} \bar{n}_o^{(l)} \partial^j E_{kl}' \, d\eta. \tag{7.3.32b}$$

Readers familiarized with CMB lensing literature can, in the above, recognize the standard CMB lensing potentials, apart from time derivatives.

7.3.4 Summary

This section provides the first derivation of the direction drift and redshift drift in a perturbed FLRW universe. As such each has two contributions arising from scalar and tensor modes. Recall that the contribution to vector modes can be found simply by making the identification $E_{ij} \to \partial_{(i} E_{j)}$ in the tensor modes' results. The scalar part of our expression (7.3.24) corrects a mistake in the only expression proposed in the literature so far and first published in ref. [89]. Such an expression plays an important role in estimating the expected cosmological variance of the redshift drift. Note that the scalar mode contribution has to be combined with eq. (7.2.25) to include the effect of the motions of the observer and the sources.

Concerning gravity waves, several results [111, 112, 113, 114, 115, 116] have been used, in particular by Pulsar Timing Array experiments. The result of ref. [113] gives the

[5] This trick is given by $\int_{\eta_o}^{\eta_s} \left(\int_{\eta_o}^{\eta'} f(\eta) d\eta \right) d\eta' = \int_{\eta_o}^{\eta_s} (\eta_s - \eta') f(\eta') d\eta'$, which results from an integration by parts with $u = F(\eta') = \int_{\eta_o}^{\eta'} f(\eta) d\eta$ and $dv = d\eta'$.

perturbed values of z and $n^{(i)}$ with respect to the background $\bar{z}$ and $\bar{n}^{(i)}$ values. Their equation (28) is directly comparable to eq. (7.3.14b), and the results match considering the relationship between the parameters λ and η and that eq. (28) is for a pure Minkowski spacetime. To compute the perturbation of $n^{(i)}$ in our framework, we start from eq. (7.3.31) and split it into its background and perturbed values. Noticing that the positions of the source and observer are fixed in the "straight geodesic" approximation, we take the perpendicular projection of eq. (7.3.31) to find

$$\delta n^{(i)} = -\perp^i_j \bar{n}^{(k)} E^j_{\text{o}k} - \frac{2\perp^i_j \bar{n}^{(k)}}{\chi_{so}} \int_{\eta_\text{o}}^{\eta_s} E^j_k \,\mathrm{d}\eta + \perp^i_j \int_{\eta_\text{o}}^{\eta_s} \frac{(\eta_s - \eta)}{\chi_{so}} \bar{n}^{(k)}_\text{o} \bar{n}^{(l)}_\text{o} \partial^j E_{kl} \,\mathrm{d}\eta. \quad (7.3.33)$$

This matches with the eq. (56) of ref. [113] if we take into account our opposite sign in defining the direction vector and again, the relationship between the parameters ζ and η. Thus, the plane wave expansion used in refs. [111, 112, 113, 114, 115, 116] is compatible with our analysis. Indeed, their results only provide the deviation of z and $n^{(i)}$ from their background values and do not provide their drifts computed here.

7.4 Spatially homogeneous and anisotropic spacetimes: Bianchi I case

The Bianchi spacetimes are a class of solutions to the Einstein's equation with strong cosmological appeal. Letting go of the assumption of isotropy one arrives at this class of spatially anisotropic spacetimes, which are classified according to the Lie algebra of their Killing vectors. In the first part of this thesis, we evaluated the 2pcf for two types of Bianchi models and showed how one can use our results to assess primordial anisotropies. As we shall see, the drifts of the redshift and direction vector will give means to test late time anisotropies. Here, we evaluate these drifts for the Bianchi I model, the simplest solution which arises when requiring space to be anisotropic, but still homogeneous.

Being homogeneous, these spaces still enjoy three Killing vectors associated with the three spatial translations. The spatial sections are also Euclidean, and its spacetime metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad (7.4.1)$$

with

$$\gamma_{ij} = e^{2\beta_i(t)}\delta_{ij} \quad (7.4.2)$$

where the $\beta_i(t)$ are three directional scale factors which satisfy $\sum_i \beta_i = 0$, and $a(t)$ is the average scale factor defined by the volume expansion. The directional scale factors β_i are not components of a vector, and thus, in the above the index i is not summed.

7.4.1 Null geodesics

Thanks to the three translational Killing vectors, we still have

$$k_i = \text{const} \tag{7.4.3}$$

along any null geodesic for a photon with 4-momentum $k^\mu = \omega(u^\mu - n^\mu)$, as long as we use the Cartesian coordinates introduced in eq. (7.4.1). Raising the index of the above equation leads to

$$a^2 e^{2\beta_i} k^i = \text{const}\,, \tag{7.4.4}$$

again, with no summation over i (this notation is explained below). The worldline of a photon is again given by

$$\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} = k^i, \tag{7.4.5}$$

where λ is the parameter along the geodesic. The orthonormal tetrads for Bianchi I are explicitly given by

$$\boldsymbol{e}_{(0)} = \partial_t, \quad \boldsymbol{e}_{(i)} = a^{-1}e^{-\beta_i}\partial_i. \tag{7.4.6}$$

Once more, the right side of eq. (7.4.4) is fixed at the point of observation. Then, in terms of tetrad components and conformal time it becomes

$$a\omega e^{2\beta_i}\frac{\mathrm{d}x^i}{\mathrm{d}\eta} = -a_\mathrm{o}\omega_\mathrm{o}e^{\beta_i^0}n_\mathrm{o}^{(i)} \tag{7.4.7}$$

where $n_\mathrm{o}^{(i)}$ are the tetrad basis components of n_o^i. We also define the standard shear tensor by

$$\sigma_{ij} = \gamma'_{ij} = \beta'_i \mathrm{e}^{2\beta_i}\delta_{ij}, \tag{7.4.8}$$

which is a symmetric trace-free tensor. Again, there is no sum in the above, and a prime denotes derivative with respect to conformal time η (defined through $\mathrm{d}t = a\mathrm{d}\eta$).

7.4.2 Direction and redshift drifts to first order in shear

We shall treat only static observers in Bianchi I since, as a perturbation, we already know the contribution from a moving observer. To start, we must find how ω evolves along the geodesic, i.e., it is needed to solve the 0-component of the geodesic equation. The non-zero relevant Christoffel symbols for this case are

$$\Gamma^0_{ij} = a^2\left(H\gamma_{ij} + \frac{1}{2}\dot{\gamma}_{ij}\right), \tag{7.4.9}$$

where H is defined as before, i.e., $H = \frac{\dot{a}}{a}$. Then, the 0-component of the geodesic equation reads

$$\frac{1}{a\omega}\frac{\mathrm{d}(a\omega)}{\mathrm{d}t} + \frac{\dot{\beta}_i a^2\gamma_{ij}k^ik^j}{\omega^2} = 0. \tag{7.4.10}$$

In conformal time and in terms of tetrad components it is then given by

$$\frac{1}{a\omega}\frac{\mathrm{d}(a\omega)}{\mathrm{d}\eta} + \beta_i' n^{(i)} n_{(i)} = 0, \tag{7.4.11}$$

a solution of which is

$$a\omega = a_{\mathrm{o}}\omega_{\mathrm{o}} \exp\left(-\int_{\eta_{\mathrm{o}}}^{\eta} \beta_i' n^{(i)} n_{(i)} \mathrm{d}\eta\right). \tag{7.4.12}$$

Now, in the above equations there is summation in i, i.e., $\beta_i' n^{(i)} n_{(i)} = \left(\beta_x' n^{(x)} n_{(x)} + ...\right)$. For simplicity, we shall not explicitly denote these sums. To avoid confusion, however, one should keep in mind that an expression like $\beta_i n^{(i)}$ is not summed, while $\beta_i n^{(i)} n_{(i)}$ is.

Observational data shows that our universe is very close to isotropic. Thus, we shall now perform a small shear approximation and consider only the lowest order terms in σ_{ij} or, equivalently, in β_i' (and in β_i). In this limit, eq. (7.4.12) becomes

$$\frac{\omega}{\omega_{\mathrm{o}}} \simeq \frac{\bar{\omega}}{\bar{\omega}_{\mathrm{o}}}\left[1 - \left(\beta_i - \beta_i^{\mathrm{o}}\right) \bar{n}^{(i)} \bar{n}_{(i)}\right], \tag{7.4.13}$$

where an overbar denotes the FLRW value since Bianchi I spacetime can be thought as a homogeneous perturbation of FLRW spacetime. To find the above we used that, for the background quantities, $a\bar{\omega} = a_{\mathrm{o}}\bar{\omega}_{\mathrm{o}}$. Since $\bar{n}^{(i)}$ is constant along the geodesic, the small shear approximation is related to a "straight geodesic" approximation.

In possession of eq. (7.4.13), we plug it into eq. (7.4.7) to find, after some simple algebra,

$$x_s^i - x_{\mathrm{o}}^i \simeq -\int_{\eta_{\mathrm{o}}}^{\eta_s} \left(1 - \beta_i\right) n_{\mathrm{o}}^{(i)} \mathrm{d}\eta + \int_{\eta_{\mathrm{o}}}^{\eta_s} \perp_j^i \left(\beta_j - \beta_j^{\mathrm{o}}\right) \bar{n}^{(j)} \mathrm{d}\eta \tag{7.4.14}$$

where $\perp_j^i = \delta_j^i - \bar{n}^{(i)}\bar{n}_{(j)}$ is the perpendicular projector with respect to $\bar{n}^{(i)}$ (both of which are constant along the geodesic). The above is the solution of the geodesic equation to first order in shear for Bianchi I. To find the direction drift, we now just have to follow the same procedure as before, i.e., write (7.4.14) for a second nearby geodesic and determine the relation between $\delta\eta_s$ and $\delta\eta_{\mathrm{o}}$. To find the latter according to eq. (B.0.9), we need the redshift z which can be easily read from eq. (7.4.13) to be

$$1 + z = \frac{\omega_s}{\omega_{\mathrm{o}}} = \frac{a_{\mathrm{o}}}{a_s}\left[1 - \left(\beta_i^s - \beta_i^{\mathrm{o}}\right) \bar{n}^{(i)} \bar{n}_{(i)}\right]. \tag{7.4.15}$$

Then, the relation between the lapses is given by

$$\frac{\delta\eta_s}{\delta\eta_{\mathrm{o}}} = \left[1 + \left(\beta_i - \beta_i^{\mathrm{o}}\right) \bar{n}^{(i)} \bar{n}_{(i)}\right]. \tag{7.4.16}$$

The second geodesic is written as

$$\begin{aligned} x_s^i - x_{\mathrm{o}}^i \simeq &-\int_{\eta_{\mathrm{o}}+\delta\eta_{\mathrm{o}}}^{\eta_s+\delta\eta_{\mathrm{o}}} \left(1 - \beta_i\right) \left(n_{\mathrm{o}}^{(i)} + \delta_{12} n_{\mathrm{o}}^{(i)}\right) \mathrm{d}\eta + \int_{\eta_{\mathrm{o}}+\delta\eta_{\mathrm{o}}}^{\eta_s+\delta\eta_s} \perp_j^i \left(\beta_j - \beta_j^{\mathrm{o}}\right) \bar{n}^{(j)} \mathrm{d}\eta \\ &+ \left(\eta_{\mathrm{o}} - \eta_s\right) \perp_j^i (\beta_j^{\mathrm{o}})' \bar{n}^{(j)} \delta\eta_{\mathrm{o}}. \end{aligned} \tag{7.4.17}$$

Now, it is just a matter of algebra to expand eq. (7.4.17) to first order in $\delta\eta_{s,\mathrm{o}}$ and subtract it from eq. (7.4.14). Then, using the relation (7.4.16), one can arrive at the direction drift, which is

$$\frac{\delta_{12}n^{(i)}}{\delta t_{\mathrm{o}}} = -\frac{2}{a_{\mathrm{o}}\chi_{s\mathrm{o}}}\perp^i_j\left(\beta^s_j - \beta^{\mathrm{o}}_j\right)\bar{n}^{(j)} - \perp^i_j\dot{\beta}^{\mathrm{o}}_j\bar{n}^{(j)}, \tag{7.4.18}$$

where $\chi_{s\mathrm{o}} = \eta_{\mathrm{o}} - \eta_s$ is the observer-source radial distance in the FLRW background space. Notice that this expression is similar to the one for a general observers in a FLRW spacetime, containing a parallax type (the first term above) and an aberration type (the second term) contribution. Although these terms have a similar behavior to the moving observer case, they are related to anisotropies of the universe, not any type of peculiar motion. Note also that the first term is a redshift-dependent one, whereas the second is redshift-independent.

To evaluate the redshift drift, we start from eq. (7.4.15) to write

$$1 + z + \delta_{12}z = \frac{a_{\mathrm{o}}}{a_s}\left(H_{\mathrm{o}}\delta t_{\mathrm{o}} - H_s\delta t_s\right)\left[1 - \left(\beta^s_i - \beta^{\mathrm{o}}_i\right)\bar{n}^{(i)}\bar{n}_{(i)} - \left(\dot{\beta}^s_i\delta t_s - \dot{\beta}^{\mathrm{o}}_i\delta t_{\mathrm{o}}\right)\bar{n}^{(i)}\bar{n}_{(i)}\right]. \tag{7.4.19}$$

After some simple algebraic manipulations one is able to arrive at the redshift drift

$$\frac{\delta_{12}z}{\delta t_{\mathrm{o}}} = (1+z)\left(H_{\mathrm{o}} + \dot{\beta}^{\mathrm{o}}_i\bar{n}^{(i)}\bar{n}_{(i)}\right) - \left(H_s + \dot{\beta}^s_i\bar{n}^{(i)}\bar{n}_{(i)}\right), \tag{7.4.20}$$

which is in agreement with the analysis of ref. [90]. Again, one can note the similarity between this result and the one for general observers in FLRW. One manner to distinguish between them, say, as an observer measures them, is through their multipolar decomposition, as explained below.

7.4.3 Decomposition in multipoles

As done previously, we can decompose the redshift and direction drifts in terms of multipoles, the decomposition being made with respect to $\bar{n}^{(i)}$. Defining

$$\mathcal{B}_{ij} \equiv \delta_{ij}\beta_i \qquad \dot{\mathcal{B}}_{ij} \equiv \delta_{ij}\dot{\beta}_i \tag{7.4.21}$$

we read from eq. (7.4.20) that the Bianchi I spacetime brings a quadrupolar structure to the redshift drift,

$$\mathcal{W}_{ij} = (1+z)\dot{\mathcal{B}}^{\mathrm{o}}_{ij} - \dot{\mathcal{B}}^s_{ij}\,. \tag{7.4.22}$$

Note however that it adds no dipolar contribution. It is thus in principle distinguishable from the peculiar velocity of the observer in a FLRW spacetime.

The direction drift in Bianchi I structure also inherits a quadrupolar contribution in $\mathcal{E}(\bar{n}^{(i)})$, but not in $\mathcal{H}(\bar{n}^{(i)})$. From eq. (7.4.18), it is

$$\mathcal{E}_{ij} = -\frac{1}{a_{\mathrm{o}}\chi_{s\mathrm{o}}}\left(\mathcal{B}^s_{ij} - \mathcal{B}^{\mathrm{o}}_{ij}\right) - \frac{1}{2}\dot{\mathcal{B}}^{\mathrm{o}}_{ij}\,. \tag{7.4.23}$$

7.4.4 Discussion

Equations (7.4.18) and (7.4.20) provide the expressions of the direction and redshift drifts in a Bianchi I spacetime, in small shear approximation. Indeed when $\beta_i = 0$ for all i, we recover the FLRW expressions for a comoving observer.

Moreover, since the infinite wavelength limit of a gravitational wave is equivalent to a homogeneous shear (i.e., $E_{ij} \to \beta_i(t)\delta_{ij}$), it can be checked that the tensor part of eq. (7.3.24) is equivalent to eq. (7.4.20) for the redshift drift and that eq. (7.3.32b) is equivalent to eq. (7.4.18) for the aberration drift.

Let us now compare to existing results, and in particular refs. [26, 78] that assume a "straight geodesic approximation" and in which the results are obtained through a time derivative of the angular separation. In these works, the Bianchi I metric is parametrized as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \tilde{a}^2(t)dx^2 + \tilde{b}^2(t)dy^2 + \tilde{c}^2(t)dz^2, \tag{7.4.24}$$

where we have put tildes over the directional scale factors not to confuse them with the average scale factor $a(t)$. From this metric, the authors define directional expansion rates $H_x = \dot{\tilde{a}}/\tilde{a}$ and so on, the average scale factor $a = (\tilde{a}\tilde{b}\tilde{c})^{1/3}$, and shear components, $\Sigma_i = H_i/H - 1$, where $H = \dot{a}/a$. On comparison with the parametrization in the present work, we find the identification $(\tilde{a}, \tilde{b}, \tilde{c}) = (ae^{\beta_x}, ae^{\beta_y}, ae^{\beta_z})$. This allows us to also identify $\Sigma_i = \dot{\beta}_i/H$. Then, we decompose the direction of observation as $n_{\mathrm{o}}^{(i)} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. With these considerations in mind, we can show that the z-component of the aberration-like drift term $\perp^i_j \dot{\beta}^{\mathrm{o}}_j \bar{n}^{(j)}$ gives

$$\sin\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\beta}^{\mathrm{o}}_z \cos\theta - \dot{\beta}^{\mathrm{o}}_i \bar{n}^{(i)} \bar{n}_{(i)}. \tag{7.4.25}$$

After some algebraic manipulations, and recalling that $\sum_i \beta_i = 0$, one is able to rewrite this as

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\sin 2\theta}{4}\left[3\left(\dot{\beta}^{\mathrm{o}}_x + \dot{\beta}^{\mathrm{o}}_y\right) + \cos 2\phi \left(\dot{\beta}^{\mathrm{o}}_x - \dot{\beta}^{\mathrm{o}}_y\right)\right], \tag{7.4.26}$$

which is exactly the result in eq. (34) of ref. [26] (or eq. (7) of ref. [78]) given that $\dot{\beta}^{\mathrm{o}}_i = H_{\mathrm{o}}\Sigma_{\mathrm{o}i}$. Thus, their results are equivalent to $\perp^i_j \dot{\beta}^{\mathrm{o}}_j \bar{n}^{(j)}$, which is only the aberration-like effect of eq. (7.4.18). They do not contain the parallax-like term included in our results. Indeed, it cannot be obtained by deriving the expression of separation of angles with respect to time since it arises from the fact that the end points of the two geodesics are different.

Lastly, we can give some crude estimates of the level to which measurements of the direction drift can constrain the spatial anisotropy of Bianchi I spacetimes (i.e., the shear). For simplicity let us write $\dot{\beta}_{\mathrm{o}}$ as a fraction of H_{o}, i.e., $\dot{\beta}_{\mathrm{o}} = \epsilon H_{\mathrm{o}}$. Then, we can separate the constraints in two types: early and late anisotropies. For the first type, CMB data severely constrains the value of the shear today to no larger than the observed

CMB quadrupole, i.e., $\epsilon < 10^{-5}$. This would lead to an aberration drift of the order of $10^{-4}\,\mu$as/yr or smaller. As has been pointed out, this is three order of magnitudes smaller than the expected peculiar velocities in the standard (FLRW) model, so very unlikely to be detected with future experiments [79]. Late type anisotropies are however more promising. Future surveys of weak-lensing shear such as Euclid could constrain the late anisotropy of the cosmic flow to order $\dot{\beta}_o/H_o = 1\%$ [12]. This would imply a signal in the aberration drift of the order $0.1\,\mu$as/yr per source, or $1.0\,\mu$as over ten years. This is compatible with the figure of 0.4μas in ten years coming from the Local Group proper motion with respect to the CMB frame found in ref. [80]. Indeed, as we found previously, assuming $v_{\rm LG} \sim 620\,$km/s leads to a drift of the order of $0.3\,\mu$as in ten years for the Local Group's proper motion contribution. Moreover, since the shear contribution to the drift is coming from a quadrupole, it can in principle be distinguished from the velocity contribution.

8 Conclusions and perspectives

Our analysis proposes a general method to compute the direction and redshift drifts, making clear the importance of considering two nearby geodesics connecting the observer and the source. Two effects have to be considered, the first related to the integral along the line of sight and the second related to the end points of the geodesics. In a non-static spacetime or for non-inertial observer and sources the lapses connecting the geodesics at the source and at the observer differ.

This general method allowed us to first recover the standard formula for the direction and redshift drifts for a general observer in a FLRW spacetime. The multipolar decomposition shows that for both the redshift and the direction drifts, the observer's velocity only induces a dipole. We were able to separate the contribution of the velocity to the direction drift into two effects, the parallax drift (which is z-dependent) and aberration drift. Comparing the two for various combined motions, we showed that the two do contribute significantly to the total drift. However, the parallax drift is usually considered as a noise to be removed, which can be accomplished using the z-dependence. The drift caused by the motion of the Sun around the galactic center must also be removed if one is interested in the cosmological aberration drift.

We also provided an expression for the direction and redshift drifts in a perturbed FLRW universe for scalar, vector and tensor perturbations. Concerning the scalar part of the redshift drift, we corrected a mistake in the literature. The tensor contribution to both the redshift and direction vector were compared to existing results for a plane (gravitational) wave.

To finish, we provided an expression for the direction and redshift drifts in Bianchi I universes in the "straight geodesic approximation". A multipolar decomposition shows that the shear contributes as a quadrupole, which makes it distinguishable from the effects of the velocity. We also showed that the results published in the literature were actually missing a parallax-like drift contribution. We then estimated how well the shear can be constrained using our results. Future experiments with Gaia satellite could, in principle, detect late time anisotropies according to our results.

One extension of this work is to use the connection between perturbed FLRW in the long wavelenght limit and Bianchi spacetimes. This connection was explored in ref. [23], where Bianchi models with FLRW limit are expressed as linear perturbations over an FLRW background. This allowed the authors to connect the Bianchi models to the usual cosmological perturbation theory in the limit of long wavelenghts. For example, the Bianchi IX model is equivalent to a positively curved FLRW plus a gravitational wave of longest wavelenght [117]. However, it is not yet known precisely how to directly express the homogeneous perturbations in terms of the usual scalar, vector and tensor modes. Given that this relationship is found, one can in principle compare the drifts of the other Bianchi models with FLRW limit (namely, Bianchi VII_0, V, VII_h and IX) to our results for perturbed FLRW.

Another possibility is to use the results on perturbed FLRW to evaluate the power spectrum of the drifts. Possible measurements of the drifts with Gaia satellite would include the time variation of the position of more than 1 billion stars and celestial objects [95], thus requiring these results to be treated statistically. Correlation functions (and hence, the power spectrum) are fundamental tools for doing so. One would need then to use our results to find the correlation function and power spectrum of the drifts. This would allow for a full multipolar and statistical analysis of the scalar, vector and tensor contribution to the drifts.

APPENDIX A – Power spectrum from 2pcf

We would like to show it is possible to obtain the power spectrum from both 2pcf we found for Bianchi I and VII_0, eqs. (3.4.25) and (3.4.35). This can be used, e.g., to build temperature fluctuation maps and further analyze the impact of the anisotropic corrections found here on CMB temperature maps. Let us illustrate this for the case of Bianchi I. If we use the results from Table 2, the 2pcf expanded in terms of spherical harmonics is

$$\xi_I = \xi_{FL}(r_-) + r_- \frac{d\xi_{FL}}{dr_-}\left(C_1 Y_{20}(\hat{\boldsymbol{r}}_-) + C_2\left[Y_{2-2}(\hat{\boldsymbol{r}}_-) + Y_{22}(\hat{\boldsymbol{r}}_-)\right]\right), \tag{A.0.1}$$

where

$$C_1 = \beta_{33}\sqrt{\frac{4\pi}{5}}, \qquad C_2 = (\beta_{11} - \beta_{22})\sqrt{\frac{2\pi}{15}}. \tag{A.0.2}$$

Using the definition of the power spectrum, eq. (3.4.45), we can write $P_I(\boldsymbol{k})$ (the invariant Bianchi I power spectrum) as

$$P_I(\boldsymbol{k}) = P_{FL}(k) + \int d^3\boldsymbol{r}_- e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_-}\left(C_1 Y_{20}(\hat{\boldsymbol{r}}_-) + C_2\left[Y_{2-2}(\hat{\boldsymbol{r}}_-) + Y_{22}(\hat{\boldsymbol{r}}_-)\right]\right) r_- \frac{d\xi_{FL}}{dr_-}, \tag{A.0.3}$$

where clearly $P_{FL}(k) = \int d^3\boldsymbol{r}_- e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_-}\xi_{FL}(r_-)$. We can invert the above to write the 2pcf of flat FLRW as

$$\xi_{FL}(r_-) = \frac{1}{2\pi^2}\int q^2 dq P_{FL}(q)\, j_0(qr_-) \Rightarrow \frac{d\xi_{FL}(r_-)}{dr_-} = -\frac{1}{2\pi^2}\int q^3 dq P_{FL}(q)\, j_1(qr_-). \tag{A.0.4}$$

Using eq. (A.0.4) and the Rayleigh expansion

$$e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_-} = 4\pi\sum_{\ell m} i^{-\ell} j_\ell(kr_-)\, Y_{\ell m}\left(\hat{\boldsymbol{k}}\right) Y^*_{\ell m}(\hat{\boldsymbol{r}}_-) \tag{A.0.5}$$

in eq. (A.0.3) and performing the angular integrals allows us to write the integral in (A.0.3) as

$$\int d^3\boldsymbol{r}_-(\cdots) = \frac{2}{\pi}\int q^3 dq P_{FL}(q)\int r_-^3 j_2(kr_-)\, j_1(qr_-)\, dr_- \times \\ \times\left(C_1 Y_{20}\left(\hat{\boldsymbol{k}}\right) + C_2\left[Y_{22}\left(\hat{\boldsymbol{k}}\right) + Y_{2-2}\left(\hat{\boldsymbol{k}}\right)\right]\right) \tag{A.0.6}$$

The integration over dr_- can be solved using the following relations for spherical Bessel functions:

$$r_- j_2(kr_-) = \frac{3}{k} j_1(kr_-) - r_- j_0(kr_-), \qquad -r_- j_1(qr_-) = \frac{dj_0(qr_-)}{dq}. \tag{A.0.7}$$

Recalling that $\int r^2 j_\ell\left(kr_-\right) j_\ell\left(qr_-\right) dr_- = \frac{\pi}{2kq}\delta(k-q)$, we have that the integration over dr_- is given by

$$\int r_-^3 j_2\left(kr_-\right) j_1\left(qr_-\right) dr_- = \frac{3}{k}\int r_-^2 j_1\left(kr_-\right) j_1\left(qr_-\right) dr_- + \frac{d}{dq}\int r_-^2 j_0\left(kr_-\right) j_0\left(qr_-\right) dr_- \tag{A.0.8}$$

$$= \frac{\pi}{2}\left[\frac{3\delta\left(k-q\right)}{k^2 q} - \frac{\delta\left(k-q\right)}{kq^2} + \frac{\delta'\left(k-q\right)}{kq}\right] \tag{A.0.9}$$

Plugging this result back in the integral (A.0.6) gives

$$\int\left(\cdots\right) d^3\boldsymbol{r}_- = -k\frac{dP_{FL}}{dk}\left(C_1 Y_{20}\left(\hat{\boldsymbol{k}}\right) + C_2\left[Y_{22}\left(\hat{\boldsymbol{k}}\right) + Y_{2-2}\left(\hat{\boldsymbol{k}}\right)\right]\right). \tag{A.0.10}$$

Therefore, the invariant power spectrum for Bianchi I is

$$P_I\left(\boldsymbol{k}\right) = P_{FL}\left(k\right) - k\frac{dP_{FL}}{dk}\left(C_1 Y_{20}\left(\hat{\boldsymbol{k}}\right) + C_2\left[Y_{22}\left(\hat{\boldsymbol{k}}\right) + Y_{2-2}\left(\hat{\boldsymbol{k}}\right)\right]\right). \tag{A.0.11}$$

The same can easily be made for Bianchi VII_0 following the same steps, which yields the result

$$P_{VII_0}(\boldsymbol{k}) = P_{FL}(k) - C_1' Y_{00}\left(\hat{\boldsymbol{k}}\right)\left(3P_{FL}\left(k\right) + k\frac{dP_{FL}}{dk}\right) - C_2' Y_{20}\left(\hat{\boldsymbol{k}}\right) k\frac{dP_{FL}}{dk}, \tag{A.0.12}$$

where here $C_1' = -\frac{\sqrt{4\pi}}{3}\beta_{33}$ and $C_2' = \frac{4}{3}\sqrt{\frac{4\pi}{5}}\beta_{33}$.

APPENDIX B – Relationship between time lapses and redshift

Here we show the validity of the relation

$$\frac{\delta\tau_s}{\delta\tau_{\rm o}} = \frac{\omega_{\rm o}}{\omega_s} = \frac{1}{1+z} \tag{B.0.1}$$

where $\tau_{s,\rm o}$ are the proper times of the source and the observer, respectively. In the geometrical optics approximation (also known as eikonal approximation), the electromagnetic vector potential satisfies (in vacuum) [17]

$$\nabla^\mu \nabla_\mu A_\nu = 0. \tag{B.0.2}$$

Let us consider a solution of the form

$$A_\mu = C_\mu e^{i\varphi} \tag{B.0.3}$$

with approximately constant amplitude. Then, by neglecting derivatives of C_μ, eq. (B.0.3) can be rewritten as

$$\nabla^\mu \varphi \nabla_\mu \varphi = 0 \tag{B.0.4}$$

$$\nabla^\mu \nabla_\mu \varphi = 0. \tag{B.0.5}$$

The vector $k^\mu = \nabla^\mu \varphi$ is normal to the surfaces of constant φ. Differentiating (B.0.4) gives the geodesic equation

$$k^\mu \nabla_\mu k_\nu = 0. \tag{B.0.6}$$

Hence k^μ is also tangent to a null geodesic. Thus, in the eikonal approximation, null geodesics are curves of constant phase φ, since k^μ, being a null vector, is both tangent and normal to the null geodesic it generates.

The frequency of the wave as measured by an observer with 4-velocity u^μ is precisely (minus) the rate of change of the phase of the wave with respect to his proper time. That is,

$$\omega = -\frac{\delta\varphi}{\delta\tau} = -u^\mu \nabla_\mu \varphi = -u^\mu k_\mu. \tag{B.0.7}$$

Thus, for a source and observer connected by a null geodesic, we have that

$$\frac{\omega_s}{\omega_{\rm o}} = \frac{\delta\varphi/\delta\tau_s}{\delta\varphi/\delta\tau_{\rm o}}. \tag{B.0.8}$$

Since null geodesics are curves of constant phase, we thus have that

$$\frac{\delta\tau_s}{\delta\tau_{\rm o}} = \frac{\omega_{\rm o}}{\omega_s} = \frac{1}{1+z}. \tag{B.0.9}$$

For the cases of inertial FLRW and Bianchi I, the observer and source are both comoving, so we had that $\delta\tau_{s,\mathrm{o}} = \delta t_{s,\mathrm{o}}$. For moving observers and source, the proper time and coordinate time coincide to first order in velocity; however, as explained in the text, they differ from the static case, and thus we write $\delta\tau_{s,\mathrm{o}} = \delta\tilde{t}_{s,\mathrm{o}}$. Note that eq. (B.0.9) can be easily converted to conformal time in these cases, giving

$$\frac{\delta\eta_s}{\delta\eta_\mathrm{o}} = \frac{a_s}{a_\mathrm{o}}\frac{1}{1+z}, \tag{B.0.10}$$

which can ease calculations, since in most cases above the redshift is of the form

$$1 + z = \frac{a_s}{a_\mathrm{o}}(\cdots), \tag{B.0.11}$$

thus elimination the scale factors. In perturbed FLRW, the proper time and cosmic time differ, as explained in the text, and this must be taken into account when using eq. (B.0.9). After doing so, one can then write it in terms of conformal time using $\mathrm{d}t = a\mathrm{d}\eta$.

APPENDIX C – Perturbed FLRW formulae

We gather here the basic necessary formulae for perturbed FLRW spacetimes. The metric of a perturbed FLRW spacetime satisfies

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{C.0.1}$$

where $\bar{g}_{\mu\nu}$ is the background (FLRW) metric and $\delta g_{\mu\nu}$ is a perturbation. To first order in perturbations, the inverse of the perturbed metric is given by $g^{\mu\nu} = \bar{g}^{\mu\nu} + \delta g^{\mu\nu}$, where

$$\delta g^{\mu\nu} = -\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}\delta g_{\rho\sigma}, \tag{C.0.2}$$

so that $g_{\mu\rho}g^{\rho\nu} = \delta^{\nu}_{\mu}$. A general decomposition of the metric to linear order is given by

$$\mathrm{d}s^2 = a^2(\eta)[-(1+2A)\mathrm{d}\eta^2 + 2B_i\mathrm{d}x^i\mathrm{d}\eta + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j]. \tag{C.0.3}$$

Both B_i and h_{ij} can be decomposed irreducibly as

$$B^i = \partial^i B + \bar{B}^i, \tag{C.0.4}$$

where $\partial^i \bar{B}_i = 0$, and

$$h_{ij} = 2C\delta_{ij} + 2\partial_i\partial_j E + 2\partial_{(i}E_{j)} + 2E_{ij}, \tag{C.0.5}$$

with $\partial^i E_i = 0$, $\partial_i E^{ij} = 0$ and $E^i_i = 0$. This decomposes the metric in 10 degrees of freedom, 4 of which can be removed through a properly chosen gauge. In the Newtonian gauge, which is the one we adopt, one has that $B = 0 = E$ and $\bar{B}^i = 0$. We then rename the scalar perturbations as $A = \Phi$ and $C = -\Psi$ to arrive at

$$\mathrm{d}s^2 = a^2\left[-(1+2\Phi)\,\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij})\,\mathrm{d}x^i\mathrm{d}x^j\right], \tag{C.0.6}$$

with

$$h_{ij} = -2\Psi\delta_{ij} + 2\partial_{(i}E_{j)} + 2E_{ij}\,. \tag{C.0.7}$$

In the text, we make use of the conformal metric $\hat{g}_{\mu\nu}$ defined through $g_{\mu\nu} = a^2\hat{g}_{\mu\nu}$, i.e.,

$$\mathrm{d}\hat{s}^2 = \left[-(1+2\Phi)\,\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij})\,\mathrm{d}x^i\mathrm{d}x^j\right], \tag{C.0.8}$$

where $\mathrm{d}\hat{s}^2$ represents the metric of $\hat{g}_{\mu\nu}$. To first order, the Christoffel symbols for the conformal metric are

$$\Gamma^0_{00} = \Phi', \tag{C.0.9}$$

$$\Gamma^i_{00} = \partial^i\Phi, \tag{C.0.10}$$

$$\Gamma^0_{0i} = \partial_i\Phi, \tag{C.0.11}$$

$$\Gamma^0_{ij} = -2\Phi\delta_{ij} + h_{ij} + \frac{1}{2}h'_{ij}, \tag{C.0.12}$$

$$\Gamma^i_{0j} = \frac{1}{2}h^{i\prime}_{j}, \tag{C.0.13}$$

$$\Gamma^i_{jk} = \partial_{(j}h^i_{k)} - \frac{1}{2}\partial^i h_{jk}. \tag{C.0.14}$$

For more information on perturbed FLRW spacetimes, see ref. [57].

APPENDIX D – Direction drift calculations in perturbed FLRW

We provide here the steps for the calculation of the direction drift in perturbed FLRW. We start from the geodesic path, eq. (7.3.31):

$$\begin{aligned} x_s^i - x_\mathrm{o}^i =& (\eta_s - \eta_\mathrm{o}) n_\mathrm{o}^{(i)} + \bar{n}_\mathrm{o}^{(i)} \int_{\eta_\mathrm{o}}^{\eta_s} \Theta \, \mathrm{d}\eta - \perp_j^i \int_{\eta_\mathrm{o}}^{\eta_s} (\eta_s - \eta) \partial^j \Theta \, \mathrm{d}\eta + (\eta_s - \eta_\mathrm{o}) \perp_j^i \bar{n}_\mathrm{o}^{(k)} E_{\mathrm{o}k}^j \\ & - \perp_j^i \bar{n}_\mathrm{o}^{(k)} \int_{\eta_\mathrm{o}}^{\eta_s} E_k^j \, \mathrm{d}\eta - \bar{n}_\mathrm{o}^{(j)} \int_{\eta_\mathrm{o}}^{\eta_s} E_j^i \, \mathrm{d}\eta + \perp_j^i \int_{\eta_\mathrm{o}}^{\eta_s} (\eta_s - \eta) \bar{n}_\mathrm{o}^{(k)} \bar{n}_\mathrm{o}^{(l)} \partial^j E_{kl} \, \mathrm{d}\eta, \end{aligned} \tag{D.0.1}$$

Let us illustrate by calculating for scalar modes; the calculation for tensor modes follows the same steps. For this case, the second geodesic is given by

$$\begin{aligned} x_s^i - x_\mathrm{o}^i =& (\eta_s + \delta\eta_s - \eta_\mathrm{o} - \delta\eta_\mathrm{o})(n_\mathrm{o}^{(i)} + \delta_{12} n_\mathrm{o}^{(i)}) + \bar{n}_\mathrm{o}^i \int_{\eta_\mathrm{o}+\delta\eta_\mathrm{o}}^{\eta_s+\delta\eta_s} \Theta[\eta, x_2^i(\eta)] \mathrm{d}\eta \\ & - \perp_j^i \int_{\eta_\mathrm{o}+\delta\eta_\mathrm{o}}^{\eta_s+\delta\eta_s} (\eta_s + \delta\eta_s - \eta) \partial^j \Theta[\eta, x_2^i(\eta)] \mathrm{d}\eta. \end{aligned} \tag{D.0.2}$$

The first integral in the above can be solved using the result from eq. (7.3.22), which gives

$$\int_{\eta_\mathrm{o}+\delta\eta_\mathrm{o}}^{\eta_s+\delta\eta_s} \Theta[\eta, x_2^i(\eta)] \mathrm{d}\eta = \int_{\eta_\mathrm{o}}^{\eta_s} \Theta[\eta, x_1^i(\eta)] \mathrm{d}\eta + \delta\eta_\mathrm{o} \int_{\eta_\mathrm{o}}^{\eta_s} \Theta'[\eta, x_1^i] \mathrm{d}\eta. \tag{D.0.3}$$

For the second integral, after some algebra and also using the result from eq. (7.3.22), we have

$$\begin{aligned} \int_{\eta_\mathrm{o}+\delta\eta_\mathrm{o}}^{\eta_s+\delta\eta_s} & (\eta_s + \delta\eta_s - \eta) \partial^j \Theta[\eta, x_2^i(\eta)] \mathrm{d}\eta = \int_{\eta_\mathrm{o}}^{\eta_s} (\eta_s - \eta) \partial^j \Theta[\eta, x_1^i(\eta)] \mathrm{d}\eta \\ & + \delta\eta_s \int_{\eta_\mathrm{o}}^{\eta_s} \partial^j \Theta[\eta, x_1^i(\eta)] \mathrm{d}\eta + \delta\eta_\mathrm{o} \int_{\eta_\mathrm{o}}^{\eta_s} (-1) \partial^j \Theta[\eta, x_1^i(\eta)] \mathrm{d}\eta + \delta\eta_\mathrm{o} \int_{\eta_\mathrm{o}}^{\eta_s} (\eta_s - \eta) \partial^j \Theta'[\eta, x_1^i(\eta)] \mathrm{d}\eta, \end{aligned} \tag{D.0.4}$$

which, when recalling that at zeroth order $\delta\eta_s = \delta\eta_\mathrm{o}$, reduces simply to

$$\begin{aligned} \int_{\eta_\mathrm{o}+\delta\eta_\mathrm{o}}^{\eta_s+\delta\eta_s} (\eta_s + \delta\eta_s - \eta) \partial^j \Theta[\eta, x_2^i(\eta)] \mathrm{d}\eta =& \int_{\eta_\mathrm{o}}^{\eta_s} (\eta_s - \eta) \partial^j \Theta[\eta, x_1^i(\eta)] \mathrm{d}\eta \\ & + \delta\eta_\mathrm{o} \int_{\eta_\mathrm{o}}^{\eta_s} (\eta_s - \eta) \partial^j \Theta'[\eta, x_1^i(\eta)] \mathrm{d}\eta, \end{aligned} \tag{D.0.5}$$

Plugging this back in the second geodesic and subtracting the first gives

$$0 = (\eta_s - \eta_\mathrm{o}) \delta_{12} n_\mathrm{o}^{(i)} + (\delta\eta_s - \delta\eta_\mathrm{o}) \bar{n}_\mathrm{o}^{(i)} + \delta\eta_\mathrm{o} \bar{n}_\mathrm{o}^{(i)} \int_{\eta_\mathrm{o}}^{\eta_s} \Theta' \mathrm{d}\eta + \delta\eta_\mathrm{o} \perp_j^i \int_{\eta_\mathrm{o}}^{\eta_s} (\eta_s - \eta) \partial^j \Theta' \mathrm{d}\eta. \tag{D.0.6}$$

Now, one must find the relationship between η_s and $\eta_{\rm o}$ to first order on perturbations. From eq. (B.0.9), and recalling that $\delta\tau = a(1+\Phi)\delta\eta$, we have

$$\frac{(1+\Phi_s)a_s\delta\eta_s}{(1+\Phi_{\rm o})a_{\rm o}\delta\eta_{\rm o}} = \frac{1}{1+z}. \tag{D.0.7}$$

Now, using eq. (7.3.14a), expanding to first order gives

$$\frac{\delta\eta_s}{\delta\eta_{\rm o}} = 1 - \int_{\eta_{\rm o}}^{\eta_s} \Theta' \mathrm{d}\eta. \tag{D.0.8}$$

Plugging this back into (D.0.6), after some simple algebra, gives the final result

$$\frac{\delta_{12} n_{\rm o}^{(i)}{}^{(S)}}{\delta\tau_{\rm o}} = -\frac{\perp^i_j}{a_{\rm o}} \int_{\eta_{\rm o}}^{\eta_s} \frac{(\eta_s - \eta)}{\chi_{s{\rm o}}} \partial^j \Theta' \mathrm{d}\eta \tag{D.0.9}$$

where $\chi_{s{\rm o}} = \eta_{\rm o} - \eta_s$. For the tensor modes, the steps to evaluate each term are similar. The only difference is in the relationship between the conformal and proper times, which does not get a tensor contribution to first order in perturbation, giving, in this case, simply

$$\frac{a_s\delta\eta_s}{a_{\rm o}\delta\eta_{\rm o}} = \frac{1}{1+z}. \tag{D.0.10}$$

With this in mind, following the same steps as above will then lead to the result of eq. (7.3.32b).

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